Generalised Operational Calculus Approach for Fractional Differential Equations

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ABSTRACT

Mikusiński's operational calculus is a method for interpreting and solving fractional differential equations, formally similar to Laplace transforms but more rigorously justified. This formalism was established for Riemann–Liouville and Caputo fractional calculi in the 1990s, and more recently for other types of fractional calculus. In this thesis, we consider the operators of Riemann–Liouville and Caputo fractional differentiation of a function with respect to another function, and discover that the approach of Luchko can be followed, with small modifications, in the more general settings too. We establish all the function spaces, formalisms, and identities required to build the versions of Mikusiński's operational calculus which cover Riemann–Liouville and Caputo derivatives with respect to functions. In the process, we gain a deeper understanding of some of the structures involved in applying Mikusiński's operational calculus to fractional calculus, such as the existence of a group isomorphic to $\mathbb{R}$. The mathematical structure established here is used to solve fractional differential equations using Riemann–Liouville and Caputo derivatives with respect to functions, the solutions being written using multivariate Mittag-Leffler functions, in agreement with the results found in other recent work.

It is useful to understand how the various operators of fractional calculus relate to each other, especially relations between newly defined operators and classical well-studied ones. In this work, we also focus on an important type of such relationship, namely conjugation relations, also called transmutation relations. We define a general abstract setting in which such relations are relevant, and indicate how they can be used to prove many results easily in general settings such as fractional calculus with respect to functions and weighted fractional calculus.

Keywords: fractional differential equations, Mikusiński's operational calculus, fractional calculus with respect to functions, algebraic conjugation, weighted fractional calculus

ÖZ

Mikusiński'nin operasyonel kalkülüs metodu, kesirli diferansiyel denklemleri yorumlama ve çözme yöntemi olup, biçimsel olarak Laplace dönüşümlerime benzese de daha detaylı doğrulanmıştır. Bu formalizm Riemann-Liouville ve Caputo kesirli kalkülüsleri için 1990'lı yıllarda belirlenmiş olup, şimdilerde diğer kesirli kalkülüs çeşitleri için de kullanılmaktadır. Bu tezde, bir fonksiyonun Riemann-Liouville ve Caputo kesirli türevinin diğer bir fonksiyona göre operatörleri ele alınmıştır ve Luchko'nun yaklaşımının ufak değişikliklerle daha genel durumlarda da kullanılabileceği keşfedilmiştir. Aynı zamanda, Mikusiński'nin operasyonel kalkülüs metodunun, fonksiyonlara bağlı olarak Riemann-Liouville ve Caputo türevlerini kapsayan türlerini oluşturmak için gereken tüm fonksiyon alanları, formalizmler ve özdeşlikler belirlenmiştir. Bu süreçte, Mikusiński'nin işlemsel kalkülüsünü kesirli kalkülüse uygulamada, reel sayılar kümesine izomorfik bir grubun varlığı gibi yer alan yapılar daha derin biçimde kavranabilmiştir. Burada belirlenenen matematiksel yapı, Riemann-Liouville ve Caputo türevlerini fonksiyonlara göre kullanarak kesirli diferansiyel denklemleri çözmek için kullanılmıştır. Çok değişkenli Mittag-Leffler fonksiyonlarını kullanarak yazılan çözümler, yakın zamanda diğer çalışmalarla ortaya çıkan neticelerle uyumludur.

Çeşitli kesirli kalkülüs operatörlerinin, özellikle yeni tanımlanmış ve hâlihazırda iyice çalışılmış olanların, birbiriyle olan ilişkisini anlamak faydalıdır. Bu makale, bir diğer adı dönüşüm ilişkileri olan, operatörler arası ilişkinin önemli bir çeşidi konjugasyon ilişkilerine odaklanmaktadır. Bu ilişkilerin geçerli olduğu genel bir soyut durumu tanımlanarak, bunların fonksiyonlara bağlı kesirli kalkülüs ve ağırlıklı kesirli kalkülüs gibi genel durumlarda birçok neticeyi kolayca ıspatlayabilmek için nasıl

kullanılabilecekleri belirtilmiştir.

Anahtar Kelimeler: kesirli diferansiyel denklemler, Mikusiński'nin operasyonel kalkülüs, fonksiyonlara göre kesirli kalkülüs, cebirsel konjugasyon, ağırlıklı kesirli kalkülüs

Dedicated to my loving wife, parents and brothers

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Chapter 1

INTRODUCTION

The birth of fractional calculus is accredited to the communication between Leibniz and de l'Hôpital which took place when the 17th century was drawing to its end. Thereafter, researchers worked little on the subject in the following three centuries, but notable strides in the field became conspicuous in the last few decades [1–5]. Despite the physical or geometrical complications involved in understanding the meaning of fractional operators [6], there lies deep interest among researchers to work upon the applications of fractional calculus in various disciplines [7–10], including dynamics [11, 12] and continuum mechanics [13, 14]. The multi-faceted nature of fractional calculus and its applications makes it vital to study the methods and techniques which can be applied to solving fractional differential equations.

Unlike in classical calculus, fractional derivative and integral operators can be defined in various different non-equivalent ways. The Riemann–Liouville fractional derivative is the most well-established and historically the most used definition. However, it has some innate disadvantages: for fractional differential equations in this model, the required initial conditions are intrinsically fractional, which makes the model less useful for applications. For this reason, the Caputo fractional derivative arose in the late 20th century to challenge Riemann–Liouville, being more suitable for modelling physical phenomena due to requiring initial conditions in the classical form [8].

Many other operators have been proposed and named as fractional integrals or fractional derivatives [3, 4, 15, 16], many of them being motivated by and studied for the sake of their applications in modelling. From the mathematical viewpoint, it is logical to study fractional calculus in a generalised framework, defining general classes of operators rather than proving the same results over and over again for every single operator [17].

Several of the operators defined in the past fall into the general class of fractional operators with respect to functions [3, 18], although it is itself a subset of the class of fractional operators with analytic kernels with respect to functions [15, 19]. Riemann–Liouville fractional calculus with respect to functions was first defined by Osler in 1970 [18], and further studied in several textbooks [2–4]. From this definition it is simple to obtain analogously a Caputo version [20] and a Hilfer version [21], while further studies of fractional operators with respect to functions, using operational calculus, can be found in [22, 23].

The major focus for applications of fractional calculus lies in fractional differential equations, which relate a function to some of its fractional derivatives, and which can be used in the modelling and understanding of many real-world systems, especially those involving hereditary or intermediate effects. Both ordinary and partial differential equations can be extended to a fractional setting, and some classical methods for solving ordinary or partial differential equations can be extended to fractional differential equations: for example, iteration methods [24], series methods [2, Section 8.6], transform methods [25], etc. But all these and other techniques are not without certain innate shortcomings: for example, the Laplace transform method cannot be used for solving equations with too swiftly increasing forcing functions [3,5]. Meanwhile, some methods work efficiently only for relatively simple equations, such as those with rational order [1,26].

In the 1950s, Jan Mikusiński [27] proposed a new algebraic construction for an operational calculus which can be used to understand the operator of differentiation from a new perspective and to rewrite differential equations in a formal symbolic way for easier solving. His approach was based on treating the convolution as a multiplication operation forming a ring of continuous functions, extending the ring to its quotient field, and interpreting derivatives and differential equations symbolically within this field. The structure that emerges is formally equivalent to that of the Laplace transform, but it has been argued [28] that Mikusiński's operational calculus is easier to make mathematically rigorous, more approachable than the distribution theory required for a rigorous formulation of the Laplace transform, and can be applied in certain situations where the Laplace transform cannot. Since Mikusiński, several other researchers [29–32] have investigated other types of operational calculus inspired by Mikusiński's and its further extensions.

The fractional-order version of Mikusiński's operational calculus has been developed since the 1990s, largely by Luchko [33] together with various collaborators [34–37]. To the best of our knowledge, the first work on this topic was in 1994 by Yakubovich and Luchko [34], in a Russian-language paper dealing with Erdélyi–Kober operators and some associated fractional differential equations. This was rapidly followed by papers establishing Mikusiński's operational calculus for Riemann–Liouville operators [35, 38, 39], and then for Caputo operators [36] as well as other types of fractional-calculus operators [40, 41] and other types of operational calculus [42], with a survey paper published by Luchko in 1999, [33]. The development of

Mikusiński's operational calculus for various fractional-calculus operators has continued into the 2000s [37, 43, 44], always with applications to fractional integro-differential equations of various types.

The extension of Mikusinski's operational calculus to the setting of fractional-calculus operators of a function with respect to another function was first achieved in the papers on which two chapters of this thesis are based [45, 46], since fractional differential equations using fractional derivatives with respect to functions are gathering considerable attention in recent mathematical research, e.g. [23, 47–49]. In Chapter 3, following closely the methodology used in [35] with the necessary modifications to deal with the operators being taken with respect to a function, we construct an adaptation of Mikusiński's operational calculus which is applicable to Riemann–Liouville fractional-calculus operators with respect to functions. Using this operational calculus, we obtain solutions of fractional differential equations in the framework of the Riemann–Liouville fractional differential operator of a function with respect to another function.

Just as the original formalism was adapted from Riemann–Liouville derivatives to Caputo derivatives by Luchko and Gorenflo in 1999 [36], in Chapter 4 we seek to adapt it from Riemann–Liouville derivatives with respect to functions to Caputo derivatives with respect to functions. As mentioned above, Caputo-type derivatives are often more useful than Riemann–Liouville-type derivatives, and cases such as Caputo–Hadamard have already received attention and applications in the literature [50–52], while more general Caputo fractional differential equations with respect to functions are also being studied mathematically [47–49]. Therefore, we expect this work to be considered useful by researchers in both pure and applied fields of study.

In recent decades, the study of fractional calculus has branched out to include many other operators in addition to the Riemann–Liouville and Caputo ones, sparking debate on what conditions should be satisfied by a fractional derivative or integral [53]. One important way to establish an operator as part of fractional calculus is to show that it is somehow related to the classical differintegral operators of Riemann–Liouville and Caputo. For some operators [17, 54], this has been done by means of infinite series formulae; for others [22, 55], the relationship takes the form of conjugation relations, expressing the new operators as conjugations of the classical ones by some invertible functional operator.

Unfortunately, such connections between different fractional operators have been greatly underappreciated in much of the recent literature. For example, conjugation relations for fractional operators with respect to functions have been shown in the classical textbooks [3, 4] and, under the name of transmutations, their power in solving fractional differential equations has been noted in Erdélyi–Kober [56, 57] and more general settings [58], but they have been entirely ignored in the vast majority of recent papers on fractional operators with respect to functions and the associated differential equations. This has led to a lot of wasted effort, since many papers have provided full proofs (parallel to the classical proofs) of mathematical facts about fractional operators with respect to functions, when in fact the classical results together with the conjugation relations would be enough for very short proofs.

The same process applies for any such mathematical connection, whether a series formula, a conjugation relation, or anything else: they grant the ability to deduce many facts in a generalised setting immediately from the corresponding known facts in the classical setting, without need to waste time reproducing the proofs. The only challenge remaining is to find the appropriate generalised setting and set up the framework for extending known results there. A class of operators has been defined [15] in which the methodology of series formulae can be used in its most general possible setting. Chapter 5 of this thesis is based on a paper [59] which established the most general possible setting for conjugation relations in fractional calculus. We define a general abstract setting in which these relations are relevant, and indicate how they can be used to prove many results easily in general settings such as fractional calculus with respect to functions and weighted fractional calculus.

Chapter 2

PRELIMINARIES

Fractional calculus is defined as the study of derivatives and integrals taken to non-integer orders, extending the concept of repeated differentiation and integration to a more general "differintegration" with a continuous (real or complex) order parameter. Here in this chapter, we recall some classical as well as fractional calculus definitions which we will need to use in the subsequent chapters.

Definition 2.1: The field of fractional calculus revolves around the Riemann–Liouville integral [3,4,8], defined by

$$ {}_{a}^{R}I_{x}^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_{a}^{x} (x-t)^{\mu-1} f(t)\,\mathrm{d}t, \qquad x \in (a,b), \tag{2.1} $$

for $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) > 0$ (or $\mu \geq 0$ if we assume real order) and $f \in L^1(a,b)$, and two competing fractional derivatives, usually named after Riemann–Liouville and Caputo, defined respectively as follows:

$$ {}_{a}^{R}D_{x}^{\mu} f(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left({}_{a}^{R}I_{x}^{n-\mu} f(x) \right), \qquad x \in (a,b), \tag{2.2} $$

$$ {}_{a}^{C}D_{x}^{\mu} f(x) = {}_{a}^{R}I_{x}^{n-\mu} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) \right), \qquad x \in (a,b), \tag{2.3} $$

for $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) \geq 0$ (or $\mu \geq 0$ if we assume real order) and $n := \lfloor \mathrm{Re}(\mu) \rfloor + 1$. Interpolating smoothly between these two definitions of fractional derivatives is a more general one, usually named after Hilfer [7], which is defined as follows:

$$ {}_{a}^{H}D_{x}^{\mu,\nu} f(x) = {}_{a}^{R}I_{x}^{\nu(n-\mu)} \left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} \left({}_{a}^{R}I_{x}^{(1-\nu)(n-\mu)} f(x) \right) \right), \qquad x \in (a,b), \tag{2.4} $$

for μ, n, f as before and $0 < \mathrm{Re}(\nu) < 1$ (or $0 \leq \nu \leq 1$ if we assume real order). Note that the case $\nu = 0$ gives the Riemann–Liouville derivative and the case $\nu = 1$ gives the Caputo derivative. A further extension of the Hilfer derivative has been studied recently by Luchko [60], defined as follows for any $m \in \mathbb{N}$:

$$ {}^{mL}_{a}D^{\mu,\gamma_1,\cdots,\gamma_m}_{x} f(x) = \left(\prod_{k=1}^{m} {}^{R}_{a}I^{\gamma_k}_{x} \frac{\mathrm{d}}{\mathrm{d}x} \right) \left({}^{R}_{a}I^{m-\mu-\gamma_1-\cdots-\gamma_m}_{x} f(x) \right), $$

for $0 < \mu \leq 1$ and $\gamma_1, \cdots, \gamma_m \geq 0$ such that $\mu + \gamma_1 + \cdots + \gamma_k \leq k$ for $k = 1, 2, \cdots, m$, and for f in some suitable function space, such as the space X^1_{mL} defined in [60, Equation (49)]. Note that the case $m = 1$ gives the Hilfer derivative, while the case $\gamma_k = 0$ for all k gives the Riemann–Liouville derivative. (It is presumed that this definition can also be extended to general $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) \geq 0$, just like the Hilfer derivative, but this was not done in Luchko's paper [60] and attempting it here would take us too far out of our way.)

