Generalized Momentum Operator

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ABSTRACT

In this research, we propose a generalized momentum operator upon imposing the spatial expansion provided that the EUP relation is confirmed. According to the EUP algebra, we use $A(x)$ to be an auxiliary function (or $1+\mu(x)$) in the construction of our new momentum. Thus, we provide a formalism containing the auxiliary function in the real or complex domain upon which one has the freedom to define a hermitian or non-hermitian momentum operator. We start with introducing a generalized Lagrangian density affected by the proposed formalism. Our investigation is continued on the motion of a quantum particle under the variation of such Lagrangian density. Besides, we demonstrate the $\mathscr{P}\mathscr{T}$-symmetry field theory including the $\Psi(x,t)$, $\Psi^{*}(x,t)$ or $\Psi^{\mathscr{P}\mathscr{T}}(x,t)$ in the structure of the extended Lagrangian density. Having applied the principle of least action, the Euler-Lagrange equations lead to the corresponding Schrödinger equations. Upon finding the generalized Lagrangian density, we obtain the Hamiltonian density, momentum density and energy flux which are known to be the elements of the stress-energy tensor. The expectation value of the generalized Hamiltonian density for the hermitian and $\mathscr{P}\mathscr{T}$-symmetric fields are determined where it leads to the energy of the system. Thereupon, we extend the probability and particle current densities which significantly satisfy the continuity equation. Next, we develop further the concept of generalized momentum operator and elucidate the significance of our proposal considering once the real definition and once the $\mathscr{P}\mathscr{T}$-symmetric structure. With solving the eigen-value problems for the two combinations, we represent the eigen-values and eigen-functions of some examples of the generalized momentum operator. In accordance with the generalized Schrödinger equation, the kinetic energy operator is rebuilt and consequently the Hamiltonian operator is identified based on the imposed potential energy and the new kinetic energy. The corresponding differential equations are declared and the exact solutions are computed using the variable transformation method. Accordingly, we transform the extended Schrödinger equation from x-space into the target space, here z-space, with the manner whose energy spectrum remains identical. Later, we demonstrate an illustrative examples based on the idea of a step momentum operator which is flexible to have a hermitian or the $\mathscr{PT}$-symmetric Hamiltonian operator. We employ the formalism such that expresses a sudden change in the momentum of a system at a specific point. To show that, we use a step auxiliary function and develop the Schrödinger equation considering a quantum particle inside a square well. The outcomes yield infinite bound states with real energy spectrum for the particles with hermitian step momentum, it is finite for a particle with $\mathscr{PT}$-symmetric momentum. Afterwards, we study the $\mathscr{PT}$-symmetric Hamiltonian in two dimensions. Having considered standard kinetic energy, a two dimensional complex harmonic oscillator potential is introduced which is invariant under the parity and time reversal operator. The Schrödinger equation yields real eigen-values with complex eigen-functions. We also construct the coherent state of the system by using a superposition of 12 eigen-functions. Utilizing the complex correspondence principle for the probability density, we investigate the possible modifications in the probability densities due to the non-hermitian aspect of the Hamiltonian.

Keywords: Exact-Solution, Non-Hermitain Quantum Physics, Generalized Lagrangian, $\mathscr{PT}$-Symmetry, Non-Hermitian Field Theory.

ÖZ

Bu araştırmada, EUP ilişkisinin doğrulanması koşuluyla, mekansal genişlemeyi dayatarak genelleştirilmiş bir momentum operatörü öneriyoruz. EUP cebrine göre, yeni momentumumuzun inşasında yardımcı fonksiyon (veya $1+\mu(x)$) olarak $A(x)$ kullanıyoruz. Böylece, bir hermityen veya hermityen olmayan momentum operatörü tanımlama özgürlüğüne sahip olan gerçek veya karmaşık alandaki yardımcı fonksiyonu içeren bir formalizm sağlıyoruz. Önerilen formalizmden etkilenen genelleştirilmiş bir Lagrange yoğunluğu tanıtarak başlıyoruz.Araştırmamız, böyle bir Lagrange yoğunluğunun değişimi altında bir kuantum parçacığının hareketi üzerinde devam ediyor.Ayrıca, aşağıdakileri içeren $\mathscr{PT}$-simetri alan teorisini gösteriyoruz: $\Psi(x,t)$, $\Psi^*(x,t)$ or $\Psi^{\mathscr{PT}}(x,t)$ genişletilmiş Lagrange yoğunluğunun yapısında. En az etki ilkesini uygulayan Euler-Lagrange denklemleri, karşılık gelen Schrödinger denklemlerine yol açar. Genelleştirilmiş Lagrange yoğunluğunu bulduktan sonra, gerilim-enerji tensörünün elemanları olarak bilinen Hamilton yoğunluğu, momentum yoğunluğu ve enerji akışını elde ederiz. Hermitian ve $\mathscr{PT}$-simetrik alanlar için genelleştirilmiş Hamiltonian yoğunluğunun beklenen değeri, sistemin enerjisine götürdüğü yerde belirlenir. Bunun üzerine, süreklilik denklemini önemli ölçüde sağlayan olasılık ve parçacık akım yoğunluklarını genişletiyoruz. Daha sonra, genelleştirilmiş momentum operatörü kavramını daha da geliştireceğiz ve bir kez gerçek tanım ve bir kez de $\mathscr{PT}$-simetrik yapıyı göz önünde bulundurarak teklifimizin önemini açıklayacağız. İki kombinasyon için öz değer problemlerini çözerek, genelleştirilmiş momentum operatörünün bazı örneklerinin öz değerlerini ve öz fonksiyonlarını temsil ediyoruz. Genelleştirilmiş Schrödinger denklemine göre, kinetik enerji operatörü yeniden oluşturup, uygulanan potansiyel enerjiye ve yeni kinetik enerjiye dayalı olarak Hamilton operatörü tanımlıyoruz. Karşılık gelen diferansiyel denklemler bildirilir ve kesin çözümler değişken dönüştürme yöntemi kullanılarak hesaplanır. Buna göre, genişletilmiş Schrödinger denklemini x-space'den hedef uzaya, burada z-space'e, enerji spektrumu aynı kalacak şekilde dönüştürüyoruz. Daha sonra, bir hermitiyen veya $\mathscr{PT}$-simetrik Hamilton operatörüne sahip olmak için esnek olan bir adım momentum operatörü fikrine dayanan açıklayıcı bir örnek göstereceğiz. Belirli bir noktada sistemin momentumundaki ani bir değişikliği ifade eden formalizmi kullanacağız. Bunu göstermek için, bir adım yardımcı fonksiyonu kullanıyoruz ve kare bir kuyu içindeki bir kuantum parçacığını dikkate alarak Schrödinger denklemini geliştiriyoruz. Sonuçlar, hermit adım momentumlu parçacıklar için gerçek enerji spektrumlu sonsuz bağlı durumlar verir,$\mathscr{PT}$-simetrik momentumlu bir parçacık için sonludur. Daha sonra, $\mathscr{PT}$-simetrik Hamiltoniyeni iki boyutta inceleyeceğiz. Standart kinetik enerji göz önüne alındığında, parite ve zaman ters çevirme operatörü altında değişmez olan iki boyutlu karmaşık harmonik osilatör potansiyelini tanıtacağız. Schrödinger denklemi, karmaşık öz fonksiyonlara sahip gerçek öz değerleri verir. Ayrıca 12 öz fonksiyonun üst üste binmesini kullanarak sistemin tutarlı durumunu da oluşturacağız. Yoğunluk olasılığı için karmaşık yazışma ilkesini kullanıp Hamiltonian'ın hermityen olmayan yönü nedeniyle olasılık yoğunluklarındaki olası değişiklikleri araştıracağız.

Anahtar Kelimeler: Kesin Çözüm, Hermityen Olmayan Kuantum Fiziği, Genelleştirilmiş Lagrange, $\mathscr{PT}$-Simetrik, Hermityen Olmayan Alan Teorisi.

Dedicated to my beloved parents

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Π	Alternative Parity Operator
Θ	Metric Operator
Ω	Hermitian, Invertible and Linear Operator in the Reference Hilbert Space
ρ	Probability Density
$\hbar$	Planck Constant
$\mathscr{C}$	Charge Operator
$\mathscr{H}$	Reference Hilbert Space
$\mathfrak{h}$	Hermitized Hamiltonian
$\mathscr{L}$	Target Hilbert Space
$\mathscr{P}$	Parity Operator
$\mathscr{S}$	Energy Flux Density
$\mathscr{T}$	Time reversal Operator
$A(x)$	Auxiliary Function
c	Speed of Light in a Vacuum Inertial Frame
j	Particle Current Density
m	Mass of a Quantum Particle
P	Momentum Density
EUP	Extended Uncertainty Principle
GMO	Generalized Momentum Operator
GUP	Generalized Uncertainty Principle
PDM	Position Dependent Mass

Chapter 1

INTRODUCTION

In classical mechanics books, the meaning of the momentum is significantly defined as a mass in motion and derived by multiplying mass and velocity. The mass and velocity manifestly are discussed as distinctive physical quantities of an object [1]. In quantum mechanics, observables are identified in the structure of operators. Momentum is an operator and conceptually signified based on the classical implication. In this approach, it has been expressed that the momentum is generated by the translation operator and corresponded to the geometry of the space [2]. Our proposal over the significance of momentum operator concerns its generalization upon breaking the Lorentz symmetry under the context of extended Heisenberg uncertainty principle (EUP) [3,4]. Besides, the proposed momentum may broaden the boundary of quantum mechanics beyond the real domain discussing non-hermitian systems. A limited form of such a generalization has been already considered in Ref. [5], where its non-hermitian version has been discussed in Ref. [6].

1.1 Heisenberg Uncertainty Principle

The measurement in quantum theory reveals uncertainty for incompatible physical quantities such as position and momentum or time and energy. To measure simultaneously the position and momentum, where the HUP states

$$\Delta x \Delta p \geq \frac{1}{2} \left| \langle [\hat{x}, \hat{p}] \rangle \right| , \tag{1.1}$$

the more precisely one measures the position of a particle the less information in momentum one gains [7, 8]. In order to modify the HUP, the minimum length is proposed in quantum gravity [9], string theory [10] and non-commutative space-time due to quantum field in a quantized space-time [11, 12]. Correspondingly, the generalization of the uncertainty principle for minimum position and momentum has been presented in the form of the extended and generalized uncertainty principle (EGUP) [7, 13] given by

$$\Delta\hat{x}\Delta\hat{p} \geq \frac{1}{2}\left(1+\alpha\Delta x^2+\beta\Delta\hat{p}^2\right). \tag{1.2}$$

The implementation of quantum mechanics in gravity limits the measurement by a minimal length in which β is related to the Planck's length [13–17] in accordance with the generalized uncertainty principle (GUP). In the identical manner, the extended uncertainty principle (EUP) discusses the cutoff in the minimum momentum in the (anti) de sitter space-time with the term α representing the radius of the curved space-time [18]. In Ref. [19, 20], upon using an EUP on the primordial perturbation spectrum during the inflation due to quantum fluctuations is investigated. The extended form of the uncertainty principle is obtained on the (anti) de sitter space-time considering the deformation of the space-time in Ref. [3]. The authors in the latter research also by taking the gravitational interactions into account and ignoring the curved space-time have found the corrections to the temperature of black holes. In Ref. [21], the corrections of energy spectra for harmonic oscillator and Hydrogen atom is acquired inserting the extended form of the momentum operator in the de sitter background. Again in Ref. [22], the quantum theory is investigated for the (anti) de sitter space-time in which the special case of EUP is applied. Accordingly, the deformed exponential and trigonometric functions are attained using the deformed-derivative and deformed-integral. Furthermore, in Ref. [23] the classical approach to the (anti) de sitter background, having deformed-mechanics, is comprehensively discussed in one and D-dimensions. The study on the corrections to the Bekenstein-Hawking entropy in the Schwarzschild black holes has been done in Ref. [24] for the EUP and GUP cases with considerable outcomes. Interestingly, in Ref. [25], the exact solutions of the Klein Gorden and Dirac equations in the deformed space for one dimensional harmonic oscillator are obtained. Additionally, the impact of EUP on thermodynamics of the system at high-temperature has been determined. In Ref. [26], in the context of EUP, the exact solutions of some D-dimensional potentials including an infinite box, a harmonic and pseudo-harmonic oscillators have been obtained. In this respect, the minimum momentum uncertainty imposed by the corresponding EUP, due to the (anti) de sitter space-time, has been found. The Ramsauer Townsend effect has been studied in the curved space-time with corresponding EUP correction in Ref. [27]. In the latter work, authors considered the WKB approximation for one dimensional Schrödinger equation upon utilizing various potentials. In Ref. [28], the effects of EUP and GUP on the entropy-area relation on the apparent horizon of the FRW have been studied. Costa Filho, et al. in Ref. [29] showed that the translation operator relates the metric of space-time to the generalized momentum. This, ultimately, gives the deformed Hamiltonian as a consequence of curved space-time. Upon considering the EUP corrections very recently Hamil in Ref. [30] obtained the one dimensional harmonic oscillation AdS and dS background. Recently, Costa Filho et al. in Ref. [31] have found connections between the quantum harmonic oscillator in a deform space - due to its EUP corrections - and the Morse potential in the regular space.

1.2 Non-Hermitian Quantum Mechanics

Fundamental cornerstones in quantum theory are based on two important axioms; i) Operators are hermitian, for instance a hermitian Hamiltonian $\hat{H} = \hat{H}^{\dagger}$ demonstrates an isolated or closed system which is in equilibrium and assures the reality of the energy spectrum. ii) The conservation of the probability density, which implies that the time evolution of states of a system is unitary. Violation of either of the above principles invalidates the traditional quantum physics. The thought of the non-hermitian Hamiltonian was discussed initially for a one-dimensional harmonic-oscillator-like model in Ref. [32–34]. Yet, Bender and Boettcher have initiated $\mathscr{PT}$-symmetric quantum physics with complex extension of a set of one-dimensional harmonic oscillator potentials as a toy model in Ref. [35] and later in Ref. [36].

1.2.1 $\mathscr{PT}$-Symmetric Quantum Mechanics

$\mathscr{PT}$-symmetry is a less general form of non-hermitian quantum physics which is defined based on parity and time reversal operators. Parity is the space reflector operator with linearity property changes the sign of the coordinate and momentum operators $\hat{x}$ and $\hat{p}$, respectively. But, the anti-linear time operator merely reverses the sign of the imaginary part. In short one writes

$$\mathscr{P}\hat{x}\mathscr{P} = -\hat{x},\ \mathscr{P}\hat{p}\mathscr{P} = -\hat{p}, \tag{1.3}$$

and

$$\mathscr{T}i\mathscr{T} = -i,\ \mathscr{T}\hat{p}\mathscr{T} = -\hat{p} \tag{1.4}$$

in which $\mathscr{P}$ and $\mathscr{T}$ are the parity and time symmetry operators, respectively. In the non-hermitian quantum theory, it is necessary for the Hamiltonian to be *invariant* under the $\mathscr{P}$arity and $\mathscr{T}$ime transformation which is called $\mathscr{PT}$-symmetry. The $\mathscr{PT}$-symmetric Hamiltonian is defined as

$$\begin{aligned}\mathscr{PT}\hat{H}(\mathscr{PT})^{-1} &= \mathscr{PT}\hat{H}\mathscr{TP} \\ &= \hat{H}.\end{aligned} \tag{1.5}$$

Or briefly

$$\hat{H} = \hat{H}^{\mathscr{PT}} \tag{1.6}$$

Bender suggested a family of Hamiltonians represented by

$$\hat{H} = \hat{p}^2 + \hat{x}^2 (i\hat{x})^{\varepsilon} \tag{1.7}$$

where the term ε is real. Having approved Eq. (1.6), the introduced Hamiltonians are $\mathscr{PT}$-symmetric and for $\varepsilon \geq 0$ the energy spectrum is obtained real. On the other hand, it is numerically computed for $\varepsilon < 0$ such that the obtained eigen-values are complex [35–37]. Hence, the energy spectrum is found to be real as long as the $\mathscr{PT}$-symmetry is unbroken and it is complex for the cases with broken $\mathscr{PT}$-symmetry [38–40]. As we mentioned before, there exists one more condition for any feasible quantum theory to be satisfied which states that the norm of the wave functions must be positive and invariant in time [41]. In non-hermitian quantum theory, however, the associated probability density may attain complex values. With this contradiction, three conditions for the relevant probability density in the complex plane for a particle with harmonic potential were imposed [42]: i) the infinitesimal measurement of the imaginary part of the probability should be zero i.e., $\Im(\rho(z)\,dz) = 0$, ii) the real part of the probability must be positive i.e., $\Re(\rho(z)\,dz) \geq 0$, and finally iii) its integral over the whole space must be one i.e., $\int_C \rho(z)\,dz = 1$. A $\mathscr{PT}$-symmetric Hamiltonian which satisfies (1.6) describes a combination of two open systems, dynamically in equilibrium upon holding gain and loss in balanced [43]. In such a balanced system the concept of probability density safely is conserved since the incoming and outgoing flux of probability density are accordingly conformable. Having considered the Hamiltonian operator

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(x), \tag{1.8}$$

$\hat{H}$ represents the energy of a system. To recognise if the Hamiltonian is $\mathscr{PT}$-symmetric, one has to see the potential, $V(x)$, is parity and time invariant ($\mathscr{PT}$-symmetric). In this respect, the potential approves the parity and time reversal symmetries provided that $\mathscr{PT}V(x) \equiv V^{\star}(-x) = V(x)$ [44]. It is worth to note that the canonical momentum operator i.e., $\hat{p} = -i\hbar\frac{d}{dx}$ is not only hermitian but also a $\mathscr{PT}$-symmetric operator. However, the kinetic energy operator is obviously a hermitian operator. There exist several remarkable studies in the sense of the non-hermitian Hamiltonian concerning $\mathscr{PT}$-symmetric potential in the literature [45–57]. Furthermore, Znojil in [58], and relevantly in [59] studied real energy spectra for $\mathscr{PT}$-symmetric Hamiltonians and their generalizations as nonlinear Hamiltonians. Moreover, the conservation of pseudo-norms is expressed in $\mathscr{PT}$-symmetry with the spontaneous breaking symmetry. In this respect, a complex Lie algebra to examine non-hermitian systems have been studied in Ref. [60, 61]. More importantly, Bagchi et. al. in Ref. [62] have modified the continuity equation and found the normalization of $\mathscr{PT}$-symmetric quantum mechanics. Besides, the $\mathscr{PT}$-asymmetry has been reconstructed upon consideration of charge $\mathscr{C}$ operator in $\mathscr{CPT}$-symmetry model in Ref. [63]. Furthermore, in [64], it is demonstrated that in the polar coordinate system a $\mathscr{PT}$-symmetry can be considered as a combination of a hermitian and a non-hermitian ($\mathscr{PT}$-symmetric) Hamiltonian.

