Geodesics of BTZ Black Hole and Black Holes in Minimal Massive Gravity

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ABSTRACT

In this thesis, we study to derive the equation of motion for the geodesics of the 3 dimensional (3D) charged BTZ black hole and also geodesics of the black holes in the minimal massive gravity (MMG) theory at its merger point. The motions of the massless and massive particles are going to be study by using the Lagrangian equation. Then we are going to focus on null and time-like geodesics solutions.

After finding the Euler-Lagrange equations, we are going to investigate the radial motions of the geodesics. For this purpose, we are going to study to find the exact analytical solutions of the geodesic equations.

At the end, we perform some numerical simulations to plot graphs for displaying the geodesics.

Keywords: General Relativity, BTZ, Minimal Massive Gravity, Black Hole, Geodesics

ÖZ

Bu tezde, 3 boyutlu yüklü BTZ kara deliğinin ve ayrıca birleşme noktasında minimal massif kütle çekim teorisindeki kara deliklerin jeodezikleri için hareket denklemlerini araştırıyoruz. Kütlesiz ve kütleli parçacıkların hareketleri Lagrange denklemi kullanılarak incelenecektir. Daha sonra ışık ve zaman benzeri jeodezik çözümlere odaklanacağız.

Euler-Lagrange denklemlerini bulduktan sonra jeodeziklerin radyal hareketlerini inceleyeceğiz. Bu amaçla jeodezik denklemlerin kesin analitik çözümlerini bulmaya çalışacağız.

En sonunda, jeodeziklerin görüntülerini oluşturmaya yardım edecek grafikler çizmek için bazı sayısal simülasyonlar yapıyoruz.

Anahtar Kelimeler: Genel Görelilik, BTZ, Minimal Masif Yerçekimi Modeli, Kara Delik, Jeodezikler

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INTRODUCTION

The mysteries of the universe have always fascinated mankind. Nicolaus Copernicus, one of the history's greatest mathematicians and astronomers, was an advanced contributor to the scientific revolution [1]. The movements of the planets and stars have been observed by mankind for thousands of years, aroused their interest and curiosity. Although precocious cultures already had a precise knowledge of the movement of the stars, the starting points of the modern scientific world is based on the works of Johannes Kepler (1571-1630), Isaac Newton (1643-1727) and Albert Einstein (1879-1955).

In 1609 Kepler acknowledged the existence of a force radiated by the Sun in his work Astronomica Nova. This force decreases with distance and also it causes planets to move faster when they are closer to the Sun. After looking his assumption and examining the orbital data of Mars, he realized that the planets move on elliptical instead of circular orbits and developed the first and second law of planetary motion. Kepler explained his third law in 1618, which he used the connection between the length of a planet's semi-major axis and its orbital period. This connection allowed him to accurately calculate the orbital velocity of a planet. With these three laws Kepler became one of the founders and greatest names of the modern astronomy [2].

At first, Kepler believed that a force from the sun was pushing the planets in their orbits, but he could not identify this force. Later on, Newton's work on gravity showed us why planets orbit this way. When Newton applied his universal gravitation law to the Sun and planets, he can predicts the motion of the planets correctly. Newton laid down his laws in Philosophiae Naturalis Principia Mathematica [3].

1.1 General Relativity

Einstein started to thinking about gravity after his work on Special Relativity. His main idea was how to give gravity a relativistically invariant formulation. After all trials and errors, Einstein presented one of his biggest work General Theory of Relativity in 1915 [4]. In Newton's principle, gravity causes an attractive force between large objects. But in general relativity, the gravitational effect is caused by the distortion of space-time around masses. As time passed, Einstein's definition of gravity was shown to explain various effects that could not be drawn from Newton's law, such as the orbits of the planets and the effect of gravity on light. Mass is an important property in determining the gravitational effect of matter. But in general relativity, mass cannot be the only source of gravity by itself. Relativity unifies mass and energy. Also relativity unifies energy and momentum. Today, Einstein's General Relativity Theory is the key of the scientists' best understanding of gravity.

1.2 Geodesics

We know that the shortest distance between two points which are on a plane is a straight line. So how to define the shortest distance between two points which are on a sphere? The shortest distance between these two points expressed by the segment of the arc where its center is the center of this sphere and also passes through these two points. We give a name to this curve, which shows the shortest distance between two points on a surface. This name is geodesic curve [5].

In geometry, a geodesic is commonly a curve representing in some sense the shortest path between two points in a surface, or more generally in a Riemannian manifold.

In general relativity, a geodesic generalizes the notion of a straight line to curved space time. Importantly, the world line of a particle free from all external, non-gravitational forces is a particular type of geodesic. In other words, a freely moving or falling particle always moves along a geodesic.

1.3 Black Hole

Black hole is one of the most impressive predictions of general theory of relativity which has long been tempting for physicists for a long time. And unfortunately we are not fully understand it, still it has some unknown parts to work on it [6]. Black hole is a region of space time where the gravitational field is so strong that nothing (even light) can escape from it [7]. The general theory of relativity shows that a sufficiently compact mass can form a black hole by deforming the space-time [8]. It has an event horizon whose total area does not decrease in any physical process. Also, in curved space time, quantum field theory shows that event horizons emit Hawking radiation with the same spectrum as a black body at a temperature inversely proportional to its mass.

1.4 BTZ Black Hole

The name BTZ comes from its founders. Banados, Teitelboim and Zanelli found that there is a solution for black hole in (2+1) dimensional space time in 1992 [9]. In local space time, they need a constant curvature for solution of the gravitational field equation [10]. Also, this solution is similar with the solution of Anti-de Sitter (AdS)-Maxwell gravity in three dimension [11]. The BTZ black hole has a connection with the string theory and this connection makes it more interesting [12]. The BTZ black hole has another usage which is studying black holes in quantum scale [13].

In this thesis, my main aim is to observe the motion of the both massive and massless particles around the BTZ black hole. To achieve this aim, in chapter 2, we first described the metric and then solved the geodesic equations in BTZ black hole by using Lagrangian method. After that, in chapter 3, we found the analytical solutions of the geodesic equations for both null and time-like geodesics.

1.5 Minimal Massive Gravity

In physics one of the main problem is to have a theory in which gravitational effects brought together the quantum mechanical principles, namely getting a unitary theory. By unitary it is meant both bulk and boundary unitarity since the inception of AdS/conformal field theory (CFT) correspondence. In order to test some aspects of quantum gravity three dimensional space time is a useful theoretical background. Even in three dimensional background, it is not easy task to form a theory which is bulk and boundary unitary, for example, the first theory come to mind is cosmological Einstein's theory which has no propagating degrees of freedom and that makes the theory locally trivial even though it has positive central charges in 3D. The simplest way to introduce propagating degrees of freedom, and get rid of the local triviality, to the theory, the Einstein's theory can be modified by introducing mass to the gravitation [14].

Massive gravity theories are studied for a long while. Here properties of some of the relevant massive gravity theories are given. The cosmological Topologically Massive Gravity (TMG) is one of these theories. TMG is the most experienced modification of the general relativity in 3D. It completes the Einstein-Hilbert action with the Chern-Simons term [15]. But it has a bulk vs boundary clash problem. So to get rid of this problem, a new model was constructed. This new model is known as New Massive Gravity (NMG). This theory also includes usual Einstein-Hilbert term but it has additional quadratic curvature terms. But still NMG has the same problem with TMG [16]. Later on Minimal Massive Gravity (MMG) comes up, which is a newest version of NMG, it is obtained by unifying the Chern-Simmons term with the NMG's action. Finally, Generalized Massive Gravity (GMG) is introduced. It is important for us, because in a certain range of parameters it can get rid of the bulk vs boundary clash problem [17].

In this thesis, my other main aim is to observe the motion of the massless particles around the black hole of MMG theory. To achieve this aim, in chapter 3, we first described the metric and then solved the geodesic equations in MMG by using Lagrangian method. After that, in chapter 4, we found the analytical solution of the geodesic equations for null particles.

Chapter 2

METRIC AND GEODESIC EQUATIONS OF CHARGED BTZ BLACK HOLE

In this study, we take the Minkowski metric with mostly plus signature. The coordinates are defined as $x^\mu = (t, r, \phi)$. Greek indices run from 0 to 3 and Latin indices run from 1 to 3.

The Maxwell's power law theory is given by the following Lagrangian [18]

$$\mathcal{L} = -\alpha(kF)^s. \tag{2.1}$$

Then the action is [19]

$$I = \frac{1}{16\pi}\int d^3x\sqrt{-g}[R - 2\Lambda + (k\mathcal{F})^s], \tag{2.2}$$

where R is the scalar curvature, $\Lambda = -\frac{1}{l^2}$ is the cosmological constant, $\mathcal{F}$ is the Maxwell invariant acts as a source in the theory, which is equal to $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor field with A_μ as the gauge potential, and s is an arbitrary positive nonlinearity parameter.

For radial electric field the gauge potential is given as $A^\mu = h(r)\delta_0^\mu$ then the non-vanishing components of electromagnetic field tensor becomes

$$F^{10} = \partial^1 A^0 - \partial^0 A^1, \tag{2.3}$$

$$F^{10} = \partial^1 A^0 = \partial^r \phi(r) = -E(r), \tag{2.4}$$

$$F^{01} = \partial^0 A^1 - \partial^1 A^0, \tag{2.5}$$

$$F^{01} = -\partial^1 A^0 = -\partial^r h(r) = E(r), \tag{2.6}$$

where we have used the definition of electric field $E^i = \partial^i A^0 - \partial^0 A^i$.

