Existence and Controllability of Conformable Impulsive Equations

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ABSTRACT

In this study, the focus is directed towards the explanation of existence and uniqueness phenomena in impulsive dynamical systems governed by conformable fractional nonlinear differential equations. Concurrently, the analysis encompasses the controllability of systems delineated by linear and semilinear conformable fractional impulsive control mechanisms. By utilizing the mathematical apparatus of conformable fractional derivatives, salient constructs such as the conformable controllability operator and the conformable controllability Gramian matrix are introduced. These instrumentalities facilitate the derivation of both necessary and sufficient conditions that are requisite for achieving comprehensive controllability in linear impulsive systems within the framework of conformable fractional calculus. Moreover, an assemblage of rigorously formulated sufficient criteria is prefered to ascertain the controllability of semilinear impulsive systems in the domain of conformable fractional calculus.

Keywords: Existence, Controllability, Conformable Derivative, Impulsive Equation

ÖZ

Bu çalışmada, odak noktası, uyumlu kesirli tip dürtüsel olmayan diferansiyel denklemlerle yönetilen sistemlerde varlık ve benzersizlik olgularının açıklanmasına yöneliktir. Aynı anda, analiz, doğrusal ve yarı doğrusal uyumlu kesirli ani kontrol mekanizmaları tarafından belirlenen sistemlerin kontrol edilebilirliğini kapsar. Uyumlu kesirli türevlerin matematik araçlarını kullanarak, uyumlu kontrol edilebilirlik operatörü ve uyumlu kontrol edilebilirlik Gramian matrisi gibi önemli yapılar tanıtılmıştır. Bu araçlar, uyumlu kesirli hesaplamalar çerçevesinde doğrusal sistemlerde kapsamlı kontrol edilebilirlik için gerekli olan hem gerekli hem de yeterli koşulların türetilmeyi kolaylaştırır. Ayrıca, uyumlu kesirli hesaplamalar alanında yarı doğrusal sistemlerin kontrol edilebilirliğini belirlemek için sıkıca formüle edilmiş yeterli kriterlerin bir derlemesi sunulmaktadır.

Anahtar Kelimeler: Varlık, Kontrol Edilebilirlik, Uyumlu Türev, Dürtüsel Denklem

DEDICATION

Dedicated to the Akgün Family

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TABLE OF CONTENTS

LIST OF SYMBOLS AND ABBREVIATIONS

I	The Set of nxn Identity Matrix
L	Pre-fixed Positive Number
Q	The Set of Q-operator
X	Banach Space
∞	The Infinite Symbol
M^*	The Adjoint Operator
Θ	The Set of nxn Null Matrix
$\mathbb{R}$	The Set of Real Number
$\mathbb{N}$	The Set of Natural Number
$\lVert\cdot\rVert$	Norm of
$\Gamma(k)$	Gamma Function
$\beta(n,m)$	Beta Function
$E_0^v y$	Caputo Conformable Derivative
${}^cE_{0^+}^b$	Caputo Fractional Derivative
$(I_v^b h)(r)$	The Riemann-Liouville Fractional Derivative
$(R_v^b h)(r)$	The Fractional Derivative
FDE	Fractional Differential Equation
FPT	Fixed Point Theorem
FTS	Finite-Time Stability

Chapter 1

INTRODUCTION

Within the purview of fractional calculus, diverse definitional frameworks—such as Riemann-Liouville and Caputo—provide the mathematical foundations for solving different classes of problems. The former, grounded in the principles of repeated integration, proves particularly efficacious for confronting problems framed by initial conditions. The latter, based on initial value problems, exhibits greater suitability when boundary conditions are central to the inquiry. Notably, these formulations are not exhaustive, as other specialized definitions like Grunwald-Letnikov, Weyl, and Riesz exist, each with its own application-specific features, as corroborated by existing literature [2].

The pertinence of fractional derivatives extends beyond theoretical mathematics, permeating multiple scientific domains such as physics, engineering, economics, and biology. Consequently, the selection of an appropriate definitional framework is contingent upon the domain-specific requirements and the nature of the problem under consideration.

Introduced in the seminal works [3]-[5], the notion of the conformal derivative has found applications in augmenting Newtonian mechanics [6], logistic modeling frameworks [7], and web models [8]. This mathematical concept is established based on a fundamental limit associated with the traditional concept of a derivative.

Significantly, the conformal derivative possesses various mathematical properties, including those related to multiplication, rules of composition, and division. Consequently, it functions as an extension of the standard derivative that does not depend on previous data and memory.

Recently, many research papers have been published concerning the Caputo fractional derivative. Some of them presented additional results on caputo fractional derivative and others solved differential equations modeled on caputo fractional derivative sense. Some researchers have tried to demonstrate that the caputo fractional derivative may be alternative to the caputo fractional derivative in many fractional equations due to the ease of calculating the caputo fractional derivative an others. Many researchers presented new definitions of non-local caputo fractional derivative and published several articles that included applications to new definitions.

There is an extensive body of research that has investigated a wide range of deterministic and stochastic differential equations, including linear, semi-linear, and non-linear ones. These studies have considered both conformable and classical derivatives in their analyses and have been well-documented in the academic literature [9-26]. Likewise, equations featuring Caputo derivatives have been the focus of investigations in [27-29]. The utility of the conformal derivative transcends theoretical discourse; it is particularly germane in the realm of nonlinear control systems. Within this context, the conformal derivative aids in the articulation of system behaviors, facilitating the derivation of targeted control strategies.

Semilinear impulsive differential equations serve as descriptive mathematical paradigms that capture the dynamical evolution of systems subject to both continuous and abrupt state-variable alterations. Real-world phenomena, including biological systems with threshold behaviors, economic models requiring optimal control, and periodically modulated systems, often manifest impulsive effects. Hence, impulsive differential equations offer a compelling mathematical representation for the nuanced behaviors observed in these complex systems.

In control theory, the notion of controllability—the capacity to govern a system's state to reach a specified objective via control inputs—holds significant importance. It serves as a fundamental principle in designing controllers that can efficaciously guide a system toward a desired state. Recently, the area of impulsive control systems has garnered substantial scholarly attention owing to its applicability across diverse disciplines. Pioneering research by Muni and George [30], Han et al. [31], Guan et al. [32, 33], Zhao and Sun [34, 35], Xie and Wang [36, 37], George et al. [38], and Benzaid and Sznaier [39] has enriched our understanding of the controllability attributes of impulsive systems. These theoretical advancements have found practical applications in an array of system types, such as those exhibiting fractal behaviors, polynomial architectures, switched configurations, index function setups, and rational function designs, thereby broadening the scope for control algorithm development.

However, the realm of impulse differential equations employing a conformable derivative remains largely unexplored. Motivated by the prior research mentioned, the present study aims to investigate the presence, distinctiveness, and the extent to which solutions can be controlled for the following set of semilinear impulsive differential equations that employ a conformable derivative:

$$\begin{cases} E_0^v y(r)=Ay(r)+Bu(r)+h(r,y(r)), \ r\in[0,R]\backslash\{r_1,r_2,r_3,\dots,r_p\}, \ 0<v<1, \\ y(r_k^+)=(I+C_k)y(r_k^-), \ k\in\mathbb{K}:=\{1,2,3,\dots,p\}, \ r_0=0, \ r_{p+1}=R, \\ y(0)=y_0. \end{cases} \tag{1}$$

Where $E_0^v y$ is the conformable derivative having a lower index of 0 applied to the function y, $A, C_k \in \mathbb{R}^{d\times d}$ are matrices, $B\in\mathbb{R}^{r\times d}$ is a matrix,

$y(r_k)=y(r_k^-)$, $h:[0,R]\times\mathbb{R}^d\to\mathbb{R}^d$,

$u:[0,R]\to\mathbb{R}^r$ is a control function that belong to $L^2([0,R],\mathbb{R}^r)$.

The organization of this study is outlined as follows: In Chapter 3, we revisit and provide a summary of the fundamental principles of conformable fractional derivatives and integrals, as well as a review of previously established findings. Moving on to Chapter 4, we dive into the examination of the conformable linear impulsive Cauchy problem, which is presented as:

$$\begin{cases} E_0^v y(r)=Ay(r)+Bu(r)+h(r), \ r\in[0,R], \ 0<v<1, \ h\in C([0,R],\mathbb{R}^d), \\ y(r_k^+)=(I+C_r)y(r_k^-), \quad k\in\mathbb{K}:=\{1,2,,\dots,p\}, \quad r_0=0, \quad r_{p+1}=R, \\ y(0)=y_0. \end{cases} \tag{2}$$

Herein we develop an expression for the solution to the linear impulsive problem involving a conformable derivative (equation 2). Chapter 5 then shifts its focus towards investigating the presence and singularity of solutions for impulsive semilinear and nonlinear differential equations that incorporate the conformable derivative. Our analytical approaches involve iterative methods and the application of the Schauder fixed point theorem. Finally, Chapter 6 is dedicated to the discourse on controllability, this section elucidates on the linear and semilinear impulsive equations within the framework of conformable calculus.

Summarizing the main contributions of this study: Firstly, we establish a method to represent the solution for the nonhomogeneous system described in equation (2) and subsequently deduce its general solution. Next, we conduct a thorough investigation into the characteristics of existence and uniqueness of the solution for the semilinear system outlined in equation (1). Additionally, the manuscript introduces two important concepts: the conformable controllability operator and the conformable controllability Gramian matrix. These tools aid in determining both the necessary and sufficient conditions required for achieving full controllability in linear impulsive systems with conformable dynamics. In conclusion, the study concludes by outlining a set of sufficient conditions that are crucial for ensuring the controllability of the semilinear impulsive system described in equation (1).

Chapter 2

CONFORMABLE FRACTIONAL DERIVATIVE

2.1 Basic Definitions of Fractional Derivative

Definition 2.1 [(2)] The(left) fractional derivative originating from of a function $h:[d,\infty)\to\mathbb{R}$ of order $0<v\le 1$ is defined by:

$$\left(R_v^d h\right)(r)=\lim_{\varepsilon\to 0}\frac{h\left(r+\varepsilon(r-d)^{1-v}\right)-h(r)}{\varepsilon}.$$

When $d=0$ we write R_v. If $\left(R_v h\right)(r)$ exists on (d,e) then,

$$\left(R_v^d h\right)(d)=\lim_{r\to d^+}\left(R_v^d h\right)(r).$$

The right fractional derivative with an order of $0<v\le 1$ concluding at e of h is defined by,

$$\left({}_v^e Rh\right)(r)=-\lim_{\varepsilon\to 0}\frac{h(r+\varepsilon(e-r)^{1-v})-h(r)}{\varepsilon}.$$

If $\left({}^e R_d h\right)(r)$ exists on, (d,e) then,

$$\left({}^e R_v h\right)(e)=\lim_{r\to e^-}\left({}^e R_v h\right)(r).$$

Note that if h is differentiable then

$$\left(R_{v}^{d}h\right)(r)=\left(r-d\right)^{1-v}h'(r)$$

and

$$\left({}_{v}^{e}Rh\right)(r)=-(e-r)^{1-v}h'(r).$$

It is evident that, for a constant function, the value of its conformable fractional derivative converges to zero. Conversely, if $R_v h(r)=0$ on a specific interval (d,e) can be demonstrated with the assistance of the conformable fractional mean value theorem, as established in [10]. It can be that $h(x)=0$ for all $x\in(d,e)$. Furthermore, by the fractional mean value theorem allows us to establish that when the conformable fractional derivative of a function h over an interval (d,e) assumes a positive (negative) value, then the graph of h is an increase (decrease) within that interval.

Notation. $\left(I_{v}^{d}h\right)(r)=\int_{d}^{r}h(x)dv(x,d)=\int_{d}^{r}(x-d)^{v-1}h(x)dx.$

When $d=0$ we write $dv(x).$ In the right-case, we have

$$\left({}^{e}I_{v}h\right)(r)=\int_{r}^{e}h(x)dv(e,x)=\int_{r}^{e}(e-x)^{v-1}h(x)dx.$$

The operators I_{v}^{d} and ${}^{e}I_{v}$ called the conformable left and right fractional integrals of a given order $0<v\leq 1.$

We can generalised as below:

Definition 2.2 ([3]) Let $v \in (m, m+1]$, and set $\beta = v - m$.

Then, the (left) fractional derivative starting from d of a function $h:[d,\infty) \to \mathbb{R}$ of a order v, where $h^{(m)}(r)$ exists, is defined by,

$$\left(R_v^d h\right)(r) = \left(R_\beta^d h^{(m)}\right)(r).$$

When $d = 0$ we write R_v.

