Properties of Block Matrices

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ABSTRACT

In this master thesis, we study the block matrices and their properties. After giving a general overview on matrices, block matrices, different types of block matrices, and multiplication of two block matrices are discussed. In the inverse section, we first examine inverses of 2×2 block diagonal and block triangular matrices, ideas of proofs here can be extended to a general $n \times n$ block diagonal or a block triangular matrix. Then we give the inverse formula for 2×2 block matrix, in the case that one of the blocks is invertible. We then generalise this to any $n \times n$ block matrix by splitting it into 4 blocks (by producing a 2×2 block matrix). Determinant chapter is covered by two different methods, existing in the literature. First we revise a formulae for determinant of a block matrix where the blocks (matrices) belong to a commutative subring of $M_{n \times n}(F)$, where F is a field or a commutative ring. Then we give the general formula which would work for any block matrix, without any commutativity condition between the blocks. We also present formulas for the determinant of tensor product of two given matrices.

Keywords: block matrix, inverses, determinants, tensor products.

ÖZ

Bu yüksek lisans tezinde, blok matrisler ve özellikleri incelenmiştir. Matrislere genel bir bakış verildikten sonra, blok matrisler, farklı blok matris türleri ve iki blok matrisin çarpımı ele alınmıştır. Blok matrislerin tersleri bölümünde, önce 2×2 blok köşegen ve blok üçgensel matrislerin tersi incelenmiştir. Buradaki ispat yöntemleri genel bir $n \times n$ blok köşegen veya blok üçgensel matrisine genişletilebilir. Daha sonra bloklarin herhangi birinin tersinin olması koşuluna dayanarak 2×2 block matrislerinin terslerinin formülü verilmiştir. Ayrıca bu formül $n \times n$ blok matrisini 4 tane bloğa bölerek genelleştirilebilir (2×2 blok matris üreterek). Determinant bölümü, literatürde var olan iki farklı yöntemle ele alınmıştır. İlk olarak blokların(matrislerin), $M_{n\times n}(F)$' nin değişme özelliği olan alt-halkasına ait olması durumunda (buradaki F bir cisim veya değişme özelliği olan bir halkadır) blok matrisin determinant formülü revize edilmiştir. Bunun yanında bloklar arasında herhangi bir değişme koşulu olmaksızın determinant formülü incelenmiştir. Ayrıca verilen iki matrisin tensör çarpımının determinantı formülleri sunulmuştur.

Anahtar Kelimeler: blok matris, tersler, determinantlar, tensör çarpımlar.

My dear and precious parents; I would like to express my feelings from my heart to you. You have always taught me that success is not about money, but is about pleasure. I love you so much because of the great effort you have put on me from the kindergarten to the university period, and also because you have always been by my side in good and bad times throughout this journey, until the end of my master's period.

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Chapter 1

INTRODUCTION

Block matrices are obtained by dividing a matrix into partitions (blocks) that fit together to form a rectangle or a square, so in a way they can be thought as more advanced versions of matrices. These blocks help to simplify certain algebraic properties on matrices, like taking determinants or inverses, as normally these operations would be very time consuming if a matrix has very large size. Therefore one of the most important points of this thesis is that it explains with formulas how to find the inverses and determinants of large-dimensional matrices in an easier way. Working with block matrices enables us to reach the results of algebraic operations faster as it facilitates operations on large dimensional matrices. There are many published works in literature concerning inverses and determinants of block matrices, for example, inverses of 2×2 block matrices have been studied by Lu and Shiou in [11], determinants of these matrices have been examined by Powell, Silvester and Ali&Khan in [15], [17], [2] respectively, generalized inverses and ranks for 2×2 block matrices are given in [12, 13]. It is very important to first obtain formula for the inverse or determinant of a 2×2 block matrix, because of the following reasons: In the inverse case, any general $n \times n$ block matrix can be split into 4 blocks (by creating a 2×2 block matrix) and in the determinant case determinant of an $n \times n$ block matrix can always be expressed by using determinants of 2×2 matrices via cofactor expansion, see [8].

Block matrices are commonly used in pure mathematics, more specifically in linear algebra, in proofs of many theorems [8], in particular block diagonal matrices appear a lot in canonical forms [4]. These matrices are a common subject of study not only in mathematics but they also have applications in other fields; i.e. physics, computer science, engineering and economy, just to name a few examples. In physics, they are used in electrical networks, dynamical systems, approximation theory of solutions of differential equations, magnetohydrodynamics, fluid mechanics [6], and circulant structure [5], in computer science; in faster block matrix multiplication algorithms with reinforcement learning, quantum coding and puzzle game [7,10,14], and finally in engineering in Stieltjes transform, R-transform, S-transform, free central limit theorem and electric power systems.

This thesis consists of 5 different chapters in total. Block matrices are introduced in Chapter 2, which should make it easier to understand the notions in the coming chapters. Then, in Chapter 3, the inverses of the block matrices are examined. We give theorems which are related to inverse of the 2×2 block diagonal matrices, and 2×2 block triangular matrices, and prove them by the block Gaussian elimination method. After delivering inverse theorems on block diagonal and block triangular matrices, we give the inverse formula for the 2×2 block diagonal matrix, in the case that one of the blocks is invertible. Furthermore, we define a J matrix here, the main task of this block matrix is to change the positions of the columns of the block matrix when it is multiplying it on the right. Conversely, when this matrix J is multiplying a block matrix on the left, it reverses the order of rows in the block matrix. There are different conditions for each inverse theorem, the theorems of inverses of the 2×2 block matrices are given first, and some of them are proved with the block Gaussian elimination method and the rest with the J matrix towards the end of this chapter.

In Chapter 4, we start by talking about what the determinant of an $n \times n$ matrix is and how it can be found by cofactor expansion. The aim of the determinant notion is to convert a matrix into a real number and it is denoted by $|G|$ or by $det(G)$. First, determinants of 2×2 block diagonal matrix and 2×2 block triangular matrices are given. We provide examples for the determinants of 2×2 block diagonal matrices, 2×2 block lower triangular matrices and 2×2 block upper triangular matrices. Here, the more important case is the proof of the determinant of a 2×2 block matrix. Moreover, in this part we define the basic properties of a ring and a field because we provide formula for the determinant of an $n \times n$ block matrix in the case that blocks belong to a commutative subring of $M_{n \times n}(F)$, where F is a field or a commutative ring. Afterwards, we define what a tensor product between two block matrices is and compute the determinants of tensor products. For example, in the case that U is a 2×2 matrix, and W is an $n \times n$ matrix, $U \otimes W$ would be a $2n \times 2n$ matrix. Another important point in this section is to give the proof of the determinant of tensor product of two matrices (under a field) by using mathematical induction. Finally, the determinant of the $N \times N$ block matrix is given without looking for the commutativity condition between the blocks in the matrix. The determinant formula given here is explicitly calculated for the cases $N = 2$ and $N = 3$.

Inverse and determinant theories covered in this thesis can also be found in our recently published paper [16].

Chapter 2

PRELIMINARIES

2.1 Matrices and Operations on Matrices

In this first section, we give a general overview on matrices, and operations on matrices, for details please refer to [1].

Definition 2.1: In linear algebra, a matrix is a rectangular grid of numbers arranged into rows and columns which is an $m \times n$ array of scalars in given field. In other words, the horizontal and vertical lines of entries in a matrix are called rows and columns, respectively. The individual values in the matrix are called entries.

$$T = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mn} \end{pmatrix}, T \in M_{m\times n}(F)$$

The upper case T denotes the name of this matrix. The size of matrix T above is $m \times n$ $(m, n \in Z^{+})$. Subscripts denote the number of rows and the number of columns of matrix T respectively.

Definition 2.2: In linear algebra, matrix multiplication is one of the main operations of the matrices. Let $U \in R_{m\times n}(F)$, $W \in R_{n\times p}(F)$. In order to perform matrix multiplication, the number of columns of the matrix on the left should match the number of rows of the one on the right. If U is an $m \times n$ and W is an $n \times p$, then T will be of size $m \times p$. Otherwise, multiplication operation is not defined. Definition of matrix multiplication is as follows:

$$(UW)_{jk} = \sum_{r=1}^{n} U_{jr} W_{rk}$$

Thus, the entry in row j, column k , of UW is computed by multiplying row j of U , with column k of W .

In other words, if $T = UW$, then, T_{ij} is the dot product of the i^{th} row of U with the j^{th} column of W.

Theorem 2.1: Let K, L, M be matrices such that the following operations are defined. Then

1. Commutativity property in general does not hold.
2. Zero matrix on multiplication

   If $KL = 0$ then it can be that $K \neq 0$ and $L \neq 0$
3. Associativity Property: $(KL)M = K(LM)$
4. Distributivity Properties: For $\beta,\ \gamma \in \mathbb{R}$

   i. $K(L+M) = KL + KM$

   ii. $(K+L)M = KM + LM$

   iii. $\beta(K+L) = \beta K + \beta L$

   iv. $(\beta+\gamma)K = \beta K + \gamma K$
5. Multiplicative Identity:

   For any square matrix K, $KI = IK = K$, where I is the identity matrix of the same order as K .

We will prove the selected items.

Proof of 3:

$$[(KL)M]_{mn} = \sum_k (KL)_{mk} M_{kn} = \sum_k (\sum_s K_{ms} L_{sk}) M_{kn} = \sum_k \sum_s (K_{ms} L_{sk} M_{kn}) =$$

$$\sum_s \sum_k (K_{ms} L_{sk} M_{kn}) = \sum_s K_{ms} (\sum_k L_{sk} M_{kn}) = \sum_s K_{ms} (LM)_{sn} = [K(LM)]_{mn}$$

Proof of 4:

i.

$$[K(L+M)]_{mn} = \sum_k (K)_{mk} (L+M)_{kn} = \sum_k K_{mk} (L_{kn} + M_{kn}) =$$

$$\sum_k (K_{mk} L_{kn} + K_{mk} M_{kn}) = \sum_k K_{mk} L_{kn} + \sum_k K_{mk} M_{kn} = (KL)_{mn} + (KM)_{mn} = [KL + KM]_{mn}$$

Similar proof also works for the next case which is the right distributivity property. Remaining two items can easily be seen.

For item 1, we may give the counter example below.

Example 2.1: Let $K = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $L = \begin{bmatrix} 0 & 2 \\ 4 & 5 \end{bmatrix}$

$$KL = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 16 & 26 \end{bmatrix}$$

but

$$LK = \begin{bmatrix} 0 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 19 & 28 \end{bmatrix}$$

2.2 Block Matrices

In this section, we define block matrices, introduce different types of block matrices, and we define block matrix multiplication. Our main references here are [**?**, 9, 18].

Definition 2.3: A block matrix (partitioned matrix) is a matrix that is clarified as a result of split sections called blocks or submatrices. Intuitively, a matrix interpreted as a block matrix can be visualized as the original matrix with a collection of horizontal and vertical lines, which break it up or partition it, into a collection of smaller matrices.

Example 2.2: Let $L = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ have 4 submatrices A, B, C and D.

Example 2.3: Let $D = \left[\begin{array}{c} W \\ \hline V \end{array}\right]$ have 2 submatrices W and V.

Example 2.4: Let $C = \begin{bmatrix} 4 & 3 & 2 & -9 & -6 \\ 3 & 4 & 12 & 14 & 15 \\ 15 & 17 & 21 & 6 & 8 \\ 13 & 15 & 0 & 6 & 2 \end{bmatrix}$

We can partition this matrix in different ways. We can create maximum 20 different submatrices. Maximum number of submatrices can be equal to the total number of entries in this matrix, in other words, the size of the matrix. First possible way: Let's create submatrices of matrix C with their different sizes as

$$c_1 = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 4 & 12 \end{bmatrix}, c_2 = \begin{bmatrix} -9 & -6 \\ 14 & 15 \end{bmatrix}, c_3 = \begin{bmatrix} 15 & 17 & 21 \\ 13 & 15 & 0 \end{bmatrix}, c_4 = \begin{bmatrix} 6 & 8 \\ 6 & 2 \end{bmatrix}$$

$$C = \left(\begin{array}{c|c} c_1 & c_2 \\ \hline c_3 & c_4 \end{array} \right) = \left(\begin{array}{ccc|cc} 4 & 3 & 2 & -9 & -6 \\ 3 & 4 & 12 & 14 & 15 \\ \hline 15 & 17 & 21 & 6 & 8 \\ 13 & 15 & 0 & 6 & 2 \end{array} \right)$$

Second Possible Way: Let's create submatrices of this matrix with equal size

$$c_1 = \begin{bmatrix} 4 \end{bmatrix}, c_2 = \begin{bmatrix} 3 \end{bmatrix}, c_3 = \begin{bmatrix} 2 \end{bmatrix}, c_4 = \begin{bmatrix} -9 \end{bmatrix}, c_5 = \begin{bmatrix} -6 \end{bmatrix}, c_6 = \begin{bmatrix} 3 \end{bmatrix}, c_7 = \begin{bmatrix} 4 \end{bmatrix},$$

$$c_8 = \begin{bmatrix} 12 \end{bmatrix}, c_9 = \begin{bmatrix} 14 \end{bmatrix}, c_{10} = \begin{bmatrix} 15 \end{bmatrix} c_{11} = \begin{bmatrix} 15 \end{bmatrix}, c_{12} = \begin{bmatrix} 17 \end{bmatrix} c_{13} = \begin{bmatrix} 21 \end{bmatrix}, c_{14} = \begin{bmatrix} 6 \end{bmatrix},$$

$$c_{15} = \begin{bmatrix} 8 \end{bmatrix}, c_{16} = \begin{bmatrix} 13 \end{bmatrix}, c_{17} = \begin{bmatrix} 15 \end{bmatrix}, c_{18} = \begin{bmatrix} 0 \end{bmatrix}, c_{19} = \begin{bmatrix} 6 \end{bmatrix}, c_{20} = \begin{bmatrix} 2 \end{bmatrix}$$

$$C = \left(\begin{array}{c|c|c|c|c} c_1 & c_2 & c_3 & c_4 & c_5 \\ \hline c_6 & c_7 & c_8 & c_9 & c_{10} \\ \hline c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ \hline c_{16} & c_{17} & c_{18} & c_{19} & c_{20} \end{array} \right) = \left(\begin{array}{c|c|c|c|c} 4 & 3 & 2 & -9 & -6 \\ \hline 3 & 4 & 12 & 14 & 15 \\ \hline 15 & 17 & 21 & 6 & 8 \\ \hline 13 & 15 & 0 & 6 & 2 \end{array} \right)$$

There can be other possibilities to divide this matrix into submatrices.