In order to take the indices of electromagnetic field tensor down we multiply equation (2.3) with the metric

$$F_{01} = g_{00} g_{11} F^{01}, \tag{2.7}$$

$$F_{01} = -F^{01} = -E(r), \tag{2.8}$$

$$-E(r) = \partial_0 A_1 - \partial_1 A_0, \tag{2.9}$$

$$-E(r) = -\partial_1 A_0, \tag{2.10}$$

$$F_{0r} = -E(r) = -\partial_r h(r). \tag{2.11}$$

The electromagnetic field tensor can be written in matrix form as follows

$$F^{\mu\nu} = \begin{pmatrix} 0 & E(r) & 0 \\ -E(r) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.12}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E(r) & 0 \\ E(r) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.13}$$

where the raising and lowering of the indices are done by the following metric ansatz [20]

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\phi^2, \tag{2.14}$$

which can be written in component form

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{g(r)} & 0 & 0 \\ 0 & g(r) & 0 \\ 0 & 0 & \frac{1}{r^2} \end{pmatrix}, \tag{2.15}$$

$$g_{\mu\nu} = \begin{pmatrix} -g(r) & 0 & 0 \\ 0 & \frac{1}{g(r)} & 0 \\ 0 & 0 & r^2 \end{pmatrix}. \tag{2.16}$$

Since we need the following tensor in our calculation we write it in the matrix form whose detailed derivation can be found in Appendix-A

$$F_{\nu}{}^{\alpha} = g^{\alpha\beta} F_{\nu\beta}, \tag{2.17}$$

$$F_{\nu}{}^{\alpha} = \begin{pmatrix} 0 & -g(r)E(r) & 0 \\ -\frac{E(r)}{g(r)} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.18}$$

We also need to calculate the following identities

$$\mathcal{F} = F_{\mu\nu} F^{\mu\nu}, \tag{2.19}$$

$$\mathcal{F} = F_{00} F^{00} + F_{0i} F^{0i} + F_{i0} F^{i0} + F_{ij} F^{ij}, \tag{2.20}$$

$$\mathcal{F} = F_{01} F^{01} + F_{10} F^{10} = -2E^2(r), \tag{2.21}$$

and

$$F_{\mu\alpha} F_{\nu}{}^{\alpha} = F_{\mu 0} F_{\nu}{}^{0} + F_{\mu 1} F_{\nu}{}^{1} + F_{\mu 2} F_{\nu}{}^{2} = \mathcal{A}_{\mu\nu}, \tag{2.22}$$

$$F_{\mu\alpha} F_{\nu}{}^{\alpha} = \begin{pmatrix} g(r)E^2(r) & 0 & 0 \\ 0 & -\frac{E^2(r)}{g(r)} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.23}$$

The detailed derivation of equation (2.23) can be found also in Appendix-A.

2.1 Maxwell Power Law Action

Varying the cosmological Einstein-Hilbert-Maxwell Power Law action (2.2) with respect to $g_{\mu\nu}$ (the metric tensor) we can obtain the equation of gravitational field as

$$\delta I = \frac{1}{16\pi} \int d^3x \delta[\sqrt{-g}(R - 2\Lambda + (k\mathcal{F})^s)], \tag{2.24}$$

$$\delta I = \frac{1}{16\pi} \int d^3x [\left(\delta\sqrt{-g}\right)(R - 2\Lambda + (k\mathcal{F})^s) + \sqrt{-g}(\delta R + \delta(k\mathcal{F})^s)], \tag{2.25}$$

where $\delta\sqrt{-g} = -\frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} \sqrt{-g}$ and $\delta R = \delta g^{\mu\nu} R_{\mu\nu}$.

We can take the variation of the term $\delta(kF)^s$ in equation (2.24) as

$$\delta(k\mathcal{F})^s = s(k\mathcal{F})^{s-1}k\delta\left(F_{\mu\nu}F^{\mu\nu}\right), \tag{2.26}$$

$$\delta(k\mathcal{F})^s = 2sk(k\mathcal{F})^{s-1}F_{\mu\alpha}F_{\nu}{}^{\alpha}\delta g^{\mu\nu}. \tag{2.27}$$

The steps between two equations are shown in Appendix-B.

Then, the variation of the action becomes

$$\delta I = \frac{1}{16\pi}\int d^3x\sqrt{-g}[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda - \frac{1}{2}g_{\mu\nu}(k\mathcal{F})^s + 2sk(k\mathcal{F})^{s-1}F_{\mu\alpha}F_{\nu}{}^{\alpha}]\delta g^{\mu\nu}, \tag{2.28}$$

and setting $\delta I = 0$ we get

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda - \frac{1}{2}g_{\mu\nu}(k\mathcal{F})^s + 2sk(k\mathcal{F})^{s-1}F_{\mu\alpha}F_{\nu}{}^{\alpha} = 0, \tag{2.29}$$

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = \frac{1}{2}g_{\mu\nu}(k\mathcal{F})^s - 2sk(k\mathcal{F})^{s-1}F_{\mu\alpha}F_{\nu}{}^{\alpha}. \tag{2.30}$$

Finally, the gravitational field becomes

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = T_{\mu\nu}, \tag{2.31}$$

where the energy-momentum tensor is defined as

$$T_{\mu\nu} \equiv -2\left[sk(k\mathcal{F})^{s-1}F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}g_{\mu\nu}(k\mathcal{F})^s\right]. \tag{2.32}$$

Now varying the action with respect to the gauge potential A_μ we can obtain the equation of electromagnetic field as

$$\frac{\partial\mathcal{L}}{\partial A_\alpha} - \partial_\beta\frac{\partial\mathcal{L}}{\partial\partial_\beta A_\alpha} = 0, \tag{2.33}$$

$$\frac{\partial\mathcal{L}}{\partial\partial_\beta A_\alpha} = 4\sqrt{-g}sk(k\mathcal{F})^{s-1}F^{\beta\alpha}, \tag{2.34}$$

whose calculations are shown in Appendix-B.

Then, one can write the second field equation in the following form

$$4\text{sk}\frac{1}{\sqrt{-g}}\partial_\beta(\sqrt{-g}(k\mathcal{F})^{s-1}F^{\beta\alpha}) = 0. \tag{2.35}$$

Finally, the electromagnetic field becomes

$$\frac{1}{\sqrt{-g}}\partial_\beta(\sqrt{-g}(k\mathcal{F})^{s-1}F^{\beta\alpha}) = 0. \tag{2.36}$$

This equation determines the electromagnetic field which has the electric field as the non-vanishing component. At this stage the discussion bifurcates into two parts according to the power of the Maxwell term in equation (2.36). For $s = 1$, $\sqrt{-g} = r$ and $F^{rt} = -E(r)$, equation (2.36) can be solved for the electric field:

$$\partial_r(rF^{r0}) = 0, \tag{2.37}$$

$$\partial_r(r\partial_r h) = 0, \tag{2.38}$$

$$\partial_r h(r) = \frac{q}{r}. \tag{2.39}$$

By multiplying and dividing the right hand side with $\frac{1}{l}$ we get

$$\partial_r h(r) = q\frac{\frac{1}{l}}{r\frac{1}{l}}, \tag{2.40}$$

$$\partial_r h(r) = q\frac{d}{dr}\ln\left(\frac{r}{l}\right), \tag{2.41}$$

$$h(r) = q\ln\left(\frac{r}{l}\right). \tag{2.42}$$

Then, the electric field becomes

$$F_{0r} = \frac{q}{r}, \tag{2.43}$$

where we defined the integration constant as q. Then the electric field becomes

$$E(r) = \partial_r h(r) = \frac{q}{r}. \tag{2.44}$$

For the general power of the Maxwell term in equation (2.36), $s \neq 1$ the gauge potential takes the following form

$$h = -qr^{\frac{2(s-1)}{2s-1}}, \tag{2.45}$$

and the electric field becomes

$$E(r) = \partial_r h(r) = -\frac{2q(s-1)}{2s-1} r^{\frac{-1}{2s-1}}. \tag{2.46}$$

2.2 Solution of EH-Maxwell Power Law Gravity

In this part we are going to find the solution of the equation (2.31) for the metric ansatz (2.14).

Both sides of the equation (2.31) can be written in matrix form

$$G_{\mu\nu} + g_{\mu\nu}\Lambda$$
$$= \begin{pmatrix} -\frac{g(r)(2r\Lambda + g'(r))}{2r} & 0 & 0 \\ 0 & \frac{2r\Lambda + g'(r)}{2rg(r)} & 0 \\ 0 & 0 & \frac{r^2(2\Lambda + g''(r))}{2} \end{pmatrix}, \tag{2.47}$$

and

$$T_{\mu\nu}$$
$$= \begin{pmatrix} -2sk(-2kE^2)^{s-1}g(r)E^2(r) - \frac{(-2kE^2)^s g(r)}{2} & 0 & 0 \\ 0 & 2sk(-2kE^2)^{s-1}\frac{E^2(r)}{g(r)} + \frac{(-2kE^2)^s}{2g(r)} & 0 \\ 0 & 0 & \frac{(-2kE^2)^s r^2}{2} \end{pmatrix}. \tag{2.48}$$

From the equality of the matrices (2.47) and (2.48) one can get three equations

$$-\frac{g(r)\left(2r\Lambda + g'(r)\right)}{2r} = -2sk(-2kE^2)^{s-1}g(r)E^2(r) - \frac{(-2kE^2)^s g(r)}{2}, \tag{2.49}$$

$$\frac{2r\Lambda + g'(r)}{2rg(r)} = 2sk(-2kE^2)^{s-1}\frac{E^2(r)}{g(r)} + \frac{(-2kE^2)^s}{2g(r)}, \tag{2.50}$$

$$2\Lambda + g''(r) = (-2kE^2)^s. \tag{2.51}$$

The first two differential equations are the same which can be seen easily and the third one is related to the first two equations by an integral. This equivalence can be seen easily for $s = 1$. In this case equation (2.49) reduces to

$$2r\Lambda + g'(r) = 2rkE^2, \tag{2.52}$$

and equation (2.51) becomes

$$2\Lambda + g''(r) = -2kE^2. \tag{2.53}$$

After writing left hand side of equation (2.53) as a total derivative and using $E = \frac{q}{r}$ one can easily integrate equation (2.53) and get

$$2r\Lambda + g'(r) = 2rkE^2, \tag{2.54}$$

which is the same equation with (2.52). Therefore we have just one independent equation. The unknown function $g(r)$ can be found separately for $s = 1$ and $s \neq 1$.