The fractional derivative on the right-hand side, with a specified order v terminating at e of h formally characterized by

$$\left({}_v^e R h\right)(r) = (-1)^{m+1}\left({}_\beta^e R h^{(m)}\right)(r).$$

Note that if $v = m+1$ then $\beta = 1$ and the fractional derivative of h becomes $h^{(m+1)}(r)$. Also when $m = 0$ (or $v \in (0,1)$) then $\beta = v$ and the definition coincides with those in definition,

$$\left(R_v^d h\right)(r) = \lim_{\varepsilon \to 0} \frac{h\left(r + \varepsilon (r-d)^{1-v}\right) - h(r)}{\varepsilon}.$$

Lemma 2.1 ([3]) *Let us suppose that* $h:[d,\infty) \to \mathbb{R}$ *is continuous and* $0 < v \le 1$. *Then, for all* $r > d$ *we have*

$$R_v^d I_v^d h(r) = h(r).$$

Lemma 2.2 ([5]) *Let us suppose that* $h:(-\infty,e]\to\mathbb{R}$ *is continuous and* $0<v\le 1.$ *Then, for all* $r<e$ *we have*

$${}^{e}R_{v}\,{}^{e}I_{v}h(r)=h(r)$$

Subsequently, we introduce the formal definitions for both left and right fractional integrals any order $v>0.$

Definition 2.3 ([6]) Let $v\in(m,m+1]$ then the left fraction integral of order *d* is defined by

$$\left(I_{v}^{d}h\right)(r)=\boldsymbol{I}_{m+1}^{d}\left((r-d)^{\beta-1}h\right)=\frac{1}{m!}\int_{d}^{r}(r-x)^{m}(x-d)^{\beta-1}h(x)dx$$

Notice that if $v=m+1$ then

$\beta=v-m=m+1-m=1$ and hence

$$\left(I_{v}^{d}h\right)(r)=\left(\boldsymbol{I}_{m+1}^{d}h\right)(r)=\frac{1}{m!}\int_{d}^{r}(r-x)^{m}h(x)dx,$$

This, through the application of the Cauchy Formula, represents an iterative integral of $h,$ $m+1$ times over $(d,r].$

Revisiting the fact that the left Riemann-Liouville fractional integral with of order $v>0$ commencing from v is defined by

$$\left(\boldsymbol{I}_{v}^{d}h\right)(r)=\frac{1}{\Gamma(v)}\int_{d}^{r}(r-s)^{v-1}h(s)ds,$$

It becomes evident that

$\left(I_{v}^{d}h\right)(r)=\left(\boldsymbol{I}_{v}^{d}h\right)(r)$ for $v=m+1,$ $\mathrm{m}=0,1,2,\ldots$

Example 2.4 Recalling that $\left(\boldsymbol{I}_v^d (r-d)^{\mu-1}\right)(x) = \dfrac{\Gamma(\mu)}{\Gamma(\mu+v)}(x-d)^{v+\mu-1},\ v,\ \mu > 0,$ the (conformable) fractional integral can be computed as $(r-d)^{\mu}$ of order $v \in (m, m+1]$. Indeed, if $\mu \in \mathbb{R}$ such that $v+\mu-m>0$ then

$$(I_v^d (r-d)^{\mu})(x) = (\boldsymbol{I}_{m+1}^d (r-d)^{\mu+v-m-1})(x) = \frac{\Gamma(v+\mu-m)}{\Gamma(v+\mu+1)}(x-d)^{v+\mu}.$$

In a similar vein, the (conformable) right fractional integral for these functions can also be determined. Namely,

$$({}^{e}I_v (e-r)^{\mu})(x) = ({}^{e}\boldsymbol{I}_{m+1} (r-d)^{\mu+v-m-1})(x) = \frac{\Gamma(v+\mu-m)}{\Gamma(v+\mu+1)}\ (e-x)^{v+\mu},$$

where $\mu \in \mathbb{R}$ such that $v+\mu-m>0$.

Based on the foregoing analysis, it is observable as described The Riemann fractional integrals and conformable fractional integrals, when applied to polynomial functions, diverge merely by a constant factor, and align precisely for orders that are natural numbers.

The subsequent semigroup property establishes a connection between the composition operator $I_\mu I_v$ and the operator $I_{v+\mu}$.

Proposition 2.3 ([41]) Let $h:[d,\infty) \to \mathbb{R}$ be a function and $0 < v,\ \mu \le 1$ be such that $1 < v+\mu \le 2.$ Then

$$(I_v I_\mu h)(r) = \frac{r^\mu}{\mu}(I_v h)(r) + \frac{1}{\mu}(I_{v+\mu} h)(r) - \frac{r}{\mu}\int_0^r s^{v+\mu-2} h(s)\,ds.$$

Proof By exchanging the sequence of integration and taking into account that

$$(I_{\nu+\mu}h)(r) = \left(\boldsymbol{I}_2 s^{\nu+\mu-2}h(s)\right)(r) = \int_0^r (r-s)s^{\nu+\mu-2}ds,$$

It is evident that

$$\begin{aligned}(I_\nu I_\mu h)(r) &= \int_0^r \left(\int_0^{r_1} h(s)s^{\nu-1}ds \right) r_1^{\mu-1}dr_1 \\ &= \int_0^r h(s)s^{\nu-1}\left(\int_s^r r_1^{\mu-1}dr_1 \right) ds \\ &= \int_0^r h(s)s^{\nu-1}\left[\frac{r^\mu}{\mu} - \frac{s^\mu}{\mu} \right] ds \\ &= \frac{r^\mu}{\mu}\left(I_\nu h\right)(r) + \frac{1}{\mu}\left[\left(I_{\nu+\mu}h\right)(r) - r\int_0^r s^{\nu+\mu-2}h(s)ds \right].\end{aligned}$$

Notice that if in Proposition 2.13

we let $\nu,\ \mu \to 1$ we verify $\left(I_1 I_1 h\right)(r) = \left(I_2 h\right)(r).$

Reflecting upon the operational impact of the Q-operator with respect to fractional integration

$$\left(Qh(r) = h(d+e-r),\ h:[d,e] \to \mathbb{R}\right)$$

on Riemann's left and right fractional integrals, it becomes evident that,

$$Q\mathbf{I}_\nu^d h(r) = {}^e\mathbf{I}_\nu Qh(r).$$

Indeed, for $\nu \in (m, m+1]$ we take

$$\begin{aligned}QI_\nu^d h(r) &= Q\mathbf{I}_{m+1}^d((r-d)^{\nu-m-1}h(r)) \\ &= {}^d\mathbf{I}_{m+1}((e-r)^{\nu-m-1}h(d+e-r)) = {}^eI_\nu Qh(r),\end{aligned}$$

We now proceed to present a generalized form of Lemma 2.1.

Lemma 2.4 ([8]) *Assuming* $h:[d,\infty)\to\mathbb{R}$ *such that* $h^{(m)}(r)$ *is continuous and* $v\in(m,m+1]$. *Then, for all* $r>d$ *, we observe*

$\boldsymbol{R}_v^d I_v^d h(r)=h(r).$

Proof Based on the established definition, it follows that:

$$\boldsymbol{R}_v^d I_v^d h(r)=R_\beta^d\left(\frac{d^m}{dt^m}I_v^d h(r)\right)=R_\beta^d\left(\frac{d^m}{dt^m}I_{m+1}^d\left((r-d)^{\beta-1}h(r)\right)\right)$$
$$=R_\beta^d\left(I_1^d(r-d)^{\beta-1}h(r)\right)$$

That it is $\boldsymbol{R}_v^d I_v^d h(r)=\boldsymbol{R}_\beta^d I_\beta^d h(r)$ and

hence the result follows by $R_v^d I_v^d h(r)=h(r),$

Similarly, Lemma 2.2 can be generalised.

Lemma 2.5 ([13]) *Let us suppose that* $h:(-\infty,e]\to\mathbb{R}$ *with the condition that* $h^{(m)}(r)$ *exhibits continuity and* $v\in(m,m+1]$.*In such a case, for all* $r<e$ *we have*

${}^e\boldsymbol{R}_v\,{}^eI_v h(r)=h(r).$

Lemma 2.6 ([25]) *Let* $h,w:[d,\infty)\to\mathbb{R}$ *be a functions with the condition that* R_v^d *exists for* $r>d$, h *demonstrates the property of being differentiable over the interval of* (d,∞) *and* $R_v^d h(r)=(r-d)^{1-v}w(r)$. *In such a case* $w(r)=h'(r)$ *for all* $r>d$.

The proof follows by definition and setting $w=\varepsilon(r-d)^{1-v}$ so that $w\to0$ as $\varepsilon\to0.$

As a result of Lemma 2.6 the following at stated

Corollarly 2.1 Let $h:[d,e)\to\mathbb{R}$ be such that $\left(I_v^d R_v^d\right)h(r)$ exists for $e>r>d$. Then, $h(r)$ is differentiable on (d,e).

Lemma 2.8 ([9]) *Let* $h:(d,e)\to\mathbb{R}$ *be differentiable and* $0<v\le 1$. *Then, for all* $r>d$ *we have*

$$I_v^d R_v^d(h)(r)=h(r)-h(d).$$

Proof Given that h is differentiable, it can be deduced from Theorem 2.1 in [10] we have:

$$\begin{aligned} I_v^d R_v^d(h)(r) &= \int_d^r (x-d)^{v-1} R_v(h)(x)dx \\ &= \int_d^r (x-d)^{v-1}(x-d)^{1-v} h'(x)dx = h(r)-h(d) \end{aligned}$$

$I_v^d R_v^d(h)(r)=h(r)-h(d)$ can be generalized for the higher as below.

Proposition 2.9 ([18]) Let $v\in(m,m+1]$ and $h:[d,\infty)\to\mathbb{R}$ be $(m+1)$ times differentiable for $r>d$. Then, $r>d$ we observe

$$I_v^d \boldsymbol{R}_v^d(h)(r)=h(r)-\sum_{k=0}^{m}\frac{h^{(k)}(d)(r-d)^k}{k!}.$$

Proof Drawing upon both the extant definition and Theorem 2.1

$$\left(I_v^d h\right)(r)=\boldsymbol{I}_{m+1}^d\left((r-d)^{\beta-1}h\right)=\frac{1}{m!}\int_d^r (r-x)^m (x-d)^{\beta-1} h(x)dx$$

it can be concluded that and

$$\begin{aligned} I_v^d \boldsymbol{R}_v^d(h)(r) &= I_{m+1}^d\left((r-d)^{\beta-1} R_\beta^d h^{(m)}(r)\right) \\ &= I_{m+1}^d\left((r-d)^{\beta-1}(r-d)^{1-\beta} h^{(m+1)}(r)\right) \\ &= I_{m+1}^d h^{(m+1)}(r). \end{aligned}$$

Similarly, we can give the following Proposition 2.10, it can be established.

Proposition 2.10 ([16]) Let $v \in (m, m+1]$ and $h:(-\infty, e] \to \mathbb{R}$ be $(m+1)$ times differentiable $r < e$. Then for all $r < e$ we have

$$^{e}I_v\,{}^{e}\boldsymbol{R}_v(h)(r) = h(r) - \sum_{k=0}^{m} \frac{(-1)^k h^{(k)}(e)(e-r)^k}{k!}.$$

In particular, *if* $m = 0$ or $0 < v \leq 1$, Then,

$$^{e}I_v\,{}^{e}R_v(h)(r) = h(r) - h(e).$$

Theorem 2.11 ([7]) *Assume* $h, g:(d,\infty) \to \mathbb{R}$ *be (left)* $v-$ *differentiable functions, where* $0 < v \leq 1.$ *Consider* $w(r) = h\big(g(r)\big).$ *Then* $w(r)$ *is (left)* $v-$ *differentiable and for all* r *with* $r \neq d$ *and* $g(r) \neq 0$ *we have*

$$\left(R_v^d w\right)(r) = \left(R_v^d h\right)\big(g(r)\big) \cdot \left(R_v^d g\right)(r) g(r)^{v-1}.$$

If $r = d$, it follows that:

$$\left(R_v^d w\right)(d) = \lim_{r \to d^+} \left(R_v^d h\right)\big(g(r)\big) \cdot \left(R_v^d g\right)(r) g(r)^{v-1}.$$

Proof By setting $u = r + \varepsilon(r-d)^{1-v}$ within the definition and using the continuity of g it becomes evident that,

$$
\begin{aligned}
R_v^d w(r) &== \lim_{u\to r} \frac{h(g(u)) - h(g(r))}{(u-r)} r^{1-v} \\
&= \lim_{u\to r} \frac{h(g(u)) - h(g(r))}{(g(u)-g(r))} \cdot \lim_{u\to r} \frac{g(u)-g(r)}{u-r} r^{1-v} \\
&= \lim_{g(u)\to g(r)} \frac{h(g(u)) - h(g(r))}{(g(u)-g(r))} g(r)^{1-v} R_v^d g(r) \cdot g(r)^{v-1} \\
&= \left(R_v^d h\right)(g(r)) \cdot \left(R_v^d g\right)(r) \cdot g(r)^{v-1}.
\end{aligned}
$$

Proposition 2.12 ([17]) Let $h:[d,\infty) \to \infty$ be twice differentiable on (d,∞) and $0 < v,\ \beta \le 1$ such that $1 < v + \beta \le 2.$ Then,

$$\left(R_v^d R_\beta^d h\right)(r) = R_{v+\beta}^d h(r) + (1-\beta)(r-d)^{-\beta} R_v^d h(r).$$

Proof Using the fractional product rule and taking into account the fact that h is twice differentiable, we have:

$$
\begin{aligned}
\left(R_v^d R_\beta^d h\right)(r) &= r^{1-v} \frac{d}{dt}[r^{1-\beta}(r-d)^{-\beta} h^{'}(r)] \\
&= r^{1-v}[r^{1-\beta} h^{''}(r) + (1-\beta)(r-d)^{-\beta} h^{'}(r)] \\
&= R_{v+\beta}^d h(r) + (1-\beta)(r-d)^{-\beta} R_v^d h(r).
\end{aligned}
$$

Note that in

$$\left(R_v^d R_\beta^d h\right)(r) = R_{v+\beta}^d h(r) + (1-\beta)(r-d)^{-\beta} R_v^d h(r)$$

if we let $v,\ \beta \to 1$ then we have

$$R_v^d R_\beta h(r) = R_2 h(r) = h^{''}(r).$$

Subsequently, we introduce a fractional adaptation of the Gronwall inequality, which serves as a valuable instrument for assessing the the analysis involves examining the stability of (conformable) fractional systems during analysis.