Remark 2.1: The blocks (submatrices) of a block matrix must fit together to form a rectangle or square.

Example 2.5: Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 & 9 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 3 \end{bmatrix}$

$S = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ makes sense since it makes a rectangle.

Example 2.6: $T = \left(\begin{array}{c|c} B & C \\ \hline D & A \end{array}\right) = \left(\begin{array}{c|c} \begin{matrix} 7 & 8 \\ 9 & 10 \end{matrix} & \begin{matrix} 2 & 1 & 9 \end{matrix} \\ \hline \begin{matrix} 2 & 3 \end{matrix} & \begin{matrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{matrix} \end{array}\right)$

$T = \left(\begin{array}{c|c} B & C \\ \hline D & A \end{array}\right)$ does not make sense since it does not create a rectangle or a square.

Example 2.7: Let, $F = \begin{bmatrix} 4 & 6 \\ 4 & 1 \end{bmatrix}$, $G = \begin{bmatrix} 1 & 2 & 5 \end{bmatrix}$, $H = \begin{bmatrix} 5 & 8 \\ 6 & 2 \\ 1 & 3 \\ 3 & 7 \end{bmatrix}$, $I = \begin{bmatrix} 5 & 7 & 9 \\ 1 & 7 & 2 \\ 0 & 3 & 4 \\ 1 & 3 & 6 \end{bmatrix}$,

$J = \begin{bmatrix} 3 & 5 & 6 \\ 2 & 1 & 4 \\ 0 & 2 & 1 \end{bmatrix}$, $K = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $L = \begin{bmatrix} 7 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 4 & 5 & 8 & 9 \end{bmatrix}$, $N = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$R = \left(\begin{array}{c|c|c} N & K & L \\ \hline F & G & I \\ \hline H & J & M \end{array}\right)$$

does not make sense because it does not produce a rectangle or a square.

2.2.1 Types of Block Matrices

Definition 2.4: (Block Diagonal Matrices): If a matrix is block diagonal, then the matrices which are placed in the diagonal position should be square matrices and the matrices which are found in off-diagonal position should be zero matrices. Let T be a block diagonal matrix (diagonal block matrix).

$$T = \left(\begin{array}{c|c|c|c} T_{11} & 0 & \dots & 0 \\ \hline 0 & T_{22} & \dots & 0 \\ \hline . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \hline 0 & 0 & \dots & T_{mm} \end{array}\right),$$

Here $T_{11}, T_{22}, T_{33}, ..., T_{mm}$ are square matrices.

Definition 2.5: (Block Upper Triangular Matrix): A block matrix is upper triangular if all the block matrices below the main diagonal are zero matrices.

$$W = \left(\begin{array}{c|c|c|c} W_{11} & W_{12} & \dots & W_{1m} \\ \hline 0 & W_{22} & \dots & W_{2m} \\ \hline . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \hline 0 & 0 & \dots & W_{mm} \end{array}\right),$$

Definition 2.6: (Block Lower Triangular Matrix): A block matrix is lower triangular if all the block matrices above the main diagonal are zero matrices.

$$X = \left(\begin{array}{c|c|c|c} X_{11} & 0 & \dots & 0 \\ \hline X_{21} & X_{22} & \dots & 0 \\ \hline . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ \hline X_{m1} & X_{m2} & \dots & X_{mm} \end{array}\right),$$

Definition 2.7: (Block Elementary Matrix): A block matrix is called block elementary matrix if it is produced by the block identity matrix (block matrix having identity matrices on the main diagonal) after only one elementary row operation.

Example 2.8: Let us consider the elementary block matrix below

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) R_2 \to 6R_1 + R_2 \left(\begin{array}{c|c} I & 0 \\ \hline 6I & I \end{array}\right)$$

2.2.2 Block Matrix Multiplication

Definition 2.8: (Block Matrix Multiplication): The number of columns in each block must be equal to the number of rows in the corresponding block of another matrix. If matrices are partitioned compatibly into blocks, the product can be calculated by matrix multiplication using blocks as entries. When we do matrix multiplication, sizes of the block matrices must be compatible in both of the matrices. When we do multiplication operation between two block matrices the result will be a new block matrix.

Let,

$$Q = \left(\begin{array}{c|c|c|c} Q_{11} & Q_{12} & \cdots & Q_{1n} \\ \hline Q_{21} & Q_{22} & \cdots & Q_{21} \\ \hline . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \hline Q_{m1} & Q_{m2} & \cdots & Q_{mn} \end{array}\right), \quad P = \left(\begin{array}{c|c|c|c} P_{11} & P_{12} & \cdots & P_{1p} \\ \hline P_{21} & P_{22} & \cdots & P_{2p} \\ \hline . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \hline P_{n1} & P_{n2} & \cdots & P_{np} \end{array}\right)$$

The multiplication of the two block matrices will be as follows: The size of Q is $m \times n$ because there are mn submatrices. On the other hand, the size of P is $n \times p$ because there are np submatrices, so when we multiply Q and P the size of QP will be $m \times p$, hence it will contain mp matrices.

$$QP = \left(\begin{array}{c|c|c|c} Q_{11}P_{11} + Q_{12}P_{21} + \cdots + Q_{1n}P_{n1} & Q_{11}P_{12} + Q_{12}P_{22} + \cdots + Q_{1n}P_{n2} & \cdots & Q_{11}P_{1p} + Q_{12}P_{2p} + \cdots + Q_{1n}P_{np} \\ \hline Q_{21}P_{11} + Q_{22}P_{21} + \cdots + Q_{2n}P_{n1} & Q_{21}P_{12} + Q_{22}P_{22} + \cdots + Q_{2n}P_{n2} & \cdots & Q_{21}P_{1p} + Q_{22}P_{2p} + \cdots + Q_{2n}P_{np} \\ \hline . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ \hline Q_{m1}P_{11} + Q_{m2}P_{21} + \cdots + Q_{mn}P_{n1} & Q_{m1}P_{12} + Q_{m2}P_{22} + \cdots + Q_{mn}P_{n2} & \cdots & Q_{m1}P_{1p} + Q_{m2}P_{2p} + \cdots + Q_{mn}P_{np} \end{array}\right)$$

Example 2.9: Compute TS , using the indicated block partitioning.

$$T=\left(\begin{array}{ccccc|cccc} 5 & 1 & 9 & 9 & 2 & 0 & 9 & 8 & 7 \\ 6 & 7 & 1 & 0 & 0 & 6 & 5 & 4 & 3 \\ 1 & 11 & 12 & 4 & 21 & 2 & 1 & 0 & 0 \\ \hline 2 & 3 & 11 & 18 & 22 & \frac{1}{2} & -8 & 2 & 6 \end{array}\right), S=\left(\begin{array}{ccc|cc} 12 & 15 & 22 & 8 & 8 \\ \frac{1}{7} & -19 & 34 & 8 & 8 \\ -\frac{9}{29} & 7 & 4 & 9 & 9 \\ 4 & 7 & 71 & 9 & 9 \\ 32 & \frac{5}{4} & 21 & 33 & 21 \\ \hline 0 & 11 & 56 & 55 & 55 \\ 5 & \frac{4}{3} & -98 & \frac{4}{3} & 21 \\ 2 & -9 & 7 & \frac{4}{15} & 7 \\ 5 & 0 & 2 & 56 & 22 \end{array}\right)$$

$$TS=$$

$$\left(\begin{array}{c|c} \begin{bmatrix} 5 & 1 & 9 & 9 & 2 \\ 6 & 7 & 1 & 0 & 0 \\ 1 & 11 & 12 & 4 & 21 \end{bmatrix}\begin{bmatrix} 12 & 15 & 22 \\ \frac{1}{7} & -19 & 34 \\ -\frac{9}{29} & 7 & 4 \\ 4 & 7 & 71 \\ 32 & \frac{5}{4} & 21 \end{bmatrix}+\begin{bmatrix} 0 & 9 & 8 & 7 \\ 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 11 & 56 \\ 5 & \frac{4}{3} & -98 \\ 2 & -9 & 7 \\ 5 & 0 & 2 \end{bmatrix} & \begin{bmatrix} 5 & 1 & 9 & 9 & 2 \\ 6 & 7 & 1 & 0 & 0 \\ 1 & 11 & 12 & 4 & 21 \end{bmatrix}\begin{bmatrix} 8 & 8 \\ 8 & 8 \\ 9 & 9 \\ 9 & 9 \\ 33 & 21 \end{bmatrix}+\begin{bmatrix} 0 & 9 & 8 & 7 \\ 6 & 5 & 4 & 3 \\ 2 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 55 & 55 \\ \frac{4}{3} & 21 \\ \frac{4}{15} & 7 \\ 56 & 22 \end{bmatrix} \\ \hline \begin{bmatrix} 2 & 3 & 11 & 18 & 22 \end{bmatrix}\begin{bmatrix} 12 & 15 & 22 \\ \frac{1}{7} & -19 & 34 \\ -\frac{9}{29} & 7 & 4 \\ 4 & 7 & 71 \\ 32 & \frac{5}{4} & 21 \end{bmatrix}+\begin{bmatrix} \frac{1}{2} & -8 & 2 & 6 \end{bmatrix}\begin{bmatrix} 0 & 11 & 56 \\ 5 & \frac{4}{3} & -98 \\ 2 & -9 & 7 \\ 5 & 0 & 2 \end{bmatrix} & \begin{bmatrix} 2 & 3 & 11 & 18 & 22 \end{bmatrix}\begin{bmatrix} 8 & 8 \\ 8 & 8 \\ 9 & 9 \\ 9 & 9 \\ 33 & 21 \end{bmatrix}+\begin{bmatrix} \frac{1}{2} & -8 & 2 & 6 \end{bmatrix}\begin{bmatrix} 55 & 55 \\ \frac{4}{3} & 21 \\ \frac{4}{15} & 7 \\ 56 & 22 \end{bmatrix} \end{array}\right)$$

$$=\left(\begin{array}{ccc|cc} \frac{51430}{203} & \frac{249}{2} & 49 & \frac{10232}{15} & 651 \\ \frac{3500}{29} & \frac{2}{3} & 254 & \frac{9281}{15} & 642 \\ \frac{142678}{203} & -\frac{389}{12} & 1183 & \frac{3133}{3} & 812 \\ \hline \frac{160576}{203} & \frac{541}{3} & 2768 & \frac{41411}{30} & \frac{1537}{2} \end{array}\right)$$

Chapter 3

INVERSES OF BLOCK MATRICES

3.1 Inverses of Block Diagonal and Block Triangular Matrices

Proposition 3.1: Let Y be a 2×2 block diagonal matrix, where Y_{11}, Y_{22} are square and invertible blocks

$$Y = \left(\begin{array}{c|c} Y_{11} & 0 \\ \hline 0 & Y_{22} \end{array}\right). \text{ Then its inverse is } Y^{-1} = \left(\begin{array}{c|c} Y_{11}^{-1} & 0 \\ \hline 0 & Y_{22}^{-1} \end{array}\right).$$

Proof. By using the Gauss Elimination,

$$\left(\begin{array}{cc|cc} Y_{11} & 0 & I & 0 \\ \hline 0 & Y_{22} & 0 & I \end{array}\right) \quad R_1 \rightarrow Y_{11}^{-1}R_1 \quad \left(\begin{array}{cc|cc} I & 0 & Y_{11}^{-1} & 0 \\ \hline 0 & Y_{22} & 0 & I \end{array}\right) \quad R_2 \rightarrow Y_{22}^{-1}R_2$$
$$\left(\begin{array}{cc|cc} I & 0 & Y_{11}^{-1} & 0 \\ \hline 0 & I & 0 & Y_{22}^{-1} \end{array}\right)$$

$$YY^{-1} = \left(\begin{array}{c|c} Y_{11} & 0 \\ \hline 0 & Y_{22} \end{array}\right) \left(\begin{array}{c|c} Y_{11}^{-1} & 0 \\ \hline 0 & Y_{22}^{-1} \end{array}\right) = I$$

Therefore,

$$Y^{-1} = \left(\begin{array}{c|c} Y_{11}^{-1} & 0 \\ \hline 0 & Y_{22}^{-1} \end{array}\right) \qquad \square$$

Example 3.1: Let $K = \left(\begin{array}{cccc|ccc} 5 & 9 & -8 & 12 & 0 & 0 & 0 \\ 1 & 6 & 2 & 5 & 0 & 0 & 0 \\ 3 & 5 & 1 & 4 & 0 & 0 & 0 \\ 1 & 0 & 4 & 5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 7 & 8 & 8 \end{array}\right)$.