When $s = 1$ equation (2.49) reduces to

$$g'(r) = -2r\Lambda + 2rk\frac{q^2}{r^2}. \tag{2.55}$$

We can easily integrate equation (2.55) and get

$$g(r) + m = \frac{r^2}{l^2} + 2kq^2 ln\left(\frac{r}{l}\right), \tag{2.56}$$

and for $k = -1$

$$g(r) = \frac{r^2}{l^2} - m - 2q^2 ln\left(\frac{r}{l}\right), \tag{2.57}$$

which is the solution of Einstein-Maxwell theory in 3 dimensions and known as charged BTZ black hole solution, in which m and q are the mass and electric charge respectively.

For $s \neq 1$ and $k = -1$, equation (2.49) takes the following form

$$\frac{2r\Lambda + g'(r)}{2r} = -2s(2E^2)^{s-1}E^2 + \frac{(2E^2)^s}{2}. \tag{2.58}$$

The integration of (2.58) results in

$$g(r) = \frac{r^2}{l^2} - \mathrm{m} - (2s-1)^2 \left(\frac{8q^2(s-1)^2}{(2s-1)^2}\right)^s \frac{r^{\frac{2(s-1)}{2s-1}}}{2(s-1)}. \tag{2.59}$$

When $s = \frac{3}{4}$ is inserted in equation (2.59) the solution reduces to the well-known metric which is called conformally invariant Maxwell solution,

$$g(r) = \frac{r^2}{l^2} - \mathrm{m} - \left(2\frac{3}{4} - 1\right)^2 \left(\frac{8q^2\left(\frac{3}{4}-1\right)^2}{\left(2\frac{3}{4}-1\right)^2}\right)^{\frac{3}{4}} \frac{r^{\frac{2\left(\frac{3}{4}-1\right)}{2\frac{3}{4}-1}}}{2\left(\frac{3}{4}-1\right)}, \tag{2.60}$$

$$g(r) = \frac{r^2}{l^2} - \mathrm{m} - \frac{1}{2}(2q^2)^{\frac{3}{4}} r^{-1}. \tag{2.61}$$

Defining $(2q^2)^{\frac{3}{4}} \equiv K$ one can write the equation (2.61) in its simplest form

$$g(r) = \frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}. \tag{2.62}$$

Finally, the solution of conformally invariant Maxwell gravity can be written as

$$ds^2 = -\left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)dt^2 + \frac{dr^2}{\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}} + r^2 d\phi^2. \tag{2.63}$$

From now on, we are going to find the geodesic equations of (2.63)

2.3 Geodesic Equations of Conformally Invariant Maxwell Gravity

Now, we can find the geodesic equations and constants of motion. The geodesic equations can be determined with

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0, \tag{2.64}$$

where $d\lambda^2 = g_{\mu\nu}dx^\mu dx^\nu$ is the proper time and $\Gamma^\mu_{\rho\sigma}$ is the Christoffel connections given by

$$\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\nu}\left(\partial_{\rho} g_{\sigma\nu} + \partial_{\sigma} g_{\rho\nu} - \partial_{\nu} g_{\rho\sigma}\right). \quad (2.65)$$

Although, the constants of motion can be found by equation (2.64), we will use a simpler method by which the constants of motion are given by the following Lagrangian

$$L = \frac{1}{2}\sum_{\mu,\nu=0}^{3} g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = \frac{1}{2}\epsilon$$

$$= \frac{1}{2}\left[-\left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \frac{1}{\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}}\left(\frac{dr}{d\lambda}\right)^2 \right. \quad (2.66)$$

$$\left. + r^2\left(\frac{d\phi}{d\lambda}\right)^2\right],$$

where ϵ takes -1 for massive particles and 0 for massless particles and λ is an affine parameter.

Now, we can find the constants of motion by using Euler-Lagrange equations

$$P_t = \frac{\partial L}{\partial \dot{t}} = -\left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)\dot{t} = -E, \quad (2.67)$$

and equation (2.67) can be written for $\dot{t}$

$$\dot{t} = \frac{E}{\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}}, \quad (2.68)$$

where '.' represents derivative with respect to the affine parameter and E stands for the energy of the particle.

The second Euler-Lagrange equation is

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = r^2\dot{\phi} = \mathcal{L}, \tag{2.69}$$

where $\mathcal{L}$ is the angular momentum of the particle and again equation (2.69) can be written for $\dot{\phi}$

$$\dot{\phi} = \frac{\mathcal{L}}{r^2}. \tag{2.70}$$

By using these constants of motion we can obtain the geodesic equations as follows

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{\epsilon r^2}{l^2} - m\epsilon - \frac{K\epsilon}{2r} + E^2 - \frac{\mathcal{L}^2}{l^2} + \frac{m\mathcal{L}^2}{r^2} + \frac{K\mathcal{L}^2}{2r^3}, \tag{2.71}$$

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\epsilon r^6}{l^2\mathcal{L}^2} - \frac{m\epsilon r^4}{\mathcal{L}^2} - \frac{K\epsilon r^3}{2\mathcal{L}^2} + \frac{E^2 r^4}{\mathcal{L}^2} - \frac{r^4}{l^2} + mr^2 + \frac{Kr}{2} = R(r), \tag{2.72}$$

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)^2 + \frac{\epsilon\left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)^3}{E^2} - \frac{\mathcal{L}^2\left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)^3}{E^2 r^2}. \tag{2.73}$$

This set of equations determines the trajectory of the particle around the black hole (2.63).

By using equation (2.77) we can find the effective potential as

$$V_{eff} = -\frac{\epsilon r^2}{l^2} + m\epsilon + \frac{K\epsilon}{2r} + \frac{\mathcal{L}^2}{l^2} - \frac{m\mathcal{L}^2}{r^2} - \frac{K\mathcal{L}^2}{2r^3}. \tag{2.74}$$

For simplicity we define some dimensionless parameters

$$\tilde{r} = \frac{r}{m}, \qquad \tilde{l} = \frac{l}{m}, \qquad \widetilde{K} = \frac{K}{m}, \qquad \tilde{\mathcal{L}} = \frac{m^2}{\mathcal{L}^2}, \tag{2.75}$$

and rewrite equation (2.72) in the following form

$$\left(\frac{d\tilde{r}}{d\phi}\right)^2 = \frac{\epsilon\tilde{r}^6\tilde{\mathcal{L}}}{\tilde{l}^2} - m\epsilon\tilde{r}^4\tilde{\mathcal{L}} - \frac{\widetilde{K}\epsilon\tilde{r}^3\tilde{\mathcal{L}}}{2} + E^2\tilde{r}^4\tilde{\mathcal{L}} - \frac{\tilde{r}^4}{\tilde{l}^2} + m\tilde{r}^2 + \frac{\widetilde{K}\tilde{r}}{2} = R(\tilde{r}). \tag{2.76}$$

2.4 Possible Regions for Geodesic Motion

Equation (2.76) shows us that there is a condition for the existence of a geodesic which is $R(\tilde{r}) \geq 0$. The real positive zeros of $R(\tilde{r})$ are extremal values of the geodesic motion. Since $\tilde{r} = 0$ is a zero of this polynomial for all values of the parameters we can neglect it. So our equation reduces to a polynomial of degree 5 from 6

$$R^*(\tilde{r}) = \frac{\epsilon \tilde{r}^5 \tilde{\mathcal{L}}}{\tilde{l}^2} - m\epsilon \tilde{r}^3 \tilde{\mathcal{L}} - \frac{\tilde{K}\epsilon \tilde{r}^2 \tilde{\mathcal{L}}}{2} + E^2 \tilde{r}^3 \tilde{\mathcal{L}} - \frac{\tilde{r}^3}{\tilde{l}^2} + m\tilde{r} + \frac{\tilde{K}}{2}. \tag{2.77}$$

By using analytical solutions we can obtain possible orbits which depend on the parameters of the particle ϵ, l, E^2, K and $\mathcal{L}$.

Solving $R^*(\tilde{r}) = 0$ and $\frac{dR^*(\tilde{r})}{d\tilde{r}} = 0$ we can find E^2 and $\tilde{\mathcal{L}}$.