It is interesting to note that all of the above fractional derivatives are simple compositions of Riemann–Liouville fractional integrals with the ordinary derivative $D_x = \frac{\mathrm{d}}{\mathrm{d}x}$:

$$ \begin{aligned} {}^{R}_{a}D^{\mu}_{x} &= D_x \circ D_x \circ \cdots \circ D_x \circ {}^{R}_{a}I^{n-\mu}_{x}, \\ {}^{C}_{a}D^{\mu}_{x} &= {}^{R}_{a}I^{n-\mu}_{x} \circ D_x \circ D_x \circ \cdots \circ D_x, \\ {}^{H}_{a}D^{\mu,\nu}_{x} &= {}^{R}_{a}I^{\nu(n-\mu)}_{x} \circ D_x \circ D_x \circ \cdots \circ D_x \circ {}^{R}_{a}I^{(1-\nu)(n-\mu)}_{x}, \end{aligned} $$

where in each case there are n repetitions of the D_x operator, with $n = \lfloor \mathrm{Re}(\mu) \rfloor + 1$ so that $n - 1 \leq \mathrm{Re}(\mu) < n$, and similarly

$$ {}^{mL}_{a}D^{\mu,\gamma_1,\cdots,\gamma_m}_{x} = {}^{R}_{a}I^{\gamma_1}_{x} \circ D_x \circ \cdots \circ {}^{R}_{a}I^{\gamma_m}_{x} \circ D_x \circ {}^{R}_{a}I^{m-\mu-\gamma_1-\cdots-\gamma_m}_{x}. $$

Thus, the operators of fractional integration (in the Riemann–Liouville sense) and ordinary first-order differentiation can be understood as the basic building blocks used to generate several different types of fractional calculus, at levels of generality up to and including Luchko's *m*th level fractional derivative.

Throughout this thesis (unless specified otherwise), the function $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is assumed to be a non-negative C^∞ function with $\psi' > 0$ everywhere (therefore ψ monotonically increasing) and $\psi(0) = 0$. The reason for these restrictions is that we shall be using ψ as a substitution in integrals, so ψ should be bijective, and we shall be dealing with fractional powers of $\psi(x)$ and their relationships with fractional operators, so we want $\psi(0) = 0$ and $\psi(x) \geq 0$ for all $x \geq 0$.

Definition 2.2 ([3, 18, 20]): The μth Riemann–Liouville fractional integral of a function $f(x)$ with respect to $\psi(x)$ is defined as

$$ {}_aI^{\mu}_{\psi(x)} f(x) = \frac{1}{\Gamma(\mu)} \int_a^x \big(\psi(x) - \psi(t)\big)^{\mu-1} f(t)\psi'(t)\,\mathrm{d}t, \qquad x \in (a,b), \tag{2.5} $$

where the order of integration can be real, $\mu > 0$, or complex, $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) > 0$.

The μth Riemann–Liouville fractional derivative of a function $f(x)$ with respect to $\psi(x)$ is defined as

$$ {}^{R}_{a}D^{\mu}_{\psi(x)} f(x) = \left(\frac{1}{\psi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right)^n {}_aI^{n-\mu}_{\psi(x)} f(x), \qquad x \in (a,b), \tag{2.6} $$

where the order of integration can be real, $\mu > 0$ with $n-1 \leq \mu < n \in \mathbb{N}$, or complex, $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) \geq 0$ and $n-1 \leq \mathrm{Re}(\mu) < n \in \mathbb{N}$.

The μth Caputo fractional derivative of a function $f(x)$ with respect to $\psi(x)$ is defined as

$$ {}^{C}_{a}D^{\mu}_{\psi(x)} f(x) = {}_aI^{n-\mu}_{\psi(x)} \left(\frac{1}{\psi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right)^n f(x), \qquad x \in (a,b), \tag{2.7} $$

where the order of integration can be real, $\mu > 0$ with $n-1 \leq \mu < n \in \mathbb{N}$, or complex,

$\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) \geq 0$ and $n-1 \leq \mathrm{Re}(\mu) < n \in \mathbb{N}$.

Definition 2.3 ([23, 61]): Given two functions f and g defined on the positive reals, their ψ-convolution, or generalised Laplace convolution, is the function $f *_{\psi} g$ defined as follows:

$$\left(f *_{\psi} g\right)(x) = \int_0^x f\left(\psi^{-1}\left(\psi(x) - \psi(t)\right)\right) g(t) \psi'(t) \, \mathrm{d}t, \; x > 0,$$

provided that this expression is well-defined (e.g. if the functions f and g are piecewise continuous and of ψ-exponential order). Note that here the condition $\psi(0) = 0$ is required to make this convolution operation work in a natural and desired way.

Chapter 3

RIEMANN–LIOUVILLE FRACTIONAL CALCULUS WITH RESPECT TO FUNCTIONS

Our work in this Chapter will consist of following closely the methodology of Luchko [33,35] and adapting as necessary to replace integration and differentiation with respect to x by integration and differentiation with respect to $\psi(x)$.

3.1 Function spaces for Riemann–Liouville operators

We begin by defining the function space $C_{\alpha,\psi}$, $\alpha \in \mathbb{R}$, which will be used in our investigation.

Definition 3.1: For any given $\alpha \in \mathbb{R}$, a function $f(x)$, $x > 0$, is said to be in the space $C_{\alpha,\psi}$ if there exists a real number p, $p > \alpha$, such that

$$f(x) = \{\psi(x)\}^p f_1(x),$$

for some function $f_1(x)$ in $C[0,\infty)$.

Clearly, $C_{\alpha,\psi}$ is a vector space and the set of spaces $C_{\alpha,\psi}$, $\alpha \in \mathbb{R}$, is ordered by inclusion according to

$$C_{\alpha,\psi} \subset C_{\beta,\psi} \iff \alpha \geq \beta. \tag{3.1}$$

These function spaces are a straightforward analogue of the well-known spaces C_α (those defined by Dimovski [29,30] and used by Luchko in his works on Mikusiński's operational calculus for fractional derivatives such as [33]), which are a suitable setting for consideration of the fractional integral and derivative operators with respect to the function ψ [3,4,18].

Theorem 3.1: The fractional integral of a function with respect to another function, namely the operator ${}_0I^{\mu}_{\psi(x)}$ with $\mu \geq 0$, is a linear map of the space $C_{\alpha,\psi}$ into itself, for any $\alpha \geq -1$. More specifically,

$$ {}_0I^{\mu}_{\psi(x)} : C_{\alpha,\psi} \to C_{\mu+\alpha,\psi} \subset C_{\alpha,\psi}. $$

Proof. The case $\mu = 0$ is trivial. In the case $\mu > 0$, setting $\psi(t) = \psi(x)\tau$ in (2.5), we obtain

$$ {}_0I^{\mu}_{\psi(x)} f(x) = \frac{\{\psi(x)\}^{\mu+p}}{\Gamma(\mu)} \int_0^1 \tau^p (1-\tau)^{\mu-1} f_1\left(\psi(x)\tau\right) \mathrm{d}\tau = \{\psi(x)\}^{\mu+p} f_2(x), $$

where $p > \alpha$ and $f_1 \in C[0,\infty)$. The last integral is uniformly convergent with respect to x in every closed interval $[0,X]$, $X > 0$, since $p > -1$ and $\mu > 0$; consequently, we have $f_2 \in C[0,\infty)$ and ${}_0I^{\mu}_{\psi(x)} f \in C_{\mu+\alpha,\psi}$. □

The fractional integral operator ${}_0I^{\mu}_{\psi(x)}$, $\mu > 0$, has a ψ-convolution representation in the space $C_{\alpha,\psi}$, $\alpha \geq -1$:

$$ \left({}_0I^{\mu}_{\psi(x)} f\right)(x) = \left(h_{\mu,\psi} *_{\psi} f\right)(x), \quad h_{\mu,\psi}(x) := \frac{\{\psi(x)\}^{\mu-1}}{\Gamma(\mu)}, \;\; f \in C_{\alpha,\psi}, \tag{3.2} $$

where $*_{\psi}$ is the generalised convolution operation defined [23,61,62] by

$$ \left(g *_{\psi} f\right)(x) = \int_0^x g\left(\psi^{-1}\left(\psi(x) - \psi(t)\right)\right) f(t)\psi'(t)\,\mathrm{d}t, \; x > 0. \tag{3.3} $$

Using this formalism, we can generalise Theorem 3.1 as follows.

Theorem 3.2: Given two functions $f \in C_{\alpha,\psi}$ and $g \in C_{\beta,\psi}$ with $\alpha, \beta \geq -1$, their ψ-convolution satisfies

$$g *_\psi f \in C_{\alpha+\beta+1,\psi} \subset C_{\alpha,\psi} \cap C_{\beta,\psi}.$$

Proof. We use the same technique as in the proof of Theorem 3.1, writing $f(x) = \{\psi(x)\}^p f_1(x)$ and $g(x) = \{\psi(x)\}^q g_1(x)$ with $p > \alpha$, $q > \beta$, and $f_1, g_1 \in C[0,\infty)$, and then setting $\psi(t) = \psi(x)\tau$ as an integral substitution:

$$\begin{aligned}
\left(g *_\psi f\right)(x) &= \int_0^x g\left(\psi^{-1}\left(\psi(x)-\psi(t)\right)\right) f(t)\psi'(t)\,\mathrm{d}t \\
&= \int_0^x \left(\psi(x)-\psi(t)\right)^q g_1\left(\psi^{-1}\left(\psi(x)-\psi(t)\right)\right)\left(\psi(t)\right)^p f_1(t)\psi'(t)\,\mathrm{d}t \\
&= \{\psi(x)\}^{p+q+1} \\
&\qquad \times \int_0^1 (1-\tau)^q \tau^p g_1\left(\psi^{-1}\left((1-\tau)\psi(x)\right)\right) f_1\left(\psi^{-1}\left(\tau\psi(x)\right)\right)\mathrm{d}\tau.
\end{aligned}$$

The last integral is uniformly convergent with respect to x in every closed interval $[0,X]$, $X > 0$, since $p, q > -1$; consequently, we have $g *_\psi f \in C_{\alpha+\beta+1,\psi}$. □

Corollary 3.1: The space $C_{-1,\psi}$ is preserved under the operation of ψ-convolution: namely, if $f, g \in C_{-1,\psi}$, then $g *_\psi f \in C_{-1,\psi}$.

Using the representation (3.2) and the commutativity and associativity properties of the generalised Laplace convolution 3.3, we obtain the well-known commutativity relation for fractional integrals with respect to functions, in the space $C_{\alpha,\psi}$:

$$\left({}_0I^{\mu}_{\psi(x)}\,{}_0I^{\nu}_{\psi(x)}f\right)(x) = \left({}_0I^{\nu}_{\psi(x)}\,{}_0I^{\mu}_{\psi(x)}f\right)(x), \quad f \in C_{\alpha,\psi}, \quad \alpha \geq -1,$$

for any $\mu > 0$, $\nu > 0$. Moreover, using the Euler integral of the first kind (together with a ψ-substitution in the integral) for the evaluation of $\left(h_{\mu,\psi} *_\psi h_{\nu,\psi}\right)(x)$, we obtain

$$\left({}_0I^{\mu}_{\psi(x)}\,{}_0I^{\nu}_{\psi(x)}f\right)(x) = \left({}_0I^{\mu+\nu}_{\psi(x)}f\right)(x), \quad f \in C_{\alpha,\psi}, \quad \alpha \geq -1, \tag{3.4}$$

for any $\mu > 0$ and $\nu > 0$. From (3.4), it follows that

$$\left(\underbrace{{}_0I^{\mu}_{\psi(x)} \cdots {}_0I^{\mu}_{\psi(x)}}_{n} f\right)(x) = \left({}_0I^{n\mu}_{\psi(x)} f\right)(x), \quad f \in C_{\alpha,\psi}, \quad \alpha \geq -1, \tag{3.5}$$

for any $\mu > 0$ and $n \in \mathbb{N}$.

Theorem 3.3: The fractional integral of a function with respect to another function ${}_0I^{\mu}_{\psi(x)}$ is a right inverse of the fractional derivative of a function with respect to another function ${}_0^RD^{\mu}_{\psi(x)}$ on the function space $C_{\alpha,\psi}$, for any $\alpha \geq -1$ and $\mu > 0$.

Proof. Consider $f \in C_{\alpha,\psi}$, $\alpha \geq -1$, and $n-1 \leq \mu < n \in \mathbb{Z}^+$. Using (3.4), we obtain

$$\begin{aligned}\left({}_0^RD^{\mu}_{\psi(x)} {}_0I^{\mu}_{\psi(x)} f\right)(x) &= \left(\frac{1}{\psi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left({}_0I^{n-\mu}_{\psi(x)} {}_0I^{\mu}_{\psi(x)} f\right)(x) \\ &= \left(\frac{1}{\psi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left({}_0I^{n}_{\psi(x)} f\right)(x) = f(x),\end{aligned}$$

since n is a positive integer. □

Definition 3.2: A function $f(x)$, $x > 0$ is said to be in the space $\Omega^{\mu}_{\alpha,\psi}$, $\mu \geq 0$, if we have the inclusion

$$\left({}_0^RD^{\nu}_{\psi(x)} f\right)(x) \in C_{\alpha,\psi},$$

for all ν with $0 \leq \nu \leq \mu$.

Due to the inclusion (3.1), the spaces $\Omega^{\mu}_{\alpha,\psi}$ are ordered by inclusion in the parameter α as follows:

$$\Omega^{\mu}_{\alpha,\psi} \subset \Omega^{\mu}_{\beta,\psi}, \ \alpha \geq \beta,$$

and also trivially in the parameter μ as follows:

$$\Omega^{\mu}_{\alpha,\psi} \subset \Omega^{\nu}_{\alpha,\psi}, \ \mu \leq \nu.$$

Remark 3.1: Consider a function $g \in C_{\alpha,\psi}$ and its fractional integral with respect to

ψ,

$$f(x) = \left({}_0I^{\mu}_{\psi(x)}g\right)(x). \tag{3.6}$$

Making use of property (3.4) of the fractional integral of a function with respect to another function and Theorems 3.1 and 3.3, we get the inclusion $f \in \Omega^{\mu}_{\alpha,\psi}$ and the formula

$$\left({}_0I^{\mu}_{\psi(x)}{}^R_0D^{\mu}_{\psi(x)}f\right)(x) = f(x).$$

This means that the fractional integral operator ${}_0I^{\mu}_{\psi(x)}$ is a left inverse of the Riemann–Liouville fractional differential operator ${}^R_0D^{\mu}_{\psi(x)}$ on some subspace of $\Omega^{\mu}_{\alpha,\psi}$ which contains in particular the functions $f \in \Omega^{\mu}_{\alpha,\psi}$ that are representable in form (3.6).

Remark 3.2: It is important to note that several of the results above do not represent new formulae in the theory of fractional calculus with respect to functions, but only new function spaces in which these formulae are valid. Results such as (3.4) and (3.5), for example, are well known from the studies in classical textbooks such as [4, §18.2] and [3, §2.5], but in those sources the results were stated using different function spaces instead of the $C_{\alpha,\psi}$ spaces, which here we have established, in results such as Theorem 3.1, as interacting in a natural way with the operators of fractional integration with respect to ψ.