1.2.2 Amended Hilbert Space

According to the conjectures of $\mathscr{PT}$-symmetric quantum physics in Ref. [35], the mathematical modification has been carried out upon a series of studies. Particularly, Mostafazadeh has given a wider view of non-hermitian quantum theory entitled as pseudo-hermitian [65–73] where it was shown that, being $\mathscr{PT}$-symmetry for a Hamiltonian is neither a sufficient nor necessary condition to have the spectrum real. Instead, the non-hermitian Hamiltonians which admit real spectrum are called pseudo-hermitian. A pseudo-hermitian Hamiltonian is defined as

$$H^{\dagger}\Theta = \Theta H \tag{1.9}$$

in which $\Theta = \Omega^{\dagger}\Omega$, $\Omega : \mathscr{H} \to \mathscr{H}$ is a hermitian invertible linear operator and $\mathscr{H}$ is a complex Hilbert space with the inner product defined by

$$\left(\psi_i, \psi_j\right)_{\mathscr{H}} = \langle \psi_i | \Theta \left| \psi_j \right\rangle. \tag{1.10}$$

Having $\Theta = \Omega^{\dagger}\Omega$ suggests that $\Omega \left| \psi_j \right\rangle = \left| \psi_j \succ \right.$ is a map from the reference Hilbert space $\mathscr{H}$ into a target Hilbert space $\mathscr{L}$ where the inner products yield by

$$\left(\psi_i, \psi_j\right)_{\mathscr{H}} = \left(\psi_i, \psi_j\right)_{\mathscr{L}} = \prec \psi_i | \psi_j \succ. \tag{1.11}$$

Subsequently, the pseudo-hermitian Hamiltonian $\hat{H}$, originally defined in the Hilbert space $\mathscr{H}$, is transferred to a hermitian Hamiltonian $\hat{\mathfrak{h}}$ in the Hilbert space $\mathscr{L}$, such that

$$\hat{\mathfrak{h}} = \Omega \hat{H} \Omega^{-1}. \tag{1.12}$$

Let's add that $\hat{\mathfrak{h}}$ is hermitian in the Hilbert space $\mathscr{L}$ due to

$$\begin{aligned} \hat{\mathfrak{h}}^{\dagger} &= \left(\Omega \hat{H} \Omega^{-1}\right)^{\dagger} \\ &= \left(\Omega^{-1}\right)^{\dagger} \left(\Omega \hat{H}\right)^{\dagger} \\ &= \left(\Omega^{-1}\right)^{\dagger} \hat{H}^{\dagger} \Omega^{\dagger} \\ &= \left(\Omega^{-1}\right)^{\dagger} \Omega^{\dagger} \Omega \hat{H} \Omega^{-1} \\ &= \Omega \hat{H} \Omega^{-1} = \hat{\mathfrak{h}}. \end{aligned} \tag{1.13}$$

Accordingly, Znojil has explained the time dependent non-hermitian operators which are hermitized defining an inner product mapping to the invented physical Hilbert space [74] and the need for defining the third Hilbert space has been clarified spaces in distinct use in [75–80] while the mathematical concept and applications of the non-self-adjoint operators have been expanded in [77].

1.2.3 $\mathscr{P}\mathscr{T}$-Symmetry Field Theory

In non-relativistic quantum physics, we deal with the Schrödinger equation who corresponds to the equation of motion of a quantum particle. The alternative hatch finding the latter equation is the principle of the least action. The principle says that for an infinitesimal variation in a given end-fixed path, the action is an extremum. Having obtained the functional derivation of the action zero, one calculates the Lagrangian, and correspondingly the Hamiltonian, of an object. Besides, the conservation of the laws of motion in a physical system can be found upon searching a symmetry in the Lagrangian, had been discovered by Noether [81]. For further investigations on the motion of a quantum particle experiencing the introduced momentum, we express the calculation of the equation of motion upon the Lagrangian dynamics in the context of quantum field theory. However, it is important to notice that the corresponding Lagrangian is expressed in the complex domain, provided that the momentum is non-hermitian - subjected to the $\mathscr{P}\mathscr{T}$-symmetric. Consequently, the approach of field theory is argued under the discourse of $\mathscr{P}\mathscr{T}$-symmetric quantum physics. Bender et. al. in the field approach have discussed the idea of broken symmetry of a separated parity and time reversal operators, although the field is $\mathscr{P}\mathscr{T}$-symmetric, in Ref. [82]. Over the latter study, the idea has been expanded in a two-dimensional supersymmetric quantum field theory according to a defined potential $-ig\,(i\phi)^{1+\delta}$ for $\delta > 0$. Besides, in Ref. [83], the authors have applied the technique of truncating the Schwinger-Dyson equations in a set of fields with the non-hermitian Hamiltonian families, $\frac{g}{N}\,(i\phi)^{N}$, provided that $N \geq 2$. In the latter paper, the corresponding solution and renormalization have been obtained and the properties of the scalar quantum field $-g\phi^4$ in four-dimensional space-time discussed. Later on, in Ref. [62], Bagchi et. al. have argued meticulously the $\mathscr{P}\mathscr{T}$-symmetric field theory upon the variation of Lagrangian to find the generalized continuity equation, using the

Schrödinger equation and its $\mathscr{PT}$-symmetric conjugate. Furthermore, the modified normalization constant has been obtained on the real x-axis. A significant study has been carried out in Ref. [84] in which presents a perturbative method to find the $\mathscr{C}$ operator in quantum mechanics, included systems with higher degrees of freedom, particularly in quantum field theory. The same authors in a short letter in Ref. [85] have presented the successful physical confirmation of $\mathscr{PT}$-symmetric quantum field theory corresponding to the field $i\phi^3$. Although in Ref. [85], the need of redefining the inner product in Hilbert space has been denoted using the perturbation method to build $\mathscr{C}$ operator. The Hamiltonian of a free fermionic field theory in Ref. [86] has been investigated and shown that its $\mathscr{PT}$ invariance depends on the corresponding mass term. According to a study in Ref. [86], it has been confirmed that the $\mathscr{PT}$-symmetric massive Thirring and the scalar sine-Gordon models are dual to each other and equivalent to their hermitian version. Regarding the importance of field ϕ^3 in the context of $\mathscr{PT}$-symmetric quantum field theory, in Ref. [87] the authors have compared the renormalization-group properties of hermitian field $g\phi^3$ and $\mathscr{PT}$-symmetric $ig\phi^3$ field theories. Following the prior study, in Ref. [88], the critical behavior of $\mathscr{PT}$-symmetric $i\phi^3$ quantum field theory has been studied in 6-ε dimensions around the exceptional points. Furthermore, the calculation on the critical exponent has been carried out employing the mean-field approximation and the renormalization-group technique. In Ref. [89], having considered $\mathscr{PT}$-symmetric quantum theory, the logarithmic timelike Liouville Lagrangian has been discussed. Accordingly, the authors have found the energy of a quantum mechanical system assuming the semi-classical limit. Recently, Alexandre et. al. in a remarkable study in Ref. [90] have argued Noether theorem considering complex scalar and fermionic $\mathscr{PT}$-symmetric field theories for hermitian and anti-hermitian mass using the variation of Lagrangian. Very recently, Mazharimousavi rigorously in Ref. [91] has declared that one field is sufficient to find a nonlinear or generalized Schrödinger equation related to the position dependent mass (PDM). Besides, in Ref. [91], the author has obtained the corresponding probability and particle current densities applying one field Ψ and its complex or $\mathscr{PT}$-symmetry conjugates. Later, we bring a different sight of a non-hermitian system with a complex potential in two dimensions where upon the concept of coherent state, the superposition of several given eigen-functions are obtained.

1.3 Coherent State

Coherent state is mostly a subject of interest in quantum optics and is used to express the superposition of a certain number of states in a bounded quantum mechanical system. In fact, the average of energy of several eigenstates in a quantum mechanical system represents the energy of the correspondence classical model. Besides, the expectation value of position and momentum of coherent states in quantum approach demonstrate the classical behavior of the particle. The coherent state for a harmonic oscillator was denoted first by Schrödinger in quantum mechanics where he was investigating the correspondence principle. Schrödinger described a coherent state as the summation of several states underlying the annihilation operator exertion. In Dirac notation, for a real harmonic oscillator a coherent state may be shown as $|\beta\rangle = |\beta|e^{i\theta}$ where $|\beta|$ denotes the amplitude and θ is the phase of $|\beta\rangle$ such that

$$\hat{a}\,|\beta\rangle = \beta\,|\beta\rangle\,, \langle\beta|\beta\rangle = 1, \tag{1.14}$$

in which $\hat{a}$ is the annihilation non-hermitian operator and β is its complex eigenvalue. In terms of the energy eigenkets of the harmonic oscillator (say $|n\rangle$) , the representation of the coherent state is found to be

$$|\beta\rangle = e^{-\frac{|\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{\hat{a}^n}{\sqrt{n!}}\,|n\rangle\,. \tag{1.15}$$

We note that, two different coherent states are not orthogonal, i.e., $\langle\beta|\beta'\rangle \neq 0$. The classical aspect of the harmonic oscillator is used to analogize coherent states of two dimensional harmonic oscillator in vortex structure, in [92]. Glauber, who was awarded for the Nobel prize, et.al. in 2006 in [93–96], expressed the coherent states as the classical analogy of the radiation in quantum optics. Gerry and Knight in [97] represented the properties of the so-called Schrödinger-cat states and clarified the field states in the electrodynamics aspect of the quantum optics. The minimum uncertainty in time evolutionary form is presented in [98] by Howard and Roy who introduced the coherent state of a harmonic oscillator [99]. In [100], the minimum uncertainty and likeliness of classically equation of motion corresponding to the coherent states of a damped harmonic oscillatoris examined. The superposition of the harmonic oscillator considering two different position dependent mass models has been investigated in [101]. Furthermore, let's mention that the coherent states of the $\mathscr{P}\mathscr{T}$-symmetric quantum systems have been studied in [102, 103]. Finally to complete the introduction, we would like to add that, most of the lower dimensional quantum problems are considered as toy models which shed light on the more complicated problems in the real three dimensional quantum systems. Nevertheless, there are systems in three dimensions which effectively can be reduced to two dimensions. For a two dimensional harmonic oscillator, we refer to the work of Li and Sebastian [104] where the Landau quantum theory of a charged particle in a uniform magnetic field has been considered. In their work, with a specific magnetic vector potential, the problem is reduced to a two dimensions isotropic harmonic oscillator. Has been considered two dimensional harmonic oscillator, rationalizing method is employed to demonstrate the two dimensional complex harmonic oscillator in the extended phase space in [105].

Chapter 2

GENERALIZED PROBABILITY AND CURRENT DENSITIES: A FIELD THEORY APPROACH

2.1 Lagrangian Density in Real Domain

We begin with introducing a form of momentum operator and named generalized momentum operator (GMO), i.e

$$\hat{p} = -i\hbar\left(A\partial_x + \frac{A'}{2}\right), \tag{2.1}$$

where the auxiliary function $A(x)$ is a real function of x, we introduce the generalized Lagrangian density to be

$$\mathscr{L} = i\hbar\dot{\Psi}\Psi^* - \frac{\hbar^2}{2m}A^2\Psi^{*\prime}\Psi' - \left[-\frac{\hbar^2}{4m}AA'' - \frac{\hbar^2}{8m}A'^2 + V(x)\right]\Psi\Psi^*. \tag{2.2}$$

Herein, a dot and prime stand for the derivative with respect to t and x, respectively, and * implies the complex conjugate. By Applying the variation of the action with respect to $\Psi^*(x,t)$ and $\Psi(x,t)$ and choosing the interaction potential $V(x)$ to be real such that the field equations are expressed by

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2m}\left(\left(A^2\Psi'\right)' + \frac{A''A}{2}\Psi + \frac{A'^2}{4}\Psi\right) + V(x)\Psi \tag{2.3}$$

and

$$-i\hbar\dot{\Psi}^* = -\frac{\hbar^2}{2m}\left(\left(A^2\Psi^{*\prime}\right)' + \frac{A''A}{2}\Psi^* + \frac{A'^2}{4}\Psi^*\right) + V(x)\Psi^*, \tag{2.4}$$

respectively. Eqs. (2.3) and (2.4) are the generalized Schrödinger equation and its complex conjugate, upon employing the field,

$$\Psi(x,t) = \psi(x)\exp\left(-\frac{iE}{\hbar}t\right), \tag{2.5}$$

into Eqs. (2.3) and (2.4) which turns to the stationary Schrödinger equation

$$E\psi = -\frac{\hbar^2}{2m}\left(\left(A^2\psi'\right)' + \frac{A''A}{2}\psi + \frac{A'^2}{4}\psi\right) + V(x)\,\psi. \tag{2.6}$$

Hence, the Hamiltonian operator $\hat{H}$ is defined to be

$$\hat{H} = \hat{H}^\dagger = -\frac{\hbar^2}{2m}\left(A^2\partial_x^2 + 2AA'\partial_x + \frac{A''A}{2} + \frac{A'^2}{4}\right) + V(x)\,. \tag{2.7}$$

Next, we use the Lagrangian density (2.2) and the definition of Hamiltonian density

$$\mathscr{H} = \Sigma_\sigma \pi_\sigma \dot{f}_\sigma - \mathscr{L}, \tag{2.8}$$

where $f_\sigma \in \{\Psi, \Psi^*\}$ and π_σ is the momentum-density conjugate to f_σ, calculate the explicit form of $\mathscr{H}$. To do so, let's calculate π_σ as

$$\pi_\Psi = \frac{\partial \mathscr{L}}{\partial \dot{\Psi}} = i\hbar\Psi^*, \tag{2.9}$$

and

$$\pi_{\Psi^*} = \frac{\partial \mathscr{L}}{\partial \dot{\Psi}^*} = 0. \tag{2.10}$$

After the substitution into Eq. (2.8), one-dimensional Hamiltonian density is obtained to be

$$\mathscr{H} = \frac{\hbar^2}{2m}A^2\Psi^{*\prime}\Psi' - \frac{\hbar^2}{4m}AA''\Psi^*\Psi - \frac{\hbar^2}{8m}A'^2\Psi^*\Psi + V(x)\,\Psi^*\Psi. \tag{2.11}$$

Having Hamiltonian density found, we apply $E = \int \mathscr{H}dx$ to find the energy of the system. Explicitly, one finds

$$E = \int\left[\frac{-\hbar^2}{2m}\Psi^*\left(A^2\partial_x^2 + 2A'A\partial_x + \frac{AA''}{2} + \frac{A'^2}{4}\right)\Psi + \Psi^*V(x)\,\Psi\right]dx \tag{2.12}$$

in which according to (2.7) and (2.12) provided that $\Psi(x,t) = \Psi^*(x,t)$ and H is hermitian, the two terms become identical and the energy reduces to

$$E = \int \Psi\hat{H}\Psi dx = \int \Psi^*\hat{H}\Psi dx = \langle \hat{H} \rangle\,. \tag{2.13}$$

Whereas, $\langle \hat{H} \rangle$ is the expectation value of $\hat{H}$. The next quantity which can be discussed is the stress-energy tensor. In this line, first, we obtain the energy flux density corresponding to the Cartesian coordinate which is defined by

$$\mathscr{S} = \dot{\Psi}\frac{\partial \mathscr{L}}{\partial \Psi'} + \dot{\Psi}^* \frac{\partial \mathscr{L}}{\partial \Psi^{*\prime}}, \tag{2.14}$$

such that the explicit calculation reveals the latter equation is given by

$$\mathscr{S} = -\frac{\hbar^2}{2m} A^2 \left(\dot{\Psi}\Psi^{*\prime} + \dot{\Psi}^*\Psi'\right). \tag{2.15}$$

Second, we obtain the momentum density which is defined by

$$P = \Psi' \frac{\partial \mathscr{L}}{\partial \dot{\Psi}} + \Psi^{*\prime} \frac{\partial \mathscr{L}}{\partial \dot{\Psi}^*}, \tag{2.16}$$

while our detailed calculations lead to

$$P = i\hbar \left(\Psi'\Psi^*\right). \tag{2.17}$$

Let's remind that the momentum density implies the amount of energy in a unit of volume passing through a surface in the unit of time. It represents the physical spatial momenta corresponding to a field which is different from the canonical momentum of a quantum particle. Next, we utilize Eqs. (2.3) and (2.4) to find the continuity equation. Let's multiply by Ψ and Ψ^* from the left the Eqs. (2.3) and (2.4), respectively. Then by subtraction of the two equations, we obtain

$$i\hbar\partial_t \left(\Psi\Psi^*\right) = -\frac{\hbar^2}{2m}\partial_x \left(A^2 \left(\Psi\Psi^{*\prime} - \Psi^*\Psi'\right)\right). \tag{2.18}$$

This is the continuity equation provided we define

$$\rho = \Psi\Psi^*, \tag{2.19}$$

to be the probability density and

$$j_x = \frac{\hbar}{2im} \left(A^2 \left(\Psi^*\Psi' - \Psi\Psi^{*\prime}\right)\right), \tag{2.20}$$

to be the particle current density. Hence the continuity equation

$$\partial_t \rho\,(x,t) + \partial_x j\,(x,t) = 0 \tag{2.21}$$

holds. We would like to comment that the present outcomes in Eqs. (2.19) and (2.20) are significant since the conservation of probability density is confirmed.