For massive particles, we take $\epsilon = -1$ and get equations for the angular momentum and energy as follows

$$\tilde{\mathcal{L}} = -\frac{\tilde{l}^2\left(4m\tilde{r} + 3\tilde{K}\right)}{\tilde{r}^2\left(\tilde{K}\tilde{l}^2 + 4\tilde{r}^3\right)}, \tag{2.78}$$

$$E^2 = -\frac{4\tilde{l}^4 m^2 \tilde{r}^2 + 4\tilde{K}\tilde{l}^4 m\tilde{r} - 8\tilde{l}^2 m\tilde{r}^4 + \tilde{K}^2\tilde{l}^4 - 4\tilde{K}\tilde{l}^2\tilde{r}^3 + 4\tilde{r}^6}{\tilde{l}^4\tilde{r}\left(4m\tilde{r} + 3\tilde{K}\right)}. \tag{2.79}$$

For massless particles, we take $\epsilon = 0$ and we get

$$\tilde{\mathcal{L}} = \left(-\frac{64m^3}{27\tilde{K}^2} + \frac{1}{\tilde{l}^2}\right)\frac{1}{E^2}. \tag{2.80}$$

Chapter 3

ANALYTICAL SOLUTIONS OF BTZ

This part is devoted to the analytical solutions of the geodesic equations. First, we introduce a new parameter that is $u = \frac{1}{\tilde{r}}$ and use it in order to simplify equation (2.76) as

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{\tilde{r}^4}\left(\frac{d\tilde{r}}{d\phi}\right)^2, \tag{3.1}$$

$$\left(\frac{du}{d\phi}\right)^2 = \frac{\epsilon\tilde{\mathcal{L}}}{u^2\tilde{l}^2} - m\epsilon\tilde{\mathcal{L}} - \frac{u\tilde{K}\epsilon\tilde{\mathcal{L}}}{2} + E^2\tilde{\mathcal{L}} - \frac{1}{\tilde{l}^2} + u^2 m + \frac{u^3\tilde{K}}{2}. \tag{3.2}$$

For massless particles, $\epsilon = 0$, equation (3.2) reduces to the following form

$$\left(\frac{du}{d\phi}\right)^2 = E^2\tilde{\mathcal{L}} - \frac{1}{\tilde{l}^2} + u^2 m + \frac{u^3\tilde{K}}{2} = P_3(u) = \sum_{i=0}^{3} a_i u^i, \tag{3.3}$$

which is an elliptic type function. To get a Weierstrass form function we use another substitution which is

$$u = \frac{1}{a_3}\left(4y - \frac{a_2}{3}\right) = \frac{2}{\tilde{K}}\left(4y - \frac{m}{3}\right), \tag{3.4}$$

then equation (3.3) becomes

$$\left(\frac{dy}{d\phi}\right)^2 = 4y^3 - \alpha y - \gamma = P_3(y), \tag{3.5}$$

where

$\alpha = \frac{{a_2}^2}{12} - \frac{a_1 a_3}{4} = \frac{m^2}{12}$ and $\gamma = \frac{a_1 a_2 a_3}{48} - \frac{a_0 {a_3}^2}{16} - \frac{{a_2}^3}{216} = -\frac{(E^2\tilde{\mathcal{L}}\tilde{l}^2 - 1)\tilde{K}^2}{64\tilde{l}^2} - \frac{m^3}{216}$

are Weierstrass constants. Equation (3.5) is an elliptic type function whose solution is given by Weierstrass function [21]

$$y(\phi) = \wp(\phi - \phi_{in}; \alpha, \gamma), \tag{3.6}$$

where

$$\phi_{in} = \phi_0 + \int_{y_0}^{\infty} \frac{dy}{\sqrt{4y^3 - \alpha y - \gamma}}, \tag{3.7}$$

and

$$y_0 = \frac{1}{4}\left(\frac{a_3}{\tilde{r}_0} - \frac{a_2}{3}\right) = \frac{\widetilde{K}}{8\tilde{r}_0} + \frac{m}{12}. \tag{3.8}$$

Then the analytical solution of equation (2.82) is

$$\tilde{r}(\phi) = \frac{a_3}{4\wp(\phi - \phi_{in}; \alpha, \gamma) - \frac{a_2}{3}} = \frac{\widetilde{K}}{2\left[4\wp(\phi - \phi_{in}; \alpha, \gamma) - \frac{m}{3}\right]}. \tag{3.9}$$

For massive particles, $\epsilon = -1$, equation (3.2) can be written as follows

$$\left(u\frac{du}{d\phi}\right)^2 = -\frac{\tilde{\mathcal{L}}}{\tilde{l}^2} + u^2 m\tilde{\mathcal{L}} + \frac{u^3\widetilde{K}\tilde{\mathcal{L}}}{2} + u^2E^2\tilde{\mathcal{L}} - \frac{u^2}{\tilde{l}^2} + u^4 m + \frac{u^5\widetilde{K}}{2} = P_5(u)$$
$$= \sum_{i=0}^{5} a_i u^i, \tag{3.10}$$

which is a polynomial with a degree of 5 and its analytical solution is [22]

$$u(\phi) = -\frac{\sigma_1}{\sigma_2}(\phi_\sigma), \tag{3.11}$$

where σ_i is the i-th derivative of the Kleinian sigma function which is

$$\sigma(z) = Ce^{-\frac{1}{2}z^t\eta\omega^{-1}z}\theta[g,h]((2\omega)^{-1}z;\tau), \tag{3.12}$$

here $\tau = \omega^{-1}\acute{\omega}$ is the symmetric Riemann matrix and $\theta[g,h]$ is the Riemann theta-function

$$\theta[g,h](z;t) = \sum e^{i\pi(m+g)^t(\tau(m+g)+2z+2h)}. \tag{3.13}$$

Then the analytical solution of $\tilde{r}$ is

$$\tilde{r} = -\frac{\sigma_2}{\sigma_1}(\phi_\sigma). \tag{3.14}$$

Chapter 4

METRIC AND GEODESICS OF CIRCULARLY SYMMETRIC BLACK HOLE IN MMG

4.1 Topological Massive Gravity

The action of cosmological TMG is [23]

$$I_{TMG} = \int d^3x\sqrt{-g}(R - 2\Lambda_0) + I_{GCS}, \tag{4.1}$$

where Λ_0 is the bare cosmological constant and I_{GCS} is the gravitational Chern-Simons action

$$I_{GCS} = \frac{1}{2\mu}\int d^3x\sqrt{-g}\epsilon^{\sigma\mu\nu}\Gamma^{\rho}{}_{\sigma\tau}\left(\partial_\mu\Gamma^{\tau}{}_{\rho\nu} + \frac{2}{3}\Gamma^{\tau}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\rho}\right), \tag{4.2}$$

where μ is a mass parameter.

Varying the action (4.1) with respect to the metric tensor $g_{\mu\nu}$ we can obtain the source-free equations of motion

$$G_{\mu\nu} + \Lambda_0 g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0, \tag{4.3}$$

where $C_{\mu\nu}$ is the symmetric-traceless Cotton-York tensor which can be defined in terms of Schouten tensor $S_{\sigma\nu}$ as

$$C^{\mu}{}_{\nu} = \epsilon^{\mu\rho\sigma}\nabla_\rho S_{\sigma\nu}\,, \qquad S_{\sigma\nu} = R_{\sigma\nu} - \frac{1}{4}Rg_{\sigma\nu}\,, \qquad S = g^{\mu\nu}S_{\mu\nu} = \frac{R}{4}, \tag{4.4}$$

where $\epsilon^{\mu\rho\sigma}$ is the Levi-Civita pseudo tensor defined as $\epsilon_{\mu\rho\sigma} = \sqrt{-g}\varepsilon_{\mu\rho\sigma}$ and the convention $\varepsilon_{012} = +1$.

For the following computations we need to find the components of Cotton-York tensor

$$g_{\mu\alpha}C^{\mu}{}_{\nu} = g_{\mu\alpha}\epsilon^{\mu\rho\sigma}\nabla_{\rho}S_{\sigma\nu}, \tag{4.5}$$

$$C_{\alpha\nu} = g_{\mu\alpha}\epsilon^{\mu\rho\sigma}\nabla_{\rho}S_{\sigma\nu}, \tag{4.6}$$

$$C_{00} = C_{11} = C_{22} = 0, \tag{4.7}$$

$$C_{01} = C_{10} = C_{12} = C_{21} = 0, \tag{4.8}$$

and the non-vanishing terms are

$$C_{02} = g_{00}\epsilon^{0\sigma\rho}\nabla_{\rho}S_{\sigma 2}, \tag{4.9}$$

$$C_{02} = \frac{g_{00}}{\sqrt{-g}}(\nabla_1 S_{22} - \nabla_2 S_{12}), \tag{4.10}$$

$$C_{20} = g_{22}\epsilon^{2\sigma\rho}\nabla_{\rho}S_{\sigma 0}, \tag{4.11}$$

$$C_{20} = \frac{g_{22}}{\sqrt{-g}}(\nabla_1 S_{02} - \nabla_0 S_{12}), \tag{4.12}$$

Then,

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{g_{00}}{\sqrt{-g}}(\nabla_1 S_{22} - \nabla_2 S_{12}) \\ 0 & 0 & 0 \\ \frac{g_{22}}{\sqrt{-g}}(\nabla_1 S_{02} - \nabla_0 S_{12}) & 0 & 0 \end{pmatrix}. \tag{4.13}$$

4.2 Minimal Massive Gravity

The field equation of source-free MMG is

$$E_{\mu\nu} = \bar{\sigma}G_{\mu\nu} + \bar{\Lambda}_0 g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} + \frac{\gamma}{\mu^2}J_{\mu\nu} = 0, \tag{4.14}$$

where $\bar{\sigma}$ and γ are dimensionless parameters and $\bar{\Lambda}_0$ is the bare cosmological constant.