Theorem 3.4: Let $f \in \Omega^{\mu}_{\alpha,\psi}$, $0 < \mu \leq 1$, $\alpha \geq -1$. Then,

$$\left(\left(E - {}_0I^{\mu}_{\psi(x)}{}^R_0D^{\mu}_{\psi(x)}\right)f\right)(x) = \frac{\{\psi(x)\}^{\mu-1}}{\Gamma(\mu)}\lim_{z\to 0}\left({}_0I^{1-\mu}_{\psi(x)}f\right)(z), \tag{3.7}$$

where E is the identity operator on the space $\Omega^{\mu}_{\alpha,\psi}$. The operator $E - {}_0I^{\mu}_{\psi(x)}{}^R_0D^{\mu}_{\psi(x)}$ is called the projector of the fractional integral operator with respect to ψ.

Proof. Define a function ϕ (well-defined since $f \in \Omega^{\mu}_{\alpha,\psi}$) by

$$\phi(x) = \left({}_0I^{\mu}_{\psi(x)}{}^R_0D^{\mu}_{\psi(x)}f\right)(x). \tag{3.8}$$

Using Theorem 3.1 and Remark 3.1, we obtain

$$\phi \in C_{\alpha+\mu,\psi} \cap \Omega^{\mu}_{\alpha,\psi}. \tag{3.9}$$

Applying ${}^R_0D^{\mu}_{\psi(x)}$ to the function $\phi(x)$ and using Theorem 3.3, we get

$$\left({}^R_0D^{\mu}_{\psi(x)}\phi\right)(x) = \left({}^R_0D^{\mu}_{\psi(x)}{}_0I^{\mu}_{\psi(x)}{}^R_0D^{\mu}_{\psi(x)}f\right)(x) = \left({}^R_0D^{\mu}_{\psi(x)}f\right)(x),$$

so $f-\phi$ is in the kernel of the operator ${}^R_0D^{\mu}_{\psi(x)}$, which means

$$f(x) = \phi(x) + k\{\psi(x)\}^{\mu-1}, \tag{3.10}$$

for some constant k. Applying ${}_0I^{1-\mu}_{\psi(x)}$ to both sides of (3.10), we obtain

$$\left({}_0I^{1-\mu}_{\psi(x)}f\right)(x) = \left({}_0I^{1-\mu}_{\psi(x)}\phi\right)(x) + k\Gamma(\mu). \tag{3.11}$$

From the inclusion (3.9), Theorem 3.1, and the condition $\alpha \geq -1$, we know that the function $\left({}_0I^{1-\mu}_{\psi(x)}\phi\right)(x)$, and therefore, due to relation (3.11), the function $\left({}_0I^{1-\mu}_{\psi(x)}f\right)(x)$ too, are in the function space $C_{0,\psi}$ and thus continuous on the interval $[0,\infty)$. Furthermore,

$$\begin{aligned}\left({}_0I^{1-\mu}_{\psi(x)}\phi\right)(x) &= \left({}_0I^{1-\mu}_{\psi(x)}{}_0I^{\mu}_{\psi(x)}{}^R_0D^{\mu}_{\psi(x)}f\right)(x) = \int_0^x {}^R_0D^{\mu}_{\psi(z)}f(z)\,\mathrm{d}z \\ &= \left({}_0I^{1-\mu}_{\psi(x)}f\right)(x) - \lim_{z\to 0}\left({}_0I^{1-\mu}_{\psi(x)}f\right)(z).\end{aligned}$$

Comparing this with (3.11), we find

$$d = \frac{\lim_{z\to 0}\left({}_0I^{1-\mu}_{\psi(x)}f\right)(z)}{\Gamma(\mu)},$$

which, in view of (3.10), leads us to the desired relation (3.7). □

3.2 Operational calculus for Riemann–Liouville operators

For the sake of simplicity we shall consider in our further discussions the case of the space $C_{-1,\psi}$, which turns out to be the most interesting one for applications, largely due to the result of Corollary 3.1. Similarly to the original Mikusiński's type operational calculus, we have the following theorem.

Theorem 3.5: The space $C_{-1,\psi}$ with the operations of ψ-convolution and ordinary addition becomes a commutative rng (ring without identity) without zero divisors: $\left(C_{-1,\psi}, *_\psi, +\right)$.

Proof. Addition and ψ-convolution are known to be commutative and associative, and ψ-convolution is distributive over addition. The set $C_{-1,\psi}$ is closed under both operations by Corollary 3.1. The zero function gives an additive identity, and the negation of any function in $C_{-1,\psi}$ is in $C_{-1,\psi}$. □

Following Mikusiński's reasoning, the rng $C_{-1,\psi}$ can be extended to its quotient field $M_{-1,\psi}$ by quotienting the set $C_{-1,\psi} \times \left(C_{-1,\psi} - \{0\}\right)$ with respect to the equivalence relation

$$(f,g) \sim (f_1,g_1) \iff \left(f *_\psi g_1\right)(x) = \left(g *_\psi f_1\right)(x). \tag{3.12}$$

For the sake of convenience, the elements of the field $M_{-1,\psi}$ can be formally considered as convolution quotients $\frac{f}{g}$, where the operations of addition and multiplication are defined in $M_{-1,\psi}$ as follows:

$$\frac{f}{g} + \frac{f_1}{g_1} = \frac{f *_\psi g_1 + g *_\psi f_1}{g *_\psi g_1}, \qquad \frac{f}{g} \cdot \frac{f_1}{g_1} = \frac{f *_\psi f_1}{g *_\psi g_1}. \tag{3.13}$$

Theorem 3.6: The space $M_{-1,\psi}$ with the operations of addition and multiplication given by (3.13) is a field $\left(M_{-1,\psi},\cdot,+\right)$.

Proof. The proof of the theorem follows the same lines as the standard derivation from an integral domain (commutative ring without zero divisors) of its quotient field. The only difference here is that $C_{-1,\psi}$ is not a true ring since it does not have a multiplicative identity, but the quotient field $M_{-1,\psi}$ does have a multiplicative identity, namely $I_\psi = \frac{f}{f}$ for any $f \in C_{-1,\psi}$. This element in the quotient field is well-defined, according to the equivalence relation (3.12), and is an identity under multiplication according to (3.13). □

It can easily be seen that the rng $C_{-1,\psi}$ and the field of complex numbers $\mathbb{C}$ can be embedded in the field $M_{-1,\psi}$ by the following maps:

$$f \mapsto \frac{f *_\psi h_{\mu,\psi}}{h_{\mu,\psi}}, \tag{3.14}$$

$$z \mapsto \frac{z h_{\mu,\psi}}{h_{\mu,\psi}}, \tag{3.15}$$

respectively, where $\mu > 0$ is arbitrary, the function $h_{\mu,\psi}$ is defined in (3.2), and these embeddings are well-defined because of the equivalence relation (3.12).

In view of the ψ-convolution formulation (3.2) of the fractional integral of a function with respect to another function, we can identify the operator ${}_0I^{\mu}_{\psi(x)}$ with the element $h_{\mu,\psi}$ of the rng $C_{-1,\psi} \subset M_{-1,\psi}$. Within the quotient field, it is possible to find an inverse to this element, which can therefore be formally identified with the inverse of the fractional integral, namely the fractional derivative of a function with respect to another function. We formalise this concept in the following definition.

Definition 3.3: The algebraic inverse of the fractional integral of a function with respect to another function ${}_0I^{\mu}_{\psi(x)}$ is said to be the element $S_{\mu,\psi}$ of the field $M_{-1,\psi}$

which is reciprocal to the element $h_{\mu,\psi}$ in the field $M_{-1,\psi}$; that is,

$$S_{\mu,\psi} = \frac{I_\psi}{h_{\mu,\psi}} \equiv \frac{h_{\mu,\psi}}{h_{\mu,\psi} *_\psi h_{\mu,\psi}} \equiv \frac{h_{\mu,\psi}}{h_{2\mu,\psi}}, \tag{3.16}$$

where $I_\psi = \frac{h_{\mu,\psi}}{h_{\mu,\psi}}$ denotes the multiplicative identity element of the field $M_{-1,\psi}$.

As we have already seen, the operator ${}_0I^{\mu}_{\psi(x)}$ can be represented as a convolution in the rng $C_{-1,\psi}$ with the function $h_{\mu,\psi}$. This fact can now be rewritten in terms of the algebraic inverse as follows:

$$\left({}_0I^{\mu}_{\psi(x)}\right) f(x) = \frac{I_\psi}{S_{\mu,\psi}} \cdot f.$$

We can also define fractional powers of these operators. The behaviour of the functions $h_{\mu,\psi}$ under convolution is well known, or follows from equation (3.5): for $\alpha > 0, n \in \mathbb{N}$, we have

$$h^n_{\mu,\psi}(x) = \left(\underbrace{h_{\mu,\psi} *_\psi \cdots *_\psi h_{\mu,\psi}}_{n}\right)(x) = h_{n\mu,\psi}(x).$$

Extending this relation to an arbitrary positive real power of $h_{\mu,\psi}(x)$, we can define:

$$h^{\lambda}_{\mu,\psi}(x) = h_{\lambda\mu,\psi}(x),\ \lambda > 0. \tag{3.17}$$

Therefore, $h^{\lambda}_{\mu,\psi} \in C_{-1,\psi}$ for all $\lambda > 0$, and the following relations can be easily checked:

$$h^{\alpha}_{\mu,\psi} *_\psi h^{\beta}_{\mu,\psi} = h_{\alpha\mu,\psi} *_\psi h_{\beta\mu,\psi} = h_{(\alpha+\beta)\mu,\psi} = h^{\alpha+\beta}_{\mu,\psi},\ \alpha > 0,\ \beta > 0. \tag{3.18}$$

The above relations motivate the following definition of powers of the element $S_{\mu,\psi}$ with an arbitrary real power exponent $\lambda \in \mathbb{R}$:

$$S_{\mu,\psi}^{\lambda} = \begin{cases} h_{\mu,\psi}^{-\lambda}, & \lambda < 0, \\ I_{\psi}, & \lambda = 0, \\ \frac{I_{\psi}}{h_{\mu,\psi}^{\lambda}}, & \lambda > 0. \end{cases}$$

Using this definition and the semigroup relation (3.18), we get:

$$S_{\mu,\psi}^{\alpha} \cdot S_{\mu,\psi}^{\beta} = S_{\mu,\psi}^{\alpha+\beta}, \ \alpha, \beta \in \mathbb{R}. \tag{3.19}$$

In the following theorem, we find the relationship between the Riemann–Liouville fractional differential operator of a function with respect to another function and elements of the field $M_{-1,\psi}$.

Theorem 3.7: For any $\mu > 0$, the Riemann–Liouville fractional differential operator of a function with respect to another function ${}_{0}^{R}D_{\psi(x)}^{\mu}$ may be represented in the field $M_{-1,\psi}$ in the following form, for $f \in \Omega_{-1,\psi}^{\alpha\mu}$:

$${}_{0}^{R}D_{\psi(x)}^{\mu} f = S_{\mu,\psi} \cdot f - S_{\mu,\psi} \cdot P_{\mu,\psi} f, \tag{3.20}$$

where $P_{\mu,\psi} = E - {}_{0}I_{\psi(x)}^{\mu} {}_{0}^{R}D_{\psi(x)}^{\mu}$ is the projector of the operator ${}_{0}I_{\psi(x)}^{\mu}$. This means that the Riemann–Liouville fractional differential operator of a function with respect to another function is reduced to an operator of multiplication in the field $M_{-1,\psi}$, with an extra initial value term.

Proof. Given any $f \in \Omega_{-1,\psi}^{\mu}$, we have by definition of the projector

$$\begin{aligned} f(x) &= \left(P_{\mu,\psi} f\right)(x) + \left({}_{0}I_{\psi(x)}^{\mu} {}_{0}^{R}D_{\psi(x)}^{\mu} f\right)(x) \\ &= \left(P_{\mu,\psi} f\right)(x) + h_{\mu,\psi} \cdot \left({}_{0}^{R}D_{\psi(x)}^{\mu} f\right)(x). \end{aligned}$$

Multiplying both sides of the last relation by $S_{\mu,\psi}$ and using the definition of $S_{\mu,\psi}^{\lambda}$ as the inverse of $h_{\mu,\psi}$, we obtain the required result. $\square$

For the application of Mikusiński's operational calculus to solving fractional differential equations in the setting of Riemann–Liouville fractional derivatives with respect to functions, it is important to identify those elements of $M_{-1,\psi}$ which can be represented by means of functions in the rng $C_{-1,\psi}$. One useful class of such functions is given by the following theorem.

Theorem 3.8: Consider a multiple power series defining a function of several complex variables $z=(z_1,\dots,z_n)$ with complex coefficients, and let $z_0=(z_{10},\dots,z_{n0})\neq 0$ be a point at which this series is convergent. That is,

$$F(z_0)=\sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1,\dots,i_n} z_{10}^{i_1}\times\cdots\times z_{n0}^{i_n}=K\in\mathbb{C}.$$

Then, for any $\mu>0$, and for any $\nu>0$ and $\lambda_1,\dots,\lambda_n>0$, the formal power series

$$F\left(S_{\mu,\psi}\right)=S_{\mu,\psi}^{-\nu}\sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1,\dots,i_n}\left(S_{\mu,\psi}^{-\lambda_1}\right)^{i_1}\times\cdots\times\left(S_{\mu,\psi}^{-\lambda_n}\right)^{i_n}$$

can be represented as an element of the ring $C_{-1,\psi}$, via the following representation:

$$F\left(S_{\mu,\psi}\right)=\sum_{i_1,\dots,i_n=0}^{\infty} a_{i_1,\dots,i_n} h_{(\nu+\lambda_1 i_1+\cdots+\lambda_n i_n)\mu,\psi}(x). \tag{3.21}$$

Proof. By using the definition of $h_{\mu,\psi}(x)$, we have the following formula for the function on the right-hand side of equation (3.21):

$$g(x):=\sum_{i_1,\dots,i_n=0}^{\infty} b_{i_1,\dots,i_n} h_{(\nu+\lambda_1 i_1+\cdots+\lambda_n i_n)\mu,\psi}(x)=\{\psi(x)\}^{\nu\mu-1}g_1(x),$$

where

$$g_1(x):=\sum_{i_1,\dots,i_n=0}^{\infty}\frac{b_{i_1,\dots,i_n}\left(\{\psi(x)\}^{\lambda_1\mu}\right)^{i_1}\times\cdots\times\left(\{\psi(x)\}^{\lambda_n\mu}\right)^{i_n}}{\Gamma(\nu\mu+\lambda_1 i_1\mu+\cdots+\lambda_n i_n\mu)}.$$

This function g_1 is the composition of ψ with the power series function f_1 considered in

[35, Theorem 6]. The estimates proved in [35, Theorem 6] imply that f_1 is continuous on $[0,\infty)$, its series uniformly convergent on every bounded closed interval $[0,X]$, $0 < X < \infty$. Taking compositions with ψ, and using the fact that ψ is monotonic, we obtain the same convergence and continuity results for the function g_1. Hence, $g_1 \in C[0,\infty)$.