Example 2.1: Let's choose the auxiliary function

$$A(x) = (1-\beta x), \tag{2.22}$$

where $\beta \in \mathbb{R}$ and constant, plugging into the stationary Schrödinger equation (2.6) one finds

$$E\psi = -\frac{\hbar^2}{2m}\left((1-\beta x)^2\psi'\right)' - \frac{\hbar^2}{2m}\left[\frac{\beta^2}{4} - \frac{2m}{\hbar^2}V(x)\right]\psi. \tag{2.23}$$

Simplifying the latter equation leads to

$$\frac{d}{dx}\left((1-\beta x)^2\psi'(x)\right) + \left[\kappa^2 + \frac{\beta^2}{4} - \frac{2m}{\hbar^2}V(x)\right]\psi(x) = 0, \tag{2.24}$$

in which we denote that $\kappa^2 = \frac{2Em}{\hbar^2}$. We assign a simple one-dimensional harmonic oscillator to be the potential

$$V(x) = \frac{1}{2}m\omega^2x^2, \tag{2.25}$$

where the implementation of the given potential in Eq. (2.24) provides a differential equation

$$(1-\beta x)^2\psi'' - 2\beta(1-\beta x)\psi' + \left(\kappa^2 + \frac{\beta^2}{4} - \frac{m^2\omega^2}{\hbar^2}x^2\right)\psi = 0. \tag{2.26}$$

One can find the exact solution of the obtained differential equation, first, considering the change of variable admitted by

$$y = (1-\beta x). \tag{2.27}$$

Second, imposing $\psi(y) = \frac{u(y)}{y}$ and the differential equation is transformed to

$$u''(y) + \left[\frac{\frac{\kappa^2}{\beta^2} + \frac{1}{4} - \eta^2}{y^2} + \frac{2\eta^2}{y} - \eta^2\right]u(y) = 0, \tag{2.28}$$

in which we call $\eta^2 = \frac{m^2\omega^2}{\hbar^2\beta^4}$. Provided that we impose $u(y) = e^{-\eta y}y^{z+1/2}f(y)$ in the latter equation, one can find the exact solution of the differential equation given by

$$u(y) \sim e^{-\eta y} y^{z+1/2} \times {}_2F_0\left(\sqrt{\eta^2 - \frac{\kappa^2}{\beta^2}} - 2\eta^2 + \frac{1}{2}, -\sqrt{\eta^2 - \frac{\kappa^2}{\beta^2}} - 2\eta^2 + \frac{1}{2}, -\frac{1}{y}\right), \tag{2.29}$$

where $z = \sqrt{\eta^2 - \frac{\kappa^2}{\beta^2}}$. It is important to note that the solution is Whittaker function of the second kind in which asymptotically converges, then, the wave function is represented by

$$\psi(1-\beta x) \sim e^{-\eta(1-\beta x)} (1-\beta x)^{\eta-1} \times {}_2F_0\left(\sqrt{\eta^2 - \frac{\kappa^2}{\beta^2}} - 2\eta^2 + \frac{1}{2}, -\sqrt{\eta^2 - \frac{\kappa^2}{\beta^2}} - 2\eta^2 + \frac{1}{2}, -\frac{1}{(1-\beta x)}\right). \tag{2.30}$$

According to the second kind Whittaker function, the corresponding polynomial terminates when

$$\pm\sqrt{\eta^2 - \frac{\kappa^2}{\beta^2}} - 2\eta^2 + \frac{1}{2} = -n, \tag{2.31}$$

where n is an integer number, having obtained the energy from Eq. (2.31) is given by

$$E = \frac{\hbar^2 \beta^2}{2m}\left[\eta^2 + \left(2\eta^2 - \left(\frac{1}{2} + n\right)\right)^2\right]. \tag{2.32}$$

Thus, it is appropriate to mention that $A(x)$ can show the effect of the geometry of space in form of effective potential which is , here, real according to the domain of defined auxiliary function.

2.2 Lagrangian Density in Complex Domain : Standard Momentum Operator

Now, we suppose that a non-relativistic quantum particle with the standard form of momentum operator i. e. $\hat{p} = -i\hbar\frac{d}{dx}$, is interacting with a complex potential, $V(x) \in \mathbb{C}$. Considering $V(x) = V^*(-x)$, the potential is $\mathscr{P}\mathscr{T}$-symmetric and shown by $V(x) =$

$V^{\#}(x)$. The corresponding Lagrangian density is obtained, based on fields $\Psi(x,t)$ and $\Psi^*(-x,t)$, and represented by

$$\mathscr{L} = i\hbar\dot{\Psi}\Psi^{\#} - \frac{\hbar^2}{2m}\Psi^{\#\prime}\Psi' - V(x)\Psi\Psi^{\#}, \tag{2.33}$$

in which # stands for $\mathscr{PT}$-symmetry conjugate of field Ψ. Again, the corresponding Schrödinger equations are obtained implementing the Euler-Lagrange equations for the applied fields given by

$$-i\hbar\dot{\Psi}^{\#} = -\frac{\hbar^2}{2m}\Psi^{\#\prime\prime} + V(x)\Psi^{\#} \tag{2.34}$$

and

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2m}\Psi'' + V(x)\Psi. \tag{2.35}$$

Different $\mathscr{PT}$-symmetric potentials [45–57] can be applied on Eq. (2.35) and find the real energy spectrum. Utilizing Eqs. (2.34) and (2.35), we multiply Ψ and $\Psi^{\#}$ from left Eqs. (2.35) and (2.34), respectively. Subtracting the two obtained equations, we find

$$\partial_t\left(\Psi\Psi^{\#}\right) + \frac{\hbar}{2im}\partial_x\left(\Psi\Psi^{\#\prime} - \Psi^{\#}\Psi'\right) = 0, \tag{2.36}$$

which demonstrates the probability density and particle current density

$$\rho_{\mathscr{PT}} = \Psi\Psi^{\#} \tag{2.37}$$

and

$$j_{\mathscr{PT}} = \frac{\hbar}{2im}\left(\Psi\Psi^{\#\prime} - \Psi^{\#}\Psi'\right), \tag{2.38}$$

respectively. Eqs. (2.37) and (2.38) satisfy the conservation of probability density, i. e.

$$\partial_t\rho_{\mathscr{PT}} + \partial_x j_{\mathscr{PT}} = 0, \tag{2.39}$$

according to Eq. (2.36). Next, we apply (2.14) and (2.16) in accordance with the energy flux and the momentum density which are given by

$$\mathscr{S} = -\frac{\hbar^2}{2m}\left(\dot{\Psi}\Psi^{\#\prime} + \dot{\Psi}^{\#}\Psi'\right) \tag{2.40}$$

and

$$P = i\hbar\left(\Psi'\Psi^{\#}\right), \tag{2.41}$$

respectively. Let's keep this in our minds that Eqs. (2.37), (2.40) and (2.41) are not real due to opting a $\mathscr{P}\mathscr{T}$-symmetric field. Furthermore, we demonstrate the Hamiltonian density using (2.33) and (2.8) for $f_\sigma \in \left\{\psi, \psi^{\#}\right\}$ presented by

$$\mathscr{H} = \frac{\hbar^2}{2m}\Psi^{\#\prime}\Psi' + V(x)\Psi^{\#}\Psi. \tag{2.42}$$

One finds the energy spectrum upon applying the given Hamiltonian density with

$$E = \int \Psi^{\#}\left(-\frac{\hbar^2}{2m}\partial_x^2 + V(x)\right)\Psi dx, \tag{2.43}$$

in which $x \in \mathbb{R}$, although, we note that the integrant is complex. The obtained relation in Eq. (2.43) manifests the expectation value of the energy.

2.3 Lagrangian Density in Complex Domain : Generalized Momentum Operator

We continue our discussion upon which the proposed momentum operator and the potential belong to the complex domain, $V(x)$, $A \in \mathbb{C}$, provided that both items be $\mathscr{P}\mathscr{T}$-symmetric. Thus, the Lagrangian density is given by, applying (2.1),

$$\mathscr{L} = i\hbar\dot{\Psi}\Psi^{\#} - \frac{\hbar^2}{2m}A^2\Psi^{\#\prime}\Psi' - \left[V(x) - \frac{\hbar^2}{4m}AA'' - \frac{\hbar^2}{8m}A'^2\right]\Psi\Psi^{\#} \tag{2.44}$$

where $\Psi^{\#} = \mathscr{P}\mathscr{T}\Psi$. Accordingly, we calculate the the Schrödinger equations using the Euler-Lagrange equations with respect to $\Psi^{\#}$ and Ψ given by

$$i\hbar\dot{\Psi} = -\frac{\hbar^2}{2m}\left(A^2\Psi'\right)' + \left[V(x) - \frac{\hbar^2}{4m}AA'' - \frac{\hbar^2}{8m}A'^2\right]\Psi \tag{2.45}$$

and

$$-i\hbar\dot{\Psi}^{\#} = -\frac{\hbar^2}{2m}\left(A^2\Psi^{\#\prime}\right)' + \left[V(x) - \frac{\hbar^2}{4m}AA'' - \frac{\hbar^2}{8m}A'^2\right]\Psi^{\#}, \tag{2.46}$$

respectively. Moreover, the corresponding Hamiltonian is $\mathscr{P}\mathscr{T}$-symmetric which is admitted in Eq. (2.7). Following up the earlier discussion upon the conservation of probability density, one can approve the similar method and represent the generalized continuity equation in

$$\partial_t \left(\Psi\Psi^{\#}\right) + \frac{\hbar}{2im}\partial_x \left(A^2 \left(\Psi^{\#}\Psi' - \Psi\Psi^{\#\prime}\right)\right) = 0. \tag{2.47}$$

According to the generalized continuity equation, the probability density is demonstrated by

$$\rho = \Psi\Psi^{\#} \tag{2.48}$$

and the generalized current density admitted in

$$j_x = \frac{\hbar}{2im}\left(A^2 \left(\Psi^{\#}\Psi' - \Psi\Psi^{\#\prime}\right)\right). \tag{2.49}$$

We note that with $\Psi(x,t) = \Psi^{\#}(x,t)$, the particle current density vanishes and consequently,

$$\frac{d}{dt}\int dx\rho = 0, \tag{2.50}$$

which is the conservation of the total probability on $x \in \mathbb{R}$ [62]. Having been obtained the Hamiltonian density in accordance with the recent Lagrangian density in

$$\mathscr{H} = \frac{\hbar^2}{2m}A^2\Psi^{\#\prime}\Psi' - \frac{\hbar^2}{4m}AA''\Psi^{\#}\Psi - \frac{\hbar^2}{8m}A'^2\psi^{\#}\psi + V(x)\,\psi^{\#}\psi, \tag{2.51}$$

where the energy is presented by $E = \int \mathscr{H} dx$ related to the $\mathscr{P}\mathscr{T}$-symmetric field theory found to be

$$E = \int \Psi^{\#}\left[\frac{-\hbar^2}{2m}\left(A^2\partial_x^2 + 2A'A\partial_x + \frac{AA''}{2} + \frac{A'^2}{4}\right) + V(x)\right]\Psi dx. \tag{2.52}$$

With the situation that $\Psi(x,t) = \Psi^{\#}(x,t)$, the energy expectation value expressed by

$$E = \int \Psi\hat{H}\Psi dx = \int \Psi^{\#}\hat{H}\Psi dx = \langle\hat{H}\rangle. \tag{2.53}$$

The elements of energy-tensor, energy flux and momentum density, accordingly, are acquired in

$$\mathscr{S} = -\frac{\hbar^2}{2m}A^2\left(\Psi^{\#\prime}\dot{\Psi} + \Psi'\dot{\Psi}^{\#}\right) \tag{2.54}$$

and

$$P = i\hbar\left(\Psi'\Psi^{\#}\right), \tag{2.55}$$

respectively, which approve the obtained equations in the earlier outcomes.

Example 2.2: Let's impose

$$A(x) = 1 + i\gamma x \tag{2.56}$$

into the time independent Schrödinger equation and consider the spacial counterpart of the wave function in Eq. (2.5), therefore, one finds

$$E\psi = -\frac{\hbar^2}{2m}\left((1+i\gamma x)^2\psi'\right)' + \left[V(x) + \frac{\hbar^2\gamma^2}{8m}\right]\psi, \tag{2.57}$$

in which multiplying $-\frac{2m}{\hbar^2}$ with [Sch-example2] and calling $\kappa^2 = \frac{2m}{\hbar^2}$, the latter equation turns to

$$-\kappa^2\psi = \left((1+i\gamma x)^2\psi'\right)' - \frac{2m}{\hbar^2}\left[V(x) + \frac{\hbar^2\gamma^2}{8m}\right]\psi. \tag{2.58}$$

In this stage, we rearrange the obtained differential equation to be

$$(1+i\gamma x)^2\psi'' + 2i\gamma(1+i\gamma x)\psi' - \left[\frac{2m}{\hbar^2}V(x) + \frac{\gamma^2}{4} - \kappa^2\right]\psi = 0, \tag{2.59}$$

we set the potential in the x-space to be

$$V(x) = \frac{1}{2}\omega\hbar(i\gamma x + 1), \tag{2.60}$$

where $x \in \mathbb{R}$. Thus we calculate the differential equation employing the potential presented by

$$(1+i\gamma x)^2\psi'' + 2i\gamma(1+i\gamma x)\psi' - \left[\frac{m\omega}{\hbar}(i\gamma x + 1) + \frac{\gamma^2}{4} - \kappa^2\right]\psi = 0. \tag{2.61}$$

Considered,

$$z = 1 + i\gamma x, \tag{2.62}$$

in which $z \in \mathbb{C}$ and applied $\psi(z) = \frac{u(z)}{z}$, we transform (2.61) to

$$u'' + \left[\frac{m\omega}{\hbar\gamma^2}\frac{1}{z} + \frac{\frac{1}{4} - \frac{\kappa^2}{\gamma^2}}{z^2} \right] u = 0. \tag{2.63}$$

The solution of the latter differential equation is given by Bessel function in which $\frac{\kappa}{\gamma}$ has to be an integer and the energy spectrum is given by

$$E = \frac{\hbar^2\gamma^2 n^2}{2m}. \tag{2.64}$$

In the next chapter, we use specifically the generalized momentum operator and the equation of motion which is found here.

Chapter 3

GENERALIZED MOMENTUM OPERATOR

3.1 Generalized Momentum Operator (GMO): Hermitian

According to the EUP the commutation relation of position and momentum operators is generalized to be

$$[\hat{x}, \hat{p}] = i\hbar \left(1 + \mu\left(x\right)\right) \tag{3.1}$$

in which $\mu(x)$ is a real well-defined function of position operator $\hat{x}$, recall that $A(x) = 1 + \mu(x)$ in Eq. (2.1). With the standard definition of the momentum operator, $\hat{p} = -i\hbar \frac{d}{dx}$, Eq. (3.1) becomes simply $[\hat{x}, \hat{p}] = i\hbar$ with $\mu(x) = 0$ while for the more interesting case where $\mu(x) = \alpha x^2$ it has been found that [19, 20, 23, 25–27, 30]

$$\hat{p} = -i\hbar \left(1 + \alpha x^2\right) \frac{d}{dx}. \tag{3.2}$$

In finding (3.2), the only requirement was

$$[\hat{x}, \hat{p}] = i\hbar \left(1 + \alpha x^2\right), \tag{3.3}$$

however adding any function of the position operator to the obtained momentum operator, mathematically, does not change the commutation relation (3.3). Hence, a generalized momentum operator (GMO) satisfying (3.3) may be written as

$$\hat{p} = -i\hbar \left(1 + \alpha x^2\right) \frac{d}{dx} + f(x), \tag{3.4}$$

in which $f(x)$ is a well-defined function of position operator x. To specify $f(x)$, we impose the hermiticity condition on $\hat{p}$ which is expected to be hold by any physical quantity. This, in turn results in

$$f(x) = -i\hbar x \tag{3.5}$$

and consequently the hermitian counterpart of the extended momentum operator (3.2) becomes

$$\hat{p} = -i\hbar\left(1+\alpha x^2\right)\frac{d}{dx} - i\hbar x. \tag{3.6}$$

Next, we generalize the above result in terms of EUP expressed in (3.1) by proposing the hermitian extended momentum operator given by

$$\hat{p} = -i\hbar\left(1+\mu(x)\right)\frac{d}{dx} - i\hbar\frac{d\mu(x)}{2dx} \tag{3.7}$$

in which $\mu(x)$ is a real function of position operator $\hat{x}$.

3.1.1 The Generalized Schrödinger Equation

Employing the generalized momentum operator (3.1), the corresponding generalized Schrödinger equation of a one-dimensional quantum particle reads as

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(\frac{\hat{p}^2}{2m} + V(x)\right)\psi(x,t), \tag{3.8}$$

which upon considering $V(x)$ a time-independent potential and $\psi(x,t) = e^{-iEt/\hbar}\phi(x)$ we find the time independent Schrödinger equation in its explicit form given by

$$\frac{-\hbar^2}{2m}\left((1+\mu)^2\frac{d^2}{dx^2} + 2(1+\mu)\mu'\frac{d}{dx} + \frac{1}{2}(1+\mu)\mu'' + \frac{1}{4}\left(\mu'\right)^2\right)\phi(x) + V(x)\phi(x) = E\phi(x). \tag{3.9}$$

Here, E is the conserved energy of the particle and a prime stands for the derivative with respect to x. The generalized Schrödinger equation (3.9) can be transformed into a more familiar shape if we apply the so-called "variable transformation" (VT), defined by

$$\phi(x) = \frac{1}{\sqrt{1+\mu(x)}}\chi(z(x)) \tag{3.10}$$

and

$$z = z(x) = \int^{x} \frac{1}{1+\mu(y)} dy + z_0, \tag{3.11}$$

in which z_0 is an integration constant. Upon applying (3.10) and (3.11), the transformed generalized Schrödinger equation becomes

$$\frac{-\hbar^2}{2m}\frac{d^2}{dz^2}\chi(z) + V(x(z))\chi(z) = E\chi(z) \tag{3.12}$$

which is in the form of the standard one-dimensional Schrödinger equation in the z-space.