Theorem 2.13 ([2]) *Let* z *be a continuous, nonnegative function on an interval* $J=[d,e]$, *let* $\delta,\ k$ *be a nonnegative constant such that*

$$z(r)\le \delta+\int_d^r kz(s)(s-d)^{\nu-1}ds \qquad (r\in J),$$

Then for all $r\in J$

$$z(r)\le \delta e^{k\frac{(r-d)^\nu}{\nu}}.$$

Proof Define

$$Z(r)=\delta+\int_d^r kz(s)(s-d)^{\nu-1}ds=\delta+I_\nu^d(kz(s))(r).$$

Then $Z(d)=\delta$ and $Z(r)\ge z(r)$ and

$$R_\nu^d Z(r)-kZ(r)=kz(r)-kZ(r)\le kz(r)-kz(r)=0.$$

Multiply,

$$R_\nu^d Z(r)-kZ(r)=kz(r)-kZ(r)\le kz(r)-kz(r)=0$$

by,

$$\mathrm{K}(r)=e^{-k\frac{(r-d)^\nu}{\nu}}.$$

Utilizing the chain rule as presented in theorem 2.11

$\left(R_\nu^d w\right)(r)=\left(R_\nu^d h\right)\left(g(r)\right)\cdot\left(R_\nu^d g\right)(r)g(r)^{\nu-1}$, we see that

$$R_\nu^d \mathrm{K}(r)=-k\mathrm{K}(r)$$

and therefore, using product rule, we can deduce that

$$R_\nu^d(\mathrm{K}(r)Z(r))\le 0.$$

Since $\mathrm{K}(r)Z(r)$ is differentiable on (d,e) in such a case, Lemma 2.8 $I_v^d R_v^d(h)(r) = h(r) - h(d)$ implies that

$$I_v^d R_v^d(\mathrm{K}(r)Z(r)) = \mathrm{K}(r)Z(r) - \mathrm{K}(d)Z(d) = \mathrm{K}(r)Z(r) - \delta \le 0.$$

Hence,

$$z(r) \le Z(r) \le \frac{\delta}{\mathrm{K}(r)} = \delta e^{k\frac{(r-d)^v}{v}}$$

which completes the proof and that

To conclude this section, we discussion of the conformable fractional derivative at d in the left side and at e in the right side case for some smooth functions. Let $m-1 < v < m$ and assume $h:[d,\infty) \to \mathbb{R}$ be such that $h^{(m)}r$ exists and continuous. Then,

$$(R_v^d h)(r) = (R_{v+1-m}^d h^{(m-1)})(r) = (r-d)^{m-v} h^{(m)}(d)$$

and thus

$$(R_v^d h)(d) = \lim_{r \to d^+} (r-d)^{m-v} h^{(m)}(r) = 0.$$

Simirlarly, in the right case we have

$$({}^e R_v h)(e) = \lim_{r \to e^-} (e-r)^{m-v} h^{(m)}(r) = 0, \text{ for } (-\infty, e] \to \mathbb{R}$$

with $h^{(m)}r$ exists and continuous.

At the same time, let $0 < v < 1$ and $m \in \{1,2,3,...\}$ then, the left or the right sequential conformable fractional derivative of order m is defined by

$${}^{(m)}R_v^d h(r) = \underbrace{R_v^d R_v^d R_v^d R_v^d R_v^d ... R_v^d}_{m-times} h(r)$$

and

$${}^{e}R_{v}^{(m)}h(r)=\underbrace{{}^{e}R_{v}\,{}^{e}R_{v}\,{}^{e}R_{v}\,{}^{e}R_{v}\,{}^{e}R_{v}\ldots{}^{e}R_{v}}_{m-times}h(r),$$ respectively.

If $h:[d,\infty)\to\mathbb{R}$ is second continuously differentiable and $0<v\le\frac{1}{2}$ then direct calculations then

$${}^{(2)}R_{v}^{d}(r)=R_{v}^{d}R_{v}^{d}h(r)=\begin{cases}(1-v)(r-d)^{1-2v}h'(r)+(r-d)^{2-2v}h''(r) & if\ r>d,\\ 0 & if\ r=d.\end{cases}$$

If $h:(-\infty,e]\to\mathbb{R}$ is second continuously differentiable and $0<v\le\frac{1}{2}$ then direct calculation then

$${}^{e}R_{v}^{(2)}(r)={}^{e}R_{v}\,{}^{e}R_{v}h(r)=\begin{cases}(1-v)(e-r)^{1-2v}h'(r)+(e-r)^{2-2v}h''(r) & if\ r<e,\\ 0 & if\ r=e.\end{cases}$$

This observation under goes that the second order equential conformable fractional derivative may not be continuously, that h is second continuously differentiable for $\frac{1}{2}<v<1.$ If we proceeding inductively, it becomes apparent that if h is $m-$ continuously differentiable and $0<v\le\frac{1}{m}$ then, the sequential conformable fractional derivative of $m-$th order continuous and equals zero at the endpoints (d in the left case, e in the right case).

2.2 Integration by Parts

Theorem 2.2.1 ([11]) *Let* $h,g:[d,e]\to\mathbb{R}$, *be two functions such that* hg *is differentiable. Then,*

$$\int_{d}^{e}h(x)R_{v}^{d}g(x)dv(x,d)=hg\,|_{d}^{e}-\int_{d}^{e}g(x)R_{v}^{d}(h)(x)dv(x,d)$$

The subsequent proof, which was then presented lemma 2.8 applied to hg and

$$\left({}_{v}^{e}Rh\right)(r) = -\lim_{\varepsilon\to 0}\frac{h(r+\varepsilon(e-r)^{1-v}-h(r)}{\varepsilon}.$$

The subsequent formula for integration by parts is presented through the utilization of fractional integrals performed on both the left and right sides.

Proposition 2.2.2([16]) Let $h, g:[d,e]\to\mathbb{R}$ be functions and $0<v\leq 1.$ Then,

$$\int_d^e (I_v^d h)(r)g(r)d_v(e,r) = \int_d^e h(r)({}^eI_v g)(r)d_v(r,d).$$

Proof From the definition we get

$$\int_d^e (I_v^d h)(r)g(r)d_v(r,d) = \int_d^e (\int_d^r (x-d)^{v-1}h(x)dx)g(r)(e-r)^{v-1}dr.$$

When the order of integrals is interchanged, we arrive at

$$\int_d^e (I_v^d h)(r)g(r)d_v(e,r) = \int_d^e h(x)({}^eI_v g)(x)d_v(x,d)$$

Completes the proof.

Following this, proposition 2.2.2

$\int_d^e (I_v^d h)(r)g(r)d_v(e,r) = \int_d^e h(r)({}^eI_v g)(r)d_v(r,d)$ to prove an integration by parts formula by means of left and right fractional derivatives.

Theorem 2.2.3 ([22]) *Let* $h, g:[d,e]\to\mathbb{R}$, *differentiable functions and* $0<v\leq 1.$ *Then,*

$$\int_d^e (R_v^d h)(r)g(r)d_v(r,d) = \int_d^e h(r)({}^eR_v g)(r)d_v(e,r) + h(r)g(r)|_d^e\ .$$

Proof By Proposition 2.10 and that g is differentiable , we have

$$\int_d^e (R_v^d h)(r)g(r)d_v(r,d) = \int_d^e (R_v^d h)(r)\,{}^eI_v\,{}^eR_v g(r)d_v(r,d) + g(e)\int_d^e (R_v^d h)(t)d_v(r,d).$$

Applying proposition 2.2.2 leads to

$$\int_d^e (R_v^d h)(r)g(r)d_v(r,d) = \int_d^e (I_v^d R_v^d h)(r)\,{}^eR_v g(r)d_v(e,r) + g(e)(I_v^d R_v^d h)(d).$$

The proof is completed by the help Lemma 2.8 by substituting

$(I_v^d R_v^d h)(r) = h(r) - h(d).$ using that h is differentiable and by the help of proposition 2.10, and that g is differentiable by substituting $({}^eI_v\,{}^eR_v g)(r) = g(r) - g(e).$

Remark 2.2.1 Notice that if in theorem 2.2.1 or theorem 2.2.3 we get $v \to 1$, then we obtain the integration by parts formula in usual calculus, where we have to note that

$d_v(r,d) \to dr,\ \mathrm{d}_v(e,r) \to dr,\ \mathrm{R}_v^d h(r) \to h'(r)$

and

${}^eR_v h(r) \to -h'(r)$ as $v \to 1.$

In theorems 2.2.1 and 2.2.3, certain differentiability conditions were necessary. In the subsequent discussion, we proceed to define specific function spaces within which the derived integration by parts formulas remains valid.

Definition 2.2.1 ([37]) For $0<v\leq 1$ and an interval $[d,e]$ define

$$I_v([d,e])=\{h:[d,e]\to\mathbb{R}: h(x)=(I_v^d\psi)(x)+h(d),$$

for some

$$\psi\in L_v(d)\},$$

and

$$^{v}I([d,e])=\{g:[d,e]\to\mathbb{R}: g(x)=(^{e}I_v\varphi)(x)+g(e),$$

for some

$$\psi\in L_v(e)\},$$

where

$L_v(d)=\{\psi:[d,e]\to\mathbb{R}\}:(I_v^d\psi)(x)$ exists for all $x\in[d,e]\}$,

and

$L_v(e)=\{\varphi:[d,e]\to\mathbb{R}\}:(^{e}I_v\varphi)(x)$ exists for all $x\in[d,e]\}$.

Lemma 2.2.4 ([19]) *Let $h,g:[d,e]\to\mathbb{R}$ be functions and $0<v\leq 1.$ Then*

(a) *If h is left (g is right) $v-$ differentiable then*

$$h\in I_v([d,e])(g\in {}_vI([d,e])).$$

(b) *If $h\in I_v([d,e])$ with $h(x)=(I_v^d\psi)(x)+h(d)$*

where ψ is continuous then

$\psi(x)=R_v^d h(x)$ *and* $(I_v^d R_v^d h)(x)=h(x)-h(d).$

(c) *If $g\in {}_vI([d,e])$ with $g(x)=(^{e}I_v\varphi)(x)+g(e)$*

where φ is continuous then $\varphi(x)={}^{e}R_v g(x)$ *and*

$$(^{e}I_v\,{}^{e}R_v g)(x)=g(x)-g(e).$$

Proof The proof of (a) follows by lemma 2.8 and proposition 2.10 by choosing $\psi(r)=R_v^d h$ and $\varphi(r)={}^e R_v g$.

The proof of (b) follows by lemma 2.1 and the fact that the left $v-$ derivative of constant function is zero. The proof of (c) follows by lemma 2.2 and the fact that the right $v-$ derivative of constant function is zero.

Theorem 2.2.5 ([39]) *Let $h,g:[d,e]\to\mathbb{R}$ be functions such that $h\in I_v([d,e])$ with $\psi(r)$ is continuous and $g\in {}_vI([d,e])$ with $\varphi(r)$ is continuous and $0<v\le 1$. Then*

$$\int_d^e (R_v^d h)(r)g(r)d_v(r,d)=\int_d^e h(r)({}^eR_v g)(r)d_v(e,r)+h(r)g(r)|_d^e .$$

Proof The proof is similar to that in theorem 2.2.3, where we make use of (b) and (c) in lemma 2.2.4.

2.3 Fractional Power Series Expansions

In this section we set the fractional power series expansions so that those functions will have fractional power series expansions. Certain functions, being not infinitely differentiable at some point, do not have Taylor power series expansion there.

Theorem 2.3.1 ([24]) *Assume h is an infinitely $v-$ differentiable function, for some $0<v\le 1$ at a neighborhood of a point r_0. Then h has the fractional power series expansion:*

$$h(r)=\sum_{k=0}^{\infty}\frac{(R_v^{r_0}h)^{(k)}(r_0)(r-r_0)^{kv}}{v^k k!},\ r_0<r<r_0+Z^{1/v},\ \text{Z}>0.$$

Here, $(R_v^{r_0}h)^{(k)}(r_0)$ means the application of the fractional derivative k times.

Proof Assume

$$h(r) = c_0 + c_1(r - r_0)^v + c_2(r - r_0)^{2v} + c_3(r - r_0)^{3v} + c_4(r - r_0)^{4v} + c_5(r - r_0)^{5v} + ...,$$
$$r_0 < r < r_0 + Z^{1/v}, Z > 0.$$

Then, $h(r_0) = c_0$.

Apply $R_v^{r_0}$ to h and evaluate at r_0 we see that $(R_v^{r_0}h)(r_0) = c_1 v$ and hence $c_1 = \frac{(R_v^{r_0}h)(r_0)}{v}$. Proceeding inductively and applying $R_v^{r_0}$ to h $m-$ times and evaluating at r_0 we see that

$(R_v^{r_0}h)^{(m)}(r_0) = c_m v(2v)(3v)(4v)...(mv) = v^m \cdot m!$ and hence

$$c_m = \frac{(R_v^{r_0}h)^{(m)}(r_0)}{v^m \cdot m!}.$$

Hence theorem 2.3.1 $h(r) = \sum_{k=0}^{\infty} \frac{(R_v^{r_0}h)^{(k)}(r_0)(r - r_0)^{k\alpha}}{v^k k!}$, $r_0 < r < r_0 + Z^{1/v}$, Z>0

is derived, this concludes the proof.