Therefore the function on the right-hand side of equation (3.21) is a well-defined element of $C_{-1,\psi}$. Then by using the definition of $S_{\mu,\psi}$ and its powers, the formal power series for $F\left(S_{\mu,\psi}\right)$ must be equivalent to this function, and therefore can be identified as an element of $C_{-1,\psi}$. □

Using Theorem 3.8, we can write various specific elements of the field $M_{-1,\psi}$ using representations as functions in the rng $C_{-1,\psi}$. Before stating our next result, we need to pause briefly and introduce a family of Mittag-Leffler type functions, to be used further.

Definition 3.4 ([63]): The original Mittag-Leffler function, with one parameter and of one variable, is defined as follows:

$$E_{\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\mu+1)}, \qquad \mathrm{Re}(\mu) > 0.$$

Its generalisations to a two-parameter Mittag-Leffler function and three-parameter Mittag-Leffler function, also functions of one variable, are defined respectively as follows:

$$E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\mu+\nu)}, \qquad \mathrm{Re}(\mu) > 0;$$
$$E_{\mu,\nu}^{\rho}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k z^k}{k!\Gamma(k\mu+\nu)}, \qquad \mathrm{Re}(\mu) > 0.$$

Definition 3.5: A multivariate Mittag-Leffler function, with $n+1$ parameters and applied to n variables for any $n \in \mathbb{N}$, is defined as follows [35, 36]:

$$E_{(\mu_1,\ldots,\mu_n),\nu}(z_1,\ldots,z_n) = \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!} \cdot \frac{z_1^{k_1}\ldots z_n^{k_n}}{\Gamma(\mu_1 k_1+\cdots+\mu_n k_n+\nu)}. \quad (3.22)$$

Note that this is not the only possible way of defining a multivariate Mittag-Leffler function; a separate definition has also been proposed by Saxena et al. [64], which is independent of (3.22), neither being a special case of the other.

In the special cases of $n=2$ and $n=3$, bivariate and trivariate Mittag-Leffler functions have also been defined which are not special cases of (3.22), as they take account of an upper parameter appearing in a Pochhammer symbol like the three-parameter univariate Mittag-Leffler function defined above. These functions respectively have four parameters and two variables [65] or five parameters and three variables [66], and they have been used in solving systems of fractional differential equations [67].

Lemma 3.1: The following relations hold true between (on the left-hand side) elements of the field $M_{-1,\psi}$ and (on the right-hand side) explicit functions of x in the function space $C_{-1,\psi}$:

(a) For any $\mu>0$ and $\rho \in \mathbb{R}$,

$$\frac{I_\psi}{S_{\mu,\psi}-\rho} = \{\psi(x)\}^{\mu-1} E_{\mu,\mu}\left(\rho\,\{\psi(x)\}^{\mu}\right).$$

(b) For any $\mu>0$ and $n \in \mathbb{N}$,

$$\frac{I_\psi}{\left(S_{\mu,\psi}-\rho\right)^n} = \{\psi(x)\}^{n\mu-1} E^{n}_{\mu,n\mu}\left(\rho\,\{\psi(x)\}^{\mu}\right).$$

(c) For any $\mu>0$ and $\mu_1,\ldots,\mu_n>0$, $\beta \in \mathbb{R}$,

$$\frac{S_{\mu,\psi}^{-\beta}}{I_\psi - \sum_{i=1}^{n} \lambda_i S_{\mu,\psi}^{-\mu_i}}$$
$$= \{\psi(x)\}^{\beta\mu-1} E_{(\mu_1\mu,\ldots,\mu_n\mu),\beta\mu}\left(\lambda_1 \{\psi(x)\}^{\mu_1\mu}, \ldots, \lambda_n \{\psi(x)\}^{\mu_n\mu}\right).$$

Proof. The relation (a) can be obtained from the geometric series expansion as follows:

$$\frac{I_\psi}{S_{\mu,\psi} - \rho} = S_{\mu,\psi}^{-1} \cdot \frac{I_\psi}{I_\psi - \rho S_{\mu,\psi}^{-1}} = S_{\mu,\psi}^{-1} \sum_{i=0}^{\infty} \rho^i S_{\mu,\psi}^{-i} = \sum_{i=0}^{\infty} \rho^i h_{(i+1)\mu,\psi}(x)$$
$$= \sum_{i=0}^{\infty} \frac{\rho^i \{\psi(x)\}^{(i+1)\mu-1}}{\Gamma(\mu i + \mu)} = \{\psi(x)\}^{\mu-1} E_{\mu,\mu}\left(\rho \{\psi(x)\}^{\mu}\right).$$

For (b), we have an infinite binomial series:

$$\frac{I_\psi}{\left(S_{\mu,\psi} - \rho\right)^n} = S_{\mu,\psi}^{-n} \cdot \frac{I_\psi}{\left(I_\psi - \rho S_{\mu,\psi}^{-1}\right)^n} = S_{\mu,\psi}^{-n} \sum_{i=0}^{\infty} \frac{(m)_i \rho^i}{i!} S_{\mu,\psi}^{-i}$$
$$= \sum_{i=0}^{\infty} \frac{(m)_i \rho^i}{i!} h_{(n+i)\mu,\psi}(x)$$
$$= \{\psi(x)\}^{n\mu-1} \sum_{i=0}^{\infty} \frac{(m)_i \left(\rho \{\psi(x)\}^{\mu}\right)^i}{i!\Gamma(\mu i + n\mu)}$$
$$= \{\psi(x)\}^{n\mu-1} E_{\mu,n\mu}^{n}\left(\rho \{\psi(x)\}^{\mu}\right).$$

Making use of the technique demonstrated in (a) and (b), it is easy to derive (c). So we omit the straightforward details. □

The following relation can be verified easily, either by direct calculation using the fact that $h_{\mu,\psi} *_\psi h_{\nu,\psi} = h_{\mu+\nu,\psi}$, or by using part (c) of the above Lemma. In our further discussions, we will use this formula:

$$\left(\{\psi(t)\}^{\nu-1} E_{(\mu_1,\ldots,\mu_n),\nu}\left(\lambda_1 \{\psi(t)\}^{\mu_1}, \ldots, \lambda_n \{\psi(t)\}^{\mu_n}\right) *_\psi \frac{\{\psi(t)\}^{\gamma}}{\Gamma(1+\gamma)}\right)(x)$$
$$= \{\psi(t)\}^{\nu+\gamma} E_{(\mu_1,\ldots,\mu_n),\nu+\gamma+1}\left(\lambda_1 \{\psi(x)\}^{\mu_1}, \ldots, \lambda_n \{\psi(x)\}^{\mu_n}\right). \quad (3.23)$$

3.3 Applications to fractional differential equations

In this section, we will use the constructed operational calculus to solve Cauchy problems with constant coefficients in the setting of Riemann–Liouville fractional derivatives of a function with respect to another function. Let us begin with the following simple problem, which is suitable for illustration of our method.

Theorem 3.9: Let $\mu > 0$ and $\lambda, c \in \mathbb{R}$ be fixed, and let $f \in C_{-1,\psi}$ be a function. The unique solution of the Cauchy problem

$$\left({}_0^R D^{\mu}_{\psi(x)} y\right)(x) - \lambda y(x) = f(x), \;\; x > 0, \tag{3.24}$$

$$\lim_{x \to 0} \left({}_0 I^{1-\mu}_{\psi(x)} y\right)(x) = c, \tag{3.25}$$

in the space $\Omega^{\mu}_{-1,\psi}$ is given by:

$$y(x) = \int_0^x \left(\psi(x) - \psi(t)\right)^{\mu-1} E_{\mu,\mu}\left(\lambda\left(\psi(x) - \psi(t)\right)^{\mu}\right) f(t)\,\psi'(t)\,\mathrm{d}t$$
$$+ c\left\{\psi(x)\right\}^{\mu-1} E_{\mu,\mu}\left(\lambda\left\{\psi(x)\right\}^{\mu}\right). \tag{3.26}$$

Proof. Making use of relations (3.20) and (3.7), we can write the Cauchy problem (3.24) – (3.25) in the form of an algebraic equation in the field $M_{-1,\psi}$:

$$S_{\mu,\psi} \cdot y - \lambda y = f + S^{\alpha}_{\mu,\psi} \cdot y_0, \text{ where } y_0(x) = \frac{c}{\Gamma(\mu)}\left\{\psi(x)\right\}^{\mu-1}. \tag{3.27}$$

The unique solution of the algebraic equation (3.27) in the field $M_{-1,\psi}$ is as follows:

$$y = \frac{I_{\psi}}{S_{\mu,\psi} - \lambda} \cdot f + \frac{S_{\mu,\psi}}{S_{\mu,\psi} - \lambda} \cdot y_0. \tag{3.28}$$

Using Lemma 3.1(a) and the embedding of the rng $C_{-1,\psi}$ in the field $M_{-1,\psi}$, we obtain:

$$\begin{aligned} y_1(x) &:= \frac{I_{\psi}}{S_{\mu,\psi} - \lambda} \cdot f \\ &= \int_0^x \left(\psi(x) - \psi(t)\right)^{\mu-1} E_{\mu,\mu}\left(\lambda\left(\psi(x) - \psi(t)\right)^{\mu}\right) f(t)\,\psi'(t)\,\mathrm{d}t. \end{aligned} \tag{3.29}$$

Using relation (3.23), for the second part of (3.28) we get:

$$\begin{aligned} y_2(x) &:= \frac{S_{\mu,\psi}}{S_{\mu,\psi}-\lambda}\cdot y_0 = y_0 + \frac{\lambda}{S_{\mu,\psi}-\lambda}\cdot y_0 \\ &= \frac{c}{\Gamma(\mu)}\{\psi(x)\}^{\mu-1} + \lambda c\{\psi(x)\}^{2\mu-1}E_{\mu,2\mu}\left(\lambda\{\psi(x)\}^{\mu}\right) \\ &= c\{\psi(x)\}^{\mu-1}E_{\mu,\mu}\left(\lambda\{\psi(x)\}^{\mu}\right). \end{aligned} \tag{3.30}$$

Combining (3.29) and (3.30), we obtain the solution (3.26). It remains to check the inclusion $y \in \Omega^{\mu}_{-1,\psi}$.

By using the following fractional relation for two-parameter Mittag-Leffler functions:

$$\left({}^{R}_{0}D^{\nu}_{\psi(t)}\{\psi(t)\}^{\mu-1}E_{\mu,\mu}\left(\lambda\{\psi(t)\}^{\mu}\right)\right)(x) = \{\psi(x)\}^{\mu-\nu-1}E_{\mu,\mu-\nu}\left(\lambda\{\psi(x)\}^{\mu}\right),$$

along with the Definitions 3.1 and 3.2, we can easily deduce the inclusion $y_2 \in \Omega^{\mu}_{-1,\psi}$. From representation (3.29) and Theorem 3.8, the inclusion $y_1 \in C_{-1,\psi}$ follows. Multiplying relation (3.29) by $\left(S_{\mu,\psi}-\lambda\right)$ and then by $h_{\mu,\psi}(x)$ and taking into account relation (3.16), we get:

$$y_1(x) = \lambda\left({}_0I^{\mu}_{\psi(x)}y_1\right)(x) + \left({}_0I^{\mu}_{\psi(x)}f\right)(x). \tag{3.31}$$

Using (3.31) and Remark 3.1, we conclude the inclusion $y_1 \in \Omega^{\mu}_{-1,\psi}$. Summing y_1 and y_2, we finally obtain $y \in \Omega^{\mu}_{-1,\psi}$. □

Corollary 3.2: Consider a special case of the initial value problem (3.24)-(3.25):

$$\left({}^{R}_{0}D^{\mu}_{\psi(x)}y\right)(x) - y(x) = 1,\ 0<\mu\le 1,\ x>0, \tag{3.32}$$

$$\lim_{z\to 0}\left({}_0I^{1-\mu}_{\psi(x)}y\right)(z) = 1. \tag{3.33}$$

We consider three further special cases according to different choices of the function $\psi(x)$:

(a) If $\psi(x) = \sqrt{x}$, then $y(x) = x^{\frac{1}{2}(\mu-1)} E_{\mu,\mu}(x^{\frac{\mu}{2}}) + x^{\frac{\mu}{2}} E_{\mu,\mu+1}(x^{\frac{\mu}{2}})$.

(b) If $\psi(x) = x$, then $y(x) = x^{\mu-1} E_{\mu,\mu}(x^{\mu}) + x^{\mu} E_{\mu,\mu+1}(x^{\mu})$.

(c) If $\psi(x) = x^2$, then $y(x) = x^{2(\mu-1)} E_{\mu,\mu}(x^{2\mu}) + x^{2\mu} E_{\mu,\mu+1}(x^{2\mu})$.

Proof. Considering part (b), from the (3.26) found above, we have

$$\begin{aligned} y(x) &= x^{\mu-1} E_{\mu,\mu}(x^{\mu}) + \int_0^x \tau^{\mu-1} E_{\mu,\mu}(\tau^{\mu}) d\tau \\ &= x^{\mu-1} E_{\mu,\mu}(x^{\mu}) + \int_0^x \sum_{k=0}^{\infty} \frac{\tau^{\mu k+\mu-1}}{\Gamma(\mu k+\mu)} d\tau \\ &= x^{\mu-1} E_{\mu,\mu}(x^{\mu}) + \sum_{k=0}^{\infty} \frac{x^{\mu k+\mu}}{\Gamma(\mu k+\mu+1)} \\ &= x^{\mu-1} E_{\mu,\mu}(x^{\mu}) + x^{\mu} E_{\mu,\mu+1}(x^{\mu}). \end{aligned}$$

Similarly, taking compositions as appropriate, one can prove parts (a) and (c). □

Theorem 3.10: Let $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ and c be constants satisfying $0 \leq \mu_1 < \ldots < \mu_n \leq 1$, and let $f \in C_{-1,\psi}$ be a given function. The unique solution of the Cauchy problem

$$\sum_{i=1}^{n} \lambda_i \left({}_0^R D^{\mu_i}_{\psi(x)} y \right)(x) = f(x), \;\; x > 0, \tag{3.34}$$

$$\lim_{x \to 0} \left({}_0 I^{1-\mu_n}_{\psi(x)} y \right)(x) = c, \tag{3.35}$$

$$\lim_{x \to 0} \left({}_0 I^{1-\mu_i}_{\psi(x)} y \right)(x) = 0, \;\; 1 \leq i \leq n-1, \tag{3.36}$$

in the space $\Omega^{\mu}_{-1,\psi}$ is given by:

$$
\begin{aligned}
y(x) = c\frac{\{\psi(x)\}^{\mu_n-1}}{\lambda_n} E_{(\mu_n-\mu_1,\ldots,\mu_n-\mu_{n-1}),\mu_n}\left(-\frac{\lambda_i}{\lambda_n}\{\psi(x)\}^{\mu_n-\mu_i}\right)_{i=1}^{n-1} \\
+\int_0^x \frac{\{\psi(t)\}^{\mu_n-1}}{\lambda_n} E_{(\mu_n-\mu_1,\ldots,\mu_n-\mu_{n-1}),\mu_n}\left(-\frac{\lambda_i}{\lambda_n}\{\psi(t)\}^{\mu_n-\mu_i}\right)_{i=1}^{n-1} \\
\times f\left(\left(\psi(x)-\psi(t)\right)\right)\psi'(t)\,\mathrm{d}t, \quad (3.37)
\end{aligned}
$$

where, to make the notation more succinct, we have introduced the notation

$$
E_{(a_1,\ldots,a_{n-1}),b}\Big(z_i\Big)_{i=1}^{n-1} = E_{(a_1,\ldots,a_{n-1}),b}\Big(z_1,\ldots,z_{n-1}\Big).
$$