Example 3.1 ($\mu(\hat{x}) = \alpha^2 x^2$)**:** In this section we study a quantum particle whose GMO is defined by Eq. (3.1) with $\mu(x) = \alpha^2 x^2$ in which α is a real constant of dimension L^{-1}. Applying the VT (3.11) we obtain

$$z(x) = \frac{1}{\alpha}\arctan(\alpha x) \tag{3.13}$$

or inversely

$$x(z) = \frac{1}{\alpha}\tan(\alpha z), \tag{3.14}$$

where we set the integration constant z_0 to be zero. We note that, for $x \in \mathbb{R}$, the transformed coordinate z is confined i.e., $z \in \left[-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha}\right]$. For a zero-potential configuration in x-space, i.e., $V(x) = 0$, one finds the corresponding particle in an infinite well in the z-space where

$$V(z) = \begin{cases} 0 & |z| \leq \frac{\pi}{2\alpha} \\ \infty & elsewhere \end{cases}. \tag{3.15}$$

The eigen-functions and eigenvalues of the reference equation (3.12) are given by

$$\chi_n(z) = \begin{cases} \sqrt{\frac{2\alpha}{\pi}}\cos(n\alpha z), & odd-n \\ \sqrt{\frac{2\alpha}{\pi}}\sin(n\alpha z), & even-n \end{cases} \tag{3.16}$$

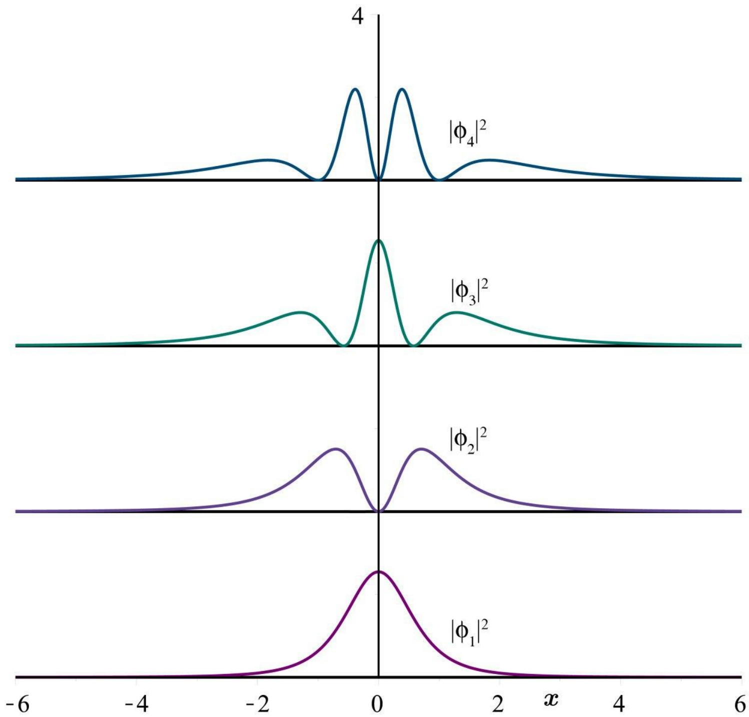

Figure 3.1: Plot of $|\phi_n(x)|^2$ in terms of x, Eq. (3.18), for $n = 1, 2, 3$ and 4 with $\alpha = 1$. The free particle is confined within its own generalized momentum such that the particle is localized around $x = 0$ where its deviation from the standard momentum remains small.

and

$$E_n = \frac{n^2\hbar^2\alpha^2}{2m} \tag{3.17}$$

respectively, where $n = 1, 2, 3, \ldots$. Furthermore, while the energy spectrum of the target Schrödinger equation, (3.9), is the same as (3.17) the normalized eigenfunctions, after applying (3.10), are given by

$$\phi_n(x) = \begin{cases} \sqrt{\frac{2\alpha}{\pi}}\frac{\cos(n\arctan(\alpha x))}{\sqrt{1+\alpha^2x^2}}, & odd-n \\ \sqrt{\frac{2\alpha}{\pi}}\frac{\sin(n\arctan(\alpha x))}{\sqrt{1+\alpha^2x^2}}, & even-n \end{cases}. \tag{3.18}$$

In Fig. (3.1) we plot the first four states corresponding to the quantum numbers $n = 1, 2, 3$ and 4 for $\alpha = 1$. Unlike a free particle with standard momentum operator which is equally likely to be found everywhere on position axis, here the extended momentum operator confines the particle to be localized around smaller x. It is remarkable to observe that, due to the zero potential i.e.,$V(x) = 0$, the generalized momentum operator, $\hat{p}$, and the Hamiltonian, $\hat{H} = \frac{\hat{p}^2}{2m}$, commute. This implies that $\phi_n(x)$ are the simultaneous eigenstates of the Hamiltonian and the momentum with eigenvalues E_n and $\pm\sqrt{2mE_n}$ respectively. The extended form of the momentum operator, i.e.,

$$\hat{p} = -i\hbar\left(1+\alpha^2x^2\right)\frac{d}{dx} - i\hbar\alpha^2 x \tag{3.19}$$

leads to a modified EUP where we expect a minimum uncertainty for the momentum operator to be observed. This can be seen from

$$\delta x\delta p \geq \frac{1}{2}\left|\langle[x,\hat{p}]\rangle\right| \tag{3.20}$$

where $[\hat{x},\hat{p}] = i\hbar\left(1+\alpha^2x^2\right)$. The latter implies

$$\delta x\delta p \geq \frac{\hbar}{2}\left(1+\alpha^2\left\langle x^2\right\rangle\right) \tag{3.21}$$

which after knowing $(\delta x)^2 = \left\langle x^2\right\rangle - \langle x\rangle^2$, one finds

$$\delta x\delta p \geq \frac{\hbar}{2}\left(1+\alpha^2\left((\delta x)^2 - \langle x\rangle^2\right)\right) \geq \frac{\hbar}{2}\left(1+\alpha^2\left((\delta x)^2\right)\right). \tag{3.22}$$

Finally the momentum uncertainty is found to satisfy

$$\delta p \geq \frac{\frac{\hbar}{2}\left(1+\alpha^2\left((\delta x)^2\right)\right)}{\delta x} \tag{3.23}$$

which obviously admits a minimum value for δp at $\delta x = \frac{1}{\alpha}$, given by

$$(\delta p)_{min} = \hbar\alpha. \tag{3.24}$$

Using the eigenfunctions of the system we discussed above, Eq. (3.19), here we shall find δx and δp, explicitly, in order to investigate the minimum value of the momentum uncertainty (3.24). From the definition,

$$(\delta p)^2 = \left\langle p^2\right\rangle - \langle p\rangle^2 \tag{3.25}$$

where

$$\left\langle p^2 \right\rangle = \left\langle \phi_n \left| p^2 \right| \phi_n \right\rangle = 2mE_n \tag{3.26}$$

and

$$\langle p \rangle = \langle \phi_n | p | \phi_n \rangle = 0 \tag{3.27}$$

which amount to $\delta p = n\hbar\alpha$. Recall that, $n = 1, 2, 3, \ldots$, the minimum uncertainty for the momentum operator occurs in the ground state where $n = 1$ and $(\delta p)_{min} = \hbar\alpha$, in agreement with (3.24). On the other hand, since $\langle x \rangle = 0$ and $\left\langle x^2 \right\rangle = \frac{2n-1}{\alpha^2}$, we obtain δx=$\frac{\sqrt{2n-1}}{\alpha}$. Finally, one gets

$$\delta x \delta p \geq n\sqrt{2n-1}\hbar \tag{3.28}$$

which clearly satisfies (3.21).

Example 3.2 ($\mu(x) = e^{-\gamma x} - 1$): In this section we study another important generalized extended momentum operator with

$$\mu(x) = e^{-\gamma x} - 1, \tag{3.29}$$

in which γ is a real constant. Similar to the previous example we apply the VT (3.11) which implies

$$z = \frac{1}{\gamma}\left(e^{\gamma x} - 1\right) \tag{3.30}$$

and

$$\phi(x) = e^{\gamma x/2}\chi(z) \tag{3.31}$$

where we set the integration constant to be $-\frac{1}{\gamma}$. Furthermore, we assume a modified-half-Morse type potential in the x-space, i.e.,

$$V(x) = \begin{cases} \infty & x \leq 0 \\ V_0\left(1 - e^{\gamma x}\right)^2 & x > 0 \end{cases}, \tag{3.32}$$

in which $V_0 > 0$. In z-space the potential becomes

$$V(z) = \begin{cases} \infty & z \le 0 \\ \frac{1}{2} m\omega^2 z^2 & z > 0 \end{cases}, \tag{3.33}$$

where $\omega^2 = \frac{2\gamma^2 V_0}{m}$. The standard solution of the quantum-half SHO is available in any text book given by $E_n = \hbar\omega\left(2n + \frac{3}{2}\right)$ and

$$\chi_n(z) = \frac{1}{\sqrt{2^{n-1} n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega z^2/2\hbar} H_n\left(\sqrt{\frac{m\omega}{\hbar}} z\right), \tag{3.34}$$

with Hermite polynomials defined by

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n}\left(e^{-t^2}\right). \tag{3.35}$$

Herein, the quantum number $n = 0, 1, 2, 3, 4, \ldots$. Finally the eigenvalues and eigenfunctions of the original system in x-space are given by

$$E_n = \hbar\omega\left(2n + \frac{3}{2}\right) \tag{3.36}$$

and

$$\phi_n(x) = \frac{1}{\sqrt{2^{n-1} n!}} \left(\frac{\gamma\sqrt{2mV_0}}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{1}{2\gamma}\left(\frac{\sqrt{2mV_0}}{\hbar}\left(e^{\gamma x} - 1\right)^2 - \gamma^2 x\right)\right]$$
$$\times H_n\left(\sqrt{\frac{\sqrt{2mV_0}}{\hbar\gamma}}\left(e^{\gamma x} - 1\right)\right), \tag{3.37}$$

respectively. In Fig. (3.2) we plot the probability density $|\phi_n(x)|^2$ in terms of x for the first three states with $V_0 = 1$, $m = 1$, $\hbar = 1$ and $\alpha = 1$. It is remarkable to observe that for higher level states, the probability density moves away from the infinite wall.

3.2 $\mathscr{PT}$-Symmetric Momentum Operator and Bound States [108]

In this section, we intend to expand our argument to the complex domain. Upon recalling Eq. (3.7) in the context of EUP in which the auxiliary function $\mu(x)$ is a real function of position operator $\hat{x}$. The GMO is hermitian, $\hat{p} = \hat{p}^\dagger$, here, in this study we propose $\mu(x)$ to be a $\mathscr{PT}$-symmetric complex function i.e., $\mathscr{PT}\mu(x) \equiv \mu^\star(-x) = \mu(x)$ which in turn makes the GMO non-hermitian but

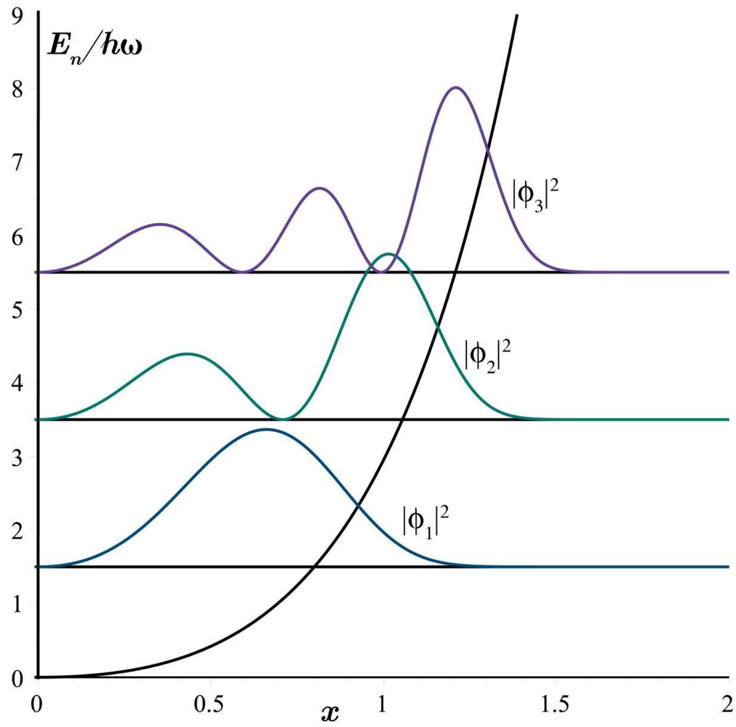

Figure 3.2: Plot of $|\phi_n(x)|^2$ Eq. (3.37) in terms of x for $n = 0, 1$ and 2 with $V_0 = 1$, $m = 1$, $\hbar = 1$ and $\alpha = 1$. The potential (3.32) is also plotted together with the energy of each state on the vertical line. We observe that in higher levels the probability densities moves away from the infinite wall located at $x = 0$.

$\mathscr{PT}$-symmetric. This means that, $\hat{p} \neq \hat{p}^{\dagger}$, but $[\mathscr{PT}, \hat{p}] = 0$. Starting from the eigenvalue equation for the $\mathscr{PT}$-symmetric momentum, i.e.,

$$\hat{p}\Phi_{\mathfrak{p}}(x) = \mathfrak{p}\Phi_{\mathfrak{p}}(x), \tag{3.38}$$

in which $\mathfrak{p}$ and $\Phi_{\mathfrak{p}}$ are the momentum eigenvalue and corresponding eigenfunction, respectively, and applying from the left, the $\mathscr{PT}$ operator, one finds

$$\mathscr{PT}\left(\hat{p}\Phi_{\mathfrak{p}}(x)\right) = \mathfrak{p}^{\star}\left(\mathscr{PT}\Phi_{\mathfrak{p}}(x)\right), \tag{3.39}$$

in which a star stands for the complex conjugate. Since, $\hat{p}$ and $\mathscr{PT}$ commute, if $\Phi_{\mathfrak{p}}(x)$ is also $\mathscr{PT}$-symmetric i.e., $\mathscr{PT}\Phi_{\mathfrak{p}}(x) = \Phi_{\mathfrak{p}}(x)$ then we obtain

$$\mathfrak{p}\Phi_{\mathfrak{p}}(x) = \mathfrak{p}^{\star}\Phi_{\mathfrak{p}}(x) \tag{3.40}$$

which implies that the eigenvalues are real, i.e., $\mathfrak{p} = \mathfrak{p}^{\star}$ and consequently the $\mathscr{PT}$-symmetric momentum is a physical observable. Next, we inspect whether the momentum eigenfunctions $\Phi_{\mathfrak{p}}(x)$ are $\mathscr{PT}$-symmetric or not. To do so we solve the generic momentum-eigenvalue equation which explicitly reads as

$$\left(-i\hbar\left(1+\mu\left(x\right)\right)\frac{d}{dx}-\frac{i\hbar}{2}\frac{d\mu\left(x\right)}{dx}\right)\Phi_{\mathfrak{p}}\left(x\right)=\mathfrak{p}\Phi_{\mathfrak{p}}\left(x\right). \tag{3.41}$$

The latter equation admits a general solution of the form

$$\Phi_{\mathfrak{p}}(x)=C\frac{exp\left(\frac{i\mathfrak{p}}{\hbar}\int\frac{dx}{1+\mu(x)}\right)}{\sqrt{1+\mu\left(x\right)}} \tag{3.42}$$

in which C is a constant. Considering C and $\mathfrak{p}$ both real and $\mu\left(x\right)$ a $\mathscr{PT}$-symmetric function, one obviously finds that $\Phi_{\mathfrak{p}}\left(x\right)$ is $\mathscr{PT}$-symmetric. Before we complete this part we would like to add that similar ambiguities as of the canonical momentum operator, such as unboundedness and continuity of the spectra occurs for the $\mathscr{PT}$-symmetric momentum which we do not intend to deal with them in this research. Furthermore, the corresponding Hamiltonian of a particle with the $\mathscr{PT}$-symmetric momentum undergoing a one-dimensional potential $V\left(x\right)$ is also given by

$$H=\frac{-\hbar^{2}}{2m}\left(\left(1+\mu\right)^{2}\frac{d^{2}}{dx^{2}}+2\left(1+\mu\right)\mu'\frac{d}{dx}+\frac{1}{2}\left(1+\mu\right)\mu''+\frac{1}{4}\left(\mu'\right)^{2}\right)+V\left(x\right) \tag{3.43}$$

in which a prime stands for the derivative with respect to x. Let's note that although $\mu\left(x\right)$ and $V\left(x\right)$ are in general complex functions but x represents the real coordinate. Herein, with a $\mathscr{PT}$-symmetric potential i.e., $\mathscr{PT}V\left(x\right)=V\left(x\right)$, H becomes $\mathscr{PT}$-symmetric which indicates

$$\left[\mathscr{PT},H\right]=0. \tag{3.44}$$

In a similar manner as of the $\mathscr{PT}$-symmetric momentum, one can prove that the eigenvalues of the Hamiltonian are real, provided the energy eigen-functions are also $\mathscr{PT}$-symmetric. In such a case, the $\mathscr{PT}$-symmetry is called exact which ensures that the energy spectrum is real [65–67, 69] i.e.,

$$H\phi(x) = E\phi(x), \tag{3.45}$$

E is real. It is worth to mention that there is no way to know if the $\mathscr{PT}$-symmetry of the Hamiltonian is exact unless one solves the Schrödinger equation explicitly [69]. Next, we apply a VT given in Eq. (3.11) , upon which we transform the Schrödinger equation (3.45) into the standard form of the Schrödinger equation in z-space given in Eq. (3.12). We observe here that, due to the VT, although the Schrödinger equation in x-space is transformed into the standard Schrödinger equation in z-space, but the energy eigenvalues remain the same. In the other words, the energy is invariant under the above transformation. Moreover, we note that, $z(x)$ is a complex coordinate due to the complex nature of the auxiliary function $\mu(x)$.

Example 3.3: Following Example. (3.1), we choose

$$\mu(x) = \alpha^2 x^2 + i2\beta x \tag{3.46}$$

in which α and β are two real constants such that $\mu(x)$ remains $\mathscr{PT}$-symmetric. After, (3.46), the $\mathscr{PT}$-symmetric momentum becomes

$$\hat{p} = -i\hbar\left(1 + \alpha^2 x^2 + i2\beta x\right)\frac{d}{dx} - i\hbar\left(\alpha^2 x + i\beta\right) \tag{3.47}$$

which satisfies the EUP relation in accordance with $[\hat{x}, \hat{p}] = i\hbar\left(1 + \alpha^2 x^2 + 2i\beta x\right)$. In Sec. (3.1) using real auxiliary function in the GMO, the uncertainty of the x and momentum are measured in the real Hilbert space. Herein, the momentum is no longer a hermitian operator, therefore, the non-commutative relation has been defined in the complex Hilbert space [76]. Referring to (3.42), the eigenfunction of the

$\mathscr{PT}$-symmetric momentum (3.47) is found to be

$$\Phi_{\mathfrak{p}}(x) = \frac{C}{\sqrt{1+\alpha^2 x^2 + i2\beta x}} \exp\left[-\frac{i\mathfrak{p}\arctan\left(\frac{\alpha^2 x + i\beta}{\sqrt{\alpha^2+\beta^2}}\right)}{\hbar\sqrt{\alpha^2+\beta^2}}\right] \tag{3.48}$$

in which $\mathfrak{p}$ is the continuous real eigenvalue of the momentum operator $\hat{p}$. Next, the $\mathscr{PT}$-symmetric Hamiltonian is obtained to be

$$H = \frac{-\hbar^2}{2m}\left(\left(1+\alpha^2 x^2 + i2\beta x\right)^2 \frac{d^2}{dx^2} + 4\left(1+\alpha^2 x^2 + i2\beta x\right)\left(\alpha^2 x + i\beta\right)\frac{d}{dx} + \alpha^2\left(1+\alpha^2 x^2 + i2\beta x\right) + \left(\alpha^2 x + i\beta\right)^2\right) + V(x). \tag{3.49}$$

Upon applying the VT, introduced in Eqs. (3.10) and (3.11), the corresponding Schrödinger equation (3.9) in x-space is transformed into z-space given in Eq. (3.12) where the transformed coordinate is found to be

$$z(x) = \frac{1}{\sqrt{\alpha^2+\beta^2}} \arctan\left(\frac{\alpha^2 x + i\beta}{\sqrt{\alpha^2+\beta^2}}\right). \tag{3.50}$$

We note that, $z(x) = \zeta + i\eta$ is a complex variable such that

$$\tan(\omega(\zeta + i\eta)) = \frac{\alpha^2}{\omega} x + i\frac{\beta}{\omega} \tag{3.51}$$

in which $\omega = \sqrt{\alpha^2+\beta^2}$. Eq. (3.51) implies

$$\frac{\alpha^2}{\omega} x = \frac{\sin(\omega\zeta)\cos(\omega\zeta)}{\cos^2(\omega\zeta)\cosh^2(\omega\eta) + \sin^2(\omega\zeta)\sinh^2(\omega\eta)} \tag{3.52}$$

and

$$\frac{\beta}{\omega} = \frac{\sinh(\omega\eta)\cosh(\omega\eta)}{\cos^2(\omega\zeta)\cosh^2(\omega\eta) + \sin^2(\omega\zeta)\sinh^2(\omega\eta)}. \tag{3.53}$$

The domain of x is $\mathbb{R}$, such that (3.52) and (3.53) describe a mapping from $\mathbb{R}$ to $\mathbb{C}$. For a finite nonzero β, at the limits $x \to \pm\infty$, we find $\omega\zeta \to \pm\frac{\pi}{2}$ and $\omega\eta \to 0$. Fig. (3.3) displays transformation (3.51) from complex xy-plane into the complex z-plane. Specifically, the line $y = \frac{\beta}{\omega}$ in the left panel is mapped into the curve shown in the right panel. It is remarkable to observe that the entire real x-axis is squeezed into $-\frac{\pi}{2} < \omega\xi < \frac{\pi}{2}$. In Eq. (3.12) we set the external potential to be zero, i.e.,

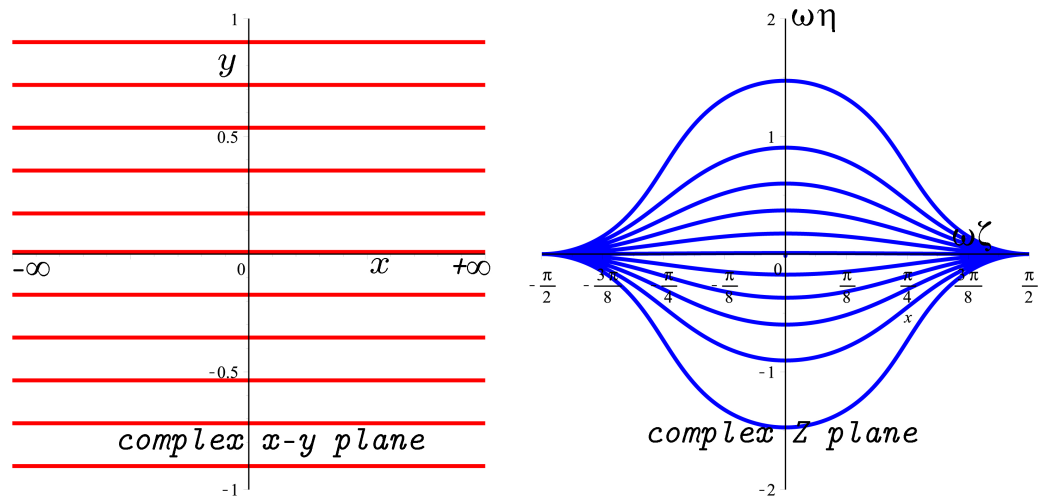

Figure 3.3: The line $y = \frac{\beta}{\omega}$ in the xy-plane (left panel) and the corresponding mapped curve in $z(x) = \zeta + i\eta$ plane (right panel) for $\frac{\beta}{\omega} = -0.90...0.90$ from the lowermost to the uppermost curves with equal steps. We note that, since $\omega = \sqrt{\alpha^2 + \beta^2}$ then $-1 < \frac{\beta}{\omega} < 1$ and $\beta = 0$ represents the hermitian momentum.