Proposition 2.3.2 (Formulation of Fractional Taylor Inequality). Assume h is a $v-$ differentiable function, for some $0 < v \leq 1$ at a neighbourhood of a point r_0, has the Taylor power series representation as denoted by theorem 2.3.1

$$h(r) = \sum_{k=0}^{\infty} \frac{(R_v^{r_0}h)^{(k)}(r_0)(r - r_0)^{kv}}{v^k k!}, \quad r_0 < r < r_0 + Z^{1/v}, \text{ Z>0}$$

such that

$|(R_v^{r_0}h)^{m+1}| \leq M$, M>0 for some $m \in \mathbb{N}$. Then, for all $(r_0, r_0 + Z)$

$$|Z_m^v(r)| \leq \frac{M}{v^{m+1}(m+1)!}(r-r_0)^{v(m+1)},$$

where

$$Z_m^v(r) = \sum_{k=m+1}^{\infty} \frac{(R_v^{r_0}h)^{(k)}(r_0)(r-r_0)^{kv}}{v^k k!} = h(x) - \sum_{k=0}^{m} \frac{(R_v^{r_0}h)^{(k)}(r_0)(r-r_0)^{kv}}{v^k k!}$$

The proof is similar to that in usual calculus, by applying $I_v^{r_0}$ instead of integration.

Example 2.3.1 Let's contemplate the fractional exponential function $h(r) = e^{\frac{(r-r_0)^v}{v}}$, where $0 < v < 1$. Notably, the function $h(r)$ is evidently devoid of differentiability at r_0, thereby negating the possibility of establishing a Taylor power series representation around r_0. Nevertheless, $(R_v^{r_0}h)^{(m)}(r_0) = 1$ holds true for all m, consequently yielding:

$$h(r) = \sum_{k=0}^{\infty} \frac{(r-r_0)^{kv}}{v^k k!}$$

The application of the ratio test shows that this series converges to h on the interval $[r_0, \infty)$.

Example 2.3.2 The functions $g(r) = \sin\frac{(r-r_0)^v}{v}$ and $h(r) = \sin\frac{(r-r_0)^v}{v}$ do not possess taylor power series expansions with respect to $r = r_0$ for $0 < v < 1$as they lack differentiability at those points. However, with the aid of equation

$$h(r) = \sum_{k=0}^{\infty} \frac{(R_v^{r_0}h)^{(k)}(r_0)(r-r_0)^{kv}}{v^k k!}, \quad r_0 < r < r_0 + Z^{1/v},\ \text{Z>0}$$

and the fact that

$$R_v^{r_0} \sin\frac{(r-r_0)^v}{v} = \cos\frac{(r-r_0)^v}{v} \text{ and } R_v^{r_0} \cos\frac{(r-r_0)^v}{v} = -\sin\frac{(r-r_0)^v}{v}$$

we can see that

$$\sin\frac{(r-r_0)^v}{v}=\sum_{k=0}^{\infty}(-1)^k\frac{(r-r_0)^{(2k+1)v}}{v^{(2k+1)}(2k+1)!},\ r\in[r_0,\infty)$$

and

$$\cos\frac{(r-r_0)^v}{v}=\sum_{k=0}^{\infty}(-1)^k\frac{(r-r_0)^{(2k)v}}{v^{(2k)}(2k)!},\ r\in[r_0,\infty).$$

Example 2.3.3 The function $h(x)=\dfrac{1}{1-\dfrac{r^v}{v}}$ is devoid of a taylor power series representation in the vicinity of $r=0$ for $0<v<1$, as they lack differentiability at those points. However, with the aid of eq.

$$h(r)=\sum_{k=0}^{\infty}\frac{(R_v^{r_0}h)^{(k)}(r_0)(r-r_0)^{kv}}{v^k k!},\quad r_0<r<r_0+Z^{1/v},\ Z>0$$

we can observe that

$$\frac{1}{1-\dfrac{r^v}{v}}=\sum_{k=0}^{\infty}r^{vk},\quad r\in[0,1)$$

Or more generally,

$$\frac{1}{1-\dfrac{(r-r_0)^v}{v}}=\sum_{k=0}^{\infty}(r-r_0)^{vk},\quad r\in[r_0,r_0+1).$$

Remark 2.3.1 In case the function h is defined over $(-\infty,d)$, and not differentiable at d we search for its conformal right fractional order derivatives ${}^dR_v$ at d for some

$0 < v \leq 1$, use it for our fractional taylor series on some interval $(d-Z,d)$, Z>0. A prime illustration of such functions includes $\frac{(d-r)^v}{v}, \sin\frac{(d-r)^v}{v}$.

2.4 The Fractional Laplace Transform

This section is devoted to introducing the fractional Laplace transform and its utility in solving linear fractional equations, resulting in the emergence of the function representing exponential growth with a fractional exponent. Following this, we utilize the successive approximation method to confirm the solution, leveraging The representation of fractional power series that was previously mentioned. Furthermore, we perform Laplace transform calculations for specific (fractional) functinos.

Definition 2.4.1 ([21]) Let $r_0 \in \mathbb{R}$, $0 < v \leq 1$ and $h:[r_0,\infty) \rightarrow$ as real valued function. Then the fractional laplace transform of order v, starting from d of h is defined by:

$$L_v^{r_0}\{h(r)\}(s) = F_v^{r_0}(s) = \int_{r_0}^{\infty} e^{-s\frac{(r-r_0)^v}{v}} h(r)dv(r,r_0) = \int_{r_0}^{\infty} e^{-s\frac{(r-r_0)^v}{v}} h(r)(r-r_0)^{v-1}dr.$$

Theorem 2.4.1 ([23]) *Let* $d \in \mathbb{R}$, $0 < v \leq 1$, *and* $h:(d,\infty) \rightarrow$ *are differentiable real-valued functions. Then,*

$$L_v^d\{R_v h(r)\}(s) = sF_v(s) - h(d).$$

Proof The proof entails a sequence of steps, commencing with the definition, followed by the utilization of theorem 2.1 equation

$$\left(I_v^d h\right)(r) = \boldsymbol{I}_{m+1}^d\left((r-d)^{\beta-1}h\right) = \frac{1}{m!}\int_d^r (r-x)^m (x-d)^{\beta-1}h(x)dx$$

in [10] and culminating with the application of the conventional integration by parts technique.

Example 2.4.2 Let's examine the conformable fractional initial value problem:

$$(R_v^d y)(r) = \lambda y(r),\ y(d) = y_0,\ r > d.$$

In this context, we make an assumption that the solution possesses differentiability within the domain on (d, ∞).

Employ the operator I_v^d to consider the equation mentioned above for the purpose of deriving

$$y(r) = y_0 + \lambda (I_v^d y)(r).$$

Then

$$y_{m+1} = y_0 + \lambda (I_v^d y_m)(r), \quad m = 0,1,2,...$$

For $m = 0$ we understand that

$$y_1 = y_0 + \lambda y_0 \frac{(r-d)^v}{v} = y_0 \left(1 + \lambda \frac{(r-d)^v}{v} \right).$$

For $m = 1$ we perceive that

$$y_2 = y_0 \left[1 + \lambda \frac{(r-d)^v}{v} + \lambda^2 \frac{(r-d)^{2v}}{v(2v)} \right].$$

By proceeding inductively, we arrive at the conclusion that:

$$y_m = y_0 \sum_{k=0}^{m} \frac{\lambda^k (r-d)^{kv}}{v^k k!}.$$

Letting $m \to \infty$ we understand that

$$y(r) = y_0 \sum_{k=0}^{m} \frac{\lambda^k (r-d)^{kv}}{v^k k!}.$$

This expression is evidently the power series representation of the fractional exponential function using Taylor series $y_0 e^{\lambda \frac{(r-d)^v}{v}}$. The subsequent lemma establishes a connection between the fractional Laplace transform and conventional Laplace transform.

Lemma 2.4.2 ([10]) *Consider* $h:[r_0,\infty) \to \mathbb{R}$ *let there exist a function for which* $L_v^{r_0}\{h(r)\}(s) = F_v^{r_0}(s)$ *exists. Then*

$$F_v^{r_0}(s) = \wp\{h(r_0 + (vr)^{1/v})\}(s),$$

where

$$\wp\{g(r)\}(s) = \int_0^{\infty} e^{-sr} g(r)\,dr.$$

The proof follows easily by setting $u = \frac{(r-r_0)^v}{v}$.

Chapter 3

PRELIMINARIES

We initiate our discussion by creating specific function spaces, introducing the notion the application of the conformable derivative and conformable integrals and providing a detailed explanation of the analytical representation of a solution to the conformable linear equation. These fundamental components are essential prerequisites for our subsequent discussions and analyses presented in this study.

- $\left(\mathbb{R}^d, \|.\|\right) - d$ dimensional Euclian space.
- $\left(C\left(\left[0,R\right],\mathbb{R}^d\right), \| \,.\, \|_\infty\right) -$ A Banach space comprising continuous functions originating from [0,R] to $\mathbb{R}^d$ with the norm in infinity (supremum) norm.
- $\mathrm{PC}\left(\left[0,R\right],\mathbb{R}^d\right) := \left\{ \begin{array}{l} y:\left[0,R\right]\to\mathbb{R}^d : y\in C\left(\left(r_k, r_{k+1}\right),\mathbb{R}^d\right), \\ k=0,1,\ldots,\ \exists y\left(r^+\right),\ y\left(r_k^-\right)=y\left(r_k\right) \end{array} \right\}$ equipped with the norm $\|y\|_{PC} := \sup\left\{\left\|y\left(r\right)\right\| : 0\le r\le R\right\}\Big\}.$
- $\mathrm{e}_{\mathrm{A}}\left(\frac{\mathrm{r}^{\mathrm{v}}}{\mathrm{v}}\right) = \exp\left(\mathrm{A}\frac{\mathrm{r}^{\mathrm{v}}}{\mathrm{v}}\right) = \sum_{m=0}^{\infty} A^m \frac{r^{vm}}{m!v^m}$

Definition 3.1 ([4]) The conformable derivative with a lower index of 0 of the function $y:[0,\infty)\to\mathbb{R}$ is defined as follows:

$$\begin{cases} E_0^v y(r) = \lim\limits_{\varepsilon\to 0}\dfrac{y\left(r+\varepsilon r^{1-v}\right)-y(r)}{\varepsilon},\ r>0,\quad 0<v<1,\\ E_0^v y(0) = \lim\limits_{r\to 0^+} E_0^v y(r). \end{cases}$$

Remark 3.1 We note that the conformable derivative $E_0^v y(r),\ r>0,$ exists if y is differentiable at r and

$$E_0^v y(r) = r^{1-v} y'(r).$$

Definition 3.2 ([4]) The conformable integral with a lower index v of a function $y:[0,\infty)\to\mathbb{R}$ is defined as follows:

$$I_0^v y(r) = \int_0^r s^{v-1} y(s)\,ds,\ r\geq 0,\ 0<v<1.$$

Lemma 3.1 ([1]) *A solution* $y\in C\left(\left[0,R\right],\mathbb{R}^d\right)$ *of the linear system*

$$\begin{cases} E_0^v y(r) = Ay(r)+h(r),\ r\in\left[r_0,R\right],\ 0<v<1,\ h\in C\left(\left[0,R\right],\mathbb{R}^d\right),\\ y(r_0)=y_0, \end{cases}$$

has the following form:

$$y(r) = e_A\left(\frac{r^v - r_0^v}{v}\right)y_0 + \int_{r_0}^r e_A\left(\frac{r^v}{v}-\frac{r_0^v}{v}\right)h(s)\,s^{v-1}ds.$$

Proof It is clear that

$$E_0^v e_A\left(\frac{r^v - r_0^v}{v}\right) = r^{1-v} e_A^{'}\left(\frac{r^v - r_0^v}{v}\right)$$

$$= r^{1-v}\sum_{m=1}^{\infty} A^n \frac{\left(r^v - r_0^v\right)^{m-1}}{(m-1)!v^{m-1}} r^{v-1}$$

$$= Ae_A\left(\frac{r^v - r_0^v}{v}\right).$$

Thus,

$$E_0^v y(r) = Ae_A\left(\frac{r^v - r_0^v}{v}\right)y_0 + A\int_{r_0}^{r} e_A\left(\frac{r^v}{v} - \frac{r_0^v}{v}\right)h(s)s^{v-1}ds + h(r)$$

$$= Ay(r).$$

Chapter 4

LINEAR SYSTEMS

In this section, our objective is to derive the analytical expression that represents the solution to equation (2).