Proof. Making use of relations (3.20) and (3.7), the Cauchy problem (3.34) – (3.36) can be reduced to the following algebraic equation in the field $M_{-1,\psi}$:

$$
\sum_{i=1}^{n} \lambda_i S_{\mu,\psi}^{\mu_i/\mu} \cdot y = f + \lambda_n S_{\mu,\psi}^{\mu_n/\mu} \cdot y_0, \text{ where } y_0(x) = \frac{c}{\Gamma(\mu_n)}\{\psi(x)\}^{\mu_n-1}. \quad (3.38)
$$

The unique solution of (3.38) in the field $M_{-1,\psi}$ is given by:

$$
y = \frac{I_\psi}{\sum_{i=1}^{n} \lambda_i S_{\mu,\psi}^{\mu_i/\mu}} \cdot f + \frac{\lambda_n S_{\mu,\psi}^{\mu_n/\mu}}{\sum_{i=1}^{n} \lambda_i S_{\mu,\psi}^{\mu_i/\mu}} \cdot y_0. \quad (3.39)
$$

Now we reduce the solution (3.39) to the form (3.37). Using Lemma 3.1(c), for the first part we obtain:

$$
\begin{aligned}
y_1(x) := \frac{I_\psi}{\sum_{i=1}^{n} \lambda_i S_{\mu,\psi}^{\mu_i/\mu}} \cdot f = \frac{S_{\mu,\psi}^{\frac{-\mu_n}{\mu}}}{\lambda_n\left(I_\psi - \sum_{i=1}^{n-1} \frac{-\lambda_i}{\lambda_n} S_{\mu,\psi}^{(\mu_i-\mu_n)/\mu}\right)} \cdot f \\
= \int_0^x \frac{\{\psi(t)\}^{\mu_n-1}}{\lambda_n} E_{(\mu_n-\mu_1,\ldots,\mu_n-\mu_{n-1}),\mu_n}\left(-\frac{\lambda_i}{\lambda_n}\{\psi(t)\}^{\mu_n-\mu_i}\right)_{i=1}^{n-1} \quad (3.40) \\
\times f\left(\left(\psi(x)-\psi(t)\right)\right)\psi'(t)\,\mathrm{d}t.
\end{aligned}
$$

Furthermore, using relation (3.23), for the second part of (3.39) we find:

$$
\begin{aligned}
y_2(x) &:= \frac{\lambda_n S_{\mu,\psi}^{\mu_n/\mu}}{\sum_{i=1}^{n} \lambda_i S_{\mu,\psi}^{\mu_i/\mu}} \cdot y_0 = y_0 - \sum_{j=1}^{n-1} \frac{\lambda_j S_{\mu,\psi}^{\mu_j/\mu}}{\sum_{i=1}^{n} \lambda_i S_{\mu,\psi}^{\mu_i/\mu}} \\
&= \frac{c}{\Gamma(\mu_n)} \{\psi(x)\}^{\mu_n - 1} - c \sum_{j=1}^{n-1} \frac{\lambda_j}{\lambda_n} \{\psi(x)\}^{2\mu_n - \mu_j - 1} \\
&\qquad \times E_{(\mu_n-\mu_1,\ldots,\mu_n-\mu_{n-1}),2\mu_n-\mu_j} \left(-\frac{\lambda_i}{\lambda_n} \{\psi(x)\}^{\mu_n-\mu_i} \right)_{i=1}^{n-1} \\
&= c \frac{\{\psi(x)\}^{\mu_n-1}}{\lambda_n} E_{(\mu_n-\mu_1,\ldots,\mu_n-\mu_{n-1}),\mu_n} \left(-\frac{\lambda_i}{\lambda_n} \{\psi(x)\}^{\mu_n-\mu_i} \right)_{i=1}^{n-1},
\end{aligned} \tag{3.41}
$$

where in the last step we used the identity discussed in [67] for summing multivariate Mittag-Leffler functions based on multinomial coefficient identities.

Combining (3.40) and (3.41), we obtain the solution (3.37). Using the same technique as in the previous result, it is easy deduce the inclusion $y \in \Omega_{-1,\psi}^{\mu}$. So we omit the straightforward details. □

Corollary 3.3: Let $A, B, C, \alpha, \beta, \gamma, c$ be constants satisfying $0 \leq \alpha < \beta < \gamma \leq 1$, and let $f \in C_{-1,\psi}$ be a given function. The unique solution of the Cauchy problem

$$
A \left({}_0^R D_{\psi(x)}^{\alpha} y \right)(x) + B \left({}_0^R D_{\psi(x)}^{\beta} y \right)(x) + C \left({}_0^R D_{\psi(x)}^{\gamma} y \right)(x) = f(x), \ \ x > 0,
$$

$$
\lim_{x \to 0} \left({}_0 I_{\psi(x)}^{1-\alpha} y \right)(x) = \lim_{x \to 0} \left({}_0 I_{\psi(x)}^{1-\beta} y \right)(x) = 0, \ \lim_{x \to 0} \left({}_0 I_{\psi(x)}^{1-\gamma} y \right)(x) = c,
$$

in the space $\Omega_{-1,\psi}^{\mu}$ is given by:

$$
\begin{aligned}
y(x) = {} & \frac{c}{C} \{\psi(x)\}^{\gamma-1} E_{\gamma-\alpha,\gamma-\beta;\gamma} \left(-\frac{A}{C} \{\psi(x)\}^{\gamma-\alpha}, -\frac{B}{C} \{\psi(x)\}^{\gamma-\beta} \right) \\
& + \frac{1}{C} \int_0^x \{\psi(t)\}^{\gamma-1} E_{\gamma-\alpha,\gamma-\beta;\gamma} \left(-\frac{A}{C} \{\psi(t)\}^{\gamma-\alpha}, -\frac{B}{C} \{\psi(t)\}^{\gamma-\beta} \right) \\
& \qquad \times f\left(\left(\psi(x) - \psi(t) \right) \right) \psi'(t) \, \mathrm{d}t,
\end{aligned}
$$

where $E_{\gamma-\alpha,\gamma-\beta;\gamma}$ is the bivariate Mittag-Leffler function defined and studied in [65].

Proof. This is simply the $n = 2$ case of Theorem 3.10. □

Chapter 4

CAPUTO FRACTIONAL CALCULUS WITH RESPECT TO FUNCTIONS

In this chapter, we set up appropriate function spaces for Caputo fractional differentiation of one function with respect to another function, and prove some properties and relationships relevant to these operators. We also define the algebraic structures and elements needed for Mikusiński's operational calculus in the context of these operators, and demonstrate how this operational calculus formalism can be used to solve different types of fractional differential equations.

4.1 Function spaces for Caputo derivatives

The Caputo fractional derivative of a function with respect to ψ (2.7) is not defined on the whole space $C_{\alpha,\psi}$, since it requires at least n times differentiability of the function with respect to ψ. Therefore, we now introduce a new function space within $C_{\alpha,\psi}$ which is suitable for dealing with this type of fractional derivative.

Definition 4.1: Let $\alpha \in \mathbb{R}$, $n \in \mathbb{Z}_0^+$, and let $\psi : \mathbb{R}^+ \to \mathbb{R}_0^+$ be as defined above. The space $C^n_{\alpha,\psi}$ is defined to be the set of all functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that

$$f_\psi^{[n]} := \left(\frac{1}{\psi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \right)^n f \in C_{\alpha\,\psi}.$$

It is immediately clear that $C^n_{\alpha,\psi}$ is a vector space, and that $C^0_{\alpha,\psi} \equiv C_{\alpha,\psi}$. We note further properties of these function spaces in the following lemmas.

Lemma 4.1 ([45]): The ψ-convolution acts on the function spaces $C_{\alpha,\psi}$, $\alpha > -1$, as follows:

$$\left(f \in C_{\alpha,\psi},\ g \in C_{\beta,\psi}\right) \Rightarrow g *_\psi f \in C_{\alpha+\beta+1,\psi} \subseteq C_{-1,\psi}, \qquad \alpha,\beta \geq -1. \tag{4.1}$$

Therefore, since the fractional integral operator with respect to ψ has a ψ-convolution representation in the space $C_{\alpha,\psi}$, namely:

$$\left({}_0I^{\mu}_{\psi(x)}f\right)(x) = \left(h_{\mu,\psi} *_\psi f\right)(x), \ \ h_{\mu,\psi}(x) := \frac{\{\psi(x)\}^{\mu-1}}{\Gamma(\mu)}, \ f \in C_{\alpha,\psi}, \ \alpha \geq -1, \tag{4.2}$$

it follows that this operator maps $C_{\alpha,\psi}$ into itself, for any $\alpha \geq -1$. More specifically,

$${}_0I^{\mu}_{\psi(x)} : C_{\alpha,\psi} \to C_{\mu+\alpha,\psi} \subset C_{\alpha,\psi}, \qquad \alpha \geq -1, \ \mu > 0.$$

Lemma 4.2: Let $\alpha \geq -1$ and $n \in \mathbb{Z}^+$.

(a) If $f \in C^n_{\alpha,\psi}$, then $f^{[k]}_\psi(0) := \lim_{x\to 0} f^{[k]}_\psi(x)$ is finite for all $k = 0,1,2,\ldots,n-1$, and the function

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x > 0, \\ f(0) & \text{if } x = 0, \end{cases}$$

is in the function space $C^{n-1}[0,\infty)$.

(b) If $f \in C^n_{\alpha,\psi}$, then $f \in C^n(0,\infty) \cap C^{n-1}[0,\infty)$.

(c) A function f is in the function space $C^n_{\alpha,\psi}$ if and only if it can be written in the following form for some function $g \in C_{\alpha,\psi}$ and some constants $c_0, c_1, \ldots, c_{n-1} \in \mathbb{R}$:

$$f(x) = \left({}_0I^{n}_{\psi(x)}g\right)(x) + \sum_{k=0}^{n-1} c_k \frac{\{\psi(x)\}^k}{k!}, \qquad x \geq 0,$$

where in fact $g = f^{[n]}_\psi$ and $c_k = f^{[k]}_\psi(0)$ for $k = 0,1,\ldots,n-1$.

(d) If $\alpha > 0$, then $C^n_{\alpha,\psi} \subset C_{n+\alpha,\psi} \subset C_{\alpha,\psi}$.

Proof. We prove the three parts of the lemma one by one as follows.

(a) By definition of the function space $C^n_{\alpha,\psi}$, we know that $f_\psi^{[n]} \in C_{\alpha,\psi}$. Let us fix $X > 0$, and note that $f_\psi^{[n]} \in C[\eta, X]$ for any $\eta \in (0, X)$, so by the Fundamental Theorem of Calculus we have

$$\int_\eta^X f_\psi^{[n]}(t)\psi'(t)\,\mathrm{d}t = f_\psi^{[n-1]}(X) - f_\psi^{[n-1]}(\eta),$$

where both sides of the above equation are continuous functions of $\eta \in (0, X]$. Moreover, since

$$\lim_{\eta\to 0} \int_\eta^X f_\psi^{[n]}(t)\psi'(t)\,\mathrm{d}t = \int_0^X f_\psi^{[n]}(t)\psi'(t)\,\mathrm{d}t < +\infty,$$

we get

$$f_\psi^{[n-1]}(0) := \lim_{\eta\to 0} f_\psi^{[n-1]}(\eta) = f_\psi^{[n-1]}(X) - \int_0^X f_\psi^{[n]}(t)\psi'(t)\,\mathrm{d}t < +\infty,$$

thus

$$f_\psi^{[n-1]}(X) = \int_0^X f_\psi^{[n]}(t)\psi'(t)\,\mathrm{d}t + f_\psi^{[n-1]}(0).$$

Now we can let $x = X$ be a free variable, and define $f_\psi^{[n-1]}(0) := f_\psi^{[n-1]}(0)$, to obtain that $f_\psi^{[n-1]}$ is continuous on $[0, \infty)$. Repeating the above argument another time, we have

$$\int_\eta^X f_\psi^{[n-1]}(t)\psi'(t)\,\mathrm{d}t = f_\psi^{[n-2]}(X) - f_\psi^{[n-2]}(\eta),$$

and

$$
\begin{aligned}
f_{\psi}^{[n-2]}(x) &= \int_0^x f_{\psi}^{[n-1]}(t)\,\psi'(t)\,\mathrm{d}t + f_{\psi}^{[n-2]}(0) \\
&= \int_0^x \psi'(t) \int_0^t \psi'(u) f_{\psi}^{[n]}(u)\,\mathrm{d}u\,\mathrm{d}t + f_{\psi}^{[n-2]}(0) + \int_0^x f_{\psi}^{[n-1]}(0)\,\psi'(t)\,\mathrm{d}t.
\end{aligned}
$$

By a process of finite descent, we finally obtain the representation

$$
f(x) = \left({}_0I^n_{\psi(x)} f_{\psi}^{[n]}\right)(x) + \sum_{k=0}^{n-1} f_{\psi}^{[k]}(0) \frac{\{\psi(x)\}^k}{k!}, \qquad x \geq 0, \tag{4.3}
$$

where

$$
f_{\psi}^{[k]}(0) := \lim_{x \to 0} f_{\psi}^{[k]}(x) < +\infty, \qquad 0 \leq k \leq n-1,
$$

and this completes the proof.

(b) Since $f_{\psi}^{[n]} \in C_{\alpha,\psi}$ and $\psi \in C^{\infty}(0,\infty)$, we have $f_{\psi}^{[n]} \in C(0,\infty)$ and therefore $f \in C^n(0,\infty)$. The fact that $f \in C^{n-1}[0,\infty)$ was already shown in part (a) above.

(c) Assuming $f \in C^n_{\alpha,\psi}$, the left-to-right implication is already proved by Eq. (4.3) above. The converse can be checked by a simple verification.

(d) If $f \in C^n_{\alpha,\psi}$, then $g = f_{\psi}^{[n]} \in C_{\alpha,\psi}$, so by Lemma 4.1, ${}_0I^n_{\psi(x)} g \in C_{n+\alpha,\psi} \subset C_{\alpha,\psi}$. Then from the representation (4.3), the result follows provided that $k > \alpha$ for $k = 0, 1, \ldots, n-1$. □

Theorem 4.1: If $n \in \mathbb{Z}_0^+$ and $f \in C^n_{-1,\psi}$, then the Caputo fractional derivative of f with respect to ψ is well-defined to any order μ with $0 \leq \mu \leq n$, and we have

$$
{}_0^C D^{\mu}_{\psi(x)} f \in \begin{cases} C_{-1,\psi}, & n-1 < \mu \leq n; \\ C^{k-1}[0,\infty) \subset C_{-1,\psi}, & n-k-1 < \mu \leq n-k,\ k = 1, \ldots, n-1. \end{cases}
$$

Proof. For $n-1 < \mu \leq n$, the μth Caputo derivative with respect to ψ is exactly the $(n-\mu)$th integral with respect to ψ of the function $f_{\psi}^{[n]}$, which is in $C_{-1,\psi}$ by definition of the $C^n_{-1,\psi}$ space, so the result follows from the last part of Lemma 4.1.