$V(x) = 0$ for the entire domain of x i.e., $x \in \mathbb{R}$. Imposing $V(x) = 0$ in Eq. (3.12), one finds

$$\frac{-\hbar^2}{2m}\frac{d^2}{dz^2}\chi(z) = E\chi(z) \tag{3.54}$$

in accordance with the transformation (3.51) which is also shown in Fig. (3.3), $\chi(z) \to 0$ when $\zeta \to \pm\frac{\pi}{2\omega}$ and $\eta \to 0$. The general solution for the complex differential equation (3.54) is obtained to be as

$$\chi(z) = C_1 \sin(Kz) + C_2 \cos(Kz) \tag{3.55}$$

in which $K^2 = \frac{2mE}{\hbar^2}$. One can easily show that imposing the boundary conditions $\chi\left(z \to \pm\frac{\pi}{2}\right) \to 0$ requires the energy E to be real given by

$$E_n = \frac{n^2\hbar^2\left(\alpha^2 + \beta^2\right)}{2m}, \tag{3.56}$$

with the corresponding eigen-functions

$$\chi_n(z) = \begin{cases} C_2 \cos\left(n\sqrt{\alpha^2+\beta^2}z\right), & odd-n \\ C_1 \sin\left(n\sqrt{\alpha^2+\beta^2}z\right), & even-n \end{cases}. \tag{3.57}$$

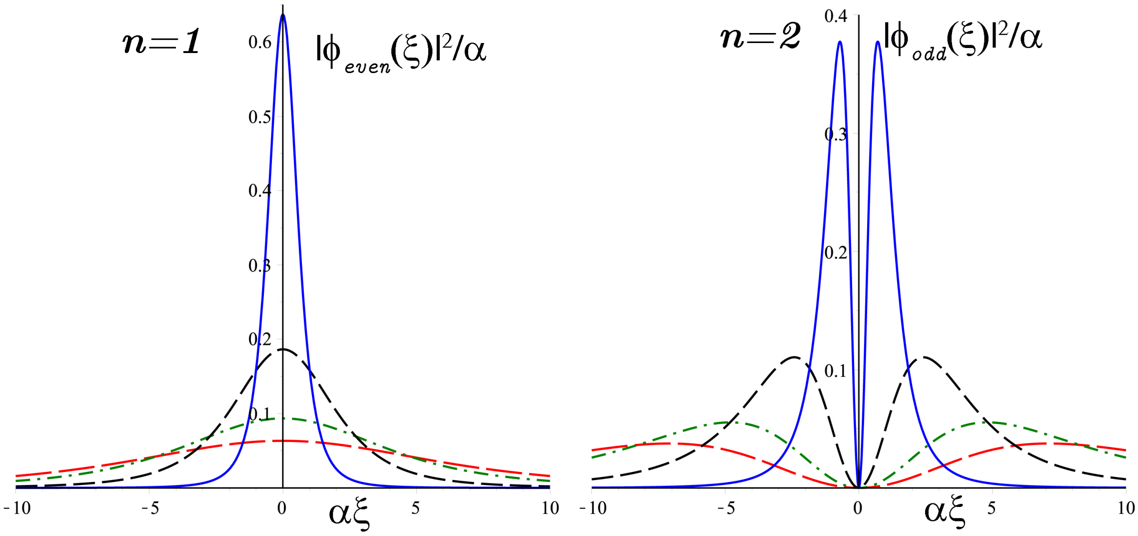

Figure 3.4: $\frac{1}{\alpha}|\phi_n(\xi)|^2$ in terms of the shifted coordinate $\alpha\xi$ in which $\xi = x + i\frac{\beta}{\alpha^2}$ for $\frac{\beta}{\alpha} = 0.0$ (blue, solid), 0.25 (black, dash), 0.50 (green, dash-dot) and 1.0 (red, long-dash) respectively. With a given value for α, the effect of the parameter $\frac{\beta}{\alpha}$ is to decrease the confinement of the particle on ξ-axis. The quantum number $n = 1$ and 2 for the left and right panels respectively.

Please note that, C_1 and C_2 are the normalization constants to be found. To find the corresponding wave-function in x-space, we write

$$\phi_n(x) = \frac{1}{\sqrt{1+\alpha^2 x^2 + 2i\beta x}}\chi_n(z) \tag{3.58}$$

which yields to

$$\phi_n(x) = \begin{cases} C_2 \dfrac{\cos\left(n \arctan\left(\frac{\alpha^2 x + i\beta}{\sqrt{\alpha^2+\beta^2}}\right)\right)}{\sqrt{1+\alpha^2x^2+2i\beta x}}, & odd - n \\ C_1 \dfrac{\sin\left(n \arctan\left(\frac{\alpha^2 x + i\beta}{\sqrt{\alpha^2+\beta^2}}\right)\right)}{\sqrt{1+\alpha^2x^2+2i\beta x}}, & even - n \end{cases} \tag{3.59}$$

with real eigenvalues given in Eq. (3.56). Unlike $\chi_n(z)$, $\phi_n(x)$ can be normalized on the specific contour $\mathfrak{C}$ such that

$$\int_{\mathfrak{C}} (\mathscr{C}\mathscr{P}\mathscr{T}\phi_n(x))\,\phi_n(x)\,dx = \int_{\mathfrak{C}} (\phi_n(x))^2\,dx \tag{3.60}$$

in which $\mathscr{C}$ operator is known to be the charge operator having eigenvalues ± 1 and commutes with $\mathscr{P}\mathscr{T}$ operator [38]. As one can see in Eq. (3.59), $(\phi_n(x))^2$ is complex function implying that it can not be the probability density. However, on the specific contour $\mathfrak{C}$ on the complex x-plane where $\mathfrak{C}$ satisfies the certain conditions, i) $Im\left[(\phi_n(x))^2 dx\right] = 0$,ii) $Re\left[(\phi_n(x))^2 dx\right] \geq 0$ and iii) $\int_{\mathfrak{C}} (\phi_n(x))^2 dx = 1$, it can be considered as the probability density [42]. Here, the contour $\mathfrak{C}$ is a line parallel to the real axis defined by $Im(x) = -\frac{\beta}{\alpha^2}$ and the normalization constants are found to be

$$C_1 = C_2 = \sqrt{\frac{2\sqrt{\alpha^2+\beta^2}}{\pi}}. \tag{3.61}$$

In Fig. (3.4), we plot the ground state and the first excited state corresponding to $n = 1$ and $n = 2$ in the solution (3.59). Increasing the value of β for a given α, the probability density admits a larger uncertainty for $\xi = x + i\frac{\beta}{\alpha^2}$. We comment that in Eq. (3.59) by shifting the coordinate x as $x = \xi - i\frac{\beta}{\alpha^2}$ the wave-function (3.59) becomes real in terms of ξ.

Chapter 4

STEP MOMENTUM OPERATOR

In this chapter, we propose a hermitian or $\mathscr{PT}$-symmetric GMO which is founded upon inserting a discrete auxiliary function into the formalism. These examples are considered to be toy models for the more realistic step momentum operator. Let's add that the physical relevance of such model may be interpreted as a quantum particle which feels a change in its velocity in an infinitesimally short time due to an impulsive external force, whilst, classically step momentum implies a particle of constant mass and given velocity. In following, we start with building the hermitian step-GMO in the real plan. Later, we expand the proposal into the complex domain and introduce its $\mathscr{PT}$-symmetric counterpart.

4.1 Hermitian Step Momentum Operator

In pursuit of investigating our introduced GMO, a step hermitian momentum is constructed using the auxiliary function $\mu(x)$ given by

$$\mu(x) = \begin{cases} \mu_0 & x < 0 \\ -\mu_0 & 0 < x \end{cases}, \tag{4.1}$$

where $\mu_0 \in \mathbb{R}^+$. Implementing the auxiliary function into Eq. (3.7) leads to

$$\hat{p} = \begin{cases} -i\hbar(1+\mu_0)\,\partial_x & x < 0 \\ -i\hbar(1-\mu_0)\,\partial_x & 0 < x \end{cases}. \tag{4.2}$$

Let's add that the momentum operator (4.2) is hermitian, i.e, $\hat{p}^\dagger = \hat{p}$ and therefore one expects to attain real eigenvalues, orthogonal eigenfunctions and positive norm. Next, we consider the corresponding eigenvalue equation expressed by

$$\hat{p}\phi(x) = \mathfrak{p}\phi(x) \Rightarrow \begin{cases} -i\hbar(1+\mu_0)\phi'_-(x) = \mathfrak{p}\phi_-(x) & x<0 \\ \\ -i\hbar(1-\mu_0)\phi'_+(x) = \mathfrak{p}\phi_+(x) & 0<x \end{cases}, \tag{4.3}$$

where $\mathfrak{p}$ is the eigenvalue of the eigenfunction $\phi_{\mathfrak{p}}(x)$ which is found to be

$$\phi_{\mathfrak{p}}(x) = N \begin{cases} \exp\left(\frac{i\mathfrak{p}x}{\hbar(1+\mu_0)}\right) & x<0 \\ \\ 1 & x=0 \\ \\ \exp\left(\frac{i\mathfrak{p}x}{\hbar(1-\mu_0)}\right) & 0<x \end{cases}, \tag{4.4}$$

in which N stands for the normalization constant. To have the solution physically accepted, $\mathfrak{p}$ has to be real and due to no additional constraint it is continuous.

4.2 Stationary Schrödinger Equation:

Upon applying Eq. (4.2) into the Hamiltonian operator, one finds the time independent Schrödinger equation in the following form

$$\begin{cases} \frac{-\hbar^2}{2m}(1+\mu_0)^2\partial_x^2\psi(x) + V(x)\psi(x) = E\psi(x) & x<0 \\ \\ \frac{-\hbar^2}{2m}(1-\mu_0)^2\partial_x^2\psi(x) + V(x)\psi(x) = E\psi(x) & 0<x \end{cases}, \tag{4.5}$$

where the potential is assumed to be an infinite potential well defined by

$$V(x) = \begin{cases} 0 & -l<x<l \\ \\ \infty & l \leq |x| \end{cases}. \tag{4.6}$$

It is appropriate to rewrite Eq. (4.5) in the form

$$\begin{cases} \psi(x)'' + \kappa^2\psi(x) = 0 & -l<x<0 \\ \\ \psi(x)'' + \bar{\kappa}^2\psi(x) = 0 & 0<x<l \end{cases}, \tag{4.7}$$

where the double prime stands for the second derivative with respect to x, $\kappa = \frac{\lambda}{(1+\mu_0)}$, $\bar{\kappa} = \frac{\lambda}{(1-\mu_0)}$ and $\lambda^2 = \frac{2mE}{\hbar^2}$. Eq. (4.7) admits a solution expressed as

$$\psi(x) = \begin{cases} Ae^{i\kappa x} + Be^{-i\kappa x} & -l < x < 0 \\ Ce^{i\bar{\kappa} x} + De^{-i\bar{\kappa} x} & 0 < x < l \end{cases}, \tag{4.8}$$

in which A, B, C and D are integration constant. Next, we apply the boundary and continuity conditions on the solution (4.8) which lead to

$$\begin{cases} \psi_-(0) = \psi_+(0), \\ \psi_+(-l) = 0, \\ \psi_-(l) = 0, \\ \partial_x \psi_-(0) = \partial_x \psi_+(0), \end{cases} \Longrightarrow \begin{cases} A + B - C - D = 0 \\ Ae^{-i\kappa l} + Be^{i\kappa l} = 0, \\ Ce^{i\bar{\kappa} l} + De^{-i\bar{\kappa} l} = 0 \\ \kappa(A - B) - \bar{\kappa}(C - D) = 0 \end{cases}. \tag{4.9}$$

The latter is a homogenous system of four equations with four unknown variables i.e., A, B, C and D. The system admits nontrivial solutions if the determinant of coefficients vanishes. This, however, implies

$$\begin{vmatrix} 1 & 1 & -1 & -1 \\ e^{-i\kappa l} & e^{i\kappa l} & 0 & 0 \\ 0 & 0 & e^{i\bar{\kappa} l} & e^{-i\bar{\kappa} l} \\ \kappa & -\kappa & -\bar{\kappa} & \bar{\kappa} \end{vmatrix} = 0 \tag{4.10}$$

or explicitly

$$(\bar{\kappa} - \kappa)\sin l(\kappa - \bar{\kappa}) + (\bar{\kappa} + \kappa)\sin l(\kappa + \bar{\kappa}) = 0. \tag{4.11}$$

We note that κ and $\bar{\kappa}$ contain the energy eigenvalue in accordance with Eq. (4.7). Hence, one can extract the energy eigenvalues, numerically, using the latter equation. To proceed solving (4.9), provided κ and $\bar{\kappa}$ satisfy the zero determinant condition

(4.11), we introduce $b=\frac{B}{A}, c=\frac{C}{A}$ and $d=\frac{D}{A}$ which yields

$$\begin{cases} b-c-d=-1, \\ be^{i\kappa l}=-e^{-i\kappa l}, \\ ce^{i\bar{\kappa}l}+de^{-i\bar{\kappa}l}=0, \\ -b\kappa-\bar{\kappa}c+\bar{\kappa}d=-\kappa \end{cases}. \tag{4.12}$$

The second equation clearly admits

$$b=-e^{-2i\kappa l} \tag{4.13}$$

which upon submitting into the other equations we obtain

$$\begin{cases} c=\frac{\kappa\cos\kappa l}{\bar{\kappa}\cos\bar{\kappa}l}e^{-i(\kappa+\bar{\kappa})l} \\ d=-ce^{2i\bar{\kappa}l} \end{cases}. \tag{4.14}$$

The main equation which gives the legitimate discrete energy eigenvalues i.e., (4.11) and identifies the value of the wave numbers κ and $\bar{\kappa}$ gives also the constants of integration with respect to A. The unknown constant A will be identified using the normalization condition. Let's introduce $\eta=\lambda l$ and recall the explicit definition of wave numbers which together with equation (4.11) imply

$$(1-\mu_0)\sin\left(\frac{\eta}{1-\mu_0}\right)\cos\left(\frac{\eta}{1+\mu_0}\right)+(1+\mu_0)\cos\left(\frac{\eta}{1-\mu_0}\right)\sin\left(\frac{\eta}{1+\mu_0}\right)=0. \tag{4.15}$$

In Fig. (4.1), we plot the left side of the latter equation in terms of η for various values of μ_0. Also, in Tab. (4.1) we present the energy eigenvalues of the first three bound states of the system. We observe that, although, the momentum distribution for negative and positive x axis apparently cancel each other but the energy eigenvalues decrease with increasing μ_0. To see the effect of the redistribution of the momentum operator on the probability density, we continue to find the explicit form of the wave functions. To do so, we substitute the value of $B=bA$, $C=cA$ and $D=dA$ in Eq. (4.8)

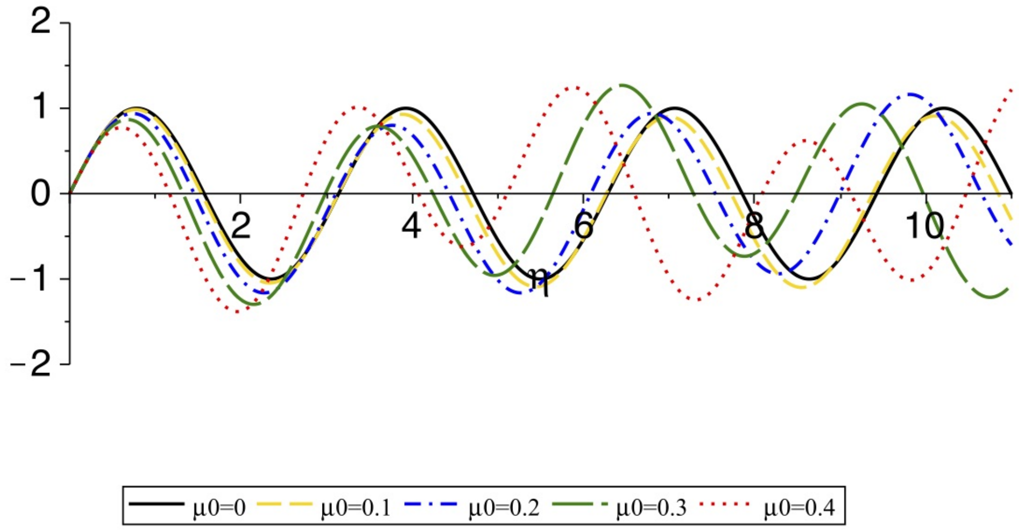

Figure 4.1: Plot of Eq. (4.15) with respect to η for $\mu_0 = 0$ (solid, black), $\mu_0 = 0.1$ (dash, yellow), $\mu_0 = 0.2$ (dash-dot, blue), $\mu_0 = 0.3$ (long-dash, green) and $\mu_0 = 0.4$ (dot, red). The zero of each curve, corresponds to the $\eta_n = \lambda_n l$ in which λ_n gives the allowed energy of the particle. From this figure, we observe that, increasing the value of μ_0, decreases the energy of each corresponding state in comparison with $\mu_0 = 0$.