Then

$$K^{-1} = \left(\begin{array}{c|c} \begin{bmatrix} 5 & 9 & -8 & 12 \\ 1 & 6 & 2 & 5 \\ 3 & 5 & 1 & 4 \\ 1 & 0 & 4 & 5 \end{bmatrix}^{-1} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{bmatrix}^{-1} \end{array}\right)$$

$$= \left(\begin{array}{cccc|ccc} -\frac{8}{461} & -\frac{341}{922} & \frac{219}{461} & \frac{29}{922} & 0 & 0 & 0 \\ -\frac{11}{461} & \frac{165}{922} & \frac{13}{461} & -\frac{133}{922} & 0 & 0 & 0 \\ -\frac{33}{461} & \frac{17}{461} & \frac{39}{461} & \frac{31}{461} & 0 & 0 & 0 \\ \frac{28}{461} & \frac{41}{922} & -\frac{75}{461} & \frac{129}{922} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\frac{8}{3} & \frac{8}{3} & -1 \\ 0 & 0 & 0 & 0 & \frac{10}{3} & -\frac{13}{3} & 2 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \end{array}\right)$$

Proposition 3.2: Let J be a 2×2 block upper triangular matrix; where J_{11}, J_{22} are square and invertible block matrices

$J = \left(\begin{array}{c|c} J_{11} & J_{12} \\ \hline 0 & J_{22} \end{array}\right)$. Then its inverse is $J^{-1} = \left(\begin{array}{c|c} J_{11}^{-1} & -J_{11}^{-1}J_{12}J_{22}^{-1} \\ \hline 0 & J_{22}^{-1} \end{array}\right)$

Proof. By using Gauss Elimination,

$$\left(\begin{array}{cc|cc} J_{11} & J_{12} & I & 0 \\ \hline 0 & J_{22} & 0 & I \end{array}\right) R_1 \to J_{11}^{-1}R_1 \left(\begin{array}{cc|cc} I & J_{11}^{-1}J_{12} & J_{11}^{-1} & 0 \\ \hline 0 & J_{22} & 0 & I \end{array}\right) R_2 \to J_{22}^{-1}R_2$$

$$\left(\begin{array}{cc|cc} I & J_{11}^{-1}J_{12} & J_{11}^{-1} & 0 \\ \hline 0 & I & 0 & J_{22}^{-1} \end{array}\right) R_1 \to -J_{11}^{-1}J_{12}R_2 + R_1 \left(\begin{array}{cc|cc} I & 0 & J_{11}^{-1} & -J_{11}^{-1}J_{12}J_{22}^{-1} \\ \hline 0 & I & 0 & J_{22}^{-1} \end{array}\right)$$

$$JJ^{-1} = \left(\begin{array}{c|c} J_{11} & J_{12} \\ \hline 0 & J_{22} \end{array}\right) \left(\begin{array}{c|c} J_{11}^{-1} & -J_{11}^{-1}J_{12}J_{22}^{-1} \\ \hline 0 & J_{22}^{-1} \end{array}\right) = I$$

So, $J^{-1} = \left(\begin{array}{c|c} J_{11}^{-1} & -J_{11}^{-1}J_{12}J_{22}^{-1} \\ \hline 0 & J_{22}^{-1} \end{array}\right)$ □

Example 3.2: Let $Z = \left(\begin{array}{ccc|cccc} 7 & -5 & 4 & 8 & 9 & -10 & 12 \\ 6 & 0 & 9 & 6 & 7 & 2 & 4 \\ 8 & -\frac{3}{4} & \frac{9}{8} & 4 & 2 & 17 & 2 \\ \hline 0 & 0 & 0 & 6 & 5 & 3 & 1 \\ 0 & 0 & 0 & 0 & 6 & 9 & 10 \\ 0 & 0 & 0 & -8 & 5 & 7 & 9 \\ 0 & 0 & 0 & 0 & 2 & 1 & 3 \end{array}\right)$

Then, Z^{-1} is as follows:

$$Z^{-1} = \left(\begin{array}{c|c} \begin{bmatrix} 7 & -5 & 4 \\ 6 & 0 & 9 \\ 8 & -\frac{3}{4} & \frac{9}{8} \end{bmatrix}^{-1} & -\begin{bmatrix} 7 & -5 & 4 \\ 6 & 0 & 9 \\ 8 & -\frac{3}{4} & \frac{9}{8} \end{bmatrix}^{-1} \begin{bmatrix} 8 & 9 & -10 & 12 \\ 6 & 7 & 2 & 4 \\ 4 & 2 & 17 & 2 \end{bmatrix} \begin{bmatrix} 6 & 5 & 3 & 1 \\ 0 & 6 & 9 & 10 \\ -8 & 5 & 7 & 9 \\ 0 & 2 & 1 & 3 \end{bmatrix}^{-1} \\ \hline \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 6 & 5 & 3 & 1 \\ 0 & 6 & 9 & 10 \\ -8 & 5 & 7 & 9 \\ 0 & 2 & 1 & 3 \end{bmatrix}^{-1} \end{array}\right)$$

$$= \left(\begin{array}{ccc|cccc} -\frac{1}{44} & -\frac{7}{792} & \frac{5}{33} & -\frac{6163}{76296} & -\frac{50791}{114444} & -\frac{541}{38148} & \frac{32263}{20808} \\ -\frac{29}{132} & \frac{193}{2376} & \frac{13}{99} & -\frac{47915}{228888} & -\frac{225143}{343332} & -\frac{28625}{114444} & \frac{230279}{62424} \\ \frac{1}{66} & \frac{139}{1188} & -\frac{10}{99} & -\frac{5849}{114444} & \frac{56467}{171666} & \frac{805}{57222} & -\frac{39667}{31212} \\ \hline 0 & 0 & 0 & -\frac{9}{578} & \frac{30}{289} & -\frac{79}{578} & \frac{20}{289} \\ 0 & 0 & 0 & \frac{4}{17} & -\frac{4}{17} & \frac{3}{17} & \frac{3}{17} \\ 0 & 0 & 0 & \frac{8}{289} & \frac{43}{289} & \frac{6}{289} & -\frac{164}{289} \\ 0 & 0 & 0 & -\frac{48}{289} & \frac{31}{289} & -\frac{36}{289} & \frac{117}{289} \end{array}\right)$$

Proposition 3.3: : Let V be a 2×2 lower triangular block matrix, where V_{11} and V_{22} are square and invertible

$$V = \left(\begin{array}{c|c} V_{11} & 0 \\ \hline V_{21} & V_{22} \end{array}\right). \text{ Then its inverse is } V^{-1} = \left(\begin{array}{c|c} V_{11}^{-1} & 0 \\ \hline -V_{22}^{-1}V_{21}V_{11}^{-1} & V_{22}^{-1} \end{array}\right).$$

Proof. By using Gauss Elimination,

$$\left(\begin{array}{cc|cc} V_{11} & 0 & I & 0 \\ \hline V_{21} & V_{22} & 0 & I \end{array}\right) R_1 \to V_{11}^{-1}R_1 \left(\begin{array}{cc|cc} I & 0 & V_{11}^{-1} & 0 \\ \hline V_{21} & V_{22} & 0 & I \end{array}\right) R_2 \to -V_{21}R_1 + R_2$$

$$\left(\begin{array}{cc|cc} I & 0 & V_{11}^{-1} & 0 \\ \hline 0 & V_{22} & -V_{21}V_{11}^{-1} & I \end{array}\right) R_2 \to V_{22}^{-1}R_2 \left(\begin{array}{cc|cc} I & 0 & V_{11}^{-1} & 0 \\ \hline 0 & I & -V_{22}^{-1}V_{21}V_{11}^{-1} & V_{22}^{-1} \end{array}\right)$$

$$VV^{-1} = \left(\begin{array}{c|c} V_{11} & 0 \\ \hline V_{21} & V_{22} \end{array}\right) \left(\begin{array}{c|c} V_{11}^{-1} & 0 \\ \hline -V_{22}^{-1}V_{21}V_{11}^{-1} & V_{22}^{-1} \end{array}\right) = I$$

So, $V^{-1} = \left(\begin{array}{c|c} V_{11}^{-1} & 0 \\ \hline -V_{22}^{-1}V_{21}V_{11}^{-1} & V_{22}^{-1} \end{array}\right)$

where V_{11} , V_{22} are invertible and square. □

Example 3.3: Let $G = \left(\begin{array}{cccc|cccc} 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 5 & 0 & 0 & 0 & 0 \\ 2 & -1 & 4 & 3 & 0 & 0 & 0 & 0 \\ \hline 2 & 3 & 8 & 5 & 2 & 1 & 0 & 0 \\ 2 & 0 & 4 & 6 & 0 & 6 & 4 & 1 \\ 1 & 4 & 2 & 9 & 5 & 3 & 4 & 5 \\ 2 & 2 & 3 & 3 & 4 & 2 & 3 & 1 \end{array}\right)$.

Then G^{-1} is as follows.

G^{-1} =

$$
\left(\begin{array}{c|c}
\begin{bmatrix} 7 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 1 & 4 & 5 \\ 2 & -1 & 4 & 3 \end{bmatrix}^{-1} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
\hline
-\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 6 & 4 & 1 \\ 5 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 3 & 8 & 5 \\ 2 & 0 & 4 & 6 \\ 1 & 4 & 2 & 9 \\ 2 & 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 7 & 1 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 1 & 4 & 5 \\ 2 & -1 & 4 & 3 \end{bmatrix}^{-1} & \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 6 & 4 & 1 \\ 5 & 3 & 4 & 5 \\ 4 & 2 & 3 & 1 \end{bmatrix}^{-1}
\end{array}\right)
$$

$$
= \left(\begin{array}{cccc|cccc}
\frac{6}{47} & \frac{2}{47} & -\frac{7}{94} & \frac{5}{94} & 0 & 0 & 0 & 0 \\
\frac{5}{47} & -\frac{14}{47} & \frac{49}{94} & -\frac{35}{94} & 0 & 0 & 0 & 0 \\
-\frac{5}{94} & -\frac{33}{94} & \frac{45}{188} & \frac{35}{188} & 0 & 0 & 0 & 0 \\
\frac{1}{47} & \frac{16}{47} & -\frac{9}{94} & -\frac{7}{94} & 0 & 0 & 0 & 0 \\
\hline
-\frac{606}{6251} & \frac{4169}{6251} & -\frac{11513}{12502} & \frac{341}{12502} & \frac{37}{133} & -\frac{11}{133} & -\frac{1}{133} & \frac{16}{133} \\
-\frac{384}{6251} & \frac{3632}{6251} & -\frac{6309}{6251} & -\frac{1006}{6251} & \frac{59}{133} & \frac{22}{133} & \frac{2}{133} & -\frac{32}{133} \\
\frac{751}{12502} & -\frac{11531}{12502} & \frac{36105}{25004} & \frac{1135}{25004} & -\frac{88}{133} & \frac{1}{133} & -\frac{12}{133} & \frac{59}{133} \\
-\frac{262}{6251} & -\frac{3252}{6251} & \frac{286}{6251} & \frac{2374}{6251} & -\frac{2}{133} & -\frac{3}{133} & \frac{36}{133} & -\frac{44}{133}
\end{array}\right).
$$

Example 3.4: Let L be a 2×2 block matrix

$$L = \left(\begin{array}{ccc|cccc} 11 & 22 & 13 & 2 & 6 & 9 & 2 \\ 22 & 44 & 6 & 0 & 0 & 1 & 8 \\ 33 & 66 & 9 & 4 & 6 & 7 & 8 \\ \hline 2 & 1 & 0 & 1 & 4 & 5 & 6 \\ -4 & -8 & 9 & 2 & 9 & 5 & 7 \\ 3 & 0 & 4 & 6 & 0 & 3 & 6 \\ 4 & 5 & 6 & 2 & 9 & 7 & 9 \end{array}\right).$$

Is L^{-1} possible?

Yes, because upper left position $\begin{bmatrix} 11 & 22 & 13 \\ 22 & 44 & 6 \\ 33 & 66 & 9 \end{bmatrix}$ is not invertible so we cannot use Theorem 3.1 but we can use theorem 3.2 because $\begin{bmatrix} 1 & 4 & 5 & 6 \\ 2 & 9 & 5 & 7 \\ 6 & 0 & 3 & 6 \\ 2 & 9 & 7 & 9 \end{bmatrix}$ is non-singular.

Therefore

$$L^{-1} = \left(\begin{array}{ccc|cccc} \frac{1817}{515139} & \frac{4877}{171713} & -\frac{120347}{515139} & -\frac{163488}{171713} & -\frac{146272}{171713} & \frac{59035}{515139} & \frac{715820}{515139} \\ -\frac{17425}{515139} & -\frac{2590}{171713} & \frac{65602}{515139} & \frac{76260}{171713} & \frac{67927}{171713} & -\frac{31457}{515139} & -\frac{337579}{515139} \\ \frac{10372}{171713} & \frac{10242}{171713} & -\frac{11821}{171713} & -\frac{27090}{171713} & -\frac{4878}{171713} & \frac{4603}{171713} & \frac{17884}{171713} \\ \hline -\frac{14028}{171713} & -\frac{23057}{171713} & \frac{28305}{171713} & \frac{20547}{171713} & \frac{36331}{171713} & \frac{17217}{171713} & -\frac{54981}{171713} \\ -\frac{24277}{343426} & -\frac{15612}{171713} & \frac{14209}{343426} & -\frac{56785}{171713} & -\frac{23346}{171713} & -\frac{7117}{171713} & \frac{71019}{171713} \\ \frac{18158}{171713} & -\frac{2201}{171713} & \frac{1920}{171713} & \frac{76740}{171713} & \frac{20664}{171713} & -\frac{4292}{171713} & -\frac{68156}{171713} \\ -\frac{14537}{343426} & \frac{14891}{171713} & -\frac{2653}{343426} & \frac{40887}{171713} & \frac{29725}{171713} & \frac{640}{171713} & -\frac{42167}{171713} \end{array}\right)$$

Example 3.5: Let S be a 2×2 upper triangular block matrix

$$S = \left(\begin{array}{ccc|ccc} 3 & 6 & 7 & 4 & 2 & 6 \\ 2 & 5 & -1 & 1 & -5 & 8 \\ 0 & 0 & 9 & 1 & 4 & -7 \\ \hline 0 & 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & 7 & 9 & 9 \\ 0 & 0 & 0 & 9 & 9 & 8 \end{array}\right).$$

Can we find S^{-1} ?

We can find S^{-1} which is possible because in the diagonal positions submatrices are invertible and in the off-diagonal positions matrix multiplication of block matrices is possible so we are ready to calculate S^{-1}.

$$S^{-1} = \left(\begin{array}{c|c} \begin{bmatrix} 3 & 6 & 7 \\ 2 & 5 & -1 \\ 0 & 0 & 9 \end{bmatrix}^{-1} & -\begin{bmatrix} 3 & 6 & 7 \\ 2 & 5 & -1 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 4 & 2 & 6 \\ 1 & -5 & 8 \\ 1 & 4 & -7 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 \\ 7 & 9 & 9 \\ 9 & 9 & 8 \end{bmatrix}^{-1} \\ \hline \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 2 & 4 & -1 \\ 7 & 9 & 9 \\ 9 & 9 & 8 \end{bmatrix}^{-1} \end{array}\right) =$$

$$\left(\begin{array}{ccc|ccc} \frac{5}{3} & -2 & -\frac{41}{27} & -\frac{377}{540} & -\frac{733}{540} & \frac{31}{36} \\ -\frac{2}{3} & 1 & \frac{17}{27} & \frac{85}{108} & \frac{65}{108} & -\frac{19}{36} \\ 0 & 0 & \frac{1}{9} & -\frac{217}{900} & \frac{67}{900} & -\frac{1}{60} \\ \hline 0 & 0 & 0 & -\frac{9}{100} & -\frac{41}{100} & \frac{9}{20} \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & -\frac{9}{50} & \frac{9}{50} & -\frac{1}{10} \end{array}\right)$$

Remark 3.1: The block Gaussian elimination methods applied to 2×2 block diagonal and block triangular matrices can easily be generalised to give inverse formulas for $n \times n$ block diagonal or block triangular matrices.