For $n-k-1<\mu\leq n-k$, $k=1,\dots,n-1$, we know $f\in C^n(0,\infty)\cap C^{n-1}[0,\infty)$ by Lemma 4.2(b), and then by mapping properties of the Riemann–Liouville fractional integral with respect to ψ, it follows that ${}_0^C D^{\mu}_{\psi(x)}f\in C^{k-1}[0,\infty)$. The inclusion $C^{k-1}[0,\infty)\subset C_{-1,\psi}$ follows from Eq. (3.1). $\square$

Theorem 4.2: If $n\in\mathbb{Z}^+$ and $f\in C^n_{-1,\psi}$, then the Riemann–Liouville and Caputo fractional derivatives of f with respect to ψ, to any order μ with $n-1<\mu\leq n$, are connected by the following relation:

$$\left({}_0^R D^{\mu}_{\psi(x)}f\right)(x)=\left({}_0^C D^{\mu}_{\psi(x)}f\right)(x)+\sum_{k=0}^{n-1}\frac{f^{[k]}_{\psi}(0)}{\Gamma(1+k-\mu)}\left\{\psi(x)\right\}^{k-\mu},\qquad x\geq 0. \quad (4.4)$$

Proof. Making use of Eq. (4.3), we obtain

$$\begin{aligned}
{}_0^R D^{\mu}_{\psi(x)}f(x)&=\left(\frac{1}{\psi'(x)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\right)^n {}_0I^{n-\mu}_{\psi(x)}f(x)\\
&=\left(\frac{1}{\psi'(x)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left({}_0I^{n-\mu}_{\psi(t)}\left\{\left({}_0I^{n}_{\psi(t)}f^{[n]}_{\psi}\right)(t)+\sum_{k=0}^{n-1}f^{[k]}_{\psi}(0)\frac{\{\psi(t)\}^k}{k!}\right\}\right)(x)\\
&=\left({}_0I^{n-\mu}_{\psi(x)}f^{[n]}_{\psi}\right)(x)+\left(\frac{1}{\psi'(x)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left({}_0I^{n-\mu}_{\psi(t)}\left\{\sum_{k=0}^{n-1}f^{[k]}_{\psi}(0)\frac{\{\psi(t)\}^k}{k!}\right\}\right)(x)\\
&=\left({}_0^C D^{\mu}_{\psi(x)}f\right)(x)+\sum_{k=0}^{n-1}\frac{f^{[k]}_{\psi}(0)}{\Gamma(1+k-\mu)}\left\{\psi(x)\right\}^{k-\mu},\qquad x\geq 0,
\end{aligned}$$

where in the last line we have used the well-known rules for fractional differintegration of power functions, and their generalisations to fractional differintegration with respect to ψ, namely:

$$\begin{aligned}
{}_0I^{\mu}_{\psi(x)}\left\{\psi(x)\right\}^{\nu}&=\frac{\Gamma(\nu+1)}{\Gamma(\nu+\mu+1)}\left\{\psi(x)\right\}^{\nu+\mu},\qquad \mu\geq 0,\nu>-1,\ x>0;\\
{}_0^R D^{\mu}_{\psi(x)}\left\{\psi(x)\right\}^{\nu}&=\frac{\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)}\left\{\psi(x)\right\}^{\nu-\mu},\qquad \mu\geq 0,\nu>-1,\ x>0.
\end{aligned}$$

$\square$

Remark 4.1: If $f\in C^n_{-1,\psi}$, then we can see from (4.4) that the Riemann–Liouville fractional derivative of f with respect to ψ is usually not in the space $C_{-1,\psi}$, since the

Caputo derivative with respect to ψ is in this space but (4.4) also involves negative powers of $\psi(x)$. There are three particular cases which are exceptions:

(a) If $\mu = n \in \mathbb{N}$, then

$$ {}_0^R D^{\mu}_{\psi(x)} f = {}_0^C D^{\mu}_{\psi(x)} f = f^{[n]}_{\psi} \in C_{-1,\psi}. $$

(b) If $f^{[k]}_{\psi}(0) = 0$ for all $k = 0, \ldots, n-1$, then

$$ {}_0^R D^{\mu}_{\psi(x)} f = {}_0^C D^{\mu}_{\psi(x)} f \in C_{-1,\psi}. $$

(c) If $0 < \mu < 1$, then ${}_0^R D^{\mu}_{\psi(x)} f \in C_{-1,\psi}$ because in this case Eq. (4.4) becomes

$$ \left({}_0^R D^{\mu}_{\psi(x)} f \right)(x) = \left({}_0^C D^{\mu}_{\psi(x)} f \right)(x) + \frac{f(0)}{\Gamma(1-\mu)} \{\psi(x)\}^{-\mu}. $$

Theorem 4.3: If $\alpha \geq -1$ and $n-1 < \mu \leq n \in \mathbb{Z}^+$, and $f \in C^n_{\alpha,\psi}$, then

$$ \left({}_0 I^{\mu}_{\psi(x)} {}_0^C D^{\mu}_{\psi(x)} f \right)(x) = f(x) - \sum_{k=0}^{n-1} f^{[k]}_{\psi}(0) \frac{\{\psi(x)\}^k}{k!}, \qquad x \geq 0. \tag{4.5} $$

Proof. This follows directly from the definition of Caputo derivatives with respect to ψ, the semigroup property for fractional integrals with respect to ψ, and the relation (4.3) proved above. □

Remark 4.2: The formulae that we have proved above, such as (4.4) and (4.5), are already seen in the existing literature on fractional calculus with respect to functions [3, 20]. Our new contribution is in proving these results in the setting of the new function spaces defined in Definition 3.1. This is important because one of the key uses of Mikusiński's operational calculus is in extending the formalism of Laplace transforms to a broader class of functions: differential equations can be solved using this method even if the functions involved do not have Laplace transforms. Therefore, it is necessary to prove that the various results concerning fractional operators hold true in the function spaces that are relevant for this work.

Theorem 4.4: Let $n \in \mathbb{Z}_0^+$. If f is a function in $C_{-1,\psi}^n$ with $f(0) = \ldots = f_\psi^{[n-1]}(0) = 0$, and g is a function in $C_{-1,\psi}^1$, then their ψ-convolution $h = f *_\psi g$ is in $C_{-1,\psi}^{n+1}$ and satisfies $h(0) = \ldots = h_\psi^{[n]}(0) = 0$.

Proof. Firstly, consider the case $n = 0$. Then $g \in C[0,\infty)$ by Lemma 4.2(b), so $h \in C[0,\infty)$ with $h(0) = 0$, and

$$\begin{aligned}
h_\psi^{[1]}(x) &= \int_0^x \frac{1}{\psi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x} g\left(\psi^{-1}\left(\psi(x)-\psi(t)\right)\right) f(t)\psi'(t)\,\mathrm{d}t + \frac{1}{\psi'(x)} g\left(\psi^{-1}(0)\right) f(x)\psi'(x) \\
&= \int_0^x g_\psi^{[1]}\left(\psi^{-1}\left(\psi(x)-\psi(t)\right)\right) f(t)\psi'(t)\,\mathrm{d}t + g(0)f(x) \\
&= g_\psi^{[1]} *_\psi f(x) + g(0)f(x), \qquad x > 0.
\end{aligned}$$

Since both f and $g_\psi^{[1]}$ are in the space $C_{-1,\psi}$, we have from (4.1) that $h_\psi^{[1]} \in C_{-1,\psi}$, so the result is proved in the case $n = 0$.

For $n = 1$, it follows from Lemma 4.2(b) that $f \in C[0,\infty)$. Using $f(0) = 0$ along with the argument used above for $h_\psi^{[1]} = \left(g *_\psi f\right)_\psi^{[1]}$, we find $h_\psi^{[1]}(0) = 0$ and

$$\begin{aligned}
h_\psi^{[2]}(x) &= \left(g_\psi^{[1]} *_\psi f + g(0)f\right)_\psi^{[1]} \\
&= g_\psi^{[1]} *_\psi f_\psi^{[1]}(x) + g_\psi^{[1]}(x)f(0) + g(0)f_\psi^{[1]}(x) \\
&= g_\psi^{[1]} *_\psi f_\psi^{[1]}(x) + g(0)f_\psi^{[1]}(x), \qquad x > 0.
\end{aligned}$$

Since $g_\psi^{[1]} \in C_{-1,\psi}$ and $f_\psi^{[1]} \in C_{-1,\psi}$, we have that $h_\psi^{[2]} \in C_{-1,\psi}$. Now the case $n = 1$ is solved.

Repeating the above methodology n times, we reach $h(0) = \ldots = h_\psi^{[n]}(0) = 0$ and the expression

$$h_{\psi}^{[n+1]}(x) = g_{\psi}^{[1]} *_{\psi} f_{\psi}^{[n]}(x) + g(0) f_{\psi}^{[n]}(x), \qquad x > 0,$$

which means that $h \in C_{-1,\psi}^{n+1}$, as required. □

4.2 Operational calculus for Caputo fractional derivatives

The function space $C_{-1,\psi}$ turns out to be a particularly suitable setting for operational calculus performed using the operators of fractional calculus with respect to the function ψ. Chapter 3 of this thesis establishes appropriate algebraic structures on this function space as part of the setup for applying Mikusiński's operational calculus to Riemann–Liouville fractional derivatives with respect to a function. The groundwork for the following theorem was also laid in Chapter 3, but we state it formally for the first time as follows.

Theorem 4.5: The elements $h_{\mu,\psi}$ and $S_{\mu,\psi}$ for $\mu > 0$, together with the identity element I_{ψ}, comprise a multiplicative group within the field $M_{-1,\psi}$ which is isomorphic to the group $(\mathbb{R}, +)$.

Proof. Firstly, the behaviour of the functions $h_{\mu,\psi}$ under ψ-convolution is well known:

$$h_{\mu,\psi}^{n} = \underbrace{h_{\mu,\psi} *_{\psi} \ldots *_{\psi} h_{\mu,\psi}}_{n} = h_{n\mu,\psi}(x), \qquad \mu > 0, n \in \mathbb{N}.$$

Therefore, it makes sense to define fractional (positive real) "powers" of $h_{\mu,\psi}$ within $C_{-1,\psi}$ as follows:

$$h_{\mu,\psi}^{\nu} := h_{\nu\mu,\psi}, \qquad \mu > 0, \nu > 0. \tag{4.6}$$

For negative powers of $h_{\mu,\psi}$, we use the multiplicative inverse $S_{\mu,\psi} \in M_{-1,\psi}$, since $h_{\mu,\psi}^{-1} = S_{\mu,\psi}$. Then, we have all real powers of both $h_{\mu,\psi}$ and $S_{\mu,\psi}$, given as follows:

$$h^{\nu}_{\mu,\psi} = \begin{cases} h_{\nu\mu,\psi}, & \nu > 0, \\ I_{\psi}, & \nu = 0, \\ \frac{I_{\psi}}{h_{-\nu\mu,\psi}}, & \nu < 0; \end{cases} \qquad S^{\nu}_{\mu,\psi} = \begin{cases} \frac{I_{\psi}}{h_{\nu\mu,\psi}}, & \nu > 0, \\ I_{\psi}, & \nu = 0, \\ h^{-\lambda}_{-\nu\mu,\psi}, & \nu < 0. \end{cases}$$

Therefore, the set of all $h_{\mu,\psi}$ and $S_{\mu,\psi}$ with $\mu > 0$, together with the identity element I_{ψ}, is exactly the set of all real powers of any one (non-identity) of these elements.

Given the composition properties or index laws as follows:

$$h_{\mu,\psi} *_{\psi} h_{\nu,\psi} = h_{\mu+\nu,\psi}, \qquad S_{\mu,\psi} *_{\psi} S_{\nu,\psi} = S_{\mu+\nu,\psi}, \qquad \mu, \nu > 0,$$

we know that this set forms a multiplicative group within the field $M_{-1,\psi}$ which is isomorphic to the group of real numbers under addition. □

Any Riemann–Liouville fractional integral with respect to ψ can be represented by an element of the field $M_{-1,\psi}$, via

$$_0I^{\mu}_{\psi(x)} f = \frac{I_{\psi}}{S_{\mu,\psi}} \cdot f \in M_{-1,\psi}, \tag{4.7}$$

but what about the Caputo fractional derivative with respect to ψ? The following theorem shows how this too can be embedded in the field $M_{-1,\psi}$ for appropriate functions f.

Theorem 4.6: Let $n \in \mathbb{Z}^+$ and $n-1 < \mu \le n$. For any $f \in C^n_{-1,\psi}$, we define a new function $f_{\mu,\psi}$ by

$$f_{\mu,\psi}(x) = \sum_{k=0}^{n-1} f^{[k]}_{\psi}(0) \frac{\{\psi(x)\}^k}{k!}, \qquad x \ge 0. \tag{4.8}$$

Then the Caputo derivative of f with respect to ψ is given by the following relation in the field $M_{-1,\psi}$:

$$ {}_0^C D^{\mu}_{\psi(x)} f = S_{\mu,\psi} \cdot f - S_{\mu,\psi} \cdot f_{\mu,\psi}. \tag{4.9}$$

Proof. From Eq. (4.5), using the new $f_{\mu,\psi}$ notation, we have

$$\left({}_0 I^{\mu}_{\psi(x)} {}_0^C D^{\mu}_{\psi(x)} f \right)(x) = f(x) - f_{\mu,\psi}(x), \qquad x \geq 0.$$

Multiplying by $S_{\mu,\psi}$ on both sides of this equation, we obtain the required result. □

For the application of Mikusiński's operational calculus to solving fractional differential equations in the setting of Caputo fractional derivatives with respect to functions, it is important to identify those elements of $M_{-1,\psi}$ which can be represented by means of functions in the rng $C_{-1,\psi}$. One useful class of such elements is given by the following result.