In equation (4.14) the symmetric curvature-squared tensor $J_{\mu\nu}$ is

$$J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\rho\sigma}\epsilon^{\nu\tau\eta}S_{\rho\tau}S_{\sigma\eta} = SS^{\mu\nu} - S^{\mu\rho}S^{\nu}{}_{\rho} + \frac{1}{2}g^{\mu\nu}\left(S^{\rho\sigma}S_{\rho\sigma} - S^2\right), \tag{4.15}$$

and the trace of $J^{\mu\nu}$ is

$$J = g_{\mu\nu} J^{\mu\nu} = \frac{1}{2}\left(S^{\rho\sigma} S_{\rho\sigma} - S^2\right). \tag{4.16}$$

4.3 The Merger Points of MMG

For an Einstein space $R_{\mu\nu} = 2\Lambda g_{\mu\nu}$ where Λ is the effective cosmological constant, the Schouten tensor becomes

$$S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}, \tag{4.17}$$

$$S_{\mu\nu} = \frac{\Lambda}{2} g_{\mu\nu}, \tag{4.18}$$

$$S = \frac{3\Lambda}{2}. \tag{4.19}$$

The tensor field $J_{\mu\nu}$ takes the following form

$$J_{\mu\nu} = S S_{\mu\nu} - S_{\mu}{}^{\rho} S_{\nu\rho} + \frac{1}{2} g_{\mu\nu}\left(S^{\rho\sigma} S_{\rho\sigma} - S^2\right), \tag{4.20}$$

$$J_{\mu\nu} = -\frac{\Lambda^2}{4} g_{\mu\nu}, \tag{4.21}$$

and the trace of $J_{\mu\nu}$ can be written as

$$J = -\frac{3\Lambda^2}{4}. \tag{4.22}$$

The Einstein tensor becomes

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{4.23}$$

$$G_{\mu\nu} = 2\Lambda g_{\mu\nu} - 3\Lambda g_{\mu\nu} = -\Lambda g_{\mu\nu}. \tag{4.24}$$

Now we can write the field equation of source-free MMG in the Einstein space as

$$g_{\mu\nu}\left(-\bar{\sigma}\Lambda + \bar{\Lambda}_0 - \frac{\gamma}{\mu^2}\frac{\Lambda^2}{4}\right) = 0. \tag{4.25}$$

Then,

$$-\bar{\sigma}\Lambda + \bar{\Lambda}_0 - \frac{\gamma}{\mu^2}\frac{\Lambda^2}{4} = 0. \tag{4.26}$$

Since we want to solve this equation for the effective cosmological constant Λ we can write this equation as

$$\Lambda^2 + \frac{4\mu^2\bar{\sigma}}{\gamma}\Lambda - \frac{4\mu^2}{\gamma}\bar{\Lambda}_0 = 0. \quad (4.27)$$

We have two possible values for Λ

$$\Lambda_{\pm} = \frac{-\frac{4\mu^2\bar{\sigma}}{\gamma} \pm \sqrt{\left(\frac{4\mu^2\bar{\sigma}}{\gamma}\right)^2 + \frac{16\mu^2\bar{\sigma}}{\gamma}\bar{\Lambda}_0}}{2}, \quad (4.28)$$

$$\Lambda_{\pm} = -\frac{2\mu^2\bar{\sigma}}{\gamma}\left(1 \pm \sqrt{1 + \frac{\gamma}{\mu^2\bar{\sigma}^2}\bar{\Lambda}_0}\right). \quad (4.29)$$

The reality condition of the cosmological constant provide the inequality $\mu^2\bar{\sigma}^2 + \gamma\bar{\Lambda}_0 \geq 0$.

In order to get the merger points we equate the roots of the equation (4.27) which are given in equation (4.29) and get

$$\mu^2\bar{\sigma}^2 + \gamma\bar{\Lambda}_0 = 0, \quad (4.30)$$

by which the bare cosmological constant can be determined by the parameters of the theory as follows

$$\bar{\Lambda}_0 = -\frac{\mu^2\bar{\sigma}^2}{\gamma}. \quad (4.31)$$

Using equation (4.31) in equation (4.29) the effective cosmological constant can be written in terms of the parameters

$$\Lambda = -\frac{2\mu^2\bar{\sigma}}{\gamma}, \quad (4.32)$$

and finally

$$\Lambda\bar{\sigma} = 2\bar{\Lambda}_0. \quad (4.33)$$

Therefore, the merger point of MMG is given by equation (4.31),(4.32) and (4.33)

4.4 Static Circularly Symmetric Solutions

Let us take the most general static circularly symmetric metric ansatz which is

$$ds^2 = -u(r)dt^2 + \frac{dr^2}{v(r)} + r^2 d\theta^2. \tag{4.34}$$

Now, we are going to find all metric forms of this metric ansatz which satisfy $E_{\mu\nu} = 0$ at the merger points. As we showed in equation (4.13), all components of Cotton tensor are vanishing except $C_{t\theta}$. Also only the diagonal components of the remaining terms in $E_{\mu\nu}$ are non-zero which can be seen from equation (4.24),(4.21) and (4.14). We are considering the combination of $E_{rr} = 0$ and $E_{\theta\theta} = 0$ which is $E^r{}_r + E^\theta{}_\theta = 0$. By use of equation (4.34) in this combination one arrives the following differential equation

$$(2\Lambda r + v')(2u^2(2\Lambda r - v') + ru'(uv' - vu') + 2uv(ru')') = 0, \tag{4.35}$$

where the prime '$'$' refers to differentiation with respect to r.

One can solve the equation (4.35) by considering the vanishing of each term separately

$$2\Lambda r + v' = 0, \tag{4.36}$$

and

$$2u^2(2\Lambda r - v') + ru'(uv' - vu') + 2uv(ru')' = 0. \tag{4.37}$$

Integrating equation (4.36) gives us

$$v(r) = v_0 - \Lambda r^2, \tag{4.38}$$

where v_0 is the integration constant.

In order to find $u(r)$ we substitute equation (4.38) in $E_{tt} = 0$ equation and find

$$r(v_0 - \Lambda r^2)((u')^2 - 2uu'') + 2v_0 uu' = 0. \tag{4.39}$$

The solution of the equation (4.39) gives us

$$u(r) = u_2\left(\sqrt{v_0 - \Lambda r^2} - u_1\right)^2, \tag{4.40}$$

where u_1 and u_2 are integration constants.

After the renaming of the integration constants and rescaling of the t-coordinate we can write the metric ansatz as

$$ds_1^2 = \Lambda\left(-r_1 + \sqrt{r^2 - r_0}\right)^2 dt^2 - \frac{dr^2}{\Lambda(r^2 - r_0)} + r^2 d\theta^2. \tag{4.41}$$

By solving the equation (4.37) for u'' and substituting back into $E_{tt} = 0$ equation we can find the following

$$\left(\frac{u}{v}\right)'(2\Lambda r u + v u') = 0. \tag{4.42}$$

Again the solution of the equation (4.42) bifurcates into two cases, which are the vanishing of the terms in equation (4.42) separately. The first one is

$$\left(\frac{u}{v}\right)' = 0, \tag{4.43}$$

and the second one is

$$2\Lambda r u + v u' = 0. \tag{4.44}$$

In order to solve the equation (4.43) we set $u(r) = v(r)$. Then equation (4.37) gives us a simple equation for u'' as $u'' = -2\Lambda$ and integrating this twice we get

$$u(r) = v(r) = -\Lambda r^2 + u_1 r + u_2, \tag{4.45}$$

where u_1 and u_2 are integration constants.

After the renaming of the integration constants we can write the metric ansatz as

$$ds_2^2 = \Lambda(r - r_-)(r - r_+)dt^2 - \frac{dr^2}{\Lambda(r - r_-)(r - r_+)} + r^2 d\theta^2. \tag{4.46}$$

Equation (4.44) gives us a constant on u' as $u' = -\frac{2\Lambda r u}{v}$. Inserting $u' = -\frac{2\Lambda r u}{v}$ into the equation (4.37) on arrives at

$$(v - \Lambda r^2)(2\Lambda r + v') = 0. \tag{4.47}$$

This equation has two solutions however the vanishing of the second term in equation (4.47) does not yield a new solution. On the other hand, the vanishing of the first term in equation (4.47) determines $v = \Lambda r^2$ and inserting $v = \Lambda r^2$ into the equation (4.44) one arrives at

$$u(r) = \frac{u_0}{r^2}, \tag{4.48}$$

where u_0 is the integration constant.

Taking $u_0 = 1$ we can write the metric ansatz as

$$ds_L^2 = -\frac{dt^2}{r^2} + \frac{dr^2}{\Lambda r^2} + r^2 d\theta^2, \tag{4.49}$$

which is the static Lifshitz spacetime [24]. However, the field equation of MMG (4.14) is not satisfied by (4.47).

We are going to study the geodesic equation of a particle around the black hole whose geometry is determined by the metric (4.46) which is well known.

Now, we can find the geodesic equations and constants of motion. The geodesic equation is by the following equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0, \tag{4.50}$$

where $d\lambda^2 = g_{\mu\nu} dx^\mu dx^\nu$ is the proper time and $\Gamma^\mu_{\rho\sigma}$ is the Christoffel connections given as

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu} \left(\partial_\rho g_{\sigma\nu} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma} \right). \tag{4.51}$$

We can obtain the geodesic equations by using Lagrangian method

$$L = \frac{1}{2}\sum_{\mu,\nu=0}^{3} g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda} = \frac{1}{2}\epsilon$$

$$= \frac{1}{2}\left[\Lambda(r-r_-)(r-r_+)\left(\frac{dt}{d\lambda}\right)^2 - \frac{1}{\Lambda(r-r_-)(r-r_+)}\left(\frac{dr}{d\lambda}\right)^2 \right. \tag{4.52}$$

$$\left. + r^2\left(\frac{d\theta}{d\lambda}\right)^2\right],$$

where ϵ is -1 for massive particles and 0 for massless particles and λ is an affine parameter.