3.2 Inverses of General 2×2 Block Matrices

Assume S is a 2×2 non-singular square block matrix $S = \left(\begin{array}{c|c} T & M \\ \hline N & E \end{array}\right)$ and its inverse is $S^{-1} = \left(\begin{array}{c|c} V & W \\ \hline X & Y \end{array}\right)$. If we want to understand S^{-1} the we need to know the sizes of T,M,N,E in S and V,W,X,Y in S^{-1} simultaneously. Let T,M,N,E be partitioned matrices in S with sizes $k \times m$, $k \times n$, $l \times m$, $l \times n$ respectively and V,W,X,Y be submatrices in S^{-1} with sizes of $m \times k$, $m \times l$, $n \times k$ and $n \times l$ respectively.

We can verify S^{-1}, here we have just 3 possible partitions.

- Square matrices in diagonal positions in S and S^{-1}, which means $k = m$ and $l = n$
- Square matrices in off-diagonal positions of S and S^{-1} where $k = n$ and $m = l$.
- Square matrices in all positions in S and S^{-1} where $k = m = n = l$.

The theorem below deals with the first case where matrices in the diagonal positions are all square.

Theorem 3.1: Let T be non-singular. Then S^{-1} exists if and only if the matrix $E - NT^{-1}M$ is invertible and

$$S^{-1} = \left(\begin{array}{c|c} T^{-1} + T^{-1}M(E-NT^{-1}M)^{-1}NT^{-1} & -T^{-1}M(E-NT^{-1}M)^{-1} \\ \hline -(E-NT^{-1}M)^{-1}NT^{-1} & (E-NT^{-1}M)^{-1} \end{array}\right)$$

Proof. By using Gauss Elimination

$$\left(\begin{array}{cc|cc} T & M & I & 0 \\ \hline N & E & 0 & I \end{array}\right) R_1 \to T^{-1}R_1 \left(\begin{array}{cc|cc} I & T^{-1}M & T^{-1} & 0 \\ \hline N & E & 0 & I \end{array}\right) R_2 \to -NR_1 + R_2$$

$$\left(\begin{array}{cc|cc} I & T^{-1}M & T^{-1} & 0 \\ \hline 0 & E-NT^{-1}M & -NT^{-1} & I \end{array}\right) R_2 \to (E-NT^{-1}M)^{-1}R_2$$

$$\left(\begin{array}{cc|cc} I & T^{-1}M & T^{-1} & 0 \\ \hline 0 & I & -(E-NT^{-1}M)^{-1}NT^{-1} & (E-NT^{-1}M)^{-1} \end{array}\right) R_1 \to -T^{-1}MR_2 + R_1$$

$$\left(\begin{array}{cc|cc} I & 0 & T^{-1}+T^{-1}M(E-NT^{-1}M)^{-1}NT^{-1} & -T^{-1}M(E-NT^{-1}M)^{-1} \\ \hline 0 & I & -(E-NT^{-1}M)^{-1}NT^{-1} & (E-NT^{-1}M)^{-1} \end{array}\right)$$

Next, we write S as a product of elementary matrices. For this we need to find inverses of standard elementary block matrices. We give these below.

$E_1^{-1} = ?$

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) R_1 \to TR_1 \left(\begin{array}{c|c} T & 0 \\ \hline 0 & I \end{array}\right)$$

$E_2^{-1} = ?$

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) R_2 \to NR_1 + R_2 \left(\begin{array}{c|c} I & 0 \\ \hline N & I \end{array}\right)$$

$E_3^{-1} = ?$

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) R_2 \to (E-NT^{-1}M)R_2 \left(\begin{array}{c|c} I & 0 \\ \hline 0 & E-NT^{-1}M \end{array}\right)$$

$E_4^{-1} = ?$

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) R_1 \to T^{-1}MR_2 + R_1 \left(\begin{array}{c|c} I & T^{-1}M \\ \hline 0 & I \end{array}\right)$$

Then is $E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = S$?

$$E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \left(\begin{array}{c|c} T & 0 \\ \hline 0 & I \end{array}\right) \left(\begin{array}{c|c} I & 0 \\ \hline N & I \end{array}\right) \left(\begin{array}{c|c} I & 0 \\ \hline 0 & E - NT^{-1}M \end{array}\right) \left(\begin{array}{c|c} I & T^{-1}M \\ \hline 0 & I \end{array}\right) =$$
$$\left(\begin{array}{c|c} T & N \\ \hline M & E \end{array}\right) = S$$

Elementary matrices provide a second way of getting the inverse matrix. We can also make sure by using product of $(VLDU)^{-1} = U^{-1}D^{-1}L^{-1}V^{-1} = S^{-1}$

$$V = E_1^{-1} = \left(\begin{array}{c|c} T & 0 \\ \hline 0 & I \end{array}\right)$$

$$L = E_2^{-1} = \left(\begin{array}{c|c} I & 0 \\ \hline N & I \end{array}\right)$$

$$D = E_3^{-1} = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & E - NT^{-1}M \end{array}\right)$$

$$U = E_4^{-1} = \left(\begin{array}{c|c} I & T^{-1}M \\ \hline 0 & I \end{array}\right)$$

$$(VLDU)^{-1} = U^{-1}D^{-1}L^{-1}V^{-1} = S^{-1} = \left(\begin{array}{c|c} I & T^{-1}M \\ \hline 0 & I \end{array}\right)^{-1} \left(\begin{array}{c|c} I & 0 \\ \hline 0 & E-NT^{-1}M \end{array}\right)^{-1}$$

$$\left(\begin{array}{c|c} I & 0 \\ \hline N & I \end{array}\right)^{-1} \left(\begin{array}{c|c} T & 0 \\ \hline 0 & I \end{array}\right)^{-1} = \left(\begin{array}{c|c} I & -(T^{-1}M) \\ \hline 0 & I \end{array}\right) \left(\begin{array}{c|c} I & 0 \\ \hline 0 & (E-NT^{-1}M)^{-1} \end{array}\right)$$

$$\left(\begin{array}{c|c} I & 0 \\ \hline -N & I \end{array}\right) \left(\begin{array}{c|c} T^{-1} & 0 \\ \hline 0 & I \end{array}\right) = S^{-1}$$

So,

$$S^{-1} = \left(\begin{array}{c|c} T^{-1}+T^{-1}M(E-NT^{-1}M)^{-1}NT^{-1} & -T^{-1}M(E-NT^{-1}M)^{-1} \\ \hline -(E-NT^{-1}M)^{-1}NT^{-1} & (E-NT^{-1}M)^{-1} \end{array}\right) \qquad \square$$

Theorem 3.2: Let now E be non-singular. Then S^{-1} exists if and only if the matrix $T-ME^{-1}N$ is invertible, and

$$S^{-1} = \left(\begin{array}{c|c} (T-ME^{-1}N)^{-1} & -(T-ME^{-1}N)^{-1}ME^{-1} \\ \hline -E^{-1}N(T-ME^{-1}N)^{-1} & E^{-1}+E^{-1}N(T-ME^{-1}N)^{-1}ME^{-1} \end{array}\right).$$

Proof. By using Gauss Elimination Method,

$$\left(\begin{array}{cc|cc} T & M & I & 0 \\ \hline N & E & 0 & I \end{array}\right) R_2 \to E^{-1}R_2 \left(\begin{array}{cc|cc} T & M & I & 0 \\ \hline E^{-1}N & I & 0 & E^{-1} \end{array}\right) R_1 \to -MR_2+R_1$$

$$\left(\begin{array}{cc|cc} T-ME^{-1}N & 0 & I & -ME^{-1} \\ \hline E^{-1}N & I & 0 & E^{-1} \end{array}\right) R_1 \to (T-ME^{-1}N)^{-1}R_1$$

$$\left(\begin{array}{cc|cc} I & 0 & (T-ME^{-1}N)^{-1} & -(T-ME^{-1}N)^{-1}ME^{-1} \\ \hline E^{-1}N & I & 0 & E^{-1} \end{array}\right) R_2 \rightarrow -(E^{-1}N)R_1 + R_2$$

$$\left(\begin{array}{cc|cc} I & 0 & (T-ME^{-1}N)^{-1} & -(T^{-1}ME^{-1}N)^{-1}ME^{-1} \\ \hline 0 & I & -E^{-1}N(T-ME^{-1}N)^{-1} & E^{-1}+E^{-1}N(T-ME^{-1}N)^{-1}ME^{-1} \end{array}\right)$$

Firstly, we can express I by using $E_4E_3E_2E_1S = I$

$E_1 = ?$

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) R_2 \rightarrow E^{-1}R_2 \left(\begin{array}{c|c} I & 0 \\ \hline 0 & E^{-1} \end{array}\right)$$

$E_2 = ?$

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) R_2 \rightarrow -MR_2 + R_1 \left(\begin{array}{c|c} I & -M \\ \hline 0 & I \end{array}\right)$$

$E_3 = ?$

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) R_1 \rightarrow (T-ME^{-1}N)^{-1}R_1 \left(\begin{array}{c|c} (T-ME^{-1}N)^{-1} & 0 \\ \hline 0 & I \end{array}\right)$$

$E_4 = ?$

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) R_2 \rightarrow -(E^{-1}N)R_1 + R_2 \left(\begin{array}{c|c} I & 0 \\ \hline -E^{-1}N & I \end{array}\right)$$

$$E_4E_3E_2E_1S = \left(\begin{array}{c|c} I & 0 \\ \hline -E^{-1}N & I \end{array}\right)\left(\begin{array}{c|c} (T-ME^{-1}N)^{-1} & 0 \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & -M \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & E^{-1} \end{array}\right)\left(\begin{array}{c|c} T & M \\ \hline N & E \end{array}\right)$$

$$= \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) = I$$

Again, to express S as a product of elementary block matrices, inverses of the original elementary blocks are required.

$E_1^{-1} = ?$

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) R_2 \to ER_2 \left(\begin{array}{c|c} I & 0 \\ \hline 0 & E \end{array}\right)$$

$E_2^{-1} = ?$

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) R_1 \to MR_2 + R_1 \left(\begin{array}{c|c} I & M \\ \hline 0 & I \end{array}\right)$$

$E_3^{-1} = ?$

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) R_1 \to (T-ME^{-1}N)R_1 \left(\begin{array}{c|c} T-ME^{-1}N & 0 \\ \hline 0 & I \end{array}\right)$$

$E_4^{-1} = ?$

$$\left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right) R_2 \to E^{-1}NR_1 + R_2 \left(\begin{array}{c|c} I & 0 \\ \hline E^{-1}N & I \end{array}\right)$$

Then is $E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = S$?

$$E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & E \end{array}\right)\left(\begin{array}{c|c} I & M \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} T-ME^{-1}N & 0 \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline E^{-1}N & I \end{array}\right) =$$

$$\left(\begin{array}{c|c} T & M \\ \hline N & E \end{array}\right) = S$$

$$D = E_1^{-1} = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & E \end{array}\right)$$

$$U = E_2^{-1} = \left(\begin{array}{c|c} I & M \\ \hline 0 & I \end{array}\right)$$

$$J = E_3^{-1} = \left(\begin{array}{c|c} T-ME^{-1}N & 0 \\ \hline 0 & I \end{array}\right)$$

$$L = E_4^{-1} = \left(\begin{array}{c|c} I & 0 \\ \hline E^{-1}N & I \end{array}\right)$$

$(DUJL)^{-1} = L^{-1}J^{-1}U^{-1}D^{-1} =$

$$\left(\begin{array}{c|c} I & 0 \\ \hline E^{-1}N & I \end{array}\right)^{-1}\left(\begin{array}{c|c} T-ME^{-1}N & 0 \\ \hline 0 & I \end{array}\right)^{-1}\left(\begin{array}{c|c} I & M \\ \hline 0 & I \end{array}\right)^{-1}\left(\begin{array}{c|c} I & 0 \\ \hline 0 & E \end{array}\right)^{-1}$$

$$= \left(\begin{array}{c|c} I & 0 \\ \hline -(E^{-1}N) & I \end{array}\right)\left(\begin{array}{c|c} (T-ME^{-1}N)^{-1} & 0 \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & -M \\ \hline 0 & I \end{array}\right)\left(\begin{array}{c|c} I & 0 \\ \hline 0 & E^{-1} \end{array}\right) = S^{-1}$$

So,

$$S^{-1} = \left(\begin{array}{c|c} (T-ME^{-1}N)^{-1} & -(T-ME^{-1}N)^{-1}ME^{-1} \\ \hline -E^{-1}N(T-ME^{-1}N)^{-1} & E^{-1}+E^{-1}N(T-ME^{-1}N)^{-1}ME^{-1} \end{array}\right) \qquad \square$$

Example 3.6: Let $S = \left(\begin{array}{ccc|cccc} 0 & -\frac{1}{2} & \frac{1}{4} & 4 & 4 & 2 & -9 \\ 0 & \frac{5}{2} & -\frac{5}{4} & 6 & 0 & 0 & 8 \\ 3 & -2 & 1 & 1 & 1 & 2 & 7 \\ \hline 3 & 1 & 2 & 1 & 1 & 2 & 8 \\ 5 & 2 & 1 & 3 & 5 & 6 & 9 \\ 0 & 0 & 3 & 0 & 0 & 1 & 11 \\ 3 & 2 & 1 & 2 & 8 & 8 & 12 \end{array}\right)$