Lemma 4.3: Let F be a function of several complex variables $\mathbf{z} = (z_1, \ldots, z_n)$ defined by a multiple power series with complex coefficients, and let $\mathbf{z_0} = (z_{10}, \ldots, z_{n0})$ be a point, with all $z_{k0} \neq 0$, at which this multiple power series is convergent, say

$$F(\mathbf{z_0}) = \sum_{i_1,\ldots,i_n=0}^{\infty} c_{i_1,\ldots,i_n} z_{10}^{i_1} \times \ldots \times z_{n0}^{i_n} = K \in \mathbb{C}.$$

Then, for any $\mu_1, \ldots, \mu_n, \nu > 0$, the formal power series

$$F\left(S_{\mu,\psi}\right) = S_{-\nu,\psi} \sum_{i_1,\ldots,i_n=0}^{\infty} c_{i_1,\ldots,i_n} \left(S_{-\mu_1,\psi}\right)^{i_1} \times \ldots \times \left(S_{-\mu_n,\psi}\right)^{i_n}$$

can be interpreted as an element of the commutative rng $C_{-1,\psi}$, namely as the following function:

$$F\left(S_{\mu,\psi}\right) = \sum_{i_1,\ldots,i_n=0}^{\infty} c_{i_1,\ldots,i_n} h_{\nu+\mu_1 i_1+\ldots+\mu_n i_n,\psi}(x). \tag{4.10}$$

Proof. This follows directly from Theorem 3.8 after a simple substitution of parameters. ☐

The result of Lemma 4.3 enables several elements of the field $M_{-1,\psi}$, expressed initially using division within this field, to be interpreted as functions in the rng $C_{-1,\psi}$. Some examples are given in Lemma 3.1, and we continue that work with the following results.

Theorem 4.7: (a) If $\mu > 0$ and $\nu, \omega \in \mathbb{R}$ and $m \in \mathbb{Z}^+$, then

$$\frac{S^{\nu}_{\mu,\psi}}{\left(S_{\mu,\psi} - \omega\right)^m} = \{\psi(x)\}^{(m-\nu)\mu-1} E^{m}_{\mu,(m-\nu)\mu}\left(\omega\{\psi(x)\}^{\mu}\right),$$

where $E^{\gamma}_{\alpha,\beta}(z)$ is the three-parameter Mittag-Leffler function due to Prabhakar [68].

(b) If $\mu, \mu_1, \mu_2, \nu > 0$ and $\omega_1, \omega_2 \in \mathbb{R}$, then

$$\frac{S^{-\nu}_{\mu,\psi}}{\left(I_{\psi} - \omega_1 S^{-\mu_1/\mu}_{\mu,\psi} - \omega_2 S^{-\mu_2/\mu}_{\mu,\psi}\right)^m} = \{\psi(x)\}^{\nu\mu-1}$$
$$\times E^{n}_{\mu_1,\mu_2,\nu\mu}\left(\omega_1\{\psi(x)\}^{\mu_1}, \omega_2\{\psi(x)\}^{\mu_2}\right),$$

where $E^{\delta}_{\alpha,\beta,\gamma}(x,y)$ is the bivariate Mittag-Leffler function due to [65].

Proof. We proceed one by one, following the methodology of Lemma 3.1 and earlier works such as [35, 36].

(a) This result is a slight generalisation of Lemma 3.1(b), and the proof is similar:

$$\begin{aligned}
\frac{S^{\nu}_{\mu,\psi}}{\left(S_{\mu,\psi}-\omega\right)^{m}} &= \frac{S^{\nu-m}_{\mu,\psi}}{\left(I_{\psi}-\omega S^{-1}_{\mu,\psi}\right)^{m}} = \sum_{i=0}^{\infty}\frac{(m)_i\omega^i}{i!}S^{-i+\nu-m}_{\mu,\psi} \\
&= \sum_{i=0}^{\infty}\frac{(m)_i\omega^i}{i!}h_{(m-\nu+i)\mu,\psi}(x) \\
&= \{\psi(x)\}^{(m-\nu)\mu-1}\sum_{i=0}^{\infty}\frac{(m)_i\left(\omega\{\psi(x)\}^{\mu}\right)^i}{i!\Gamma(\mu i+(m-\nu)\mu)} \\
&= \{\psi(x)\}^{(n-\nu)\mu-1}E^{m}_{\mu,(m-\nu)\mu}\left(\rho\{\psi(x)\}^{\mu}\right).
\end{aligned}$$

(b) This result follows from some manipulation of series, this time trinomial double series rather than binomial series. We omit the straightforward details. □

4.3 Applications to fractional differential equations

In this section, we will show how the version of Mikusiński's operational calculus constructed above can help to solve Cauchy problems with constant coefficients in the setting of Caputo fractional derivatives of a function with respect to another function.

Example 4.1: As a preliminary illustration of the method, let us begin with the following very simple problem using these fractional derivatives:

$$\left({}^{C}_{0}D^{\mu}_{\psi(x)}y\right)(x)-\lambda y(x)=f(x), \qquad x>0, \tag{4.11}$$

$$y^{[k]}_{\psi}(0)=c_k, \qquad k=0,1,\ldots,n-1, \tag{4.12}$$

where $\lambda, c_0, c_1, \ldots, c_{n-1}\in\mathbb{R}$ and $\mu>0$ with $n-1<\mu\le n$ are given constants, we assume $f\in C_{-1,\psi}$ if $\mu\in\mathbb{N}$ or $f\in C^1_{-1,\psi}$ if $\mu\notin\mathbb{N}$, and we seek a solution function y lying within $C^n_{-1,\psi}$.

By Theorem 4.6, the fractional differential equation (4.11) is equivalent to an algebraic equation in $M_{-1,\psi}$ as follows:

$$S_{\mu,\psi}\cdot y-\lambda y=S_{\mu,\psi}\cdot y_{\mu,\psi}+f,$$

where the function $y_{\mu,\psi}$ is completely given by the initial conditions (4.12):

$$y_{\mu,\psi}(x)=\sum_{k=0}^{n-1}c_k\frac{\{\psi(x)\}^k}{k!}.$$

Therefore, the unique solution in the field $M_{-1,\psi}$ can be expressed algebraically as follows:

$$y=\frac{I_\psi}{S_{\mu,\psi}-\lambda}\cdot f+\frac{S_{\mu,\psi}}{S_{\mu,\psi}-\lambda}\cdot y_{\mu,\psi}. \tag{4.13}$$

To obtain a classical solution of the initial value problem (4.11)–(4.12), we need the right-hand side of this relation to be interpretable as a function in the space $C^m_{-1,\psi}$. Using Theorem 4.7(a) and the definition of multiplication in the rng $C_{-1,\psi}$, we obtain for the first term in (4.13):

$$y_1(x):=\frac{I_\psi}{S_{\mu,\psi}-\lambda}\cdot f=\int_0^x\left(\psi(x)-\psi(t)\right)^{\mu-1}E_{\mu,\mu}\left(\lambda\left(\psi(x)-\psi(t)\right)^{\mu}\right)f(t)\psi'(t)\,\mathrm{d}t. \tag{4.14}$$

Note that this is the exact solution of the same fractional differential equation (4.11) with homogeneous initial conditions, i.e. with all $c_k=0$.

Meanwhile, the second term in (4.13) is a solution of the homogeneous version of the fractional differential equation (4.11) with the given initial conditions (4.12). It can be written as follows:

$$\begin{aligned}y_2(x)&:=\frac{S_{\mu,\psi}}{S_{\mu,\psi}-\lambda}\cdot y_{\mu,\psi}=\sum_{k=0}^{n-1}c_k\frac{S_{\mu,\psi}}{S_{\mu,\psi}-\lambda}\cdot\left\{\frac{\{\psi(x)\}^k}{k!}\right\}\\&=\sum_{k=0}^{n-1}c_k\frac{I_\psi}{I_\psi-\lambda S_{\mu,\psi}^{-1}}\cdot h_{k+1,\psi}(x)=\sum_{k=0}^{n-1}c_k\frac{S_{\mu,\psi}^{-(k+1)/\mu}}{I_\psi-\lambda S_{\mu,\psi}^{-1}}\\&=\sum_{k=0}^{n-1}c_k\{\psi(x)\}^kE_{\mu,k+1}\left(\lambda\{\psi(x)\}^{\mu}\right),\end{aligned}$$

where in the last step we have again made use of Theorem 4.7(a). Combining the expressions obtained so far for $y_1(x)$ and $y_2(x)$, we get the solution of the initial value problem (4.11)–(4.12) in the form

$$y(x) = \int_0^x \left(\psi(x)-\psi(t)\right)^{\mu-1} E_{\mu,\mu}\left(\lambda\left(\psi(x)-\psi(t)\right)^{\mu}\right) f(t)\psi'(t)\,\mathrm{d}t \\ + \sum_{k=0}^{n-1} c_k \{\psi(x)\}^k E_{\mu,k+1}\left(\lambda\{\psi(x)\}^{\mu}\right).$$

Now we consider the general linear constant-coefficient fractional differential equation using Caputo fractional derivatives with respect to a function ψ.

Theorem 4.8: Let $m \in \mathbb{Z}^+$, $\mu > \mu_1 > \ldots > \mu_m \geq 0$ with $n_i - 1 < \mu_i \leq n_i \in \mathbb{Z}^+$ for $i = 1, \ldots, m$ and $n-1 < \mu \leq n \in \mathbb{Z}^+$, and let $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and $c_1, \ldots, c_n \in \mathbb{R}$ be constants. Consider the initial value problem

$$\left({}_0^C D^{\mu}_{\psi(x)} y\right)(x) - \sum_{i=1}^{m} \lambda_i \left({}_0^C D^{\mu_i}_{\psi(x)} y\right)(x) = f(x), \qquad x > 0, \tag{4.15}$$

$$y^{[k]}_{\psi}(0) = c_k, \qquad k = 0, 1, \ldots, n-1, \tag{4.16}$$

where the function f is assumed to lie in $C_{-1,\psi}$ if $\mu \in \mathbb{N}$ or in $C^1_{-1,\psi}$ if $\mu \notin \mathbb{N}$, and the unknown function y is to be determined in the space $C^n_{-1,\psi}$. This initial value problem has a unique solution in the space $C^n_{-1,\psi}$, which can be written as

$$y(x) = y_1(x) + \sum_{k=0}^{n-1} c_k u_k(x), \qquad x \geq 0,$$

where $y_1(x)$ is the solution of the fractional differential equation (4.15) with homogeneous initial conditions (i.e. with all $c_k = 0$) and the set of functions $u_k(x)$ satisfies

$$\left(\frac{1}{\psi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right)^l u_k(0) = \delta_{kl}, \qquad k, l = 0, \ldots, n-1.$$

Explicitly, the function y_1 is given by

$$y_1(x) = \int_0^x \{\psi(x)-\psi(t)\}^{\mu-1}$$

$$\times E_{(\mu-\mu_1,\ldots,\mu-\mu_m),\mu}\left(\lambda_1\{\psi(x)-\psi(t)\}^{\mu-\mu_1},\ldots,\lambda_m\{\psi(x)-\psi(t)\}^{\mu-\mu_m}\right) f(t)\psi'(t)\,\mathrm{d}t,$$

and each function u_k is given by

$$u_k(x) = \frac{\{\psi(x)\}^k}{k!} + \sum_{i=l_k+1}^{m} \lambda_i \{\psi(x)\}^{k+\mu-\mu_i}$$

$$\times E_{(\mu-\mu_1,\ldots,\mu-\mu_m),k+1+\mu-\mu_i}\left(\lambda_1\{\psi(x)\}^{\mu-\mu_1},\ldots,\lambda_m\{\psi(x)\}^{\mu-\mu_m}\right),$$

where the numbers $l_0, l_1, \ldots, l_{n-1}$ are determined from the following condition depending on the monotonically decreasing sequence of numbers $n_i = \lfloor \mu_i \rfloor + 1 \in \mathbb{Z}^+$:

$$\begin{cases} n_{l_k} \geq k+1, \\ n_{l_k+1} \leq k, \end{cases}$$

or $l_k = 0$ if $n_i \leq k$ for all i, or similarly $l_k = m$ if $n_i \geq k+1$ for all i.

Proof. We seek a solution function $y \in C^m_{-1,\psi}$, so the initial value problem (4.15)–(4.16) is equivalent, via Theorem 4.6, to an algebraic equation in $M_{-1,\psi}$ as follows:

$$S_{\mu,\psi}\cdot y - S_{\mu,\psi}\cdot y_{\mu,\psi} - \sum_{i=1}^{m} \lambda_i \left(S_{\mu_i,\psi}\cdot y - S_{\mu_i,\psi}\cdot y_{\mu_i,\psi}\right) = f, \tag{4.17}$$

where

$$y_{\mu,\psi}(x) = \sum_{k=0}^{n-1} c_k \frac{\{\psi(x)\}^k}{k!}, \qquad y_{\mu_i,\psi}(x) = \sum_{k=0}^{n_i-1} c_k \frac{\{\psi(x)\}^k}{k!}, \ \ i = 1,\ldots,m.$$

The algebraic equation (4.17) has a unique solution in $M_{-1,\psi}$, which, using the power formalisms of Theorem 4.5, can be written as follows:

$$y = y_1 + y_2 = \frac{I_\psi}{S_{\mu,\psi} - \sum_{i=1}^{m} \lambda_i S_{\mu,\psi}^{\mu_i/\mu}} \cdot f + \frac{S_{\mu,\psi}\cdot y_{\mu,\psi} - \sum_{i=1}^{m} \lambda_i S_{\mu,\psi}^{\mu_i/\mu}\cdot y_{\mu_i,\psi}}{S_{\mu,\psi} - \sum_{i=1}^{m} \lambda_i S_{\mu,\psi}^{\mu_i/\mu}}. \tag{4.18}$$

First half: $y_1(x)$**.** We have

$$y_1 = \frac{I_\psi}{S_{\mu,\psi} - \sum_{i=1}^{m} \lambda_i S_{\mu,\psi}^{\mu_i/\mu}} \cdot f = \frac{S_{\mu,\psi}^{-1}}{I_\psi - \sum_{i=1}^{m} \lambda_i S_{\mu,\psi}^{(\mu-\mu_i)/\mu}} \cdot f, \tag{4.19}$$

so by lemma 3.1(c), we can interpret the field element $y_1 \in M_{-1,\psi}$ as the following function in the rng $C_{-1,\psi}$:

$$y_1(x) = \int_0^x \{\psi(x) - \psi(t)\}^{\mu-1}$$
$$\times E_{(\mu-\mu_1,\dots,\mu-\mu_m),\mu}\left(\lambda_1 \{\psi(x) - \psi(t)\}^{\mu-\mu_1}, \dots, \lambda_m \{\psi(x) - \psi(t)\}^{\mu-\mu_m}\right) f(t)\psi'(t)\,\mathrm{d}t.$$

In the case $\mu \notin \mathbb{N}$, we have $f \in C^1_{-1,\psi}$ by assumption, and y_1 is the ψ-convolution of f with a function in $C_{-1,\psi}$, so Theorem 4.4 gives $y_1 \in C^1_{-1,\psi}$. In the case $\mu \in \mathbb{N}$, we have $f \in C_{-1,\psi}$ by assumption, and y_1 is the ψ-convolution of f with a smooth function in $C^1_{-1,\psi}$, so again Theorem 4.4 gives $y_1 \in C^1_{-1,\psi}$.