Theorem 4.1 ([1]) *A solution* $y \in PC([0,R],\mathbb{R}^d)$ *of the solution to eq.* (2) *has the following form:*

$$y(r) = \begin{cases} e_A\left(\frac{r^\nu}{\nu}\right)y_0 + \int_{r_0}^{r} e_A\left(\frac{r^\nu}{\nu} - \frac{s^\nu}{\nu}\right)h(s)(s-r_0)^{\nu-1}\,ds, \quad 0 \le r \le r_1; \\ e_A\left(\frac{r^\nu - r_k^\nu}{\nu}\right)\prod_{j=k}^{1}\left(I + C_j\right)e_A\left(\frac{\left(r_j - r_{j-1}\right)^\nu}{\nu}\right)y_0 \\ +e_A\left(\frac{r^\nu - r_k^\nu}{\nu}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}(I + C_j)e_A\left(\frac{r_j^\nu - r_{j-1}^\nu}{\nu}\right) \\ \times\left(I + C_i\right)\int_{r_{i-1}}^{r_i} e_A\left(\frac{r_i^\nu}{\nu} - \frac{s^\nu}{\nu}\right)h(s)s^{\nu-1}ds \\ +\int_{r_k}^{r} e_A\left(\frac{r^\nu}{\nu} - \frac{s^\nu}{\nu}\right)h(s)s^{\nu-1}ds, \; r_k < r \le r_{k+1}, \; k = 1,2,\ldots,p. \end{cases} \tag{3}$$

Proof For $0 \le r \le r_1$, using lemma 3.1, we have:

$$y(r) = e_A\left(\frac{r^\nu}{\nu}\right)y(0) + \int_0^r e_A\left(\frac{r^\nu}{\nu} - \frac{s^\nu}{\nu}\right)h(s)s^{\nu-1}ds.$$

For $r = r_1^+$, we have

$$\begin{aligned} y(r_1^+) &= y(r_1^-) + C_1 y(r_1) \\ &= (I + C_1) e_A\left(\frac{r_1^v}{v}\right) y_0 \\ &+ (I + C_1) \int_0^{r_1} e_A\left(\frac{r_1^v}{v} - \frac{s^v}{v}\right) h(s) s^{v-1} ds. \end{aligned} \tag{4}$$

Moreover, for $r_1 < r \le r_2$, we use the following calculation to obtain

$$\begin{aligned} y(r) &= e_A\left(\frac{r^v - r_1^v}{v}\right) y\left(r_1^+\right) + \int_{r_1}^{r} e_A\left(\frac{r^v}{v} - \frac{s^v}{v}\right) h(s) s^{v-1} ds \\ &= e_A\left(\frac{r^v - r_1^v}{v}\right)\left(I + C_1\right) e_A\left(\frac{r_1^v}{v}\right) y_0 \\ &+ e_A\left(\frac{r^v - r_1^v}{v}\right)\left(I + C_1\right) \int_0^{r_1} e_A\left(\frac{r_1^v}{v} - \frac{s^v}{v}\right) h(s) s^{v-1} ds \\ &+ \int_{r_1}^{r} e_A\left(\frac{r^v}{v} - \frac{s^v}{v}\right) h(s) s^{v-1} ds, \end{aligned}$$

where $y(r_1^+)$ is given by equation (4). This means that theorem 4.1 holds for $k = 1$. Now, suppose that the formula (3) is true when $k = m$. Reasoning using the mathematical induction for $k = m + 1$, we have

$$\begin{aligned} y(r) &= e_A\left(\frac{r^v - r_{m+1}^v}{v}\right) y\left(r_{m+1}^+\right) + \int_{r_{m+1}}^{r} e_A\left(\frac{r^v}{v} - \frac{s^v}{v}\right) h(s) s^{v-1} ds \\ &= e_A\left(\frac{r^v - r_{m+1}^v}{v}\right)\left(I + C_{m+1}\right) y\left(r_{m+1}^-\right) + \int_{r_{m+1}}^{r} e_A\left(\frac{r^v}{v} - \frac{s^v}{v}\right) h(s) s^{v-1} ds \end{aligned}$$

$$= e_A\left(\frac{r^v - r_{m+1}^v}{v}\right)\left(I + C_{m+1}\right)e_A\left(\frac{r^v - r_m^v}{v}\right)\prod_{j=m}^{1}\left(I + C_j\right) e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right)y_0$$

$$+e_A\left(\frac{r^v - r_{m+1}^v}{v}\right)\left(I + C_{m+1}\right)e_A\left(\frac{r^v - r_m^v}{v}\right)\sum_{i=1}^{m}\prod_{j=m}^{i+1}\left(I + C_j\right) e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right)$$

$$\times\left(I + C_i\right)\int_{r_{i-1}}^{r_i} e_A\left(\frac{r_i^v}{v} - \frac{s^v}{v}\right)h(s)s^{v-1}ds$$

$$+e_A\left(\frac{r^v - r_{m+1}^v}{v}\right)\left(I + C_{m+1}\right)\int_{r_m}^{r_{m+1}} e_A\left(\frac{r_{m+1}^v}{v} - \frac{s^v}{v}\right)h(s)s^{v-1}ds$$

$$+\int_{r_m}^{r} e_A\left(\frac{r^v}{v} - \frac{s^v}{v}\right)h(s)s^{v-1}ds.$$

Consequently, it can be deduced that

$$y(r) = e_A\left(\frac{r^v - r_{m+1}^v}{v}\right)\prod_{j=m+1}^{1}\left(I + C_j\right)e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right)y_0$$

$$+e_A\left(\frac{r^v - r_{m+1}^v}{v}\right)\sum_{i=1}^{m+1}\prod_{j=m+1}^{i+1}\left(I + C_j\right)e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right)$$

$$\times(I + C_i)\int_{r_{i-1}}^{r_i} e_A\left(\frac{r_i^v}{v} - \frac{s^v}{v}\right)h(s)s^{v-1}ds$$

$$+\int_{r_m}^{r} e_A\left(\frac{r^v}{v} - \frac{s^v}{v}\right)h(s)s^{v-1}ds,\ \mathrm{r}_{m+1} < r \le r_{m+2}.$$

Hence, we can assert the validity of theorem 4.1 for $k = 1, 2, ...$ In conclusion, this marks the completion of the proof.

Theorem 4.2 ([40]) *Assume that X is a Banach space,* $B \subset PC\left([0, R], X\right)$.

Suppose that

- **(i)** *B is a uniformly bounded subset of* $PC([0, R]), X)$*;*
- **(ii)** *B is equincontinuous in* $\left(r_k, r_{k+1}\right)$*,* $k = 0,1,..., p$*;*

(iii) $B(r) := \left\{ x(r) : x \in B,\ r \in [0,R] \setminus \{r_1, r_2, ..., r_p\} \right\}$, $B(r_k^+) := \left\{ x(r_k^+) : x \in B \right\}$ and $B(r_k^-) := \left\{ x(r_k^-) : x \in B \right\}$ *are relatively compact subset of X. Then, B is a relatively compact subset of* $PC([0,R]), X)$.

Chapter 5

EXISTENCE OF SOLUTIONS

The iterative approach and the Schauder fixed point method are prevalent techniques frequently employed in the investigation of solutions within the realm of conformable impulsive semilinear/nonlinear differential equations. It's worth noting that the iterative method is a versatile approach that allows for the demonstration of both the existence and uniqueness of solutions. In contrast, the Schauder fixed point method is primarily used to establish the existence of solutions but does not typically address the issue of uniqueness.

These two methodologies stem from disparate mathematical foundations and techniques, each contributing distinctive perspectives regarding solution properties in the realm of these equations. Consequently, they synergize to yield a comprehensive comprehension of the solutions at hand.

The Picard iterative method, which is a numerical approximation technique, is employed to determine whether a solution exists and, if so, whether it is unique for initial value problems in ordinary differential equations. This method involves the iterative construction of a sequence of functions, with the aim of eventually converging to the solution of the given equation.

The Picard approximation method entails several fundamental steps, which can be outlined as follows:

- Commence with the initial value, customarily denoted as y_0
- Employ this initial value as the foundation for defining a sequence of approximations, denoted as $y_1, y_2, y_3, y_4, ...$, where each subsequent approximation depends on the prior one and the right-hand side of the given differential equation.
- Demonstrate the convergence of this sequence towards a solution of the differential equation while establishing the uniqueness of this solution.

Upon successful completion of these procedures, the Picard approximation method provides a rigorous demonstration of both the presence and distinctiveness of the solution for the specific differential equation under consideration. Therefore, in this section, we utilize the Picard approximation method as our primary analytical instrument to establish and support our core findings, particularly the theorem regarding the existence and uniqueness of solutions.

In the course of this section, we shall also rely on certain underlying assumptions, which are delineated as follows:

Hypothesis 1 $(\mathbf{H_1})$**.** $h(\cdot,\cdot) \in C\left([0,R]\times\mathbb{R}^d, \mathbb{R}^d\right).$

Hypothesis 2 $(\mathbf{H_2})$**.** $\exists L_h > 0$ *such that for any* $r \in [0,R]$ *and* $x, y \in \mathbb{R}^d$ *we have*

$$\|h(r,x) - h(r,y)\| \le L_h \|x - y\|.$$

Define

$$y_0(r)=\begin{cases} e_A\left(\frac{r^v}{v}\right)y_0,\ 0\le r\le r_1, \\ e_A\left(\frac{r^v-r_k^v}{v}\right)\prod_{j=k}^{1}(I+C_j)e_A\left(\frac{r_j^v-r_{j-1}^v}{v}\right)y_0,\ r_k<r\le r_{k+1},\ k=1,2,...p. \end{cases} \tag{5}$$

Set

$$C:=\prod_{j=p}^{1}\left(I+\|C_j\|\right),$$

$$B_z:=\left\{y\in PC([0,R],R^n:\|y-y_0\|_\infty\le z\right\},$$

where

$$z:=\left[pCe^2_{\|A\|}\left(\frac{R^v}{v}\right)+1\right]\frac{1}{\|A\|}M_h\left[e_{\|A\|}\left(\frac{R^v}{v}\right)-1\right].$$

$$z:=\left[pCe_{\|A\|}\left(\frac{2R^v}{v}\right)+1\right]\frac{1}{\|A\|}M_h\left[e_{\|A\|}\left(\frac{R^v}{v}\right)-1\right],$$

$$K(T):=C^2e_{\|A\|}\left(\frac{3R^v}{v}\right)+e_{\|A\|}\left(\frac{R^v}{v}\right).$$

It is clear that,

$$\|h(r,y(r))\|\le\|h(r,0)-h(r,y(r))\|+\|h(r,0)\|\le L_h\|y(r)\|+\|h(r,0)\|,$$

consequently,

$$M_h:=\sup\left\{\|h(r,y(r))\|:\ r\in[0,R],\ y\in B_z\right\}$$

exists.

Theorem 5.1 [(1)] *Assume that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Then, the semilinear equation* (1) *has a unique solution in the space of piecewise continuous functions* $PC([0,R],\mathbb{R}^d)$.

Proof For the initial (zeroth) approximation, we choose

$$y_0(r)=\begin{cases} e_A\left(\dfrac{r^{\nu}}{\nu}\right)y_0,\ 0\le r\le r_1; \\ e_A\left(\dfrac{r^{\nu}-r_k^{\nu}}{\nu}\right)\prod_{j=k}^{1}(I+C_j)e_A\left(\dfrac{r_j^{\nu}-r_{j-1}^{\nu}}{\nu}\right)y_0,\ \mathrm{r}_k<r\le r_{k+1},\ k=1,2,\ldots p.\end{cases}$$

n th approximation can be determined as follows:

$$y_n(r)=\begin{cases} e_A\left(\dfrac{r^{\nu}}{\nu}\right)y_0+\int_{r_0}^{r}e_A\left(\dfrac{r^{\nu}}{\nu}-\dfrac{s^{\nu}}{\nu}\right)h\left(s,y_{n-1}(s)\right)s^{\nu-1}ds,\quad 0\le r\le r_1; \\ e_A\left(\dfrac{r^{\nu}-r_k^{\nu}}{\nu}\right)\prod_{j=k}^{1}\left(I+C_j\right)e_A\left(\dfrac{r_j^{\nu}-r_{j-1}^{\nu}}{\nu}\right)y_0 \\ +e_A\left(\dfrac{r^{\nu}-r_k^{\nu}}{\nu}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}(I+C_j)e_A\left(\dfrac{r_j^{\nu}-r_{j-1}^{\nu}}{\nu}\right) \\ \times\left(I+C_i\right)\int_{r_{i-1}}^{r_i}e_A\left(\dfrac{r_i^{\nu}}{\nu}-\dfrac{s^{\nu}}{\nu}\right)h\left(s,y_{n-1}(s)\right)s^{\nu-1}ds, \\ +\int_{r_k}^{r}e_A\left(\dfrac{r^{\nu}}{\nu}-\dfrac{s^{\nu}}{\nu}\right)h(s,y_{n-1}(s))s^{\nu-1}ds,\qquad r_k<r\le r_{k+1},\ k=1,2,\ldots,p.\end{cases} \tag{6}$$

According to $\left(\mathrm{H}_1\right)$, (6) is well defined.

The firs stage: For any $n\in\mathbb{N}$, we prove that $y_n\in B_z$.