Table 4.1: The relative energy (i.e. $\frac{E_n}{E_0}$) for $n = 1, 2$ and 3 with $\mu_0 = 0.1, 0.2$ and 0.3. Note that E_0 is the ground state energy of the system with $\mu_0 = 0$.

μ_0	0	0.1	0.2	0.3
$\frac{E_1}{E_0}$	1	0.9621	0.8567	0.7112
$\frac{E_2}{E_0}$	4	3.9988	3.9211	3.6181
$\frac{E_3}{E_0}$	9	8.6796	7.9651	7.3006

and after some manipulation and redefinition of the constant A, the eigenfunctions are obtained to be

$$\psi_n(x) = N_n \begin{cases} \sin(\bar{\kappa}_n l)\sin(\kappa_n(x+l)) & -l < x < 0 \\ -\sin(\kappa_n l)\sin(\bar{\kappa}_n(x-l)) & 0 < x < l \end{cases}, \qquad (4.16)$$

in which N_n is the normalization constant and the subindex $n = 1, 2, 3, \ldots$ is the state number (the zero-number of (4.15)) for each given μ_0. We would like also to add that the normalization constant N_n is function of state. In Fig. (4.2), we plot the

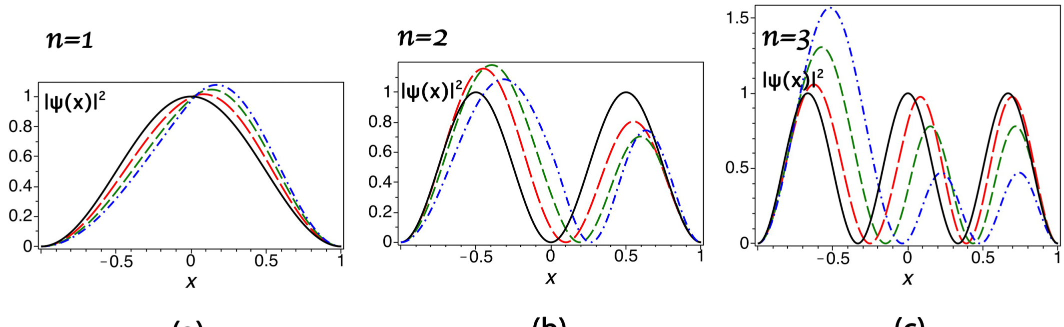

Figure 4.2: Plots of the probability density of a quantum particle with hermitian step momentum in the ground, first excited and second excited states in (a), (b), and (c) for $\mu_0 = 0$ (solid, black), $\mu_0 = 0.1$ (long-dash, red), $\mu_0 = 0.2$ (dash, green) and $\mu_0 = 0.3$ (dash-dot, blue).

probability density $|\psi_n(x)|^2$ for $n = 1,2,3$ with different values of $\mu_0 = 0, 0.1, 0.2, 0.3$. It is observed that with increasing the value of μ_0 the probability distribution changes significantly such that the particle tends to be in the negative x-axis. This is reasonable because by increasing μ_0 the momentum of the particle for $x < 0$ is smaller than for $x > 0$ and the particle will have to spend more time in $x < 0$.

4.3 $\mathscr{PT}$-Symmetric Step Momentum Operator

In this section, we follow the same steps as of the previous section and represent a $\mathscr{PT}$-symmetric step momentum operator. To do so, we set the auxiliary function $\mu(x)$ to be in the form of a finite pure imaginary step function given by

$$\mu(x) = \begin{cases} i\mu_0 & x < 0 \\ -i\mu_0 & 0 < x \end{cases}, \tag{4.17}$$

in which μ_0 is a positive real value. Inserting the latter equation into Eq. (3.7) yields a $\mathscr{PT}$-symmetric momentum operator expressed by

$$p = \begin{cases} -i\hbar(1 + i\mu_0)\,\partial_x & x < 0 \\ -i\hbar(1 - i\mu_0)\,\partial_x & 0 < x \end{cases}. \tag{4.18}$$

Employing (4.18), one finds the time-independent Schrödinger equation in the form of

$$\begin{cases} \frac{-\hbar^2}{2m}\left(1+i\mu_0\right)^2 \partial_x^2 \psi\left(x\right) + V\left(x\right)\psi\left(x\right) = E\psi\left(x\right) & x<0 \\ \frac{-\hbar^2}{2m}\left(1-i\mu_0\right)^2 \partial_x^2 \psi\left(x\right) + V\left(x\right)\psi\left(x\right) = E\psi\left(x\right) & 0<x \end{cases}, \tag{4.19}$$

where $E \in \mathbb{R}$ is the energy eigenvalue of the eigenfunction $\psi\left(x\right)$. Our choice of the potential $V\left(x\right)$ is an infinite square well given in Eq. (4.6). Hence, the Schrödinger equation turns to

$$\begin{cases} \psi\left(x\right)'' + \kappa^2 \psi\left(x\right) = 0 & -l<x<0 \\ \psi\left(x\right)'' + \tilde{\kappa}^2 \psi\left(x\right) = 0 & 0<x<l \end{cases}, \tag{4.20}$$

in which κ and $\tilde{\kappa}$ are defined to be $\kappa^2 = \frac{2mE}{\hbar^2\left(1+i\mu_0\right)^2} = \left(s-it\right)^2$ and $\tilde{\kappa}^2 = \frac{2mE}{\hbar^2\left(1-i\mu_0\right)^2} = \left(s+it\right)^2$. Introducing a new real parameter

$$\lambda = \sqrt{\frac{2mE}{\hbar^2}} \frac{1}{\left(1+\mu_0^2\right)}, \tag{4.21}$$

one obtains $s=\lambda$, $t=\lambda\mu_0$ and $\tilde{\kappa}$ and κ are the complex conjugates of each other. From Eq. (4.21) we may write

$$E = E_0 \left(\frac{2\lambda l}{\pi}\right)^2 \left(1+\mu_0^2\right)^2, \tag{4.22}$$

with $E_0 = \frac{\pi^2\hbar^2}{8ml^2}$. The solution to the Schrödinger equation (4.20) is given by

$$\psi\left(x\right) = \begin{cases} Ae^{i\kappa x} + Be^{-i\kappa x} & -l<x<0 \\ Ce^{i\tilde{\kappa} x} + De^{-i\tilde{\kappa} x} & 0<x<l \end{cases}, \tag{4.23}$$

in which A,B,C and D are four integration constants. We note that the boundary conditions for the new configuration, i.e., the $\mathscr{P}\mathscr{T}$-symmetric momentum operator other than hermitian momentum operator, are the same as Eq. (4.9). Therefore, the similar proceedings in (4.11) to (4.12) leads to a constraint equation in terms of $\eta = \lambda l$ expressed by

$$\sin\left(2\eta\right) + \mu_0 \sinh\left(2\eta\mu_0\right) = 0. \tag{4.24}$$

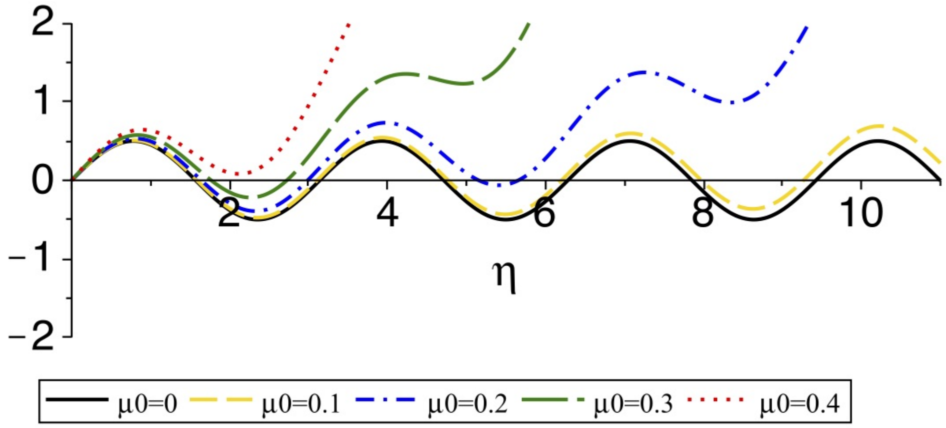

Figure 4.3: Plot of Eq. (4.24) in terms of η, for $\mu_0 = 0$ (solid, black), $\mu_0 = 0.1$ (dash, yellow), $\mu_0 = 0.2$ (dash-dot, blue), $\mu_0 = 0.3$ (long-dash, green), $\mu_0 = 0.4$ (dot, red).

Table 4.2: The relative energy (i.e. $\frac{E_n}{E_0}$) for $n = 1,2$ and 3 with $\mu_0 = 0.1, 0.2$ and 0.3 for a quantum particle with $\mathscr{PT}$-symmetric step momentum. Note that E_0 is the ground state energy of the system with $\mu_0 = 0$. The number of states with real energy is finite as for $\mu_0 = 0.3$, there exist merely two bound states.

μ_0	0	0.1	0.2	0.3
$\frac{E_1}{E_0}$	1	1.0422	1.1824	1.5035
$\frac{E_2}{E_0}$	4	3.9988	3.9205	3.5791
$\frac{E_3}{E_0}$	9	9.4073	11.6578	–

We would like to add that the complex version of equation (4.11) admits two constraint equations due to the real and imaginary part of the equation. However, the imaginary part is satisfied trivially due to our assumption of real energy. Eq. (4.24) is a transcendental equation, therefore, we plot (4.24) in terms of η for several values of μ_0 in Fig. (4.3). The energy eigenvalues are obtained numerically upon applying the zero's of (4.24) where the energy of the first three states are expressed in Tab. (4.2) for $\mu_0 = 0.1, 0.2, 0.3$. It is observed in, Fig. (4.3), that the number of bound states are finite such that for $\mu_0 > 0.377$ there is no bound state with real energy. Following the numerical value for every possible bound state, one can in principle find the wave function up to a normalization constant. The wave function, in terms of κ_n and $\tilde{\kappa}_n$, is

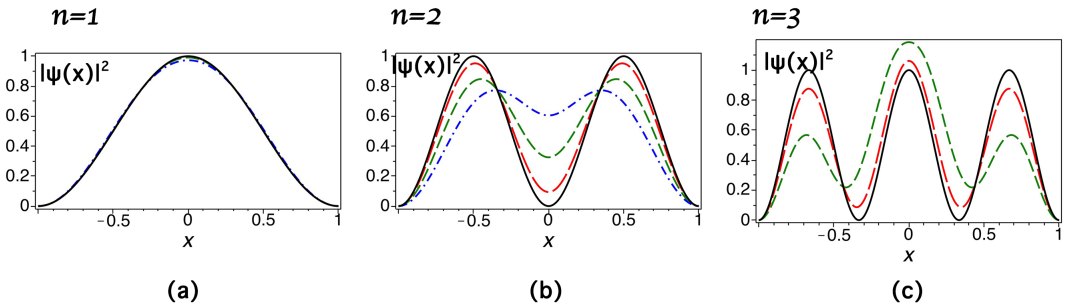

Figure 4.4: Plots of $|\psi(x)|^2$- pseudo-probability density- for a quantum particle with $\mathscr{PT}$-symmetric step momentum inside an infinite square well representing the ground, first excited and second excited states which are corresponded to $\mu_0 = 0$ (solid, black), $\mu_0 = 0.1$ (long-dash, red), $\mu_0 = 0.2$ (dash, green) and $\mu_0 = 0.3$ (dash-dot, blue).

given by

$$\psi_n(x) = N_n \begin{cases} \sin(\tilde{\kappa}_n l)\sin(\kappa_n(x+l)) & -l < x < 0 \\ \sin(\kappa_n l)\sin(\tilde{\kappa}_n(x-l)) & 0 < x < l \end{cases}, \tag{4.25}$$

in which N_n is the normalization constant for the state number $n = 1, 2, 3, \ldots$. We note that $\psi(x)\psi(x)^*$ is not a probability density when the Hamiltonian is not hermitian. We refer to Ref. [42] where three conditions i. $\Im(\psi(x)\mathscr{PT}\psi(x))\,dx = 0$, ii. $\Re(\psi(x)\mathscr{PT}\psi(x)) > 0$ and iii. $\int_C(\psi(x)\mathscr{PT}\psi(x))\,dx = 1$ are indicated, in which x is the complex plane of the particles coordinate. Therefore, instead of $\psi(x)\psi(x)^*$, $\psi(x)\mathscr{PT}\psi(x)$ represents the probability density in complex plane and $\mathbb{C}$ is the specific path where the above conditions are satisfied. Here, it is worth to mention that the $\mathscr{PT}$-symmetric is not exact in this problem. This can be seen clearly if we assume the energy to be complex where we will be able to find the wave function which satisfies all the conditions. In other words, for every specific μ_0 there are finite number of bound state with real energy and infinite number of bound states with complex energy. For the sake of comparison, we plot $|\psi(x)|^2$ in Fig. (4.4) for the first three bound states with different μ_0. The effect of μ_0 can be seen as redistribution of $|\psi(x)|^2$- We also note that $\int |\psi(x)|^2\,dx = 1$.

4.4 Remark

Here, in this section we intend to find a relation between our outcomes and the results which is acquired in Ref. [45]. M. Znojil, in Ref. [45], encounters with the ambiguity of continuation of the $\mathscr{PT}$-symmetric square well potential. It is assumed that the energy is real and defined as $E = t^2 - s^2$ with $Z = 2st$ to be the measure of non-Hermiticity. However, we rename the latter energy as E_z and the energy obtained in the present study E_μ, the subindices Z and μ stand for the factor representing the non-Hermiticity in both studies. Considering $\hbar^2 = 2m = 1$, the wave numbers are given by

$$\begin{cases} \kappa_z^2 = E_z - iZ, \quad \kappa_z^{*2} = E_z + iZ \\ \kappa_\mu^2 = \frac{E_\mu}{(1+i\mu_0)^2}, \quad \tilde{\kappa}_\mu^2 = \frac{E_\mu}{(1-i\mu_0)^2} \end{cases}. \tag{4.26}$$

Having the identical formulation corresponding to the wave functions in Eq. (4.8) and Eq. (6) in [45], we suppose the wave numbers are equal, therefore, from the equality of the latter equations one finds

$$\begin{cases} E_z - \frac{(1-\mu_0^2)}{(1+\mu_0^2)^2} E_\mu = 0 \\ Z - \frac{2\mu_0}{(1+\mu_0^2)^2} E_\mu = 0 \end{cases}. \tag{4.27}$$

Utilizing equations in Eq. (4.27) with $Z, \mu_0 > 0$, the relationship between μ_0 and Z admits

$$\mu_0 = \frac{E_z}{Z}\left(-1 + \sqrt{1 + \frac{Z^2}{E_z^2}}\right). \tag{4.28}$$

The physical approach of the latter comparison indicates that a particle with the $\mathscr{PT}$-symmetric step momentum captured in a standard infinite square well observes an effective potential of the form of $\mathscr{PT}$-symmetric square well. This is due to the corresponding variation in the non-Hermiticity factor in the momentum of the system.

Chapter 5

TWO-DIMENSIONAL $\mathscr{PT}$-SYMMETRIC HARMONIC OSCILLATOR: COHERENT STATES [110]

In this chapter, we construct a Hamiltonian operator concerning a non-hermitian potential, upon which we discuss a sample of two-dimensional $\mathscr{PT}$-symmetric Hamiltonian operator who is complexified based on the potential term. Also, a different method is presented to define a $\mathscr{PT}$-symmetric operator in two dimensions. Accordingly, we solve the corresponding Schrödinger equation and compute the exact solution of such differential equation. Later, we discuss the coherent state due to superposition of 12 obtained eigenfunctions, then, represent the outcomes of eigenfunctions.