Now, we can find the constants of motion by using Euler-Lagrange equation

$$P_t = \frac{\partial L}{\partial \dot{t}} = \Lambda(r-r_-)(r-r_+)\dot{t} = -E, \tag{4.53}$$

$$\dot{t} = -\frac{E}{\Lambda(r-r_-)(r-r_+)}, \tag{4.54}$$

$$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = r^2\dot{\theta} = \mathcal{L}, \tag{4.55}$$

$$\dot{\theta} = \frac{\mathcal{L}}{r^2}, \tag{4.56}$$

where E is energy and $\mathcal{L}$ is angular momentum of the particle.

By using these constants we can obtain the geodesic equations as follows

$$\left(\frac{dr}{d\lambda}\right)^2 = -\Lambda(r-r_-)(r-r_+)\epsilon + E^2 + \frac{\Lambda(r-r_-)(r-r_+)\mathcal{L}^2}{r^2}, \tag{4.57}$$

$$\left(\frac{dr}{d\theta}\right)^2 = -\Lambda(r-r_-)(r-r_+)\epsilon\frac{r^4}{\mathcal{L}^2} + \frac{E^2r^4}{\mathcal{L}^2} + \Lambda(r-r_-)(r-r_+)r^2, \tag{4.58}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{\left(-\Lambda(r-r_-)(r-r_+)\right)^3 \epsilon}{E^2} + \left(\Lambda(r-r_-)(r-r_+)\right)^2 + \frac{\left(\Lambda(r-r_-)(r-r_+)\right)^3 \mathcal{L}^2}{r^2 E^2}. \tag{4.59}$$

By using equation (4.57) we can find the effective potential and the effective energy as

$$V_{eff} = -\frac{1}{2}\Lambda(r-r_-)(r-r_+)\left(-\epsilon + \frac{\mathcal{L}^2}{r^2}\right), \tag{4.60}$$

$$\xi_{eff} = \frac{1}{2}\mathrm{E}^2. \tag{4.61}$$

Chapter 5

ANALYTICAL SOLUTIONS OF CIRCULARLY SYMMETRIC BLACK HOLE IN MMG

In this part we analyze the geodesic equations (4.57),(4.58) and (4.59) for massless particles for which ϵ is set to zero.

The first term in equation (4.58) vanishes for the massless particle condition and it reduces to

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{E^2 r^4}{\mathcal{L}^2} + \Lambda(r - r_-)(r - r_+)r^2. \tag{5.1}$$

Instead of using $\Lambda(r - r_-)(r - r_+)$ we are going to use $\Lambda\mathrm{r}^2 - br + \mu$ where b is the gravitational hair parameter and μ is related with the mass of the black hole.

Now, our equation becomes

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{E^2 r^4}{\mathcal{L}^2} + (\Lambda\mathrm{r}^2 - br + \mu)r^2. \tag{5.2}$$

Then the analytical solution of this equation is [24]

$$r_I(\theta) = \frac{2\mu}{b + 2\mu\kappa_I \sinh\left(\pm\sqrt{\mu}\theta + \beta\right)}, \left(Type\ I: E^2 > {V_{max}}^2\right), \tag{5.3}$$

$$r_{II}(\theta) = \frac{2\mu}{b + 2\mu\kappa_{II} \cosh\left(\pm\sqrt{\mu}\theta + \beta\right)}, \left(Type\ II: E^2 < {V_{max}}^2, r_0 < r_a\right), \tag{5.4}$$

$$r_{III}(\theta) = \frac{2\mu}{b - 2\mu\kappa_{II} \cosh\left(\pm\sqrt{\mu}\theta + \beta\right)}, \left(Type\ III: E^2 < {V_{max}}^2, r_0 > r_a\right), \tag{5.5}$$

where β is the integration constant, r_0 is the initial location of the particle, r_a is the location of the particle where it has maximum potential $r_a = \frac{2\mu}{b}$.

Also ${\kappa_I}^2 = (4\mu/\bar{D}^2 - b^2)/4\mu^2$ and ${\kappa_{II}}^2 = (b^2 - 4\mu/\bar{D}^2)/4\mu^2$