(i) For $n=1$ and $r\in[0,r_1]$, we have

$$
\begin{aligned}
\left\|y_1(r)-y_0(r)\right\| &= \left\|\int_0^r e_A\left(\frac{r^v}{v}-\frac{s^v}{v}\right)h\left(s,y_0(s)\right)s^{v-1}ds\right\| \\
&\le \int_0^r e_{\|A\|}\left(\frac{r^v}{v}-\frac{s^v}{v}\right)\left\|h\left(s,y_0(s)\right)\right\|s^{v-1}ds \\
&\le M_h\int_0^r e_{\|A\|}\left(\frac{r^v}{v}-\frac{s^v}{v}\right)s^{v-1}ds \\
&= \frac{1}{\|A\|}M_h\left[e_{\|A\|}\left(\frac{R^v}{v}\right)-1\right]\le z.
\end{aligned} \tag{7}
$$

(ii) For $n=1$ and $r\in\left(r_k,r_{k+1}\right]$, we have

$$
\begin{aligned}
&\left\|y_1(r)-y_0(r)\right\| \\
&\le \left\|e_A\left(\frac{r^v-r_k^v}{v}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}(I+C_j)e_A\left(\frac{r_j^v-r_{j-1}^v}{v}\right)\right. \\
&\quad \left.\times\left(I+C_i\right)\int_{r_{i-1}}^{r_i}e_A\left(\frac{r_i^v}{v}-\frac{s^v}{v}\right)h\left(s,y_0(s)\right)s^{v-1}ds\right\| \\
&+\left\|\int_{r_k}^{r}e_{\|A\|}\left(\frac{r^v}{v}-\frac{s^v}{v}\right)h(s,y_0(s))s^{v-1}ds\right\| \\
&\le e_{\|A\|}\left(\frac{r^v-r_k^v}{v}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}(1+\|C_j\|)e_{\|A\|}\left(\frac{r_j^v-r_{j-1}^v}{v}\right) \\
&\times\left(1+\|C_i\|\right)\int_{r_{i-1}}^{r_i}e_{\|A\|}\left(\frac{r_i^v}{v}-\frac{s^v}{v}\right)\left\|h\left(s,y_0(s)\right)\right\|s^{v-1}ds \\
&+\int_{r_k}^{r}e_{\|A\|}\left(\frac{r^v}{v}-\frac{s^v}{v}\right)\left\|h(s,y_0(s))\right\|s^{v-1}ds \\
&=\sum_{i=1}^{k}\prod_{j=k}^{i}\left(1+\left\|C_j\right\|\right)e_{\|A\|}\left(\frac{r^v-r_k^v}{v}\right)e_{\|A\|}\left(\frac{r_j^v-r_{j-1}^v}{v}\right) \\
&\times\int_{r_{i-1}}^{r_i}e_{\|A\|}\left(\frac{r_i^v}{v}-\frac{s^v}{v}\right)\left\|h\left(s,y_0(s)\right)\right\|s^{v-1}ds
\end{aligned} \tag{8}
$$

$$+\int_{r_k}^{r} e_{\|A\|}\left(\frac{r^v}{v}-\frac{s^v}{v}\right)\|h(s,y_0(s))\| s^{v-1}ds$$

$$\leq e_{\|A\|}\left(\frac{r^v-r_k^v}{v}\right)\sum_{i=1}^{k}\prod_{j=k}^{i}\left(I+\|C_j\|\right)e_{\|A\|}\left(\frac{r_j^v-r_{j-1}^v}{v}\right)\frac{1}{\|A\|}M_h\left[e_{\|A\|}\left(\frac{r_i^v-r_{i-1}^v}{v}\right)-1\right]$$

$$+\frac{1}{\|A\|}M_h\left[e_{\|A\|}\left(\frac{R^v-r_k^v}{v}\right)-1\right]$$

$$\leq\left[pCe^2_{\|A\|}\left(\frac{R^v}{v}\right)+1\right]\frac{1}{\|A\|}M_h\left[e_{\|A\|}\left(\frac{R^v}{v}\right)-1\right]$$

$$\leq z.$$

From (7) and (8), it follows that for any $r\in[0,R]$

$$\|y_1(r)-y_0(r)\|\leq z.$$

(iii) For $r\in[0,R]$ and $n=m,$ assume that $\|y_m-y_0\|_\infty\leq z.$ We have

$$\|y_{m+1}(r)-y_0(r)\|$$

$$\leq\Bigg\|e_A\left(\frac{r^v-r_k^v}{v}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}\left(I+C_j\right)e_A\left(\frac{r_j^v-r_{j-1}^v}{v}\right)$$

$$\times\left(I+C_i\right)\int_{r_{i-1}}^{r_i} e_A\left(\frac{r_i^v-s^v}{v}\right)h(s,y_m(s))s^{v-1}ds\Bigg\|$$

$$+\left\|\int_{r_k}^{r} e_A\left(\frac{r^v}{v}-\frac{s^v}{v}\right)h(s,y_m(s))s^{v-1}ds\right\|.$$

Similar to equation (8), we have

$$\|y_{m+1}(r)-y_0(r)\|\leq\left[pCe^2_{\|A\|}\left(\frac{R^v}{v}\right)+1\right]\frac{1}{\|A\|}M_h\left[e_{\|A\|}\left(\frac{R^v}{v}\right)-1\right]$$

$$\leq z.$$

It follows that for any $n \geq 1$

$$\|y_n - y_0\|_\infty \leq z.$$

The second stage: We claim that the approximating sequence $\{y_n\}$ converges uniformly on $[0,R]$.

Consider the following series

$$S(r) = y_0(r) + \sum_{m=1}^{\infty}\left(y_m(r) - y_{m-1}(r)\right),\ r \in [0,R], \tag{9}$$

and the sequence

$$y_n(r) = y_0(r) + \sum_{m=1}^{n}\left(y_m(r) - y_{m-1}(r)\right),\ r \in [0,R].$$

We show that equation (9) is uniformly convergent on the interval $[0,R]$, considering the following:

$$r \in \left[0, r_1\right]: \qquad \|y_1(r) - y_0(r)\| = \left\| \int_0^r e_A\left(\frac{r^v - s^v}{v}\right) h(s, y_0(s)) s^{v-1} ds \right\|$$

$$\leq \int_0^r e_{\|A\|}\left(\frac{r^v - s^v}{v}\right) \|h(s, y_0(s))\| s^{v-1} ds$$

$$\leq M_h \int_0^r e_{\|A\|}\left(\frac{r^v - s^v}{v}\right) s^{v-1} ds$$

$$\leq \frac{1}{v} M_h r^v e_{\|A\|}\left(\frac{r^v}{v}\right).$$

$$\left\| y_1(r) - y_0(r) \right\|$$

$$\leq e_{\|A\|}\left(\frac{r^v - r_k^v}{v}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}\left(1+\left\|C_j\right\|\right)e_{\|A\|}\left(\frac{r_j^v - r_{j-1}^v}{v}\right)$$

$$\times\left(1+\left\|C_i\right\|\right)\int_{r_{i-1}}^{r_i} e_{\|A\|}\left(\frac{r_i^v - s^v}{v}\right)\left\|h(s, y_0(s))\right\| s^{v-1}ds \tag{10}$$

$$+\int_{r_k}^{r} e_{\|A\|}\left(\frac{r^v}{v} - \frac{s^v}{v}\right)\left\|h\left(s, y_0(s)\right)\right\| s^{v-1}ds$$

$$\leq e_{\|A\|}\left(\frac{3r^v}{v}\right)C^2 M_h \sum_{i=1}^{k}\int_{r_{i-1}}^{r_i} s^{v-1}ds$$

$$+e_{\|A\|}\left(\frac{r^v}{v}\right)M_h\int_{r_k}^{r} s^{v-1}ds$$

$$\leq \max\left(e_{\|A\|}\left(\frac{3r^v}{v}\right)C^2, e_{\|A\|}\left(\frac{r^v}{v}\right)\right)M_h\frac{r^v}{v} \leq K(R)M_h\frac{r^v}{v}$$

Subsequently, by using the Lipschitz condition $\left(\mathrm{H}_2\right)$, one has:

$$\left\| y_2(r) - y_1(r) \right\|$$

$$\leq \left\| e_A\left(\frac{r^v - r_k^v}{v}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}\left(I + C_j\right)e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right) \right.$$

$$\left. \times\left(I + C_i\right)\int_{r_{i-1}}^{r_i} e_A\left(\frac{r_i^v}{v} - \frac{s^v}{v}\right)\left[h(s, y_1(s) - h(s, y_0(s))\right] s^{v-1}ds \right\|$$

$$+\left\| \int_{r_k}^{r} e_A\left(\frac{r^v}{v} - \frac{s^v}{v}\right)\left[h(s, y_1(s) - h(s, y_0(s))\right] s^{v-1}ds \right\|$$

$$\leq e_{\|A\|}\left(\frac{r^v - r_k^v}{v}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}\left(I + \left\|C_j\right\|\right)e_{\|A\|}\left(\frac{r_j^v - r_{j-1}^v}{v}\right)$$

$$\times\left(I+\|C_i\|\right)\int_{r_{i-1}}^{r_i} e_{\|A\|}\left(\frac{r_i^{v}}{v}-\frac{s^{v}}{v}\right)\|h(s,y_1(s)-h(s,y_0(s))\|s^{v-1}ds\Bigg\|$$

$$+\int_{r_k}^{r} e_{\|A\|}\left(\frac{r^{v}}{v}-\frac{s^{v}}{v}\right)\|h(s,y_1(s)-h(s,y_0(s))\|s^{v-1}ds$$

$$\leq C^2 e_{\|A\|}\left(\frac{3R^{v}}{v}\right)L_h\sum_{i=1}^{k}\int_{r_{i-1}}^{r_i}\|y_1(s)-y_0(s)\|\ s^{v-1}ds$$

$$+e_{\|A\|}\left(\frac{R^{v}}{v}\right)L_h\int_{r_k}^{r}\|y_1(s)-y_0(s)\|\ s^{v-1}ds$$

$$\leq L_h K(R)\int_0^r\|y_1(s)-y_0(s)\|\ s^{v-1}ds$$

$$\leq L_h K^2(R)M_h\int_0^r\frac{s^{v}}{v}s^{v-1}ds = L_h K^2(R)M_h^2\frac{r^{2v}}{2!v},\qquad r\in\left(r_k,r_{k+1}\right],\ \ k=1,2,\ldots$$

For $0\leq r\leq r_1$ we have the similar estimate.

Thus, for any $r\in\left[0,R\right]$

$$\|y_2(r)-y_1(r)\|\leq\frac{1}{2!v^2}L_h{}^2K^2(R)M_h r^{2v}. \tag{11}$$

By Mathematical induction, assume that

$$\|y_n(r)-y_{n-1}(r)\|\leq\frac{1}{n!v^n}K^n(R)L_h^{n-1}M_h r^{nv}$$

holds for a natural number n and $r\in\left[0,R\right]$.Then, for $r\in\left[0,R\right]$, according to $\left(\mathrm{H}_2\right)$, we have:

$$
\begin{aligned}
&\left\| y_{n+1}(r) - y_n(r) \right\| \\
&\leq CL_h \int_0^r e_{\|A\|}\left(\frac{r^v - s^v}{v}\right) \left\| y_n(s) - y_{n-1}(s) \right\| s^{v-1} ds \\
&\leq CL_h \frac{1}{n! v^n} C^n L_h^{n-1} M_h \int_0^r e_{\|A\|}\left(\frac{r^v - s^v}{v}\right) e_{\|A\|}\left(\frac{s^v}{v}\right) s^{nv} s^{v-1} ds \\
&\leq \frac{1}{(n+1)! v^{n+1}} C^{n+1} L_h^n M_h e_{\|A\|}\left(\frac{r^v}{v}\right) r^{(n+1)v}.
\end{aligned} \tag{12}
$$

Note that

$$
\begin{aligned}
\left\| S(r) \right\| &\leq \left\| y_0(r) \right\| + \sum_{m=1}^{\infty} \left\| y_m(r) - y_{m-1}(r) \right\| \\
&\leq C e_{\|A\|}\left(\frac{r^v}{v}\right) \left\| y_0 \right\| + M_h \sum_{m=1}^{\infty} \frac{K^m(R) L_h^{m-1} C^{m+1} L_h^m}{m! v^m} r^{mv}.
\end{aligned}
$$

Hence, the sequence of approximating functions $\{y_n(r)\}$ is uniformly convergent on [0,R]. So $\exists\, y \in PC\left([0,R], \mathbb{R}^d\right)$, such that $y_n(r)$ uniformly converges to $y(r)$ on [0,R].

The third stage: We claim that the limit y is a solution of the semilinear equation (1).

The sequence $y_n(r) \overset{uniformly}{\Longrightarrow} y(r)$ on $[0,R]$, so the sequence of functions $h(r, y_n(r))$ converges uniformly to the continuous function $h(r, y_n(r))$ on [0,R]. For all $r \in [0,R]$, we have:

$$\lim_{n\to\infty} y_n(r) = \begin{cases} e_A\left(\frac{r^v}{v}\right)y_0 + \lim_{n\to\infty}\int_0^r e_A\left(\frac{r^v - s^v}{v}\right)h\left(s, y_{n-1}(s)\right)s^{v-1}ds, 0 \le r \le r_1; \\ e_A\left(\frac{r^v - r_k^v}{v}\right)\prod_{j=k}^{1}\left(I + C_j\right)e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right)y_0 \\ +e_A\left(\frac{r^v - r_k^v}{v}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}\left(I + C_j\right)e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right) \\ \times\left(I + C_i\right)\lim_{n\to\infty}\int_{r_{i-1}}^{r_i} e_A\left(\frac{r_i^v - s^v}{v}\right)h\left(s, y_{n-1}(s)\right)s^{v-1}ds \\ +\lim_{n\to\infty}\int_{r_k}^{r} e_A\left(\frac{r^v - s^v}{v}\right)h\left(s, y_{n-1}(s)\right)s^{v-1}ds, \ \ r_k < r \le r_{k+1}, \ k = 1, 2, \ldots p. \end{cases}$$

$$= \begin{cases} e_A\left(\frac{r^v}{v}\right)y_0 + \int_0^r e_A\left(\frac{r^v - s^v}{v}\right)h\left(s, y(s)\right)s^{v-1}ds, \ 0 \le r \le r_1; \\ e_A\left(\frac{r^v - r_k^v}{v}\right)\prod_{j=k}^{1}\left(I + C_j\right)e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right)y_0 \\ +e_A\left(\frac{r^v - r_k^v}{v}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}\left(I + C_j\right)e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right) \\ \times\left(I + C_i\right)\int_{r_{i-1}}^{r_i} e_A\left(\frac{r_i^v - s^v}{v}\right)h\left(s, y(s)\right)s^{v-1}ds \\ +\int_{r_k}^{r} e_A\left(\frac{r^v - s^v}{v}\right)h\left(s, y(s)\right)s^{v-1}ds, r_k < r \le r_{k+1}, \ k = 1, 2, \ldots p. \end{cases}$$

$$= y(r).$$

The fourth stage: The solution is unique.