In the diagonal position, $\begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{4} \\ 0 & \frac{5}{2} & -\frac{5}{4} \\ 3 & -2 & 1 \end{pmatrix}$ is not invertible but $\begin{pmatrix} 1 & 1 & 2 & 8 \\ 3 & 5 & 6 & 9 \\ 0 & 0 & 1 & 11 \\ 2 & 8 & 8 & 12 \end{pmatrix}$ is invertible. That's why we cannot use Theorem 3.1 but we can use Theorem 3.2 so we need to use the following formula to find S^{-1}.

$$S^{-1} = \left(\begin{array}{c|c} (T-ME^{-1}N)^{-1} & -(T-ME^{-1}N)^{-1}ME^{-1} \\ \hline -E^{-1}N(T-ME^{-1}N)^{-1} & E^{-1}+E^{-1}N(T-ME^{-1}N)^{-1}ME^{-1} \end{array}\right)$$

so

$$S^{-1} = \left(\begin{array}{ccc|cccc}
\frac{38}{8421} & \frac{82}{8421} & \frac{1025}{8421} & \frac{19232}{25263} & -\frac{3533}{8421} & -\frac{1640}{3609} & \frac{3779}{25263} \\
-\frac{117}{2807} & \frac{43}{2807} & -\frac{866}{2807} & \frac{1801}{8421} & \frac{167}{2807} & -\frac{58}{1203} & -\frac{38}{8421} \\
\frac{430}{2807} & -\frac{254}{2807} & -\frac{368}{2807} & \frac{1634}{8421} & \frac{58}{2807} & \frac{268}{1203} & -\frac{820}{8421} \\
\hline
\frac{731}{8421} & \frac{691}{8421} & \frac{433}{16842} & -\frac{6766}{25263} & \frac{4127}{16842} & \frac{616}{3609} & -\frac{4201}{25263} \\
\frac{1348}{8421} & \frac{1136}{8421} & \frac{3137}{16842} & \frac{46666}{25263} & -\frac{29495}{16842} & -\frac{3472}{3609} & \frac{22366}{25263} \\
-\frac{421}{2807} & -\frac{613}{2807} & -\frac{645}{2807} & -\frac{20126}{8421} & \frac{5975}{2807} & \frac{1433}{1203} & -\frac{7814}{8421} \\
-\frac{79}{2807} & \frac{125}{2807} & \frac{159}{2807} & \frac{1384}{8421} & -\frac{559}{2807} & -\frac{94}{1203} & \frac{934}{8421}
\end{array}\right)$$

For the second class with the square blocks in the off-diagonal positions, we apply a small trick to transform S and S^{-1}, to block matrices with square diagonal positions (case 1). For this, we need the matrix J below.

Definition 3.1: J is a matrix which has 1's in the off-diagonal position and 0's elsewhere:

$$J = \begin{pmatrix}
0 & 0 & . & . & 0 & 1 \\
0 & 0 & . & . & 1 & 0 \\
0 & 0 & . & . & 0 & 0 \\
. & . & . & . & . & . \\
0 & 1 & . & . & 0 & 0 \\
1 & 0 & . & . & 0 & 0
\end{pmatrix}$$

Remark 3.2: J provides interchanging of the columns on the other matrices during the multiplication operation.

Let $T = \begin{pmatrix} t_{11} & t_{12} & . & . & . & t_{1m} \\ t_{21} & t_{22} & . & . & . & t_{2m} \\ t_{31} & t_{32} & . & . & . & t_{3m} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ t_{n1} & t_{n2} & . & . & . & t_{nm} \end{pmatrix}$.

Then

$$TJ = \begin{pmatrix} t_{11} & t_{12} & . & . & . & t_{1m} \\ t_{21} & t_{22} & . & . & . & t_{2m} \\ t_{31} & t_{32} & . & . & . & t_{3m} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ t_{n1} & t_{n2} & . & . & . & t_{nm} \end{pmatrix} \begin{pmatrix} 0 & 0 & . & . & . & 1 \\ 0 & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 0 & . & . & . & 0 \end{pmatrix} = \begin{pmatrix} t_{1m} & . & . & . & t_{12} & t_{11} \\ t_{2m} & . & . & . & t_{22} & t_{21} \\ t_{3m} & . & . & . & t_{32} & t_{31} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ t_{nm} & . & . & . & t_{n2} & t_{n1} \end{pmatrix}$$

We may also put blocks!

Example 3.7: $Z = \left(\begin{array}{c|c|c|c} Q & R & C & W \\ \hline B & L & H & A \\ \hline E & M & Y & U \\ \hline I & P & K & D \end{array}\right) \quad \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & J \\ \hline 0 & 0 & J & 0 \\ \hline 0 & J & 0 & 0 \\ \hline J & 0 & 0 & 0 \end{array}\right)$

$$= \left(\begin{array}{c|c|c|c} WJ & CJ & RJ & QJ \\ \hline AJ & HJ & LJ & BJ \\ \hline UJ & YJ & MJ & EJ \\ \hline DJ & KJ & PJ & IJ \end{array}\right)$$

Theorem 3.3: Let M now be invertible. Then S^{-1} exists if and only if $N-EM^{-1}T$ is invertible and

$$S^{-1}=\left(\begin{array}{c|c} -(N-EM^{-1}T)^{-1}EM^{-1} & (N-EM^{-1}T)^{-1} \\ \hline M^{-1}+M^{-1}T(N-EM^{-1}T)^{-1}EM^{-1} & -M^{-1}T(N-EM^{-1}T)^{-1} \end{array}\right)$$

Proof. By using Theorem 3.1

$$S^{-1}=J(SJ)^{-1}=$$

$$\left(\begin{array}{c|c} 0 & J \\ \hline J & 0 \end{array}\right)\left(\begin{array}{c|c} (MJ)^{-1}+(MJ)^{-1}(TJ)((NJ)-(EJ)(MJ)^{-1}(TJ))^{-1}(EJ)(MJ)^{-1} & -(MJ)^{-1}(TJ)((NJ)-(EJ)(MJ)^{-1}(TJ))^{-1} \\ \hline -((NJ)-(EJ)(MJ)^{-1}(TJ))^{-1}(EJ)(MJ)^{-1} & ((NJ)-(EJ)(MJ)^{-1}(TJ))^{-1} \end{array}\right)$$

$$=\left(\begin{array}{c|c} 0 & J \\ \hline J & 0 \end{array}\right)\left(\begin{array}{c|c} J^{-1}M^{-1}+J^{-1}M^{-1}T(N-EM^{-1}T)^{-1}EM^{-1} & -J^{-1}M^{-1}T(N-EM^{-1}T)^{-1} \\ \hline -J^{-1}(N-EM^{-1}T)^{-1}EM^{-1} & J^{-1}(N-EM^{-1}T)^{-1} \end{array}\right)$$

$$=\left(\begin{array}{c|c} -(N-EM^{-1}T)^{-1}EM^{-1} & (N-EM^{-1}T)^{-1} \\ \hline M^{-1}+M^{-1}T(N-EM^{-1}T)^{-1}EM^{-1} & -M^{-1}T(N-EM^{-1}T)^{-1} \end{array}\right) \qquad \square$$

Theorem 3.4: Assume N is non-singular. Then S^{-1} exists if and only if $M-TN^{-1}E$ is invertible, and

$$S^{-1}=\left(\begin{array}{c|c} -N^{-1}E(M-TN^{-1}E)^{-1} & N^{-1}+N^{-1}E(M-TN^{-1}E)^{-1}TN^{-1} \\ \hline (M-TN^{-1}E)^{-1} & -(M-TN^{-1}E)^{-1}TN^{-1} \end{array}\right)$$

Proof. By using Theorem 3.2

$$S^{-1}=J(SJ)^{-1}=$$

$$\left(\begin{array}{c|c} 0 & J \\ \hline J & 0 \end{array}\right)$$

$$\left(\begin{array}{c|c} ((MJ)-(TJ)(NJ)^{-1}(EJ))^{-1} & -((MJ)-(TJ)(NJ)^{-1}(EJ))^{-1}(TJ)(NJ)^{-1} \\ \hline -(NJ)^{-1}(EJ)((MJ)-(TJ)(NJ)^{-1}(EJ))^{-1} & (NJ)^{-1}+(NJ)^{-1}(EJ)((MJ)-(TJ)(NJ)^{-1}(EJ))^{-1}(TJ)(NJ)^{-1} \end{array}\right)$$

$$= \left(\begin{array}{c|c} 0 & J \\ \hline J & 0 \end{array}\right)\left(\begin{array}{c|c} J^{-1}(M-TN^{-1}E)^{-1} & -J^{-1}(M-TN^{-1}E)^{-1}TN^{-1} \\ \hline -J^{-1}N^{-1}E(M-TN^{-1}E)^{-1} & J^{-1}N^{-1}+J^{-1}N^{-1}E(M-TN^{-1}E)^{-1}TN^{-1} \end{array}\right)$$

$$= \left(\begin{array}{c|c} -N^{-1}E(M-TN^{-1}E)^{-1} & N^{-1}+N^{-1}E(M-TN^{-1}E)^{-1}TN^{-1} \\ \hline (M-TN^{-1}E)^{-1} & -(M-TN^{-1}E)^{-1}TN^{-1} \end{array}\right) \qquad \square$$

Remark 3.3: Theorem 3.1 and Theorem 3.3 are equivalent if T^{-1} and M^{-1} exist.

Remark 3.4: Theorem 3.1 and Theorem 3.4 are equivalent if T^{-1} and N^{-1} exist.

Remark 3.5: Theorem 3.2 and Theorem 3.3 are equivalent if E^{-1} and M^{-1} exist.

Remark 3.6: Theorem 3.2 and Theorem 3.4 are equivalent if E^{-1} and N^{-1} exist.

Remark 3.7: Obtaining an inverse formula for 2×2 block matrix is crucial, as these formulas can always be used for any $n\times n$ block matrix by splitting it into 4 blocks, i.e. by producing a 2×2 block matrix. Moreover, theorems in this section can be used to give alternate proofs for the inverses of the block diagonal and block triangular matrices.

Chapter 4

DETERMINANTS OF BLOCK MATRICES

Definition 4.1: The determinant of an $n \times n$ matrix G can be found by multiplying each element in any row or column of the matrix by its cofactor and adding these expressions for all the entries in a certain row or column. In another way, it can be thought as a function that associates a real or complex number to a given input square matrix. It is denoted by $\det(G)$ or by $|G|$.

Theorem 4.1: The determinant of an $n \times n$ matrix G can be computed as follows:

$|G| = g_{1j}C_{1j} + g_{2j}C_{2j} + \cdots + g_{nj}C_{nj}$

(cofactor expansion along the j^{th} column) and

$|G| = g_{i1}C_{i1} + g_{i2}C_{i2} + \cdots + g_{in}C_{in}$

(cofactor expansion along the i^{th} row)

4.1 Determinants of 2×2 Block Matrices (In the Case that at Least one Block is the Zero Matrix)

In this section we first provide determinants for the 2×2 block diagonal and block triangular matrices. Then we give the cases where one of the diagonal blocks is a zero matrix.

Theorem 4.2: Given the 2×2 block matrix $P = \left(\begin{array}{c|c} Q & 0 \\ \hline 0 & X \end{array}\right)$, determinant of P is given as follows:

$$|P| = \begin{vmatrix} Q & 0 \\ 0 & X \end{vmatrix} = |Q||X|$$

Proof. Let, $Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}$ and $X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$

$$\begin{pmatrix} Q & 0 \\ 0 & X \end{pmatrix} = \left(\begin{array}{cccc|cccc} q_{11} & q_{12} & \cdots & q_{1n} & 0 & 0 & \cdots & 0 \\ q_{21} & q_{22} & \cdots & q_{2n} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & 0 & \cdots & 0 & x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & x_{n1} & x_{n2} & \cdots & x_{nn} \end{array}\right)$$

We can prove the assertion above by using inductive proof on n. Let us start with base case, case 1 $(n = 1)$. Consider the block matrix Q as 1×1:

$$Q = [q_{11}], \text{ then, } \det(Q) = |q_{11}| = q_{11} X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}, \text{ giving}$$

$$|X| = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}.$$

$$\begin{bmatrix} Q & 0 \\ 0 & X \end{bmatrix} = \left(\begin{array}{c|cccc} q_{11} & 0 & 0 & \cdots & 0 \\ \hline 0 & x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 0 & x_{n1} & x_{n2} & \cdots & x_{nn} \end{array}\right), \text{ then,}$$

$$\begin{vmatrix} Q & 0 \\ 0 & X \end{vmatrix} = (-1)^{1+1} q_{11} \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} = |q_{11}| \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} = |Q||X|$$

Induction Hypothesis: $Q \to n \times n$ assume result holds for $(n-1) \times (n-1)$

$$\begin{vmatrix} Q & 0 \\ 0 & X \end{vmatrix} = \det \left(\begin{array}{cccc|cccc} q_{11} & q_{12} & \cdots & q_{1n} & 0 & 0 & \cdots & 0 \\ q_{21} & q_{22} & \cdots & q_{2n} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & 0 & \cdots & 0 & x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & x_{n1} & x_{n2} & \cdots & x_{nn} \end{array}\right)$$

$$= (-1)^{1+1}\, q_{11} \left|\begin{array}{cccc|cccc} q_{22} & q_{23} & \cdots & q_{2n} & 0 & 0 & \cdots & 0 \\ q_{32} & q_{33} & \cdots & q_{3n} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ q_{n2} & q_{n3} & \cdots & q_{nn} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & 0 & \cdots & 0 & x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & x_{n1} & x_{n2} & \cdots & x_{nn} \end{array}\right|$$

$$+(-1)^{1+2}\, q_{12} \left|\begin{array}{cccc|cccc} q_{21} & q_{23} & \cdots & q_{2n} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ q_{n1} & q_{n3} & \cdots & q_{nn} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & 0 & \cdots & 0 & x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & x_{n1} & x_{n2} & \cdots & x_{nn} \end{array}\right|$$

$$+\cdots+(-1)^{1+n}\,q_{1n}\left|\begin{array}{cccc|cccc} q_{21} & q_{22} & \cdots & q_{2(n-1)} & 0 & 0 & \cdots & 0\\ q_{31} & q_{32} & \cdots & q_{3(n-1)} & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots\\ q_{n1} & q_{n2} & \cdots & q_{n(n-1)} & 0 & 0 & \cdots & 0\\ \hline 0 & 0 & \cdots & 0 & x_{11} & x_{12} & \cdots & x_{1n}\\ 0 & 0 & \cdots & 0 & x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots\\ 0 & 0 & \cdots & 0 & x_{n1} & x_{n2} & \cdots & x_{nn} \end{array}\right|$$

$$=q_{11}\begin{vmatrix} q_{22} & \cdots & q_{2n}\\ \vdots & \cdots & \vdots\\ q_{n2} & \cdots & q_{nn}\end{vmatrix}\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{vmatrix}$$

$$-q_{12}\begin{vmatrix} q_{21} & \cdots & q_{2n}\\ \vdots & \cdots & \vdots\\ q_{n1} & \cdots & q_{nn}\end{vmatrix}\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{vmatrix}$$

$$+\cdots+(-1)^{1+n}\,q_{1n}\begin{vmatrix} q_{21} & q_{22} & \cdots & q_{2(n-1)}\\ q_{31} & q_{32} & \cdots & q_{3(n-1)}\\ \vdots & \vdots & \cdots & \vdots\\ q_{n1} & q_{n2} & \cdots & q_{n(n-1)}\end{vmatrix}\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{vmatrix}=|X|$$

$$\left[q_{11} \begin{vmatrix} q_{22} & \cdots & q_{2n} \\ \vdots & \cdots & \vdots \\ q_{n2} & \cdots & q_{nn} \end{vmatrix} - q_{12} \begin{vmatrix} q_{21} & \cdots & q_{2n} \\ \vdots & \cdots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{vmatrix} + \cdots + (-1)^{1+n} q_{1n} \begin{vmatrix} q_{21} & q_{22} & \cdots & q_{2(n-1)} \\ q_{31} & q_{32} & \cdots & q_{3(n-1)} \\ \vdots & \vdots & \cdots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{n(n-1)} \end{vmatrix} \right]$$

$$= \det(X)\det(Q)$$ □

Remark 4.1: Proof goes in a very similar way, if the sizes of the block square matrices are different.