5.1 2D-Complexified Harmonic Oscillator

We start with the two dimensional time-independent Schrödinger equation with a presumed complex potential in the polar coordinate system given by

$$-\frac{\hbar^2}{2m}\nabla^2\psi(r,\phi)+\frac{m\omega^2}{2}\left(\Lambda\frac{e^{i\phi}}{r}+1\right)r^2\psi(r,\phi)=E\psi(r,\phi), \tag{5.1}$$

in which m and ω are the mass and the angular frequency of the harmonic oscillator and Λ is a positive constant parameter. In Fig. (5.1) we plot the absolute value of the deformed harmonic oscillator potential over $\frac{m\omega^2}{2}$ in terms of ϕ for different values of Λ and $r=1$. It is observed that a nonzero Λ, particularly $0<\Lambda<1$, modifies the behavior of the potential (in terms of ϕ) significantly. To simplify and make the differential equation separable, we transfer (5.1) from the polar to the Cartesian

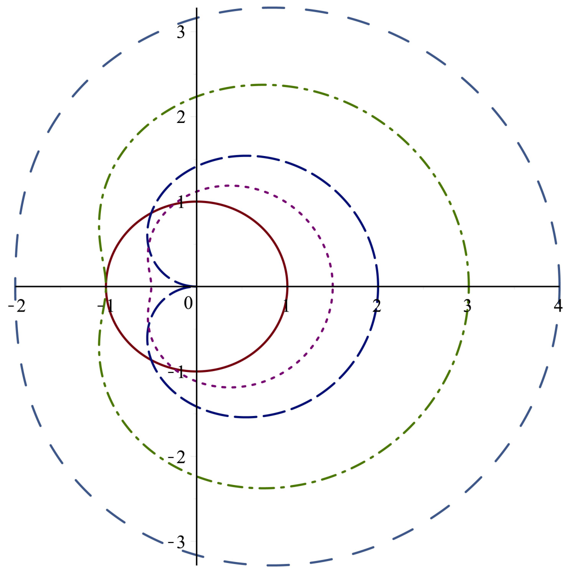

Figure 5.1: The behavior of the deformed harmonic oscillator potential (its absolute value over $\frac{m\omega^2}{2}$) in terms of the polar angle ϕ when $r = 1$ and $\Lambda = 0, 0.5, 1, 2$ and 3 corresponding to solid, dot, dash, dash-dot and dash-space respectively. At the two extremal limits where $\Lambda = 0, \infty$ the potential becomes ϕ-symmetric but for the values of Λ in between the potential depends on ϕ critically.

coordinates system where the potential becomes

$$V(x,y) = \frac{m\omega^2}{2}\left(\Lambda(x+iy)+x^2+y^2\right). \tag{5.2}$$

Based on the assumption of the $\mathscr{PT}$-symmetric Hamiltonian potential, $V(x,y)$ should not vary under the $\mathscr{PT}$-transformation. Thus, the condition $V^*(-x,-y) = V(x,y)$ has to be hold. Apparently in Eq. (5.2) the x component of the proposed harmonic oscillator is not invariant under the parity reflection while the y segment completely supports the time and space symmetries. This implies that $V^*(-x,-y) \neq V(x,y)$ and hence the potential is not $\mathscr{PT}$-symmetric. To cope with this inconsistency, one can decompose the potential into $V_x(x) + V_y(y)$ as two combined one-dimensional oscillators. This yields a hermitian and a $\mathscr{PT}$-symmetric potential given by

$$V_x(x) = x^2 + \Lambda x \tag{5.3}$$

and

$$V_y(y) = y^2 + i\Lambda y \tag{5.4}$$

respectively. A similar treatment has been used in [64] where the overall potential was called $\Pi\mathscr{T}$-symmetry (instead of $\mathscr{P}\mathscr{T}$-symmetry) such that $\Pi\mathscr{T}$-operator is an invertible, non-hermitian operator, consists of a time and phase reflectors in polar coordinates, given by

$$\Pi : \phi \to 2\pi - \phi, \mathscr{T} : i \to -i. \tag{5.5}$$

Introducing $\alpha = \frac{m\omega}{\hbar}$, after applying the separating method on (5.1), one finds

$$-X'' + \alpha^2 \left(\frac{\Lambda}{2} + x\right)^2 X = k_x^2 X \tag{5.6}$$

and

$$-Y'' + \alpha^2 \left(i\frac{\Lambda}{2} + y\right)^2 Y = k_y^2 Y. \tag{5.7}$$

in which $\psi(x,y) = X(x)Y(y)$, $k_x^2 + k_y^2 = k^2$, $k_x^2 = \frac{2mE_x}{\hbar^2}$, $k_y^2 = \frac{2mE_y}{\hbar^2}$ and $E_x + E_y = E$. Furthermore, we apply a change of variables expressed by $\tilde{x} = \left(x + \frac{\Lambda}{2}\right)\sqrt{\alpha}$, $\tilde{y} = \left(y + i\frac{\Lambda}{2}\right)\sqrt{\alpha}$ and $\tilde{k}_{x,y}^2 = k_{x,y}^2/\sqrt{\alpha}$ to simplify the above equations as

$$-X'' + \tilde{x}^2 X = \tilde{k}_x^2 X \tag{5.8}$$

and

$$-Y'' + \tilde{y}^2 Y = \tilde{k}_y^2 Y. \tag{5.9}$$

Now, we are dealing with two simple harmonic oscillators whose eigenvalues and eigenvectors are known. Referring to any standard text-book in quantum mechanics, one writes the full eigenvalues and eigenfunctions of each coordinate as given by

$$X_n = C_n H_n e^{-\tilde{x}^2/2} \tag{5.10}$$

and

$$Y_m = C_m H_m e^{-\tilde{y}^2/2}, \tag{5.11}$$

with their correspondence eigenvalues

$$E_{x,n} = \frac{\hbar\omega}{2}(2n+1) \tag{5.12}$$

and

$$E_{y,m} = \frac{\hbar\omega}{2}(2m+1) \tag{5.13}$$

in which $n, m = 0, 1, 2, ...$ and H_n/H_m are the Hermite polynomials while the constants C_n/C_m are the normalization constants. The energy of the real and complex part of the system behave as the 2D harmonic oscillator with the real potential [92]. The total eigenfunction is found to be

$$\psi_{nm}(\tilde{x},\tilde{y}) = \frac{1}{\sqrt{C_{nm}}} H_n(\tilde{x}) H_m(\tilde{y}) e^{-\left(\tilde{y}^2+\tilde{x}^2\right)/2} \tag{5.14}$$

in which $\frac{1}{\sqrt{C_{nm}}} = C_n C_m$. Since, the $\Pi\mathscr{T}$-symmetry is preserved by the Hamiltonian (i.e., $[H, \Pi\mathscr{T}] = 0$), Eq. (5.1) is simultaneous eigenstates of the Hamiltonian and $\Pi\mathscr{T}$-operators. To normalize the eigenfunctions, we refer to the redefinition of the norm in the Hilbert space due to the implication of the non-hermitian Hamiltonian [38, 42]. The inner product of the two different eigenfunctions is defined as

$$\langle \psi_{nm} | \psi_{n'm'} \rangle = \int dxdy \left(\Pi\mathscr{T}\psi_{nm}\right) \psi_{n'm'} = \delta_{nn'}\delta_{mm'}(-1)^m. \tag{5.15}$$

Applying $\Pi\mathscr{T}$-operator on ψ_{nm}, X_n and Y_m demonstrates different attribute. Based on the real argument, X_n does not vary under the $\Pi\mathscr{T}$ transformation. Thus, the normalization follows $\int_{-\infty}^{\infty} H_n^2 e^{-\tilde{x}^2} d\tilde{x} = 2^n n! \sqrt{\pi}$. In the case of Y_m, the effect of $\Pi\mathscr{T}$-operator on $H_m\left(y + i\frac{\Lambda}{2}\right)$ gives $H_m\left(-y - i\frac{\Lambda}{2}\right) = (-1)^m H_m\left(y + i\frac{\Lambda}{2}\right)$. Therefore, $\langle \psi_{nm} | \psi_{nm} \rangle$ for odd and even m is negative and positive, respectively. What is the physical interpretation of the negative norm in the Hilbert space? It is identified as the

Charge, Parity and Time symmetry ($\mathscr{C}\Pi\mathscr{T}$, hereafter) to hold the symmetry of the Hamiltonian unbroken [37]. The so-called charge operator ($\mathscr{C}$) is introduced to modify any theory with unbroken $\Pi\mathscr{T}$-symmetry. The $\mathscr{C}$-operator is linear and represented in coordinate space as a summation of simultaneous eigenfunctions of the Hamiltonian and $\Pi\mathscr{T}$-operator [38]. In fact, $\mathscr{C}$-operator reverses the negative sign of odd m in Eq. (5.15) to fulfill the positivity of the norm in Hilbert space. The $\mathscr{P}$ and $\mathscr{C}$ operators do not equate because the parity is a real operator but the charge is complex. Indeed, the Hermiticity convention in a quantum mechanical model reforms to $\mathscr{C}\Pi\mathscr{T}$, where,

$$(\mathscr{C}\Pi\mathscr{T})\,\hat{H}\,(\mathscr{C}\Pi\mathscr{T}) = \hat{H} \tag{5.16}$$

and therefore

$$\langle\psi_{nm}|\psi_{nm}\rangle_{\mathscr{C}\Pi\mathscr{T}} = \int_{-\infty}^{\infty} dx \int_{C} dy\,(\mathscr{C}\Pi\mathscr{T}\,\psi_{nm})\,\psi_{nm}. \tag{5.17}$$

Herein C is the integration contour for the complex y coordinate which is the real y axis shifted down on the imaginary axis as of $y \to y - i\frac{\Lambda}{2}$. Consequently the $\mathscr{C}\Pi\mathscr{T}$-norm becomes

$$\langle\psi_{nm}|\psi_{nm}\rangle_{\mathscr{C}\Pi\mathscr{T}} = \int_{-\infty}^{\infty} dx \int_{-\infty-i\frac{\Lambda}{2}}^{\infty-i\frac{\Lambda}{2}} dy\,(\mathscr{C}\Pi\mathscr{T}\,\psi_{nm}(x,y))\,\psi_{nm}(x,y)\,. \tag{5.18}$$

Furthermore, as $\mathscr{C}\Pi\mathscr{T}\,\psi_{nm}(x,y) = \psi_{nm}(x,y)$ the change of variable $y = \tilde{y} - i\frac{\Lambda}{2}$ yields

$$\langle\psi_{nm}|\psi_{nm}\rangle_{\mathscr{C}\Pi\mathscr{T}} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\tilde{y}\,(\psi_{nm}(x,\tilde{y}))^{2} \tag{5.19}$$

which upon the fact that $\psi_{nm}(x,\tilde{y})$ is a real function it leads to

$$\langle\psi_{nm}|\psi_{nm}\rangle_{\mathscr{C}\Pi\mathscr{T}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx d\tilde{y}\,|\psi_{nm}(x,\tilde{y})|^{2} = 1. \tag{5.20}$$

This is because the latter equation is the standard normalization relation for the two dimensional harmonic oscillator. As of the side result, the normalization relation (5.18) suggests that the probability density is defined as

$$\rho(x,y) = (\mathscr{C}\Pi\mathscr{T}\psi_{nm}(x,y))\,\psi_{nm}(x,y)\,. \tag{5.21}$$

This $\rho(x,y)$ with the real x and complex y satisfies the complex correspondence principle for the probability density on the specific contour C [38, 42].

5.1.1 The Coherent States of the Two Dimensional Harmonic Oscillator

The relation between the classical and quantum mechanical approaches has been substantially studied for decades under the discussion of correspondence principle. The coherent state is distinguished as an superposition of numerous quantum mechanical states, having minimized uncertainty with the mean energy of the correspondence states which are not orthogonal. Here we resemble a classical two dimensional harmonic oscillatory motion with the corresponding quantum mechanical circumstances. Applying variation method on the classical Lagrangian, gives the equations of motion of a particle undergoing the two dimensional complexified harmonic oscillator potential (5.1) with solutions given by

$$\tilde{x} = |\beta|\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t - \theta_x) \tag{5.22}$$

and

$$\tilde{y} = |\gamma|\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t - \theta_y)\,. \tag{5.23}$$

The classical position of the particle given in Eqs. (5.22) and (5.23) imply the oscillational behavior as described in the classical mechanical domain. In fact, these equations yield an elliptical motion depending on the amplitude and the phase difference of the correspondence components. For the real potential, it represents a real elliptical motion [92] but as the imaginary term is added into this system, it deforms the elliptical attribute. Furthermore, the coherent states of the two dimensional harmonic oscillator may be formed by a simple multiplication of the two one-dimension corresponding coherent states, namely

$$\tilde{\Psi}_{\beta\gamma}(\tilde{x},\tilde{y}) = \langle\tilde{x}|\beta\rangle\,\langle\tilde{y}|\gamma\rangle\,. \tag{5.24}$$

Herein, $|\beta\rangle$ and $|\gamma\rangle$ are the coherent states corresponding to $\tilde{x}$ and $\tilde{y}$ coordinates mentioned in Eq. (1.15), respectively. In the case of a temporal coherent state, time in exponential regime is allocated i.e.,

$$\tilde{\Psi}_{\beta\gamma}(\tilde{x},\tilde{y},t) = e^{-\frac{1}{2}\left(|\beta|^2+|\gamma|^2\right)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\beta^n\gamma^m}{\sqrt{n!m!}}\psi_{nm}(\tilde{x},\tilde{y})\,e^{-\hbar\omega(n+m+1)t}. \tag{5.25}$$

Using Cauchy product for the two partial series, the time evolutionary form of the two dimensional harmonic oscillator's coherent state, i.e., $\tilde{\Psi}_{\beta\gamma}(\tilde{x},\tilde{y},t)$ reads as

$$\tilde{\Psi}_{\beta\gamma}(\tilde{x},\tilde{y},t) = e^{-\frac{1}{2}\left(|\beta|^2+|\gamma|^2\right)} \sum_{N=0}^{\infty}\sum_{K=0}^{N} \frac{\beta^K\gamma^{N-K}e^{-\hbar\omega(N+1)t}}{\sqrt{K!\,(N-K)!}}\psi_{K,N-K}(\tilde{x},\tilde{y})\,. \tag{5.26}$$

In this respect $n=K, m=N-K$, and $\frac{\beta}{\gamma}=Ae^{i\theta}$ in which A and θ represent the relative amplitude and phase difference between $\tilde{x}$ and $\tilde{y}$ coordinates, respectively. Finally we find the coherent state up to any N given by

$$\tilde{\Psi}_{\beta\gamma}(\tilde{x},\tilde{y},t) = \sum_{N=0}^{\infty} C_N\Phi_N(\tilde{x},\tilde{y})\,e^{-\hbar\omega(N+1)t}, \tag{5.27}$$

where

$$\Phi_N(\tilde{x},\tilde{y}) = \left(\frac{1}{\sqrt{1+|A|^2}}\right)^N \sum_{K=0}^{N}\binom{N}{K}^{1/2}\left(Ae^{i\theta}\right)^K\psi_{K,N-K}(\tilde{x},\tilde{y})\,. \tag{5.28}$$

Let's comment that, $\Phi_N(\tilde{x},\tilde{y})$ expresses the elliptical stationary coherent state of an oscillator whose phase, amplitude, and also the constant Λ give its final form [92].

Normalization: The normalization procedure mimics the $\mathscr{C}\Pi\mathscr{T}$-symmetry which converts the odd terms signs. Ultimately, the time-independent part of the coherent state is normalized to unity. In the other words, application of $\mathscr{C}\Pi\mathscr{T}$ on the coherent state is found to be

$$\mathscr{C}\Pi\mathscr{T}\Phi_N = \left(\frac{1}{\sqrt{1+|A|^2}}\right)^N \sum_{K=0}^{N}\binom{N}{K}^{1/2}\left(Ae^{-i\theta}\right)^K\mathscr{C}\Pi\mathscr{T}\psi_{K,N-K}(x,y)\,, \tag{5.29}$$

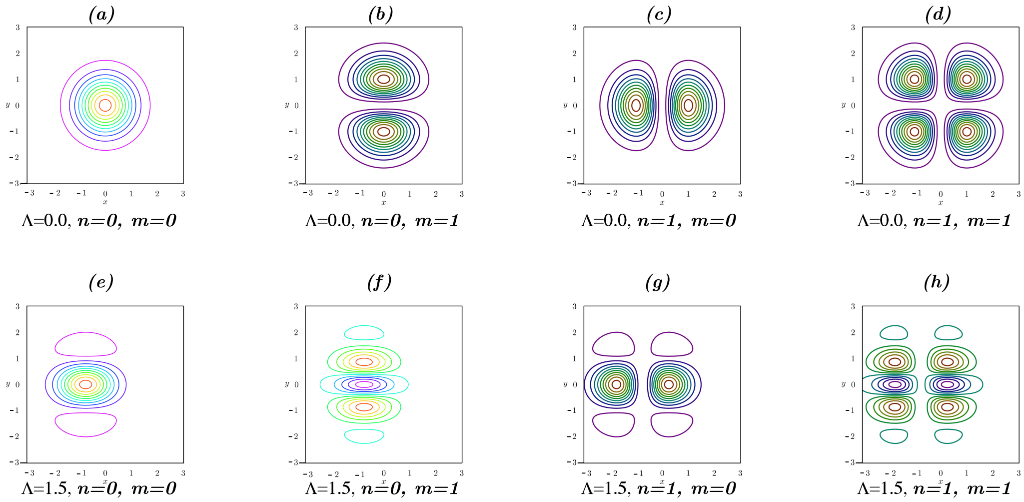

Figure 5.2: Plots of $|\psi_{00}|^2$, $|\psi_{01}|^2$, $|\psi_{10}|^2$and $|\psi_{11}|^2$ with $\Lambda = 0$, from left to right respectively in the first row. Plots of $\Re\left[(\psi_{00})^2\right]$, $\Re\left[(\psi_{01})^2\right]$, $\Re\left[(\psi_{10})^2\right]$ and $\Re\left[(\psi_{11})^2\right]$ with $\Lambda = 1.5$ from left to right respectively in the second row.

which results in

$$\int_{-\infty}^{\infty} dx \int_C dy\,(\mathscr{C}\Pi\mathscr{T}\Phi_N)\,\Phi_N dxdy = 1. \tag{5.30}$$

5.2 Results

The $\Pi\mathscr{T}$-symmetric Hamiltonian with a complexified two dimensional harmonic oscillator is considered within the time independent Schrödinger equation. Wavefunctions ascertain in complex pattern due to the complex argument of the Hermite polynomial for the y component. Normalization of the outcome wavefunctions are carried out based on the determination of the $\mathscr{C}\Pi\mathscr{T}$-symmetry which is a weaker constraint in comparison to the hermitian Hamiltonian. Based on the definition of the $\mathscr{C}\Pi\mathscr{T}$-operator and the explicit form of the $\mathscr{C}\Pi\mathscr{T}$-normalized eigenfunctions given by

$$\psi_{nm}(x,y) = \frac{e^{-\left(x^2+y^2+\Lambda(x+iy)\right)/2}}{\sqrt{2^n 2^m n! m! \pi}} H_n\left(x+\frac{\Lambda}{2}\right) H_m\left(y+i\frac{\Lambda}{2}\right) \tag{5.31}$$

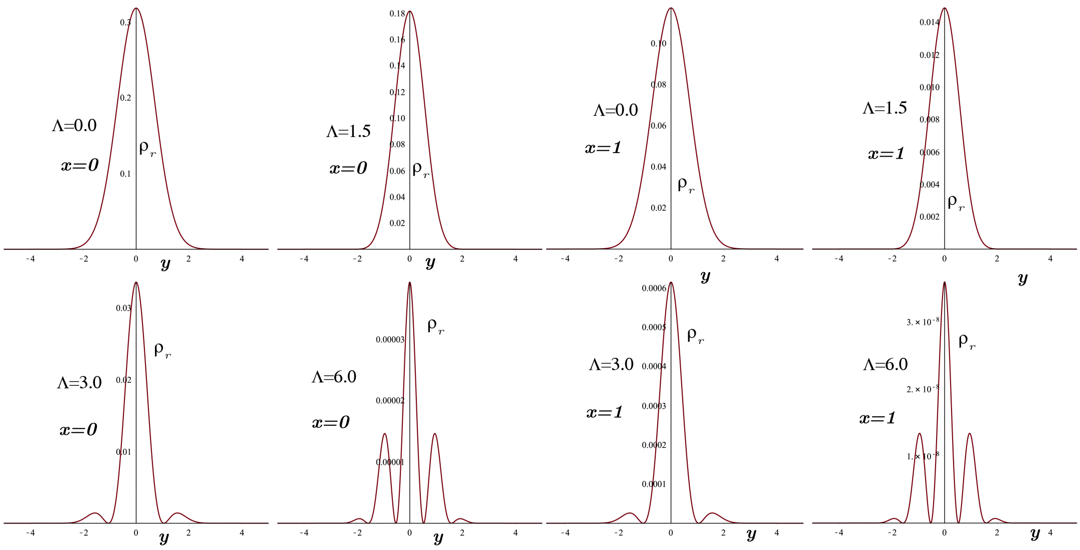

Figure 5.3: $\Re\left[(\psi_{00})^2\right]$ in terms of y for different values of Λ and two discrete values of $x = 0$ and $x = 1$.

with $n, m = \{0, 1, 2, .., 12\}$, one can easily show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\mathscr{C}\Pi\mathscr{T}\psi_{nm}(x,y))\,\psi_{nm}(x,y)\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\psi_{nm}(x,y))^2\,dxdy = 1. \quad (5.32)$$