Suppose t is another solution of (1). Using the condition $\left(\mathrm{H}_2\right)$ similar to equation (12) we have

$$\left\|y(r) - t(r)\right\| \le K(R)L_h\int_0^r e_{\|A\|}\left(\frac{r^v - s^v}{v}\right)\left\|y(s) - t(s)\right\|s^{v-1}ds.$$

Applying the Gronwall's inequality (conformable version), we get:

$$\|y(r)-t(r)\| \leq 0 \Rightarrow y(r)=t(r),\ r\in[0,R].$$

The proof is complete.

Schauder's fixed point theorem is a prominent outcome in the field of mathematical analysis. This proposition states that when a continuous and compact operator is employed on a metric space that is complete, it will always have a point that remains unchanged, known as a fixed point. This theorem has broad utility and can be used to establish the presence of solutions to various mathematical challenges, including differential equations and integral equations. However, to effectively utilize Schauder's fixed point theorem, specific conditions and prerequisites must be satisfied:

- The operator needs to demonstrate both continuity and compactness.
- The metric space that the operator maps to must possess the property of completeness.
- The result generated by the operator must stay within the boundaries of the metric space

Once these criteria are met, Schauder's fixed point theorem ensures that the operator in question will have a fixed point.

In light of these conditions, we employ Schauder's fixed point theorem to substantiate our second principal result, specifically, the existence theorem. These assertions are made under the following assumptions.

Hypothesis 3 $(\mathbf{H_3})$**.** $h:[0,R]\times\mathbb{R}^d\to\mathbb{R}^d$ *is measurable in the first variable and continuous in the second variable.*

Hypothesis 4 $(\mathbf{H}_4)$**.** *There exists a positive constant* $M_h>0$ *such that, for any* $r\in[0,R]$ *and* $y\in\mathbb{R}^d$ *we have*

$\|h(r,y)\|\le M_{h.}$

Theorem 5.2 ([1]) *Assume that* $(\mathbf{H_3})$ *and* $(\mathbf{H_4})$ *hold. Then, equation* (1) *has at least one solution in* $PC([0,R],\mathbb{R}^d)$.

Proof. Set

$$B_z:=\left\{y\in PC\left([0,R],\mathbb{R}^d\right):\|y\|_\infty\le Ce_{\|A\|}\left(\frac{R^v}{v}\right)\|y_0\|+\frac{1}{v}CM_h\left(e_{\|A\|}\left(\frac{R^v}{v}\right)-1\right)\right\}.$$

Consider the nonlinear operator H defined on B_z as follows:

$$(Hy)(r):=\begin{cases} e_A\left(\frac{r^v}{v}\right)y_0+\int_0^r e_A\left(\frac{r^v-s^v}{v}\right)h(s,y(s))s^{v-1}ds,\ 0\le r\le r_1;\\ e_A\left(\frac{r^v-r_k^v}{v}\right)\prod_{j=k}^{1}\left(I+C_j\right)e_A\left(\frac{r_j^v-r_{j-1}^v}{v}\right)y_0\\ +e_A\left(\frac{r^v-r_k^v}{v}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}\left(I+C_j\right)e_A\left(\frac{r_j^v-r_{j-1}^v}{v}\right)\\ \times\left(I+C_i\right)\int_{r_{i-1}}^{r_i}e_A\left(\frac{r_i^v-s^v}{v}\right)h(s,y(s))s^{v-1}ds\\ +\int_{r_k}^{r}e_A\left(\frac{r^v-s^v}{v}\right)h(s,y(s))s^{v-1}ds,\ r_k<r\le r_{k+1},\ k=1,2,\dots p.\end{cases}$$

Step 1. We prove that $H\left(B_z\right) \subset B_z$.

For $y \in B_z$ and any $r \in [0, R]$, we have:

$$
\begin{aligned}
\left\|(Hy)(r)\right\| \le & \prod_{j=k}^{1}\left(I+\left\|C_j\right\|\right) e_{\|A\|}\left(\frac{r^{\nu}}{\nu}\right)\left\|y_0\right\| \\
& + \prod_{j=p}^{1}\left(I+\left\|C_j\right\|\right) \int_0^r e_{\|A\|}\left(\frac{r^{\nu}-s^{\nu}}{\nu}\right)\left\|h\left(s, y(s)\right)\right\| s^{\nu-1} ds \\
& \le C e_{\|A\|}\left(\frac{r^{\nu}}{\nu}\right)\left\|y_0\right\| \\
& + C M_h \int_0^r e_{\|A\|}\left(\frac{r^{\nu}-s^{\nu}}{\nu}\right) s^{\nu-1} ds \\
& = C e_{\|A\|}\left(\frac{R^{\nu}}{\nu}\right)\left\|y_0\right\| + \frac{1}{\nu} C M_h \left(e_{\|A\|}\left(\frac{R^{\nu}}{\nu}\right) - 1\right).
\end{aligned}
$$

Step 2. We prove the continuity of the nonlinear operator *H*.

Let y_n be a sequence with $y_n \to y$ in B_z as $n \to \infty$. For any $r \in [0, R]$, we have:

$$
\left\|(Hy_n)(r) - (Hy)(r)\right\|
$$

$$
\le \prod_{j=p}^{1}\left(I+\left\|C_j\right\|\right) \int_0^r e_{\|A\|}\left(\frac{r^{\nu}-s^{\nu}}{\nu}\right)\left\|h\left(s, y_n(s)\right) - h\left(s, y(s)\right)\right\| s^{\nu-1} ds.
$$

From the assumptions (H_3) and (H_4) it follows that

$$
\max_{0 \le s \le R}\left\|h\left(s, y_n(s)\right) - h\left(s, y(s)\right)\right\| \to 0 \text{ as } n \to \infty
$$

$$
e_{\|A\|}\left(\frac{r^{\nu}-s^{\nu}}{\nu}\right)\left\|h\left(s, y_n(s)\right) - h\left(s, y(s)\right)\right\| s^{\nu-1} \le 2 M_h e_{\|A\|}\left(\frac{r^{\nu}-s^{\nu}}{\nu}\right) s^{\nu-1},
$$

$$
2 M_h e_{\|A\|}\left(\frac{r^{\nu}-s^{\nu}}{\nu}\right) s^{\nu-1}
$$

is integrable with respect to $s \in [0, R]$.

It remains to apply the Lebesgue dominated theorem to get continuity of *H*.

Step 3. We prove that the set $H(B_z)$ is equicontinuous.

Let $r', r'' \in (r_k, r_{k+1}]$, $r' < r''$, and $|r' - r''| < \delta$. For any $y \in B_z$, we have

$$\begin{aligned}
&\left\|(Hy_n)(r'') - (Hy)(r')\right\| \\
&\leq C\left[e_{\|A\|}\left(\frac{(r'')^v}{v}\right) - e_{\|A\|}\left(\frac{(r')^v}{v}\right)\right]\|y_0\| \\
&+C\left[e_{\|A\|}\left(\frac{(r'')^v}{v}\right) - e_{\|A\|}\left(\frac{(r')^v}{v}\right)\right]\int_0^{r'} e_{\|A\|}\left(\frac{-s^v}{v}\right)\|h(s, y(s))\| s^{v-1} ds \\
&+Ce_{\|A\|}\left(\frac{(r'')^v}{v}\right)\int_0^{r''} e_{\|A\|}\left(\frac{-s^v}{v}\right)\|h(s, y(s))\| s^{v-1} ds.
\end{aligned}$$

Uniform continuity of $e_{\|A\|}\left(\frac{r^v}{v}\right)$ on $[0,R]$ implies that $\left\|(Hy)(r'') - (Hy)(r')\right\| \to 0$ as $\delta \to 0$. So, $H(B_z)$ is equicontinuous.

Step 1-3 with theorem 4.2 when $X = \mathbb{R}^d$ say that the nonlinear operator $H : B_z \to B_z$ is compact. Therefore, the Schauder FPT implies that H has a fixed point in $PC([0, R], \mathbb{R}^d)$. The proof is complete.

Chapter 6

COMPLETE CONTROLLABILITY

6.1 Linear Systems

Consider

$$\begin{cases} E_0^{v} y(r) = Ay(r) + Bu(r),\ r \in [0, R],\ 0 < v < 1, \\ y(r_k^+) = (I + C_k) y(r_k^-),\ k \in \mathbb{K} := \{1, 2, , \ldots, p\},\ r_0 = 0,\ r_{p+1} = R, \\ y(0) = y_0. \end{cases} \tag{13}$$

Definition 6.3 ([1]) The system equation (13) is said to be completely controllable on [0,R] if, given an arbitrary initial vector function y_0 and v terminal state vector y_R at time R, there exists a control input $u \in L^2([0, R], \mathbb{R}^r)$, such that the condition of the system $y \in PC([0, R], \mathbb{R}^d)$ satisfies $y(R) = y_R$.

In other words, the system possesses the capability to transition from any given initial state to a predetermined terminal state through the application of an appropriate control input. Complete controllability stands as a pivotal attribute within control theory, as it guarantees the system's adeptness in being harnessed and directed to attain a desired behavior.

To define the impulsive controllability operator, we introduce the continuous linear bounded operator $M : L^2([0, R], \mathbb{R}^z) \to \mathbb{R}^d$ as follows

$$Mu = e_A\left(\frac{R^v - r_k^v}{v}\right)\sum_{i=1}^{p}\prod_{j=p}^{i+1}\left(I + C_j\right)e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right)\left(I + C_i\right)\int_{r_{i-1}}^{r_i} e_A\left(\frac{r_i^v - s^v}{v}\right)Bu(s)ds$$
$$+\int_{r_p}^{R} e_A\left(\frac{R^v - s^v}{v}\right)Bu(s)ds.$$

Before stating the controllability result, we introduce the adjoint operator M^*.

Lemma 6.2 ([1]) *The adjoint operator* $M^* : \mathbb{R}^d \to L^2\left([0,R],\mathbb{R}^z\right)$ *has the following form*

$$M^*\psi(r) = \begin{cases} B^\top e_A^\top\left(\frac{R^v - s^v}{v}\right)\varphi, & r_p < r \le R, \\ B^\top e_A^\top\left(\frac{r_k^v - r^v}{v}\right)\left(I + C_k^\top\right) & \\ \times\prod_{i=k+1}^{p} e_A^\top\left(\frac{r_i^v - r_{i-1}^v}{v}\right)\left(I + C_i^\top\right)e_A^\top\left(\frac{R^v - r_p^v}{v}\right)\varphi, & r_{k-1} < r \le r_k. \end{cases}$$

Proof Letting $y(0) = 0$ in (13) yields $\left\langle y(R),\varphi\right\rangle = \left\langle w, M^*,\varphi\right\rangle = \int_0^R \left\langle u(s), B^*\psi(s)\right\rangle ds,$

which implies

$$\left\langle y(R),\varphi\right\rangle =$$

$$\left\langle e_A\left(\frac{R^\nu - r_k^\nu}{\nu}\right)\sum_{i=1}^{p}\prod_{j=p}^{i+1}\left(I+C_j\right)e_A\left(\frac{r_j^\nu - r_{j-1}^\nu}{\nu}\right)\left(I+C_i\right)\int_{r_{i-1}}^{r_i} e_A\left(\frac{r_i^\nu - s^\nu}{\nu}\right)Bu(s)ds,\varphi\right\rangle$$

$$+\left\langle \int_{r_p}^{R} e_A\left(\frac{R^\nu - s^\nu}{\nu}\right)Bu(s)ds,\varphi\right\rangle$$

$$=\int_{r_p}^{b}\left\langle u(s), B^\top e_A^\top\left(\frac{R^\nu - s^\nu}{\nu}\right)\varphi\right\rangle ds$$

$$+\sum_{i=1}^{p}\int_{r_{i-1}}^{r_i}\left\langle u(s)ds, B^\top e_A^\top\left(\frac{r_k^\nu - r^\nu}{\nu}\right)\left(I+C_k^\top\right)\prod_{i=k+1}^{p} e_A^\top\left(\frac{r_i^\nu - r_{i-1}^\nu}{\nu}\right)\left(I+C_i^\top\right)e_A^\top\left(\frac{R^\nu - r_p^\nu}{\nu}\right)\varphi\right\rangle.$$

Lemma 6.3 ([1]) *The operator* MM^* *has the following form*

$$MM^* = \Theta_0^{r_p} + \Gamma_{r_p}^{b},$$

where $\Gamma_{r_p}^{R}, \Theta_0^{r_p} : \mathbb{R}^d \to \mathbb{R}^d$ *are non-negative matrices and defined as follows:*

$$\Gamma_{r_p}^{R} := \int_{r_p}^{R} e_A\left(\frac{R^\nu - s^\nu}{\nu}\right)BB^\top e_A^\top\left(\frac{R^\nu - s^\nu}{\nu}\right)ds,$$

$$\Theta_0^{r_p} := e_A\left(\frac{R^\nu - r_k^\nu}{\nu}\right)\sum_{i=1}^{p}\prod_{j=p}^{i+1}\left(I+C_j\right)e_A\left(\frac{r_j^\nu - r_{j-1}^\nu}{\nu}\right)$$

$$\times\left(I+C_i\right)\int_{r_{i-1}}^{r_i} e_A\left(\frac{r_i^\nu - s^\nu}{\nu}\right)BB^\top e_A^\top\left(\frac{r_k^\nu - s^\nu}{\nu}\right)ds$$

$$\times\left(I+C_i^\top\right)\sum_{k=i+1}^{p} e_A^\top\left(\frac{r_k^\nu - r_{k-1}^\nu}{\nu}\right)\left(I+C_k^\top\right)e_A^\top\left(\frac{R^\nu - r_p^\nu}{\nu}\right).$$

Proof Indeed

$$MM^*\varphi = e_A\left(\frac{R^v - r_k^v}{v}\right)\sum_{i=1}^{p}\prod_{j=p}^{i+1}\left(I + C_j\right)e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right)\left(I + C_i\right)$$

$$\times\int_{r_{i-1}}^{r_i} e_A\left(\frac{r_i^v - s^v}{v}\right)BB^{\mathsf{T}}e_A^{\mathsf{T}}\left(\frac{r_k^v - s^v}{v}\right)ds$$

$$\times\left(I + C_i^{\mathsf{T}}\right)\sum_{k=i+1}^{p} e_A^{\mathsf{T}}\left(\frac{r_k^v - r_{k-1}^v}{v}\right)\left(I + C_k^{\mathsf{T}}\right)e_A^{\mathsf{T}}\left(\frac{R^v - r_p^v}{v}\right)\varphi$$

$$+\int_{r_p}^{R} e_A\left(\frac{R^v - s^v}{v}\right)BB^{\mathsf{T}}e_A^{\mathsf{T}}\left(\frac{R^v - s^v}{v}\right)ds\varphi.$$

Obviously $\Gamma_{r_p}^{b}, \Theta_0^{r_p} : \mathbb{R}^d \to \mathbb{R}^d$ are non-negative.