Theorem 4.3: Given the 2×2 block matrix $Z = \begin{bmatrix} Y & 0 \\ G & X \end{bmatrix}$ determinant of Z is given as follows:

$$|Z| = \begin{vmatrix} Y & 0 \\ G & X \end{vmatrix} = |Y||X|$$

Proof. Let

$$Y = \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix}, \quad G = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{mn} \end{vmatrix}, \quad X = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{vmatrix}$$

$$\begin{vmatrix} Y & 0 \\ G & X \end{vmatrix} = \left[\begin{array}{cccc|cccc} y_{11} & y_{12} & \cdots & y_{1n} & 0 & 0 & \cdots & 0 \\ y_{21} & y_{22} & \cdots & y_{2n} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} & 0 & 0 & \cdots & 0 \\ \hline g_{11} & g_{12} & \cdots & g_{1n} & x_{11} & x_{12} & \cdots & x_{1m} \\ g_{21} & g_{22} & \cdots & g_{2n} & x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{mn} & x_{m1} & x_{m2} & \cdots & x_{mm} \end{array}\right]$$

Base Case $(n = 1)$: Consider the block matrix Y as 1×1

$Y = [y_{11}]$, then $\det(Y) = |y_{11}| = y_{11}$.

$$|X| = \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} Y & 0 \\ G & X \end{vmatrix} = \left[\begin{array}{c|cccc} y_{11} & 0 & 0 & \cdots & 0 \\ \hline g_{11} & x_{11} & x_{12} & \cdots & x_{1m} \\ g_{21} & x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ g_{m1} & x_{m1} & x_{m2} & \cdots & x_{mm} \end{array}\right].$$

Then,

$$\begin{vmatrix} Y & 0 \\ G & X \end{vmatrix} = (-1)^{1+1}\, y_{11} \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{vmatrix} = |y_{11}| \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{vmatrix} = |Y||X|$$

Induction Hypothesis: Now let us assume that Y is an $n \times n$ block matrix. Assume by induction that result holds for $(n-1)\times(n-1)$, to complete the proof.

$$\begin{vmatrix} Y & 0 \\ G & X \end{vmatrix} = \left|\begin{array}{cccc|cccc} y_{11} & y_{12} & \cdots & y_{1n} & 0 & 0 & \cdots & 0 \\ y_{21} & y_{22} & \cdots & y_{2n} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} & 0 & 0 & \cdots & 0 \\ \hline g_{11} & g_{12} & \cdots & g_{1n} & x_{11} & x_{12} & \cdots & x_{1m} \\ g_{21} & g_{22} & \cdots & g_{2n} & x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{mn} & x_{m1} & x_{m2} & \cdots & x_{mm} \end{array}\right| = (-1)^{1+1} \quad y_{11}$$

$$\left|\begin{array}{ccc|cccc} y_{22} & \cdots & y_{2n} & 0 & 0 & \cdots & 0 \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ y_{n2} & \cdots & y_{nn} & 0 & 0 & \cdots & 0 \\ \hline g_{12} & \cdots & g_{1n} & x_{11} & x_{12} & \cdots & x_{1m} \\ g_{22} & \cdots & g_{2n} & x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ g_{m2} & \cdots & g_{mn} & x_{m1} & x_{m2} & \cdots & x_{mm} \end{array}\right|$$

$$(-1)^{1+2} \quad y_{12} \quad \left|\begin{array}{ccc|cccc} y_{21} & \cdots & y_{2n} & 0 & 0 & \cdots & 0 \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ y_{n1} & \cdots & y_{nn} & 0 & 0 & \cdots & 0 \\ \hline g_{11} & \cdots & g_{1n} & x_{11} & x_{12} & \cdots & x_{1m} \\ g_{21} & \cdots & g_{2n} & x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ g_{m1} & \cdots & g_{mn} & x_{m1} & x_{m2} & \cdots & x_{mm} \end{array}\right| + \cdots + \quad (-1)^{1+n} \quad y_{1n}$$

$$\left|\begin{array}{cccc|cccc} y_{21} & y_{22} & \cdots & y_{2(n-1)} & 0 & 0 & \cdots & 0 \\ y_{31} & y_{32} & \cdots & y_{3(n-1)} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{n(n-1)} & 0 & 0 & \cdots & 0 \\ \hline g_{11} & g_{12} & \cdots & g_{1(n-1)} & x_{11} & x_{12} & \cdots & x_{1m} \\ g_{21} & g_{22} & \cdots & g_{2(n-1)} & x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{m(n-1)} & x_{m1} & x_{m2} & \cdots & x_{mm} \end{array}\right| = y_{11} \begin{vmatrix} y_{22} & \cdots & y_{2n} \\ \vdots & \ldots & \vdots \\ y_{n2} & \cdots & y_{nn} \end{vmatrix} \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{vmatrix} - y_{12}$$

$$\begin{vmatrix} y_{21} & \cdots & y_{2n} \\ \vdots & \ldots & \vdots \\ y_{n1} & \cdots & y_{nn} \end{vmatrix} \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{vmatrix} + \cdots + (-1)^{1+n} \; y_{1n} \begin{vmatrix} y_{21} & y_{22} & \cdots & y_{2(n-1)} \\ y_{31} & y_{32} & \cdots & y_{3(n-1)} \\ \vdots & \vdots & \ldots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{n(n-1)} \end{vmatrix} \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{vmatrix} =$$

$$\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{vmatrix} \left[y_{11} \begin{vmatrix} y_{22} & \cdots & y_{2n} \\ \vdots & \ldots & \vdots \\ y_{n2} & \cdots & y_{nn} \end{vmatrix} - y_{12} \begin{vmatrix} y_{21} & \cdots & y_{2n} \\ \vdots & \ldots & \vdots \\ y_{n1} & \cdots & y_{nn} \end{vmatrix} + \cdots + (-1)^{1+n} y_{1n} \begin{vmatrix} y_{21} & y_{22} & \cdots & y_{2(n-1)} \\ y_{31} & y_{32} & \cdots & y_{3(n-1)} \\ \vdots & \vdots & \ldots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{n(n-1)} \end{vmatrix} \right]$$

$= \det(X)\det(Y) = \det(Y)\det(X)$ □

Theorem 4.4: Given the 2×2 block matrix $B = \begin{bmatrix} D & N \\ 0 & U \end{bmatrix}$, $|B| = |D||U|$.

Proof. The induction proof works in the similar way as the proof above, it is enough for one of the off diagonal blocks to be zero. □

Remark 4.2: Given the 2×2 block matrix $K = \begin{bmatrix} S & 0 \\ L & B \end{bmatrix}$, the transpose K^T of this matrix is $K^T = \begin{bmatrix} S & L \\ 0 & B \end{bmatrix}$. Then, we have that

$$\det(K) = \begin{vmatrix} S & 0 \\ L & B \end{vmatrix} = \begin{vmatrix} S & L \\ 0 & B \end{vmatrix} = \det(K^T)$$

Therefore, to obtain the determinant of an upper triangular block matrix, one can either take transpose, and then compute the determinant of the lower triangular block matrix (as it is in Theorem 4.4) or one may also compute it directly by using cofactor expansion, not through a row this time, but through a column instead.

Example 4.1: Calculate the determinant of the following 2×2 block diagonal matrix

$$U = \left(\begin{array}{cccc|ccccc} 3 & 4 & -1 & 5 & 0 & 0 & 0 & 0 & 0 \\ 7 & 6 & 8 & 1 & 0 & 0 & 0 & 0 & 0 \\ 11 & 12 & 55 & 17 & 0 & 0 & 0 & 0 & 0 \\ 7 & 7 & 14 & 12 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 6 & 11 & 14 & 14 \\ 0 & 0 & 0 & 0 & 4 & -2 & -3 & 6 & 7 \\ 0 & 0 & 0 & 0 & 5 & 3 & 8 & 8 & 7 \\ 0 & 0 & 0 & 0 & 3 & 7 & 8 & 9 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 3 \end{array}\right)$$

$$|U| = \left|\begin{array}{cccc|ccccc} 3 & 4 & -1 & 5 & 0 & 0 & 0 & 0 & 0 \\ 7 & 6 & 8 & 1 & 0 & 0 & 0 & 0 & 0 \\ 11 & 12 & 55 & 17 & 0 & 0 & 0 & 0 & 0 \\ 7 & 7 & 14 & 12 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 6 & 11 & 14 & 14 \\ 0 & 0 & 0 & 0 & 4 & -2 & -3 & 6 & 7 \\ 0 & 0 & 0 & 0 & 5 & 3 & 8 & 8 & 7 \\ 0 & 0 & 0 & 0 & 3 & 7 & 8 & 9 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 3 \end{array}\right|$$

$$= \begin{vmatrix} 3 & 4 & -1 & 5 \\ 7 & 6 & 8 & 1 \\ 11 & 12 & 55 & 17 \\ 7 & 7 & 14 & 12 \end{vmatrix} \begin{vmatrix} 2 & 6 & 11 & 14 & 14 \\ 4 & -2 & -3 & 6 & 7 \\ 5 & 3 & 8 & 8 & 7 \\ 3 & 7 & 8 & 9 & 11 \\ 0 & 0 & 4 & 0 & 3 \end{vmatrix}$$

$$= (-2376)(8342) = -19820592$$

Example 4.2: Given the following lower triangular block matrix

$$\left[\begin{array}{ccc|cccc} 7 & 5 & 8 & 0 & 0 & 0 & 0 \\ 2 & 7 & 9 & 0 & 0 & 0 & 0 \\ 3 & 4 & -9 & 0 & 0 & 0 & 0 \\ \hline 7 & 3 & 4 & 5 & 7 & 8 & 9 \\ 5 & 5 & 5 & 3 & 2 & 1 & 1 \\ 2 & 1 & 3 & 7 & 5 & 8 & 7 \\ 3 & 4 & 4 & 9 & 2 & 4 & 8 \end{array}\right]$$

calculate the determinant of the above matrix.

$$\left|\begin{array}{ccc|cccc} 7 & 5 & 8 & 0 & 0 & 0 & 0 \\ 2 & 7 & 9 & 0 & 0 & 0 & 0 \\ 3 & 4 & -9 & 0 & 0 & 0 & 0 \\ \hline 7 & 3 & 4 & 5 & 7 & 8 & 9 \\ 5 & 5 & 5 & 3 & 2 & 1 & 1 \\ 2 & 1 & 3 & 7 & 5 & 8 & 7 \\ 3 & 4 & 4 & 9 & 2 & 4 & 8 \end{array}\right|$$

$$= \begin{vmatrix} 7 & 5 & 8 \\ 2 & 7 & 9 \\ 3 & 4 & -9 \end{vmatrix} \begin{vmatrix} 5 & 7 & 8 & 9 \\ 3 & 2 & 1 & 1 \\ 7 & 5 & 8 & 7 \\ 9 & 2 & 4 & 8 \end{vmatrix} = (-572)(-380) = 217360$$

Example 4.3: The matrix is given in the following way

$$X = \left[\begin{array}{ccccc|cc} 5 & 2 & -2 & 1 & 11 & 4 & 2 \\ 13 & 12 & 10 & 14 & 17 & 1 & 1 \\ 55 & 33 & 44 & 66 & 77 & 3 & 6 \\ 10 & 18 & 12 & 14 & 11 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 12 & 14 \\ 0 & 0 & 0 & 0 & 0 & 14 & 16 \end{array}\right].$$

Determinant of this matrix can be computed as follows:

$$\left|\begin{array}{ccccc|cc} 5 & 2 & -2 & 1 & 11 & 4 & 2 \\ 13 & 12 & 10 & 14 & 17 & 1 & 1 \\ 55 & 33 & 44 & 66 & 77 & 3 & 6 \\ 10 & 18 & 12 & 14 & 11 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 12 & 14 \\ 0 & 0 & 0 & 0 & 0 & 14 & 16 \end{array}\right|$$

$$= \begin{vmatrix} 5 & 2 & -2 & 1 & 11 \\ 13 & 12 & 10 & 14 & 17 \\ 55 & 33 & 44 & 66 & 77 \\ 10 & 18 & 12 & 14 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} 12 & 14 \\ 14 & 16 \end{vmatrix}$$

$= (0)(-4) = 0$

Lemma 4.1: If $Z = \begin{bmatrix} N & M \\ K & S \end{bmatrix}$, then $\det(Z) = \det(NS - MK)$, whenever at least one of the blocks N, M, K, S is equal to 0.