where $\bar{D}$ is the effective impact parameter and D is the impact parameter

$$D = \frac{\mathcal{L}}{E}, \tag{5.6}$$

which is connected to the effective impact parameter by the following equation

$$\bar{D}^2 = \frac{D^2}{1 + D^2\Lambda}. \tag{5.7}$$

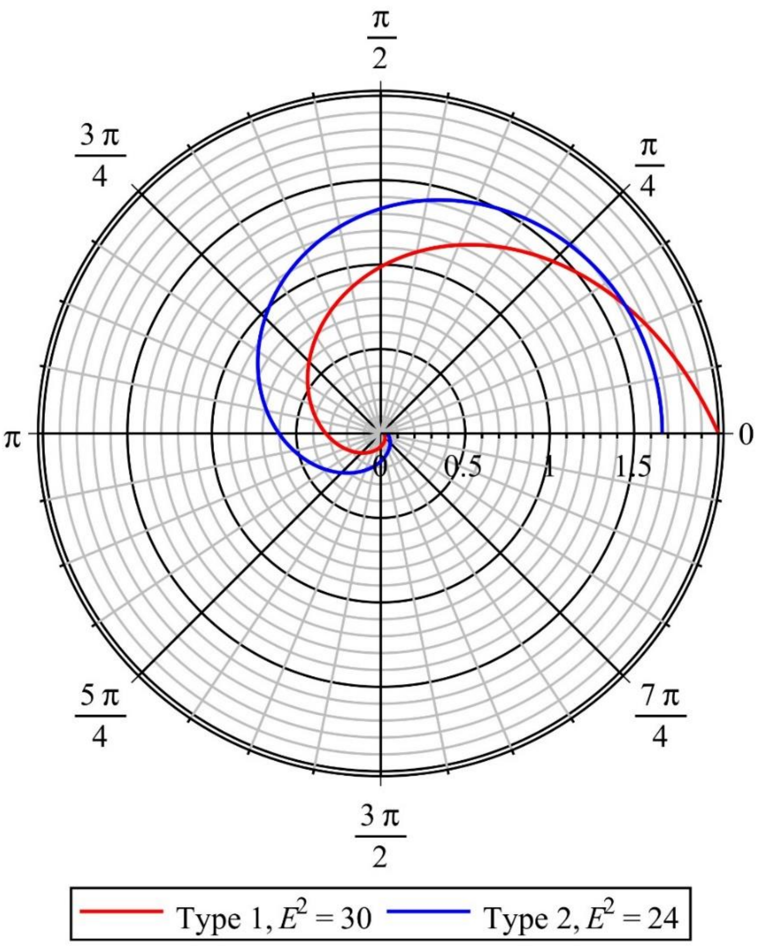

Figure 1: Behaviours of the geodesics for Type I and Type II

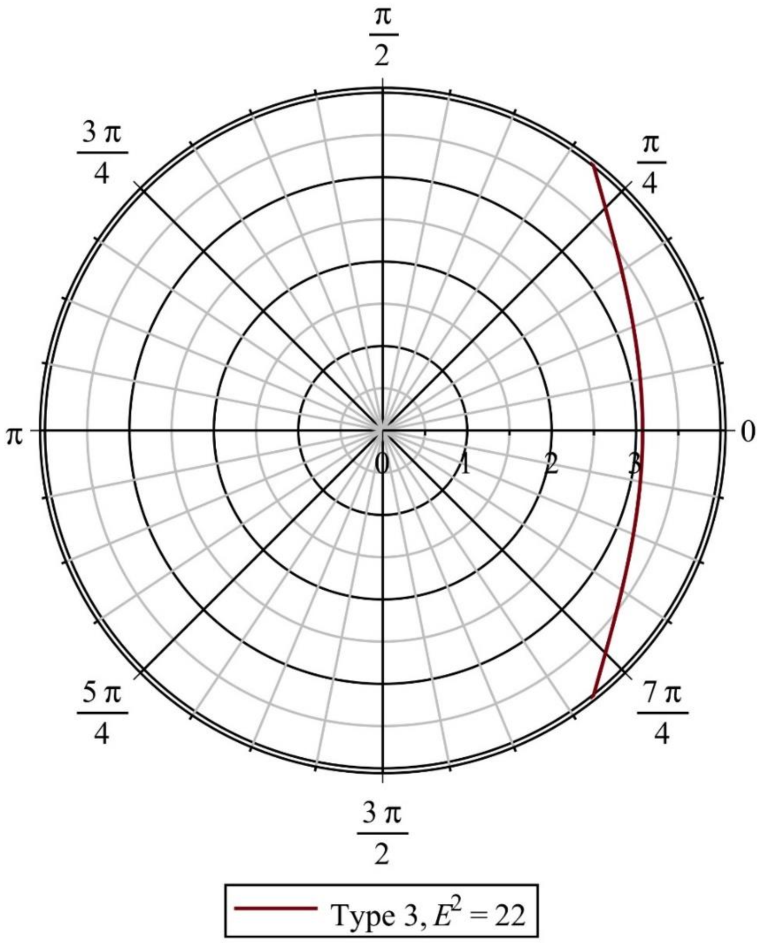

Figure 2: Behaviours of the geodesics for Type III

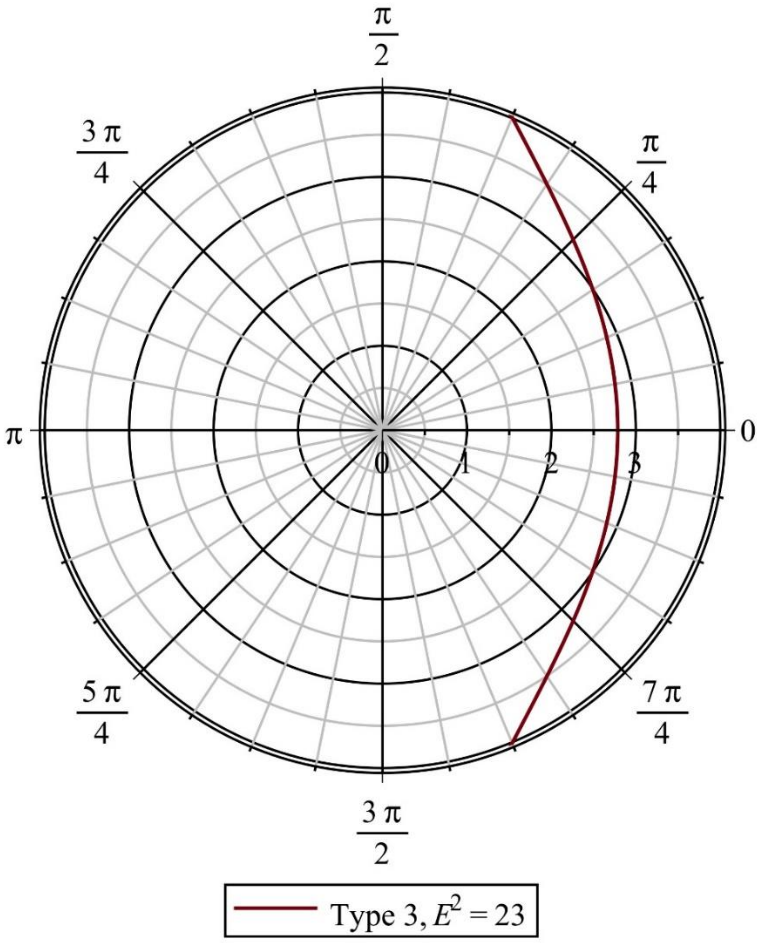

Figure 3: Behaviours of the geodesics for Type III

Chapter 6

CONCLUSION

In this thesis, our main aim is to observe the radial motions of massive and massless particles on the geodesics of both charged BTZ black hole and circularly symmetric solution of MMG theory. To achieve this aim, first we introduced the metrics then we found the geodesic equations by using Lagrangian method. Then we solved these geodesic equations to get analytical solutions. For charged BTZ black hole we got both null and time like geodesics for massless and massive particles respectively, but for circularly symmetric solution of MMG we got only null geodesics. For charged BTZ black hole we solved null geodesics in terms of Weierstrass elliptic function and time like geodesics in terms of Kleinian sigma hyper-elliptic function. Also at the end, we did some numerical simulations to plot graphs for displaying the geodesics.

I also planned that to extend my studies. In near future I want to study the higher dimensions and rotating versions of the space time. And also I want to concentrate more on less studied metrics.

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APPENDICES

Appendix A: Calculations of Some Useful Tensors

Here, I want to show the steps between the equations (2.23) and (2.24)

$$F_0{}^0 = F_1{}^1 = F_2{}^2 = 0 \tag{A1}$$

$$F_2{}^0 = F_0{}^2 = F_1{}^2 = F_2{}^1 = 0 \tag{A2}$$

and the non-vanishing terms are

$$F_1{}^0 = g^{0\beta} F_{1\beta} \tag{A3}$$

$$F_1{}^0 = g^{00} F_{10} + g^{01} F_{11} + g^{02} F_{12} = -\frac{E(r)}{g(r)} \tag{A4}$$

$$F_0{}^1 = g^{1\beta} F_{0\beta} \tag{A5}$$

$$F_0{}^1 = g^{10} F_{00} + g^{11} F_{01} + g^{12} F_{02} = -g(r)E(r) \tag{A6}$$

The steps between equations (2.28) and (2.29) are shown below

$$\mathcal{A}_{22} = \mathcal{A}_{01} = \mathcal{A}_{02} = \mathcal{A}_{10} = \mathcal{A}_{12} = \mathcal{A}_{20} = \mathcal{A}_{21} = 0 \tag{A7}$$

$$\mathcal{A}_{00} = F_{00} F_0{}^0 + F_{01} F_0{}^1 + F_{02} F_0{}^2 = g(r) E^2(r) \tag{A8}$$

$$\mathcal{A}_{11} = F_{10} F_1{}^0 + F_{11} F_1{}^1 + F_{12} F_1{}^2 = -\frac{E^2(r)}{g(r)} \tag{A9}$$

Appendix B: Calculations to Get the Metric of BTZ

Here, I want to show the steps between the equations (2.32) and (2.33)

$$\delta(k\mathcal{F})^s = sk(k\mathcal{F})^{s-1}\delta(g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}) \tag{B1}$$

$$\delta(k\mathcal{F})^s = sk(k\mathcal{F})^{s-1}[\delta g^{\mu\alpha}F_{\mu\nu}F_{\alpha}{}^{\nu} + \delta g^{\nu\beta}F_{\mu\nu}F^{\mu}{}_{\beta}] \tag{B2}$$

$$\delta(k\mathcal{F})^s = sk(k\mathcal{F})^{s-1}[\delta g^{\mu\nu}F_{\mu\alpha}F_{\nu}{}^{\alpha} + \delta g^{\nu\mu}F_{\beta\nu}F^{\beta}{}_{\mu}] \tag{B3}$$

$$\delta(k\mathcal{F})^s = \delta g^{\mu\nu}sk(k\mathcal{F})^{s-1}[F_{\mu\alpha}F_{\nu}{}^{\alpha} + F_{\nu\alpha}F_{\mu}{}^{\alpha}] \tag{B4}$$

$$\delta(k\mathcal{F})^s = \delta g^{\mu\nu}sk(k\mathcal{F})^{s-1}[F_{\mu\alpha}F_{\nu}{}^{\alpha} + F_{\nu}{}^{\alpha}F_{\mu\alpha}] \tag{B5}$$

The steps between equations (2.39) and (2.40) are shown below

$$\frac{\partial\mathcal{L}}{\partial A_\alpha} = 0 \tag{B6}$$

$$\frac{\partial\mathcal{L}}{\partial\partial_\beta A_\alpha} = \frac{\partial(\sqrt{-g}k\mathcal{F})^s}{\partial\partial_\beta A_\alpha} \tag{B7}$$

$$\frac{\partial\mathcal{L}}{\partial\partial_\beta A_\alpha} = \frac{s(k\mathcal{F})^{s-1}k\partial\mathcal{F}}{\partial\partial_\beta A_\alpha} \tag{B8}$$

$$\frac{\partial\mathcal{L}}{\partial\partial_\beta A_\alpha} = \sqrt{-g}sk(k\mathcal{F})^{s-1}\frac{\partial(F_{\mu\nu}F^{\mu\nu})}{\partial\partial_\beta A_\alpha} \tag{B9}$$

$$\frac{\partial\mathcal{L}}{\partial\partial_\beta A_\alpha} = 2\sqrt{-g}sk(k\mathcal{F})^{s-1}F^{\mu\nu}\frac{\partial(F_{\mu\nu})}{\partial\partial_\beta A_\alpha} \tag{B10}$$

$$\frac{\partial\mathcal{L}}{\partial\partial_\beta A_\alpha} = 2\sqrt{-g}sk(k\mathcal{F})^{s-1}F^{\mu\nu}\frac{\partial(\partial_\mu A_\nu - \partial_\nu A_\mu)}{\partial\partial_\beta A_\alpha} \tag{B11}$$

$$\frac{\partial\mathcal{L}}{\partial\partial_\beta A_\alpha} = 2\sqrt{-g}sk(k\mathcal{F})^{s-1}F^{\mu\nu}(\delta^{\beta}_{\mu}\delta^{\alpha}_{\nu} - \delta^{\beta}_{\nu}\delta^{\alpha}_{\mu}) \tag{B12}$$

$$\frac{\partial\mathcal{L}}{\partial\partial_\beta A_\alpha} = 2\sqrt{-g}sk(k\mathcal{F})^{s-1}(F^{\mu\nu}\delta^{\beta}_{\mu}\delta^{\alpha}_{\nu} - F^{\mu\nu}\delta^{\beta}_{\nu}\delta^{\alpha}_{\mu}) \tag{B13}$$

$$\frac{\partial\mathcal{L}}{\partial\partial_\beta A_\alpha} = 2\sqrt{-g}sk(k\mathcal{F})^{s-1}(F^{\beta\alpha} - F^{\alpha\beta}) \tag{B14}$$