Therefore, we can introduce the controllability Gram matrix as follows:

$$MM^* = \Theta_0^{r_p} + \Gamma_{r_p}^{R}.$$

Theorem 6.5 ([1]) *The linear conformable impulsive equation* (13) *is controllable on [0,R], if and only if the* $d \times d$ *matrix*

$MM^* = \Theta_0^{r_p} + \Gamma_{r_p}^{R}$ *is invertible.*

Proof Since the operator $M : L^2\left(\left[0, R\right], \mathbb{R}^z\right) \to \mathbb{R}^d$ is linear and bounded. By proposition 2.2 (iii)[41], the complete controllability of (13) is equivalent to the invertibility of the matrix MM^*.

The matrix MM^* is called the conformable controllability Gramian and it is positive semidefinite, that is,

$y^{\mathsf{T}}\left(\Theta_0^{r_p} + \Gamma_{r_p}^{R}\right)y \geq 0,$ for all $y \in \mathbb{R}^d$.

Corollary 6.1 In the context of conformable calculus, the linear equation (13) with impulsive characteristics demonstrates complete controllability within the domain on *[0,R],* if and only when the $d \times d$ conformable fractional controllability Gram matrix is positive definite.

Proof By $MM^* = \Theta_0^{r_p} + \Gamma_{r_p}^R$ The full controllability of equation (13) is synonymous with the matrix's invertibility MM^*, corresponds to the positivity of MM^*.

Corollary 6.2 The conformable impulsive linear equation (13) is completely controllable on [0,R], if $\Theta_0^{r_p}$ or $\Gamma_{r_p}^R$ *is* positive definite.

Proof By Theorem 6.5 -$MM^* = \Theta_0^{r_p} + \Gamma_{r_p}^R$ -, The linear conformable impulsive equation (13) is completely controllable on [0,R], if and only if the $d \times d$ matrix is positive definite:

$$y^{\top}\left(\Theta_0^{r_p} + \Gamma_{r_p}^R\right) y > 0, \text{ for all } 0 \neq y \in \mathbb{R}^d.$$

Since $\Theta_0^{r_p} + \Gamma_{r_p}^R$ is positive semidefinete, the positivity of $\Theta_0^{r_p} + \Gamma_{r_p}^R$ is equivalent to the positivite definity of $\Theta_0^{r_p}$ or $\Gamma_{r_p}^R$.

Corollary 6.3 The conformable impulsive linear equation (13) is controllable on $[0, R]$ if { B AB A^2B $A^{d-1}B$ } is equal to the system's dimension, denoted as *''d'.*

$\text{rank}\{B \; AB \; \mathrm{A}^2B \; A^{d-1}B\} = d.$

Proof It is known that the positivity of $\Gamma_{r_p}^{R}$ is equivalent to the Kalman rank condition:

rank { B AB A^2B $A^{d-1}B$ }= d.

Therefore, according to corollary 6.2, the conformable impulsive linear equation (13) exhibits controllability within the context on [0,R].

6.2 Semilinear Systems

We delineate the following assumptions:

Assumption 1 ($\mathbf{A_1}$). *Conformable controllability Gramian matrix* $\Theta_0^{r_p}+\Gamma_{r_p}^{R}$ *is invertible.*

Assumption 2 ($\mathbf{A_2}$). *There exists a positive constant* $M_h > 0$ *such that for any* $r \in [0, R]$ *and* $y \in \mathbb{R}^d$ *, we have*

$\|h(r, y)\| \le M_h.$

In view of (A_1), for any $y \in C\left(\left[0, R\right] \times \mathbb{R}^d\right)$, consider a control function $u\left(r; x\right)$ defined by

$$u\left(r;y\right) := M^*(\Theta_0^{r_p}+\Gamma_{r_p}^{R})^{-1}\left(y_R - e_A\left(\frac{r^v - r_k^v}{v}\right)\prod_{j=}^{1}\left(I + C_j\right)e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right)y_0\right.$$
$$-e_A\left(\frac{R^v - r_k^v}{v}\right)\sum_{i=1}^{p}\prod_{j=p}^{i+1}\left(I + C_j\right)e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right)\left(I + C_i\right)\int_{r_{i-1}}^{r_i} e_A\left(\frac{r_i^v - s^v}{v}\right)h(s(y(s))ds$$
$$\left.-\int_{r_p}^{R} e_A\left(\frac{R^v - s^v}{v}\right)h\left(s, y(s)\right)ds\right).$$

Subsequently, we prove our main result via FPT. We firstly show that, using control $u(r; y)$, the operator P: $PC\left(\left[0, R\right], \mathbb{R}^d\right) \to PC\left(\left[0, R\right], \mathbb{R}^d\right)$ defined by

$$
(Py)(r) := \begin{cases} e_A\left(\frac{r^v}{v}\right)y_0 + \int_0^r e_A\left(\frac{r^v - s^v}{v}\right)\left[h(s,y(s)) + Bu(s;y)\right]s^{v-1}ds, \ 0 \le r \le r_1; \\ e_A\left(\frac{r^v - r_k^v}{v}\right)\prod_{j=k}^{1}\left(I + C_j\right)e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right)y_0 \\ +e_A\left(\frac{r^v - r_k^v}{v}\right)\sum_{i=1}^{k}\prod_{j=k}^{i+1}\left(I + C_j\right)e_A\left(\frac{r_j^v - r_{j-1}^v}{v}\right) \\ \times\left(I + C_i\right)\int_{r_{i-1}}^{r_i} e_A\left(\frac{r_i^v - s^v}{v}\right)\left[h(s,y(s)) + Bu(s;y)\right]s^{v-1}ds \\ +\int_{r_k}^{r} e_A\left(\frac{r^v - s^v}{v}\right)\left[h(s,y(s)) + Bu(s;y)\right]s^{v-1}ds, \ r_k < r \le r_{k+1}, \ k = 1,2,...,p. \end{cases}
$$

has a fixed point y^*. It can be easily chech that $\left(Py^*\right)(R) = y_R$ and $\left(Py^*\right)(0) = y_0$.

In other word $u(r;y)$ steers system (1) from y_0 to y_R infinite time *R*. Thus, System (1) controllable on [0,*R*].

Theorem 6.6 ([1]) *Assumptions* (A_1) (A_2) *are satisfied. Then system* (1) *completely controllable on* [0,*R*].

Proof Step 1. We prove the continuity of the control variable $u(r;\cdot)$.

Let y_n be a sequence with $y_n \to y$ in B_z as $n \to \infty$. For any $r \in [0,R]$, we have:

$$
\left\|u(r;y_n) - u(r;y)\right\|
$$

$$
\le \left\|M^*\right\|\left\|\left(\Theta_0^{r_p} + \Gamma_{r_p}^{R}\right)^{-1}\right\|\prod_{j=p}^{1}\left(I + \left\|C_j\right\|\right)\int_0^R e_{\|A\|}\left(\frac{R^v - s^v}{v}\right)\left\|h(s,y_n(s)) - h(s,y(s))\right\|s^{v-1}ds.
$$

From the assumptions (A_1) and (A_2) it follows that

$$
\max_{0 \le s \le R}\left\|h(s,y_n(s)) - h(s,y(s))\right\| \to 0 \text{ as } n \to \infty,
$$

$$e^{\|A\|\left(\frac{r^{\nu}-s^{\nu}}{\nu}\right)}\|h(s,y_n(s))-h(s,y(s)\|s^{\nu-1}\le 2M_h e^{\|A\|\left(\frac{r^{\nu}-s^{\nu}}{\nu}\right)}s^{\nu-1},$$

$2M_h e^{\|A\|\left(\frac{r^{\nu}-s^{\nu}}{\nu}\right)}s^{\nu-1}$ is integrable with respect to $s\in[0,R]$.

It remains to apply Lebesgue dominated theorem to get the continuity of $u(r;\cdot)$.

Step 2. We prove that the control $u(r;y)$ is bounded.

The boundedness of $u(r;y)$ follows from the same property (A_2) of h.

Now, we can mimic the proof of theorem 6.4 to show that P has a fixed point y^* in $PC([0,R],\mathbb{R}^d)$, in other words the system (1) is completely controllable on $[0,R]$.

Chapter 7

EXAMPLES

Example 7.1 Consider the following 3-dimensional system.

$$\begin{cases} E_0^{v} y(r) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix} y(r) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} u(r),\ r \in [0,4] \setminus \{1,2,3\}, \\ \Delta y(r_i) = \frac{1}{4} y(r_i^-),\ r_i = i,\ i = 1,2,3, \\ y(0) = 0. \end{cases} \tag{14}$$

Now, let us apply our methodologies to examine the controllability of the system defined in equation (14) across the interval [0, 4]. We designate by

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},\ C_i = \frac{1}{4} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is possible to acquire

rank (B AB A^2B)

$$= \text{rank} \begin{pmatrix} 1 & 0 & 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 & 2 \end{pmatrix} = 3.$$

By corollary 6.3, the system referred to as equation (14) demonstrates controllability within the interval [0, 4].

Example 7.2 Consider the following 3-dimensional system.

$$\begin{cases} E_0^v y(r) = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 6 & 1 \\ 1 & 7 & -1 \end{pmatrix} y(r) + \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix} u(r), \ r \in [0,5] \setminus \{1,2,3,4\}, \\ \Delta y(r_i) = \frac{1}{5} y(r_i^-), \ r_i = i, \ i = 1,2,3,4, \\ y(0) = 0. \end{cases} \tag{15}$$

It is possible to acquire

$$A = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 6 & 1 \\ 1 & 7 & -1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}, \ C_i = \frac{1}{5} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

rank (B AB A^2B)

$$=\text{rank} \begin{pmatrix} 1 & 0 & -3 & * & * & * \\ 2 & 1 & 19 & * & * & * \\ 0 & 1 & 1 & * & * & * \end{pmatrix} = 3.$$

By corollary 6.3, the system denoted as equation (15) exhibits controllability within the interval [0, 5].

Example 7.3 Consider the following three-dimensional semi-linear system

$$\begin{cases} E_0^{v} y(r) = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 6 & 1 \\ 1 & 7 & -1 \end{pmatrix} y(r) + \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix} u(r) + \frac{1}{5} r \sin y(r), \ r \in [0,5] \setminus \{1,2,3,4\}, \\ \Delta y(r_i) = \frac{1}{5} y(r_i^-), \ r_i = i, \ i = 1,2,3,4, \\ y(0) = 0. \end{cases} \tag{16}$$

$$A = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 6 & 1 \\ 1 & 7 & -1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{pmatrix},$$

By referencing example 7.2, it is apparent that the linear component is controllable, and the nonlinear component remains bounded. Employing theorem 6.6, we can confidently assert that the semilinear system (16) exhibits complete controllability.

Chapter 8

CONCLUSION

Fractional impulsive differential equations represent mathematical constructs which amalgamate fractional derivatives, characterized by non-integer differentiation orders, with sudden shifts or discontinuities in the state variables. The study of controllability in the context of fractional impulsive differential equations is a rapidly developing research field. The importance of these equations is derived from their versatility, as they find application in the modeling of intricate systems within the domains of physical sciences, biology, and engineering.

The determination of controllability outcomes for fractional impulsive differential equations depends on several key elements. These elements include the particular fractional order, the characteristics of abrupt changes, and the structural properties of the control inputs. A substantial amount of additional research is required to attain a thorough understanding of the controllability dynamics that govern these complex systems.

In this research, we explore how solutions are expressed for linear systems with conformable fractional-type impulses and examine the presence and singularity of nonlinear systems with conformable fractional-type impulses. Furthermore, we assess the controllability of systems under control with conformable fractional dynamics, whether they are linear or semi-linear in nature.

By utilizing the framework of conformable fractional derivatives, we introduce a new idea known as the "conformable controllability Gramian matrix." This novel approach has the potential to provide fresh insights into the controllability features of such systems. Additionally, our investigation encompasses the controllability aspects of linear and semi-linear impulsive systems that adhere to the conformable framework, laying a valuable groundwork To support and inspire forthcoming scholarly endeavors within this particular area of study.

The results of this research are groundbreaking and hold relevance for practical use, enriching the understanding in this specific area of study. As a potential direction for future research, we suggest delving into the subjects of approximate or null controllability within the framework of conformable stochastic evolution equations and inclusions, which may involve instantaneous or non-instantaneous impulses and various stochastic disturbances, as elaborated in references [20-22].

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