Proof. $K = 0$ and $M = 0$ cases have been already studied in Theorem 4.4 and Remark 4.2.

Using

$$\begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix} \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} N & M \\ K & S \end{bmatrix} = \begin{bmatrix} -K & -S \\ N & M \end{bmatrix}$$

If $N = 0$

$$\begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix} \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & M \\ K & S \end{bmatrix} = \begin{bmatrix} -K & -S \\ 0 & M \end{bmatrix}$$

Taking determinant on both sides produces

$$\det \left[\begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix} \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & M \\ K & S \end{bmatrix} \right] = \det \begin{bmatrix} -K & -S \\ 0 & M \end{bmatrix}$$

$$\det \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \det \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix} \det \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \det \begin{bmatrix} 0 & M \\ K & S \end{bmatrix} = \det \begin{bmatrix} -K & -S \\ 0 & M \end{bmatrix}$$

$$\det \begin{bmatrix} 0 & M \\ K & S \end{bmatrix} = \det \begin{bmatrix} -K & -S \\ 0 & M \end{bmatrix}$$

$$\det(Z) = \det(-KM) = \det(-K)\det(M).$$

If $S = 0$

$$\begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix} \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} N & M \\ K & 0 \end{bmatrix} = \begin{bmatrix} -K & 0 \\ N & M \end{bmatrix}$$

Again, taking determinant on both sides gives

$$\det \left[\begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix} \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} N & M \\ K & 0 \end{bmatrix} \right] = \det \begin{bmatrix} -K & 0 \\ N & M \end{bmatrix}$$

$$\det \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \det \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix} \det \begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \det \begin{bmatrix} N & M \\ K & 0 \end{bmatrix} = \det \begin{bmatrix} -K & 0 \\ N & M \end{bmatrix}$$

$$\det \begin{bmatrix} N & M \\ K & 0 \end{bmatrix} = \det \begin{bmatrix} -K & 0 \\ N & M \end{bmatrix}$$

$$\det(Z) = \det(-KM) = \det(-K)\det(M). \qquad \square$$

4.2 Determinants of 2×2 Block Matrices (In the Case that Blocks Commute with one another)

Definition 4.2: (Ring):

A ring is an Abelian group under addition with an extra multiplication operation such that the followings are satisfied: If $c, d \in R$ then $c \cdot d \in R$, $(c+d) \cdot e = ce + de$, $c \cdot (d+e) = cd + ce$, and $(c \cdot d) \cdot e = c \cdot (d \cdot e)$.

Definition 4.3: (Field): It is a commutative ring with unity 1_R, in which every non-zero element has a multiplicative inverse.

Theorem 4.5: If $V = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$ where $W, X, Y, Z \in M_{n \times n}(F)$ and $YZ = ZY$ then

$$\det_F(V) = \det_F(WZ - XY)$$

Proof. If $YZ = ZY$, then $YZ - ZY = 0$.

Using

$$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} \begin{bmatrix} Z & 0 \\ -Y & I_n \end{bmatrix} = \begin{bmatrix} WZ - XY & X \\ YZ - ZY & Z \end{bmatrix} = \begin{bmatrix} WZ - XY & X \\ 0 & Z \end{bmatrix} \quad (1)$$

and taking determinant on both sides:

$\det_F(V)\det_F(Z) = \det_F((WZ - XY)Z) \Rightarrow$

$\det_F(V)\det_F(Z) = \det_F(WZ - XY)\det_F(Z) \Rightarrow$

$[\det_F(V) - \det_F(WZ - XY)]\det_F(Z) = 0.$

Now, we need to consider 2 cases.

Case 1: If $\det_F(Z) \neq 0$ (The case that Z is invertible), then $\det_F(V) - \det_F(WZ - XY) = 0$, which implies that $\det_F(V) = \det_F(WZ - XY)$.

Case 2: If $\det_F(Z) = 0$ (Z is not invertible), we proceed as follows.

In this case, we will need to consider the polynomial ring *F[x]*. This is a commutative ring, where the elements are in the form $a_0x^r + a_1x^{r-1} + \cdots + a_{r-1}x + a_r$ with $a_i \in F$, for all i.

Let $Z = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$ where r_{11}, r_{12}, r_{21} and r_{22} are polynomials. Then

$Z_x = xI_n + Z = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} x + r_{11} & r_{12} \\ r_{21} & x + r_{22} \end{bmatrix}$. But

$\det_F(Z_x) = \det_F(xI_n + Z) = \begin{vmatrix} x + r_{11} & r_{12} \\ r_{21} & x + r_{22} \end{vmatrix} \neq 0$. We know $YZ = ZY$ and thus $YZ_x = Z_xY$. Define $V_x = \begin{bmatrix} W & X \\ Y & Z_x \end{bmatrix}$ where Y and Z_x commute.

Using (1), substituting V_x for V and Z_x for Z, we will get

$[\det_F(V_x) - \det_F(WZ_x - XY]] \det_F(Z_x) = 0 \Rightarrow$

$\det_F(V_x) - \det_F(WZ_x - XY) = 0 \Rightarrow$

$\det_F(V_x) = \det_F(WZ_x - XY)$. □

Remark 4.3: If other blocks commute, the formulas are as follows.

i. If $WY = YX$, then $\det_F(V) = \det_F(WZ - YX)$

ii. If $XZ = ZY$, then $\det_F(V) = \det_F(ZW - XY)$

iii. If $WX = XY$, then $\det_F(V) = \det_F(ZW - YX)$

4.3 Determinants of 2×2 Block Matrices

In this part we give the determinant formula for 2×2 block matrices, where all blocks are square matrices of the same size, with the condition that lower right block matrix must be non-singular.

Theorem 4.6: If $E, F, G, H \in M_{n\times n}(F)$ and H is non-singular, then

$$\det \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \det(EH - FH^{-1}GH)$$

Proof. By using $\begin{bmatrix} E & F \\ G & H \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -H^{-1}G & I_n \end{bmatrix} = \begin{bmatrix} E - FH^{-1}G & F \\ 0 & H \end{bmatrix}$

Take determinant on both side

$$\det \left(\begin{bmatrix} E & F \\ G & H \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -H^{-1}G & I_n \end{bmatrix} \right) = \det \begin{bmatrix} E - FH^{-1}G & F \\ 0 & H \end{bmatrix}$$

$$\det \begin{bmatrix} E & F \\ G & H \end{bmatrix} \det \begin{bmatrix} I_n & 0 \\ -H^{-1}G & I_n \end{bmatrix} = \det \begin{bmatrix} E - FH^{-1}G & F \\ 0 & H \end{bmatrix}$$

$$\det \begin{bmatrix} E & F \\ G & H \end{bmatrix} \det I_n = \det(E - FH^{-1}G)\det(H)$$

$$\det \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \det(E - FH^{-1}G)\det(H)$$

$$\det \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \det(EH - FH^{-1}GH) \qquad \square$$

Example 4.4: Find E,F,G and H that satisfy the following equality

$$\det \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \det(E - FH^{-1}G)\det(H)$$

Let

$$\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \left[\begin{array}{ccc|ccc} 8 & 7 & 9 & 5 & 8 & 10 \\ 8 & 6 & 2 & 8 & 4 & 7 \\ 1 & -11 & -91 & -2 & 5 & 9 \\ \hline -12 & -49 & -13 & 12 & 21 & 35 \\ 23 & 24 & 91 & 36 & 25 & 10 \\ -14 & -41 & -61 & -15 & -16 & -14 \end{array}\right]$$

$$
\det\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \det\left[\begin{array}{ccc|ccc} 8 & 7 & 9 & 5 & 8 & 10 \\ 8 & 6 & 2 & 8 & 4 & 7 \\ 1 & -11 & -91 & -2 & 5 & 9 \\ \hline -12 & -49 & -13 & 12 & 21 & 35 \\ 23 & 24 & 91 & 36 & 25 & 10 \\ -14 & -41 & -61 & -15 & -16 & -14 \end{array}\right] =
$$

$$
\det\left(\begin{bmatrix} 8 & 7 & 9 \\ 8 & 6 & 2 \\ 1 & -11 & -91 \end{bmatrix} - \begin{bmatrix} 5 & 8 & 10 \\ 8 & 4 & 7 \\ -2 & 5 & 9 \end{bmatrix}\begin{bmatrix} 12 & 21 & 35 \\ 36 & 25 & 10 \\ -15 & -16 & -14 \end{bmatrix}^{-1}\begin{bmatrix} -12 & -49 & -13 \\ 23 & 24 & 91 \\ -14 & -41 & -61 \end{bmatrix}\right)\det\left(\begin{bmatrix} 12 & 21 & 35 \\ 36 & 25 & 10 \\ -15 & -16 & -14 \end{bmatrix}\right)
$$

$$
= \det\left(\begin{bmatrix} \frac{678}{209} & -\frac{1268}{99} & -\frac{29806}{1881} \\ \frac{5845}{209} & \frac{11047}{99} & \frac{122426}{1881} \\ -\frac{155}{19} & -\frac{614}{9} & -\frac{22180}{171} \end{bmatrix}\right)\det\left(\begin{bmatrix} 12 & 21 & 35 \\ 36 & 25 & 10 \\ -15 & -16 & -14 \end{bmatrix}\right)
$$

$$
= -\left(\tfrac{35362846}{627}\right)(-1881)
$$

$$
= 106088538.
$$

4.4 Determinants of $n \times n$ Block Matrices (When Blocks Commute with one another)

Next we state the main theorem with the commutativity condition within the blocks, for proof please refer to [17].

Theorem 4.7: Let R be a commutative subring of $M_{n\times n}(F)$, where F is a field (or a commutative ring) , and let $T \in M_{m\times m}(R)$. Then

$$
\det_F T = \det_F(\det_R T)
$$

Example 4.5: Let

$$P = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & -2 & -1 & -6 \\ 3 & 2 & 0 & 3 & 2 & 9 \\ -1 & -1 & -1 & -1 & -1 & -4 \\ \hline -2 & -1 & -6 & 1 & 2 & 3 \\ 3 & 2 & 9 & 3 & 2 & 0 \\ -1 & -1 & -4 & -1 & -1 & -1 \end{array}\right]$$

Show that $\det_F P = \det_F(\det_R P)$

First, we choose W, X, Y, Z in a way that they commute with each other.

Next, we need to calculate $\det_F P$ and $\det_F(\det_R P)$

LHS: We compute determinant directly by using cofactor expansion from any row or column.

$$\det_F P = \det_F \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & -2 & -1 & -6 \\ 3 & 2 & 0 & 3 & 2 & 9 \\ -1 & -1 & -1 & -1 & -1 & -4 \\ \hline -2 & -1 & -6 & 1 & 2 & 3 \\ 3 & 2 & 9 & 3 & 2 & 0 \\ -1 & -1 & -4 & -1 & -1 & -1 \end{array}\right] = 0$$

RHS: $\det_F(\det_R P) =$

$$\det_F \left[\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 0 \\ -1 & -1 & -1 \end{bmatrix} - \begin{bmatrix} -2 & -1 & -6 \\ 3 & 2 & 9 \\ -1 & -1 & -4 \end{bmatrix} \begin{bmatrix} -2 & -1 & -6 \\ 3 & 2 & 9 \\ -1 & -1 & -4 \end{bmatrix} \right]$$

$$= \det_F \left[\begin{bmatrix} -3 & -3 & -27 \\ 18 & 18 & 45 \\ -6 & -6 & -15 \end{bmatrix} \right] = 0.$$

Therefore, $\det_F P = \det_F(\det_R P)$.

4.5 Determinant of Tensor Product of Two Matrices

Definition 4.4: (Tensor product between two block matrices and their determinants):

Let $U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \in M_{2\times 2}(F)$ and $W \in M_{n\times n}(F)$. The tensor product $U \otimes W$ will be a $2n \times 2n$ block matrix $U \otimes W = \begin{bmatrix} u_{11}W & u_{12}W \\ u_{21}W & u_{22}W \end{bmatrix}$.

or in general , let $U \in M_{m\times m}(F)$ and $W \in M_{n\times n}(F)$. The tensor product $U \otimes W$ will be $mn \times mn$ matrix

$$U \otimes W = \begin{bmatrix} u_{11}W & u_{12}W & \cdots & u_{1m}W \\ \vdots & \vdots & \ddots & \vdots \\ u_{m1}W & u_{m2}W & \cdots & u_{mm}W \end{bmatrix}$$

Remark 4.4: From definition 4.4, we can show $\det_F(U\otimes W) = (\det_F U)^n(\det_F W)^2$

$$\begin{aligned}\det_F(U\otimes W) &= \det_F\left((u_{11}W)(u_{22}W)-(u_{12}W)(u_{21}W)\right)\\ &= \det_F\left(u_{11}u_{22}W^2-u_{12}u_{21}W^2\right)\\ &= \det_F\left((u_{11}u_{22}-u_{12}u_{21})W^2\right)\\ &= \det_F\left((\det_F U)W^2\right)\\ &= (\det_F U)^n(\det_F W)^2\end{aligned}$$

Example 4.6: Let $W=\begin{bmatrix}1&1&2\\5&3&2\\1&0&0\end{bmatrix}, S=\begin{bmatrix}1&2&1\\1&1&1\\1&1&1\end{bmatrix}$

$$W\otimes S=\left(\begin{array}{c|c|c} 1\begin{bmatrix}1&2&1\\1&1&1\\1&1&1\end{bmatrix} & 1\begin{bmatrix}1&2&1\\1&1&1\\1&1&1\end{bmatrix} & 2\begin{bmatrix}1&2&1\\1&1&1\\1&1&1\end{bmatrix}\\ \hline 5\begin{bmatrix}1&2&1\\1&1&1\\1&1&1\end{bmatrix} & 3\begin{bmatrix}1&2&1\\1&1&1\\1&1&1\end{bmatrix} & 2\begin{bmatrix}1&2&1\\1&1&1\\1&1&1\end{bmatrix}\\ \hline 1\begin{bmatrix}1&2&1\\1&1&1\\1&1&1\end{bmatrix} & 0\begin{bmatrix}1&2&1\\1&1&1\\1&1&1\end{bmatrix} & 0\begin{bmatrix}1&2&1\\1&1&1\\1&1&1\end{bmatrix}\end{array}\right)$$

$$=\left[\begin{array}{c|c|c} \begin{bmatrix}1&2&1\\1&1&1\\1&1&1\end{bmatrix} & \begin{bmatrix}1&2&1\\1&1&1\\1&1&1\end{bmatrix} & \begin{bmatrix}2&4&2\\2&2&2\\2&2&2\end{bmatrix}\\ \hline \begin{bmatrix}5&10&5\\5&5&5\\5&5&5\end{bmatrix} & \begin{bmatrix}3&6&3\\3&3&3\\3&3&3\end{bmatrix} & \begin{bmatrix}2&4&2\\2&2&2\\2&2&2\end{bmatrix}\\ \hline \begin{bmatrix}1&2&1\\1&1&1\\1&1&1\end{bmatrix} & \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix} & \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\end{array}\right]$$

Theorem 4.8: Let $K \in M_{m\times m}(F)$ and $S \in M_{n\times n}(F)$. Then $\det_F(K\otimes S) = (\det_F K)^n(\det_F S)^m$.