We note that $(\psi_{nm}(x,y))^2 = |\psi_{nm}(x,y)|^2$ for $\Lambda = 0$ - which is nothing but the standard probability density of the real two dimensional harmonic oscillator - while for $\Lambda \neq 0$, $(\psi_{nm}(x,y))^2 \neq |\psi_{nm}(x,y)|^2$. These suggest that we assume the probability density for the general case to be of the form of $(\psi_{nm}(x,y))^2$ other than $|\psi_{nm}(x,y)|^2$. Furthermore, the normalization condition (5.32) implies that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Re\left[(\psi_{nm}(x,y))^2\right] dxdy = 1 \quad (5.33)$$

while

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Im\left[(\psi_{nm}(x,y))^2\right] dxdy = 0 \quad (5.34)$$

which indicates that $Re\left[(\psi_{nm}(x,y))^2\right]$ carries information about the particle in the real space. Hence, in order to investigate the influence of the parameter Λ, one may plot $\Re\left[(\psi_{nm}(x,y))^2\right]$ with different values of Λ and the results may be compared with the actual probability density corresponding to $\Lambda = 0$. This is what we will do in

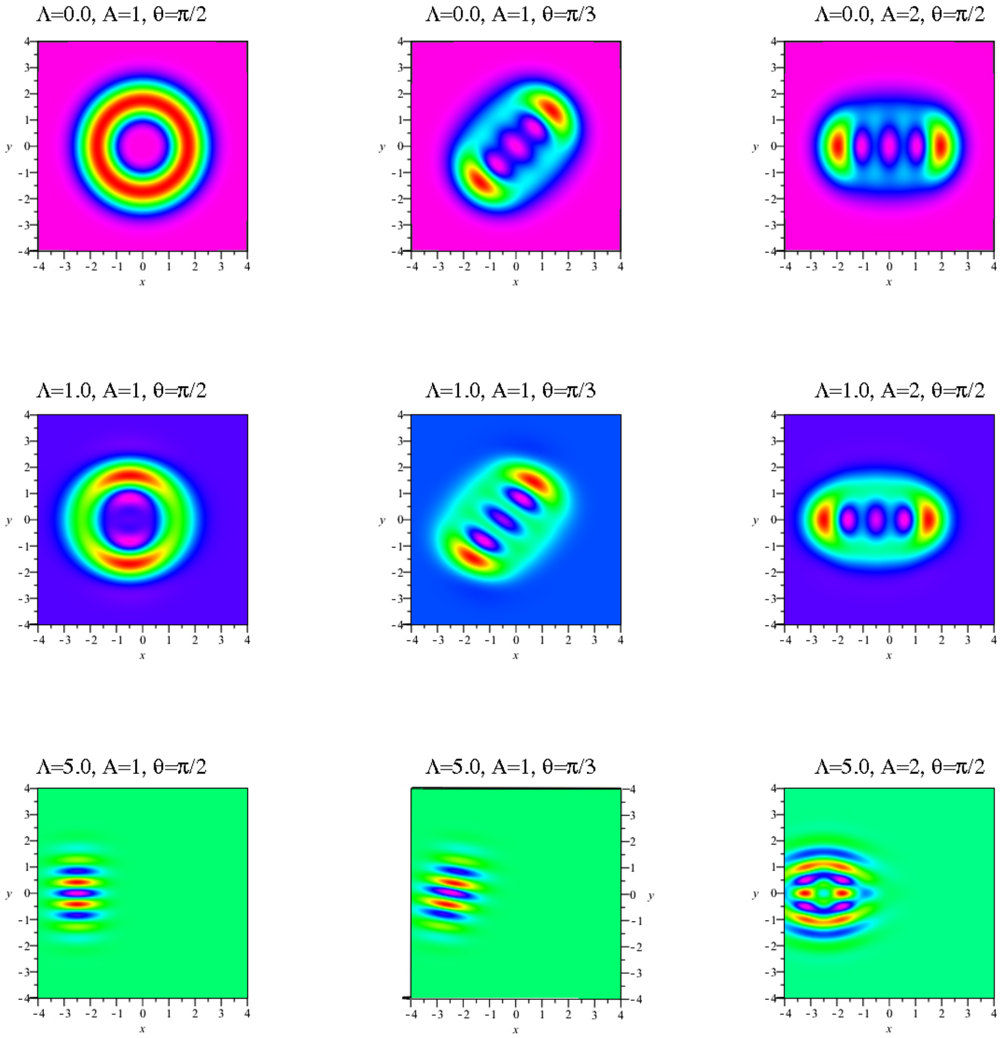

Figure 5.4: Plots of the $\Re\,(\Phi_3)^2$ in terms of x and y with various value of Λ, A and θ, in accordance with Eq. (5.28) with $N = 3$. The values of the parameters are specified on each individual graphs.

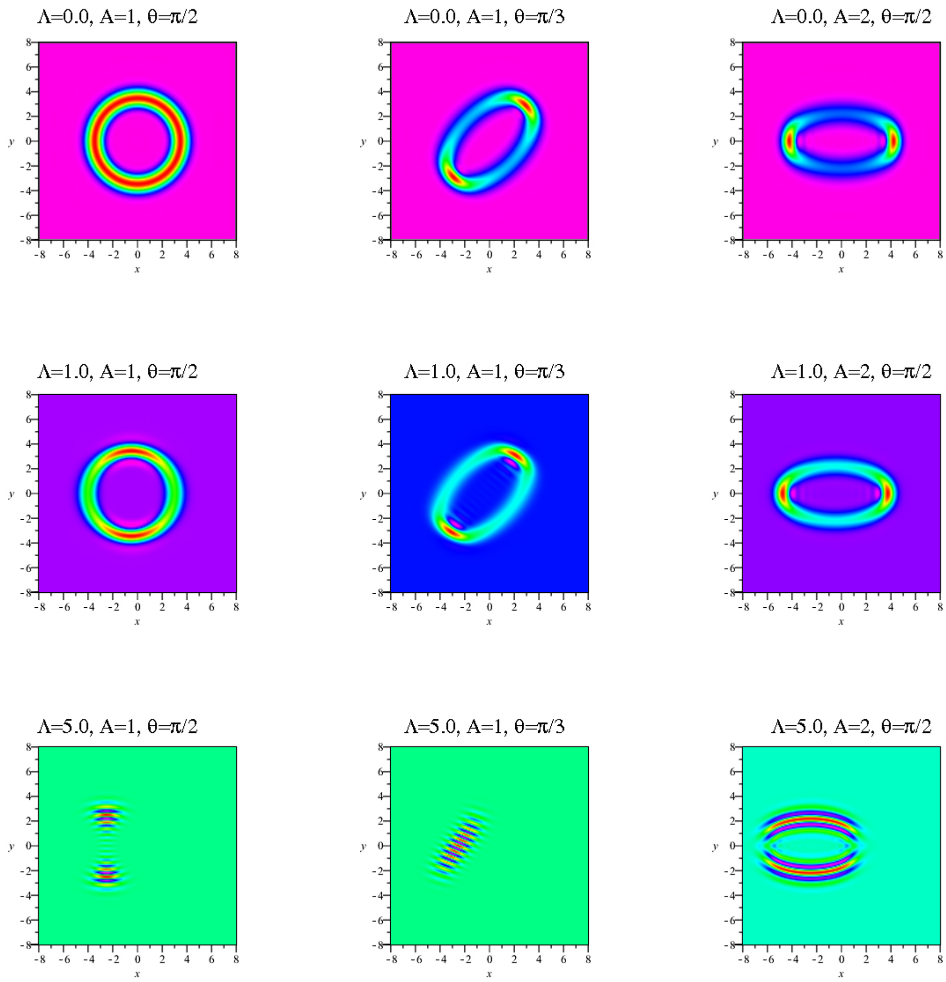

Figure 5.5: Plots of the $\Re\left(\Phi_{12}\right)^2$ in terms of x and y with various value of Λ, A and θ, in accordance with Eq. (5.24) with $N = 12$. The values of the parameters are specified on the each graph.

the sequel. Plots of $\Re\left[(\psi_{nm}(x,y))^2\right]$ in terms of x and y for different values of Λ are displayed in Fig. (5.2). The first row illustrates the probability density considering $\Lambda = 0$. The lower ones depict the deformation due to imposing the complexified potential into the system such that Λ determines the strength of the complexity. Fig. (5.3) depicts the real part of the probability density of the ground state of the complexified harmonic oscillator in terms of y with different values of Λ at $x = 0$ and $x = 1$. While the effect of Λ is easily seen in this figure, the influence of x should be found in the numerical values on the graphs. Furthermore, by superposing 12 eigenstates of the $\Pi\mathscr{T}$-symmetric Hamiltonian introduced earlier, we found the stationary coherent state of the system. Similarly, as it was mentioned for the normalization of the wavefunction $\psi_{nm}(x,y)$, $\mathscr{C}\Pi\mathscr{T}$-operator is employed to normalize the corresponding coherent states. In this respect, we assume that the probability density of Φ_N is given by $\Re\left[(\Phi_N)^2\right]$. The results are shown in Figs. (5.4) and (5.5) where we plot $\Re\left[(\Phi_N)^2\right]$ in terms of x and y for $N = 3$ and $N = 12$, respectively. It is observed that variation of the amplitude and phase difference change the form of the elliptical behavior.

Chapter 6

CONCLUSION

Momentum operator was generalized and its structure was extended during the present research. The generalization included extending the spacial coordinate in higher orders under the influence of the geometry of space. Having considered the concept of EUP in the curved space upon a cutoff in momentum, the proposed generalized momentum operator confirmed the non-commutative relation i. e., $[\hat{x},\hat{p}] = -i\hbar A(x)$, in which an auxiliary function $A(x)$ has been used in this structure. Upon our proposal, we discussed about the properties of hermitain and non-hermitian generalized momentum operator (GMO) concerning the real and complex $A(x)$.

In chapter (2), we have introduced a generalized Lagrangian density in the real domain in terms of $\Psi(x,t)$ and $\Psi^*(x,t)$ (complex conjugate) and in complex domain employing field $\Psi(x,t)$ and its $\mathscr{P}\mathscr{T}$-symmetric conjugate $\Psi^{\#}(x,t)$ [106]. By virtue of the variation of the Lagrangian density, we used the Euler-Lagrange equations and obtained the equations of motion for each case. Furthermore, we calculated the Hamiltonian density and showed that the energy of the system is the expectation value of the Hamiltonian operator. Later, we constructed the elements of stress-energy tensor such as energy flux and momentum density. Besides, the probability density and particle current density were generalized whereas the corresponding continuity equation, also, was approved. In Sec.(2.3), we used the generalized Lagrangian density which was found in Sec. (2.1) and employed fields in Sec. (2.2). Regarding the non-hermitian case, we claimed that the continuity equation was confirmed based on new definition of probability and current densities. The discussion underlying the Lagrange and Hamilton dynamics led to identify a Schrödinger equation founded by our illustration of GMO.

In chapter (3), we discussed and represented in details the structure of GMO with the hermitian and non-hermitian approaches [107, 108]. In the context of non-hermitian quantum physics, it was shown that $\mathscr{PT}$-symmetric Hamiltonians may admit real energy spectrum provided their $\mathscr{PT}$-symmetry is unbroken/exact. For such a Hamiltonian, a complex potential is usually responsible for the non-hermiticity. Here, we have denoted that, according to the definition of GMO, one may introduce a $\mathscr{PT}$-symmetric Hamiltonian through a $\mathscr{PT}$-symmetric momentum operator in virtue of a $\mathscr{PT}$-symmetric auxiliary function, $A(x) = 1 + \mu(x)$. We have also presented an explicit example consist of a free particle with a specific choice of $\mu(x)$. We obtained the normalized eigen-states on the contour $\mathfrak{C}$ within the complex plan whose energy eigenvalues were found to be real. Furthermore we have found the eigenvalues of $\mathscr{PT}$-symmetric GMO and the eigen-functions.

In chapter (4), we suggested a structure to form a step GMO justifying the EUP [109]. The paradigm was initially concerned with the hermitian operator in Sec. (4.1). Then, we modified the scheme to cover a wider realm upon generating $\mathscr{PT}$-symmetric step GMO. Moreover, we continued the investigation around an exemplar developing the step momentum inspired by the infinite non-hermitian square well potential [45]. Its eigenvalue-problem has been solved and the momentum eigenvalues and eigenfunctions have been obtained. The corresponding Schrödinger equation of such particles undergoing an infinite square well has been solved. We plot the probability densities of some of the lower states to see the effect of the main parameter μ_0 in the distribution of the particle inside the well. It is observed that the higher the step i. e., $2\mu_0$, the more deviated distribution. This redistribution is in a way that the particle tends to stay in the region with a smaller momentum operator which is in agreement with our classical experience. In Sec. (4.3), we extended the concept of step momentum and introduced the $\mathscr{PT}$-symmetric step momentum. The energy spectrum of a particle with $\mathscr{PT}$-symmetric momentum inside a square well has been calculated and we have shown that the number of bound states with real energy is finite. Also, it indicates that the $\mathscr{PT}$-symmetry of the system is broken. Besides, we plotted the pseudo-probability density for a few lower states. The plots display the modification of the probability density for $\mu_0 \neq 0$. We also found that for $\mu_0 > 0.377$ there are no bound states with real energy.

In chapter (5), we studied a two dimensional non-hermitian Hamiltonian with the complexified potential, a two dimensional harmonic oscillator [110]. We found the eigenvalues and eigenfunctions of the Hamiltonian analytically. Since the corresponding Hamiltonian is $\Pi\mathscr{T}$-symmetric, described an alternative definition of parity, the energy spectrum is real while the energy eigenfunctions are complex. To resolve the negative norm of the eigenfunctions, we employed the concepts of the charge operator $\mathscr{C}$ and the so called $\mathscr{C}\Pi\mathscr{T}$-norm. Furthermore, we carried on this work to find the coherent states of the complex wave functions of the complexified two dimensional harmonic oscillator. The time-independent part of the coherent state describes the deformed-elliptical motion of the wave packet without spreading behavior. Finally, we plotted $\Re\left[(\psi_{nm}(x,y))^2\right]$ and $\Re\left[(\Phi_N)^2\right]$ with different configurations to observe the effect of the non-Hermiticity parameter Λ. Since with $\Lambda = 0$, $\Re\left[(\psi_{nm}(x,y))^2\right]$ and $\Re\left[(\Phi_N)^2\right]$ reduce to the standard probability densities i.e., $|\psi_{nm}(x,y)|^2$ and $|\Phi_N|^2$, respectively, and also they satisfy (5.33) we have considered them to be the reasonable candidates for the probability densities in the real *xy*-plane. Figs. (5.4) and (5.5) reveal the effects of the parameter Λ in the probability density of the coherent state of the complexified two dimensional harmonic oscillator in the form of either diffusions or dispersions.

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APPENDICES

Appendix A: Derivation of $f(x)$

The GMO given in Eq. (3.7) is hermitian, here is the proof. Let's start with the general form of the GMO given by

$$\hat{p} = (1+\mu(x))\,\hat{p}_0 + f(x) \tag{A1}$$

in which $\hat{p}_0 = -i\hbar\frac{d}{dx}$ and x are the canonical momentum and position operators which are both hermitian. Next, we impose the Hermiticity condition i.e., $\hat{p} = \hat{p}^\dagger$ to find $f(x)$. In this line, we write

$$\hat{p}^\dagger = \left((1+\mu(x))\,\hat{p}_0 + f(x)\right)^\dagger \tag{A2}$$

which upon applying the identities $\left(\hat{A}\hat{B}\right)^\dagger = \hat{B}^\dagger\hat{A}^\dagger$ and $\left(f\left(\hat{A}\right)\right)^\dagger = f^\star\left(\hat{A}^\dagger\right)$ one obtains

$$\hat{p}^\dagger = \hat{p}_0\,(1+\mu(x)) + f^\star(x) \tag{A3}$$

in which the $\star$ stands for the complex conjugate. We note that although $f(x)$ is an unknown complex function $\mu(x)$ is assumed to be a real function. Furthermore, the commutating relation of the canonical momentum operator and any function of the position operator is given by $[\hat{p}_0, g(x)] = -i\hbar g'(x)$ which implies

$$\hat{p}^\dagger = (1+\mu(x))\,\hat{p}_0 - i\hbar\mu'(x) + f^\star(x)\,. \tag{A4}$$

Herein, a prime stands for the derivative with respect to x. Equating (A1) and (A4) we obtain

$$f(x) - f^\star(x) = -i\hbar\mu'(x) \tag{A5}$$

and consequently

$$2\mathrm{Im}\,(f(x)) = -\hbar\mu'(x)\,. \tag{A6}$$

This is remarkable to observe that the hermiticity condition i.e., $\hat{p} = \hat{p}^\dagger$ only identifies the imaginary part of the gauge function $f(x)$. However, to recover the canonical momentum when $\mu(x) = 0$, we set the real part of $f(x)$ to be zero and consequently

$$f(x) = i\text{Im}(f(x)) = -\frac{1}{2}i\hbar\mu'(x). \tag{A7}$$

Finally, the hermitian GMO can by written as Eq. (3.7).

Appendix B: Eigenvalues and Eigenfunctions of the GMO

The eigenvalue equation of the GMO is given by

$$\hat{p}\psi_{\mathfrak{p}}(x) = \mathfrak{p}\psi_{\mathfrak{p}}(x) \tag{B1}$$

in which $\mathfrak{p}$ and $\psi_{\mathfrak{p}}(x)$ are the eigenvalue and corresponding eigenfunction of the GMO p, respectively. With $\hat{p}$ given by (3.7) we obtain

$$-i\hbar(1+\mu(x))\frac{d\psi_{\mathfrak{p}}(x)}{dx} - i\hbar\frac{d\mu(x)}{2dx}\psi_{\mathfrak{p}}(x) = \mathfrak{p}\psi_{\mathfrak{p}}(x). \tag{B2}$$

Eq. (B2) can be simplified further in the form

$$\frac{d\psi_{\mathfrak{p}}(x)}{\psi_{\mathfrak{p}}(x)} = \frac{\left(-\frac{\mu'(x)}{2} + \frac{i\mathfrak{p}}{\hbar}\right)}{(1+\mu(x))}dx \tag{B3}$$

which after integration admits

$$\psi_{\mathfrak{p}}(x) = \frac{C}{\sqrt{1+\mu(x)}}\exp\left(\frac{i\mathfrak{p}}{\hbar}\int\frac{dx}{1+\mu(x)}\right) \tag{B4}$$

where C is an integration constant playing the role of the normalization constant. Here we observe that the eigenvalue of the GMO, $\mathfrak{p}$ is real and continuous while the eigenfunction corresponding to any eigenvalue $\mathfrak{p}$ is generally given by (B4). For example, in the first case where $\mu(x) = \alpha^2x^2$ one obtains the normalized eigenfunctions to be

$$\psi_{\mathfrak{p}}(x) = \sqrt{\frac{\alpha}{\pi}}\frac{1}{\sqrt{1+\alpha^2x^2}}\exp\left(\frac{i\mathfrak{p}}{\alpha\hbar}\tan^{-1}(\alpha x)\right). \tag{B5}$$