Calculations to get equation (2.51) as follows

$$\partial_r(\sqrt{-g}(k\mathcal{F})^{s-1}F^{r0}) = 0 \tag{B15}$$

$$-\partial_r(r(-2k(F_{0r})^2)^{s-1}F_{0r}) = 0 \tag{B16}$$

$$(-2k)^{s-1}\,\partial_r(r((F_{0r})^2)^{s-1}F_{0r}) = 0 \tag{B17}$$

$$\partial_r(r(F_{0r})^{2s-1}) = 0 \tag{B18}$$

$$r(F_{0r})^{2s-1} = C^{2s-1} \tag{B19}$$

$$(F_{0r})^{2s-1} = \frac{C^{2s-1}}{r} \tag{B20}$$

$$F_{0r} = C(\frac{1}{r})^{\frac{1}{2s-1}} \tag{B21}$$

$$-\partial_r h = C(r)^{-\frac{1}{2s-1}} \tag{B22}$$

$$h = -(r)^{-\frac{1}{2s-1}+1}\frac{C}{\frac{1}{2s-1}+1} \tag{B23}$$

Appendix C: Calculations to Get the Geodesics of BTZ

$$\frac{1}{\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}} \left(\frac{dr}{d\lambda}\right)^2 = \epsilon + \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)\left(\frac{dt}{d\lambda}\right)^2 - r^2 \left(\frac{d\phi}{d\lambda}\right)^2 \tag{C1}$$

$$\begin{aligned} \left(\frac{dr}{d\lambda}\right)^2 = \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)\epsilon + \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)^2 \left(\frac{dt}{d\lambda}\right)^2 \\ - \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right) r^2 \left(\frac{d\phi}{d\lambda}\right)^2 \end{aligned} \tag{C2}$$

$$\begin{aligned} \left(\frac{dr}{d\lambda}\right)^2 = \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)\epsilon + \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)^2 \left(\frac{E}{\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}}\right)^2 \\ - \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right) r^2 \left(\frac{\mathcal{L}}{r^2}\right)^2 \end{aligned} \tag{C3}$$

$$\begin{aligned} \frac{1}{\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}} \left(\frac{dr}{d\lambda}\right)^2 \left(\frac{d\lambda}{d\phi}\right)^2 \\ = \epsilon \left(\frac{d\lambda}{d\phi}\right)^2 + \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)\left(\frac{dt}{d\phi}\right)^2 - r^2 \left(\frac{d\phi}{d\phi}\right)^2 \end{aligned} \tag{C4}$$

$$\begin{aligned} \left(\frac{dr}{d\phi}\right)^2 = \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)\epsilon \left(\frac{d\lambda}{d\phi}\right)^2 + \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)^2 \left(\frac{dt}{d\phi}\right)^2 \\ - \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right) r^2 \end{aligned} \tag{C5}$$

$$\frac{d\phi}{d\lambda} = \frac{\mathcal{L}}{r^2} \tag{C6}$$

$$\left(\frac{d\lambda}{d\phi}\right)^2 = \frac{r^4}{\mathcal{L}^2} \tag{C7}$$

$$\frac{dt}{d\phi} = \frac{dt}{d\lambda}\frac{d\lambda}{d\phi} = \frac{\dot{t}}{\dot{\phi}} = \frac{E}{\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}} \frac{r^2}{\mathcal{L}} \tag{C8}$$

$$\left(\frac{dr}{d\phi}\right)^2 = \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)\epsilon\frac{r^4}{\mathcal{L}^2} + \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)^2\left(\frac{E}{\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}}\frac{r^2}{\mathcal{L}}\right)^2 - \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)r^2 \tag{C9}$$

$$\frac{1}{\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}}\left(\frac{dr}{d\lambda}\right)^2\left(\frac{d\lambda}{dt}\right)^2 = \epsilon\left(\frac{d\lambda}{dt}\right)^2 + \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)\left(\frac{dt}{dt}\right)^2 - r^2\left(\frac{d\phi}{dt}\right)^2 \tag{C10}$$

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)\epsilon\left(\frac{d\lambda}{dt}\right)^2 + \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)^2 - \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)r^2\left(\frac{d\phi}{dt}\right)^2 \tag{C11}$$

$$\frac{dt}{d\lambda} = \frac{E}{\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}} \tag{C12}$$

$$\left(\frac{d\lambda}{dt}\right)^2 = \left(\frac{\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}}{E}\right)^2 \tag{C13}$$

$$\frac{d\phi}{dt} = \frac{d\phi}{d\lambda}\frac{d\lambda}{dt} = \frac{\dot{\phi}}{\dot{t}} = \frac{\mathcal{L}}{r^2}\frac{\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}}{E} \tag{C14}$$

$$\left(\frac{dr}{dt}\right)^2 = \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)\epsilon\left(\frac{\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}}{E}\right)^2 + \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)^2 - \left(\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}\right)r^2\left(\frac{\mathcal{L}}{r^2}\frac{\frac{r^2}{l^2} - \mathrm{m} - \frac{K}{2r}}{E}\right)^2 \tag{C15}$$

Appendix D: Calculations to Get the Geodesics of Black Holes in MMG

$$-\frac{1}{\Lambda(r-r_-)(r-r_+)}\left(\frac{dr}{d\lambda}\right)^2 = \epsilon - \Lambda(r-r_-)(r-r_+)\left(\frac{dt}{d\lambda}\right)^2 - r^2\left(\frac{d\theta}{d\lambda}\right)^2 \tag{D1}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = -\Lambda(r-r_-)(r-r_+)\epsilon + \left(\Lambda(r-r_-)(r-r_+)\right)^2\left(\frac{dt}{d\lambda}\right)^2 + \Lambda(r-r_-)(r-r_+)r^2\left(\frac{d\theta}{d\lambda}\right)^2 \tag{D2}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = -\Lambda(r-r_-)(r-r_+)\epsilon + \left(\Lambda(r-r_-)(r-r_+)\right)^2\left(\frac{E}{\Lambda(r-r_-)(r-r_+)}\right)^2 + \Lambda(r-r_-)(r-r_+)r^2\left(\frac{\mathcal{L}}{r^2}\right)^2 \tag{D3}$$

$$-\frac{1}{\Lambda(r-r_-)(r-r_+)}\left(\frac{dr}{d\lambda}\right)^2\left(\frac{d\lambda}{d\theta}\right)^2 = \epsilon\left(\frac{d\lambda}{d\theta}\right)^2 - \Lambda(r-r_-)(r-r_+)\left(\frac{dt}{d\theta}\right)^2 - r^2\left(\frac{d\theta}{d\theta}\right)^2 \tag{D4}$$

$$\left(\frac{dr}{d\theta}\right)^2 = -\Lambda(r-r_-)(r-r_+)\epsilon\left(\frac{d\lambda}{d\theta}\right)^2 + \left(\Lambda(r-r_-)(r-r_+)\right)^2\left(\frac{dt}{d\theta}\right)^2 + \Lambda(r-r_-)(r-r_+)r^2 \tag{D5}$$

$$\frac{d\theta}{d\lambda} = \frac{\mathcal{L}}{r^2} \tag{D6}$$

$$\left(\frac{d\lambda}{d\theta}\right)^2 = \frac{r^4}{\mathcal{L}^2} \tag{D7}$$

$$\frac{dt}{d\theta} = \frac{dt}{d\lambda}\frac{d\lambda}{d\theta} = \frac{\dot{t}}{\dot{\theta}} = -\frac{E}{\Lambda(r-r_-)(r-r_+)}\frac{r^2}{\mathcal{L}} \tag{D8}$$

$$\left(\frac{dr}{d\theta}\right)^2 = -\Lambda(r - r_-)(r - r_+)\epsilon\frac{r^4}{\mathcal{L}^2} + \left(\Lambda(r - r_-)(r - r_+)\right)^2\left(-\frac{E}{\Lambda(r - r_-)(r - r_+)}\frac{r^2}{\mathcal{L}}\right)^2 + \Lambda(r - r_-)(r - r_+)r^2 \tag{D9}$$

$$-\frac{1}{\Lambda(r - r_-)(r - r_+)}\left(\frac{dr}{d\lambda}\right)^2\left(\frac{d\lambda}{dt}\right)^2 = \epsilon\left(\frac{d\lambda}{dt}\right)^2 - \Lambda(r - r_-)(r - r_+)\left(\frac{dt}{dt}\right)^2 - r^2\left(\frac{d\theta}{dt}\right)^2 \tag{D10}$$

$$\left(\frac{dr}{dt}\right)^2 = -\Lambda(r - r_-)(r - r_+)\epsilon\left(\frac{d\lambda}{dt}\right)^2 + \left(\Lambda(r - r_-)(r - r_+)\right)^2 + \Lambda(r - r_-)(r - r_+)r^2\left(\frac{d\theta}{dt}\right)^2 \tag{D11}$$

$$\frac{dt}{d\lambda} = -\frac{E}{\Lambda(r - r_-)(r - r_+)} \tag{D12}$$

$$\left(\frac{d\lambda}{dt}\right)^2 = \left(\frac{\Lambda(r - r_-)(r - r_+)}{E}\right)^2 \tag{D13}$$

$$\frac{d\theta}{dt} = \frac{d\theta}{d\lambda}\frac{d\lambda}{dt} = \frac{\dot{\theta}}{\dot{t}} = -\frac{\mathcal{L}}{r^2}\frac{\Lambda(r - r_-)(r - r_+)}{E} \tag{D14}$$

$$\left(\frac{dr}{dt}\right)^2 = -\Lambda(r - r_-)(r - r_+)\epsilon\left(\frac{\Lambda(r - r_-)(r - r_+)}{E}\right)^2 + \left(\Lambda(r - r_-)(r - r_+)\right)^2 + \Lambda(r - r_-)(r - r_+)r^2\left(\frac{\mathcal{L}}{r^2}\frac{\Lambda(r - r_-)(r - r_+)}{E}\right)^2 \tag{D15}$$