Proof. We can prove it by using inductive proof on m, size of block matrix K.

Base Case ($m = 1$): $K = [k_{11}] \in M_{1\times 1}(F)$. In this case $K\otimes S = [k_{11}S]$ and $\det_R(k_{11}S) = k_{11}S = \det_F(K)S$. Moreover, $\det_F(k_{11}S) = {k_{11}}^n \det_F S$.

Next assume by induction, that result holds for $m-1$ case, i.e. for $K \in M_{(m-1)\times(m-1)}(F)$, and prove the m case.

$$\text{Let } K = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1m} \\ k_{21} & k_{22} & \cdots & k_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ k_{m1} & k_{m2} & \cdots & k_{mm} \end{bmatrix} \in M_{m\times m}(F)$$

$$K\otimes S = \begin{bmatrix} k_{11}S & k_{12}S & \cdots & k_{1m}S \\ k_{21}S & k_{22}S & \cdots & k_{2m}S \\ \vdots & \vdots & \cdots & \vdots \\ k_{m1}S & k_{m2}S & \cdots & k_{mm}S \end{bmatrix}$$

$$\begin{aligned} \det_R(K\otimes S) &= k_{11}S\left[\det_R(K^{11}_{m-1}\otimes S)\right] - k_{12}S\left[\det_R(K^{12}_{m-1}\otimes S)\right] + \cdots + \\ &(-1)^{m+1}k_{1m}S\left[\det_R(K^{1m}_{m-1}\otimes S)\right] \\ &= k_{11}S\left((\det_F K^{11}_{m-1})S^{m-1}\right) - k_{12}S\left((\det_F K^{12}_{m-1})S^{m-1}\right) + \cdots + \\ &(-1)^{m+1}k_{1m}S\left((\det_F K^{1m}_{m-1})S^{m-1}\right) \\ &= (\det_F K)S^m \end{aligned}$$

where K_{m-1}^{ij} here is the square matrix with size $m-1$, when the ith row and jth column are deleted from K. Next,

$$\det_F(K \otimes S) = \det_F\left(\det_R(K \otimes S)\right) = \det_F\left((\det_F K)S^m\right)$$

$$= (\det_F K)^n(\det_F S^m) = (\det_F K)^n(\det_F S)^m$$

□

Example 4.7: Let

$$Y = \begin{bmatrix} 2 & 1 & 5 & 7 \\ 4 & 1 & 1 & 1 \\ 0 & 4 & -5 & -7 \\ 6 & 8 & 9 & 11 \end{bmatrix}, U = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 5 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

Find $\det_F(Y \otimes U)$.

$$\det_F(Y \otimes U) = \left(\det_F \begin{bmatrix} 2 & 1 & 5 & 7 \\ 4 & 1 & 1 & 1 \\ 0 & 4 & -5 & -7 \\ 6 & 8 & 9 & 11 \end{bmatrix}\right)^3 \left(\det_F \begin{bmatrix} 1 & 0 & 0 \\ 3 & 5 & 6 \\ 2 & 2 & 2 \end{bmatrix}\right)^4$$

$$=(-132)^3(-2)^4$$

$$=-36799488$$

4.6 Determinant Formula for General $n \times n$ Block Matrix

In this section, we present a formula for the determinant of a $n \times n$ block matrix, a result due to [15].

Theorem 4.9: Let M be an $(nN) \times (nN)$ complex matrix, which is partitioned into N^2 blocks, each of size $n \times n$:

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1N} \\ M_{21} & M_{22} & \cdots & M_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ M_{N1} & M_{N2} & \cdots & M_{NN} \end{bmatrix}$$

Then, the determinant of M is given by

$$\det(M) = \prod_{k=1}^{N} \det\left(\alpha_{kk}^{(N-k)}\right)$$

where the $\alpha^{(k)}$ are defined by

$\alpha_{ij}^{(k)} = M_{ij} - \sigma_{i,N-k+1}^{T} \widetilde{M}_k^{-1} m_{N-k+1,j}$ for $k \geq 1$, and the vectors σ_{ij}^{T} and m_{ij} are given by

$m_{ij} = (M_{ij}, M_{i+1,j}, \cdots, M_{Nj})^T$ and $\sigma_{ij}^{T} = (M_{ij}, M_{i,j+1}, \cdots, M_{iN})$.

We also let $\widetilde{M}_k$ represent the $k \times k$ block matrix formed from the lower-right corner of M:

$$\widetilde{M}_k = \begin{bmatrix} M_{N-k+1,N-k+1} & M_{N-k+1,N-k+2} & \cdots & M_{N-k+1,N} \\ M_{N-k+2,N-k+1} & M_{N-k+2,N-k+2} & \cdots & M_{N-k+2,N} \\ \vdots & \vdots & \cdots & \vdots \\ M_{N,N-k+1} & M_{N,N-k+2} & \cdots & M_{N,N} \end{bmatrix}.$$

4.6.1 Special Cases: Determinant Formulas for 2×2 and 3×3 Block Matrices

For the convenience, we give below the formula for the cases $N = 2$ and $N = 3$.

If $N = 2$

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$\det(M) = \prod_{k=1}^{2} \det(\alpha_{kk}^{(2-k)}) = \det(\alpha_{11}^{(1)}) \det(\alpha_{22}^{(0)})$$

.

For $i = j = 2, k = 0$, $\alpha_{22}^{(0)} = M_{22}$, and for $i = j = 1, k = 1$, $\alpha_{11}^{(1)} = M_{11} - \sigma_{12}^T M_{22}^{-1} m_{21}$. Hence $\alpha_{11}^{(1)} = M_{11} - \sigma_{12}^T M_{22}^{-1} m_{21} = M_{11} - M_{12} M_{22}^{-1} M_{21}$. Therefore,

$$\begin{aligned} \det(M) = \det\left(\alpha_{11}^{(1)}\right) \det\left(\alpha_{22}^{(0)}\right) &= \det\left(M_{11} - M_{12} M_{22}^{-1} M_{21}\right) \det\left(M_{22}\right) \\ &= \det\left(M_{11} M_{22} - M_{12} M_{22}^{-1} M_{21} M_{22}\right) \end{aligned}$$

If $N = 3$

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$\det(M) = \prod_{k=1}^{3} \det\left(\alpha_{kk}^{(3-k)}\right) = \det\left(\alpha_{11}^{(2)}\right)\det\left(\alpha_{22}^{(1)}\right)\det\left(\alpha_{33}^{(0)}\right)$$

.

For $i = j = 3, k = 0$, $\alpha_{33}^{(0)} = M_{33}$, for $i = j = 2, k = 1$, $\alpha_{22}^{(1)} = M_{22} - \sigma_{23}^{T}\widetilde{M}_{1}^{-1}m_{32}$, and finally for $i = j = 1$, and $k = 2, \alpha_{11}^{(2)} = M_{11} - \sigma_{12}^{T}\widetilde{M}_{2}^{-1}m_{21}$.

$$\det(M) = \det\left(\alpha_{11}^{(2)}\right)\det\left(\alpha_{22}^{(1)}\right)\det\left(\alpha_{33}^{(0)}\right) =$$
$$\det\left[M_{11} - \begin{pmatrix} M_{12} & M_{13} \end{pmatrix}\begin{pmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{pmatrix}^{-1}\begin{pmatrix} M_{21} \\ M_{31} \end{pmatrix}\right]\det\left(M_{22} - M_{23}M_{33}^{-1}M_{32}\right)\det\left(M_{33}\right).$$

Using Theorem 3.2 for the inverse of 2×2 block matrix and after some algebraic operations, we obtain the following formula.

$$\begin{aligned}\det(M) = &\Big[\det\left(M_{11} - M_{13}M_{33}^{-1}M_{31}\right) \\ &- \left(M_{12} - M_{13}M_{33}^{-1}M_{32}\right)\left(M_{22} - M_{23}M_{33}^{-1}M_{32}\right)^{-1}\left(M_{21} - M_{23}M_{33}^{-1}M_{31}\right)\Big] \\ &\times \det\left(M_{22} - M_{23}M_{33}^{-1}M_{32}\right)\det\left(M_{33}\right).\end{aligned}$$

Example 4.8: Calculate the determinant of the following 2×2 block matrix

$$A = \left[\begin{array}{ccc|ccc} 3 & 2 & 9 & -7 & 7 & 9 \\ 1 & 3 & 0 & 2 & 6 & 5 \\ 5 & 6 & 8 & -1 & -11 & 12 \\ \hline 0 & 0 & 7 & 3 & 6 & -5 \\ 6 & 6 & 8 & 2 & 2 & -4 \\ -2 & -6 & 7 & 9 & 8 & -4 \end{array}\right]$$

$$|A| = \left|\begin{array}{ccc|ccc} 3 & 2 & 9 & -7 & 7 & 9 \\ 1 & 3 & 0 & 2 & 6 & 5 \\ 5 & 6 & 8 & -1 & -11 & 12 \\ \hline 0 & 0 & 7 & 3 & 6 & -5 \\ 6 & 6 & 8 & 2 & 2 & -4 \\ -2 & -6 & 7 & 9 & 8 & -4 \end{array}\right|$$

$$= \det\left(\begin{bmatrix} 3 & 2 & 9 \\ 1 & 3 & 0 \\ 5 & 6 & 8 \end{bmatrix} - \begin{bmatrix} -7 & 7 & 9 \\ 2 & 6 & 5 \\ -1 & -11 & 12 \end{bmatrix}\begin{bmatrix} 3 & 6 & -5 \\ 2 & 2 & -4 \\ 9 & 8 & -4 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 & 7 \\ 6 & 6 & 8 \\ -2 & -6 & 7 \end{bmatrix}\right) \det\left(\begin{bmatrix} 3 & 6 & -5 \\ 2 & 2 & -4 \\ 9 & 8 & -4 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} \frac{1910}{43} & \frac{1751}{43} & \frac{1705}{43} \\ \frac{1037}{43} & \frac{1215}{43} & \frac{648}{43} \\ \frac{334}{43} & \frac{537}{43} & \frac{1026}{43} \end{bmatrix}\right) \det\left(\begin{bmatrix} 3 & 6 & -5 \\ 2 & 2 & -4 \\ 9 & 8 & -4 \end{bmatrix}\right) = \frac{264945}{43}(-86) = -529890.$$

Example 4.9: Let $C = \left(\begin{array}{cc|cc|cc} 6 & 0 & 5 & 5 & 8 & 4 \\ 2 & 9 & 8 & 8 & 9 & 8 \\ \hline 5 & 5 & 6 & 6 & 2 & 9 \\ 17 & 19 & -9 & -9 & -3 & -8 \\ \hline 7 & 7 & 9 & 6 & 3 & 7 \\ 3 & 7 & 8 & 6 & 5 & 7 \end{array}\right)$

Calculate $|C|$.

$$|C| = \left|\begin{array}{cc|cc|cc} 6 & 0 & 5 & 5 & 8 & 4 \\ 2 & 9 & 8 & 8 & 9 & 8 \\ \hline 5 & 5 & 6 & 6 & 2 & 9 \\ 17 & 19 & -9 & -9 & -3 & -8 \\ \hline 7 & 7 & 9 & 6 & 3 & 7 \\ 3 & 7 & 8 & 6 & 5 & 7 \end{array}\right|$$

$$= \det\left(\begin{bmatrix} 6 & 0 \\ 2 & 9 \end{bmatrix} - \begin{bmatrix} 5 & 5 & 8 & 4 \\ 8 & 8 & 9 & 8 \end{bmatrix} \begin{bmatrix} 6 & 6 & 2 & 9 \\ -9 & -9 & -3 & -8 \\ 9 & 6 & 3 & 7 \\ 8 & 6 & 5 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 5 \\ 17 & 19 \\ 7 & 7 \\ 3 & 7 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} 6 & 6 \\ -9 & -9 \end{bmatrix} - \begin{bmatrix} 2 & 9 \\ -3 & -8 \end{bmatrix} \begin{bmatrix} 3 & 7 \\ 5 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 9 & 6 \\ 8 & 6 \end{bmatrix} \right) \det\left(\begin{bmatrix} 3 & 7 \\ 5 & 7 \end{bmatrix} \right)$$

$$= (\tfrac{40031}{231})(\tfrac{33}{2})(-14) = -40031$$

Chapter 5

CONCLUSION

This thesis is about two fundamental algebraic properties; inverses and determinants of block matrices. It aims to serve as a primary reference for block matrices for all who are interested in the subject or would like to use block matrices in their research. We give inverse matrix formula for the 2×2 block matrix and then discuss how to generalise this to $n \times n$ case. Under determinants, two formulas existing in the literature are revised in detail, with examples. For one formula, condition is that blocks within the block matrix must commute with one another. Then the general formula is also presented. Alternate proofs for the inverses of block matrices are provided together with a new proof for the determinant of the tensor product of two given matrices.

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