We now aim to show that $y_1 \in C^n_{-1,\psi}$. Multiplying the identity (4.19) by the rightmost denominator, we obtain

$$y_1(x) = \left({}_0I^{\mu}_{\psi(x)} f\right)(x) + \sum_{i=1}^{m} \lambda_i \left({}_0I^{\mu-\mu_i}_{\psi(x)} y_1\right)(x), \tag{4.20}$$

where all the orders of integration are positive and the smallest among them is $\mu - \mu_1$. This can be rewritten as

$$y_1(x) = \left({}_0I^{\mu-\mu_1}_{\psi(x)} \phi_1\right)(x), \qquad \phi_1 \in \begin{cases} C_{-1,\psi}, & \mu \in \mathbb{N}, \\ C^1_{-1,\psi}, & \mu \notin \mathbb{N}. \end{cases} \tag{4.21}$$

Substituting the expression (4.21) for y_1 back into the right-hand side of (4.20), and using the semigroup property of Riemann–Liouville fractional integrals with respect to ψ, we achieve the following as the next stage:

$$y_1(x) = \left({}_0I_{\psi(x)}^{\min(\mu,2(\mu-\mu_1))}\phi_2\right)(x), \qquad \phi_2 \in \begin{cases} C_{-1,\psi}, & \mu \in \mathbb{N}, \\ C^1_{-1,\psi}, & \mu \notin \mathbb{N}. \end{cases} \tag{4.22}$$

Repeating the same arguments a total of $p = \lfloor \frac{\mu}{\mu-\mu_1} \rfloor + 1$ times, we arrive ultimately at the following representation for y_1:

$$y_1(x) = \left({}_0I_{\psi(x)}^{\mu}\phi_p\right)(x), \qquad \phi_p \in \begin{cases} C_{-1,\psi}, & \mu \in \mathbb{N}, \\ C^1_{-1,\psi}, & \mu \notin \mathbb{N}. \end{cases} \tag{4.23}$$

In the case $\mu = n \in \mathbb{N}$, it now follows using Lemma 4.2(c) that $y_1 \in C^n_{-1,\psi}$ and also that $y_1(0) = \ldots = (y_1)_{\psi}^{[n-1]}(0) = 0$. In the case $\mu \notin \mathbb{N}$, $n-1 < \mu < n$, we have (4.23) giving y_1 as the ψ-convolution of $\phi_p \in C^1_{-1,\psi}$ with the function $h = h_{\mu,\psi} \in C^{n-1}_{-1,\psi}$ which satisfies $h(0) = \ldots = h_{\psi}^{[n-2]}(0) = 0$. By Theorem 4.4, this means $y_1 \in C^n_{-1,\psi}$ and also $y_1(0) = \ldots = \left(\frac{1}{\psi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-1} y_1(0) = 0$.

Second half: $y_2(x)$**.** Using the definitions (4.8) for the functions $y_{\mu,\psi}(x)$, $y_{\mu_i,\psi}(x)$, we have the following expression for y_2, the second half of (4.18):

$$\begin{aligned} y_2(x) &= \frac{S_{\mu,\psi}\sum_{k=0}^{n-1} c_k \frac{\{\psi(x)\}^k}{k!} - \sum_{i=1}^{m} \lambda_i S_{\mu_i,\psi} \sum_{k=0}^{n_i-1} c_k \frac{\{\psi(x)\}^k}{k!}}{S_{\mu,\psi} - \sum_{i=1}^{m} \lambda_i S_{\mu,\psi}^{\mu_i/\mu}} \\ &= \sum_{k=0}^{n-1} c_k u_k(x), \qquad u_k(x) = \frac{S_{\mu,\psi} - \sum_{i=1}^{l_k} \lambda_i S_{\mu,\psi}^{\mu_i/\mu}}{S_{\mu,\psi} - \sum_{i=1}^{m} \lambda_i S_{\mu,\psi}^{\mu_i/\mu}} \cdot \left\{ \frac{\{\psi(x)\}^k}{k!} \right\}. \end{aligned}$$

Applying the relations from Theorem 4.5, and then Lemma 3.1(c), we obtain

$$
\begin{aligned}
u_k &= \left(I_\psi + \frac{\sum\limits_{i=l_k+1}^{m} \lambda_i S_{\mu,\psi}^{\mu_i/\mu}}{S_{\mu,\psi} - \sum\limits_{i=1}^{m} \lambda_i S_{\mu,\psi}^{\mu_i/\mu}} \right) \cdot h_{k+1,\psi} \\
&= h_{k+1,\psi} + \sum_{i=l_k+1}^{m} \lambda_i \frac{S_{\mu,\psi}^{-(k+1+\mu-\mu_i)/\mu}}{I_\psi - \sum_{i=1}^{m} \lambda_i S_{\mu,\psi}^{-(\mu-\mu_i)/\mu}} \\
&= \frac{\{\psi(x)\}^k}{k!} + \sum_{i=l_k+1}^{m} \lambda_i \{\psi(x)\}^{k+\mu-\mu_i} \\
&\qquad\qquad \times E_{(\mu-\mu_1,\ldots,\mu-\mu_m),k+1+\mu-\mu_i} \left(\lambda_1 \{\psi(x)\}^{\mu-\mu_1}, \ldots, \lambda_m \{\psi(x)\}^{\mu-\mu_m} \right).
\end{aligned}
$$

By the way the numbers l_k are defined, we have, for all $i = l_k + 1, \ldots, m$, the inequality $n_i \leq k$ and therefore $k + \mu - \mu_i \geq \mu$. This guarantees that $u_k \in C^n_{-1,\psi}$ for $k = 0, \ldots, n-1$ and also the relations

$$
\left(\frac{1}{\psi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x} \right)^l u_k(0) = \delta_{kl}, \qquad k, l = 0, \ldots, n-1.
$$

Thus, the functions $u_0(x), \ldots, u_{n-1}(x)$ generate the space of solutions for the homogeneous version of the fractional differential equation (4.15), while the solution of our initial value problem (4.15)–(4.16) is given by

$$
y(x) = y_1(x) + \sum_{k=0}^{n-1} c_k u_k(x) \in C^n_{-1,\psi},
$$

exactly as stated. □

Remark 4.3: In a recent work of Restrepo et al [49], the initial value problem (4.15)–(4.16) is considered. In fact, they consider a more general problem with variable coefficients, and in [49, §4] they consider the constant-coefficient version exactly the same to our (4.15)–(4.16). Comparing the result of our Theorem 4.8 above with their [49, Theorem 4.2], we see that both approaches end up with exactly the same solution function.

Of course this is to be expected, since both works are considering the same problem and it has a unique solution. But it acts as a useful confirmation that our work is correct and that the approach of using Mikusiński's operational calculus to solve fractional differential equations is valid. Note that, as well as the methods used here and in [49] being different, the function spaces in which uniqueness is proved are also different. Indeed, much of the difficult work in our proof above was to ensure that the obtained solution function is in the claimed function space, and this is a new contribution of ours.

Remark 4.4: In some situations, the results of Theorem 4.8 can also be used for a modified version of the initial value problem (4.15)–(4.16) in which Riemann-Liouville fractional derivatives of a function with respect to another function are used instead of Caputo ones. In particular, as we have seen in Remark 4.1, it is known that ${}^{R}_{0}D^{\mu}_{\psi(x)}y(x) = {}^{C}_{0}D^{\mu}_{\psi(x)}y(x)$ either if $\mu = n \in \mathbb{N}$ or if the following condition is valid:

$$\left(\frac{1}{\psi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right)^{k} y(0) = 0, \qquad k = 0,\ldots,n-1, n-1 < \mu \leq n.$$

In the case where $0 < \mu < 1$, again by Remark 4.1, we can use

$$\left({}^{R}_{0}D^{\mu}_{\psi(x)}y\right)(x) = \left({}^{C}_{0}D^{\mu}_{\psi(x)}y\right)(x) + \frac{y(0)}{\Gamma(1-\mu)}\left\{\psi(x)\right\}^{-\mu},$$

to reduce an initial value problem using Riemann-Liouville fractional derivatives of a function with respect to another function to a different initial value problem of the type (4.15)–(4.16) using Caputo derivatives, since the function $y(0)h_{-\mu,\psi}$ that forms the difference between these operators is a function in the space $C_{-1,\psi}$.

Chapter 5

GENERAL CONJUGATED FRACTIONAL CALCULUS

In this chapter, we focus specifically on operators which can be expressed as conjugations of classical fractional integrals and derivatives, and to define a general setting in which the methodology of conjugation relations can be applied. We shall see that any operator with a conjugation relation will naturally have varieties of Riemann–Liouville, Caputo, and Hilfer type stemming from the fractional integral type operator. As illustrative examples, we shall consider some well-known families, including fractional integrals and derivatives with respect to functions and also weighted fractional integrals and derivatives. These will help to relate our work directly to ideas and problems which are current topics of concern in the literature.

5.1 The general setup

Throughout this chapter, $[a,b]$ is a fixed interval in $\mathbb{R}$. Let $\mathscr{S}$ be an invertible linear bijection $\mathscr{S}: X \to Y$, where Y is any vector space (usually a space of functions) and X is the space of all real-valued or complex-valued functions defined on the interval $[a,b]$. We can now define a first-order "derivative" operator $\mathbb{D}$ acting on the subspace of Y which is the $\mathscr{S}$-image of the space of differentiable functions on $[a,b]$, and a "fractional integral" operator $\mathbb{I}^{\mu}$ acting on the space $\mathscr{S}\left(L^1[a,b]\right) \subset Y$, as follows:

$$\mathbb{D} := \mathscr{S} \circ \frac{\mathrm{d}}{\mathrm{d}x} \circ \mathscr{S}^{-1}, \qquad \mathbb{I}^{\mu} := \mathscr{S} \circ {}^{R}_{a}I^{\mu}_{x} \circ \mathscr{S}^{-1},$$

where μ can be any positive real number or any complex number with positive real part. Starting from these operators, we can immediately define "fractional derivative" operators of Riemann–Liouville, Caputo, and Hilfer types, on the space

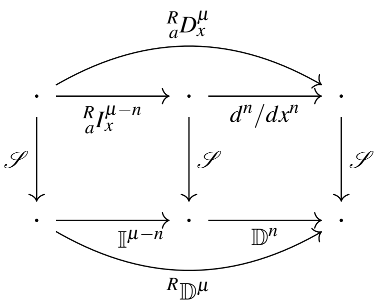
$\mathscr{S}(AC^n[a,b]) \subset Y$, as follows:

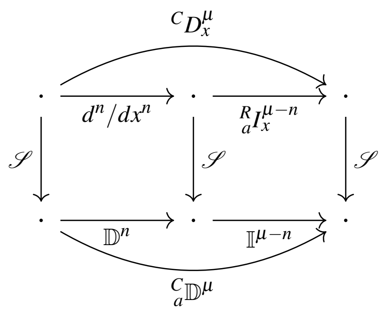
$$ {}^R\mathbb{D}^\mu := \mathbb{D} \circ \mathbb{D} \circ \cdots \circ \mathbb{D} \circ \mathbb{I}^{n-\mu} = \mathscr{S} \circ {}^R_aD^\mu_x \circ \mathscr{S}^{-1}, $$

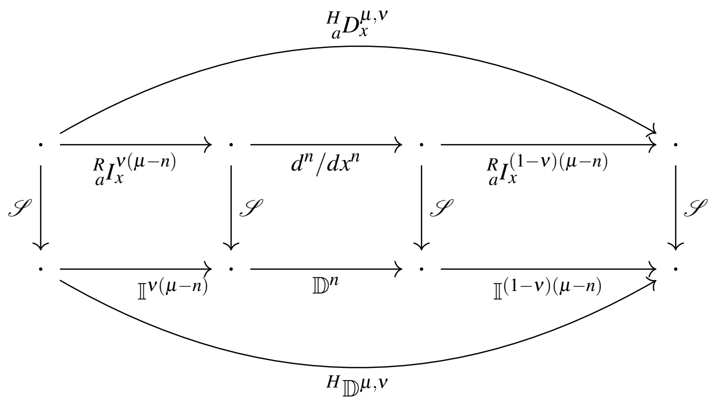
$$ {}^C\mathbb{D}^\mu := \mathbb{I}^{n-\mu} \circ \mathbb{D} \circ \mathbb{D} \circ \cdots \circ \mathbb{D} = \mathscr{S} \circ {}^C_aD^\mu_x \circ \mathscr{S}^{-1}, $$

$$ {}^H\mathbb{D}^{\mu,\nu} := \mathbb{I}^{\nu(n-\mu)} \circ \mathbb{D} \circ \mathbb{D} \circ \cdots \circ \mathbb{D} \circ \mathbb{I}^{(1-\nu)(n-\mu)} = \mathscr{S} \circ {}^H_aD^{\mu,\nu}_x \circ \mathscr{S}^{-1}, $$

where in all cases μ can be any positive real number or any complex number with non-negative real part, and $n := \lfloor \mathrm{Re}(\mu) \rfloor + 1 \in \mathbb{N}$ is the number of repetitions of the $\mathbb{D}$ operator, and in the last case ν lies in $[0,1]$ or has real part in $(0,1)$. The above conjugation relations for ${}^R\mathbb{D}^\mu$ and ${}^C\mathbb{D}^\mu$ and ${}^H\mathbb{D}^{\mu,\nu}$ follow immediately from those for $\mathbb{D}$ and $\mathbb{I}^\mu$, as we can illustrate using the notation of commutative diagrams borrowed from category theory:







Similarly, any type of operator which is defined by combining ordinary derivatives and fractional integrals can now be conjugated via $\mathscr{S}$ to the setting of Y. This includes Luchko's mth level fractional derivative, considered above; its equivalent in the new setting is the following operator:

$$ {}^{mL}\mathbb{D}^{\mu,\gamma_1,\cdots,\gamma_m} = \mathbb{I}^{\gamma_1} \circ \mathbb{D} \circ \mathbb{I}^{\gamma_2} \circ \mathbb{D} \circ \cdots \circ \mathbb{I}^{\gamma_m} \circ \mathbb{D} \circ \mathbb{I}^{m-\mu-\gamma_1-\cdots-\gamma_m}, $$

where $0 < \mu \leq 1$ and $\gamma_1, \cdots, \gamma_m \geq 0$ such that $\mu + \gamma_1 + \cdots + \gamma_k \leq k$ for $k = 1, 2, \cdots, m$. A suitable domain for this operator would be $\mathscr{S}(X^1_{mL}) \subset Y$ where X^1_{mL} is as defined in [60, Equation (49)], i.e.:

$$ \mathscr{S}(X^1_{mL}) = \left\{ f : \mathbb{I}^{\gamma_1} \circ \mathbb{D} \circ \cdots \circ \mathbb{I}^{\gamma_m} \circ \mathbb{D} \circ \mathbb{I}^{m-\mu-\gamma_1-\cdots-\gamma_m} f \in \mathscr{S}\left(AC[a,b]\right) \right\}. $$

We observe that function spaces defined using derivative and integral operators (ordinary or fractional) always have natural analogues in the space Y given by mapping them along $\mathscr{S}$. For example, it is straightforward to define the space $\mathscr{S}(X^0_{mL})$, following [60, Equation (47)], as

$$ \mathscr{S}(X^0_{mL}) = \left\{ f \in \mathbb{I}^{\mu}\left(\mathscr{S}\left(L^1[a,b]\right)\right) : \mathbb{D} \circ \mathbb{I}^{\gamma_1+\cdots+\gamma_k} f = \mathbb{I}^{\gamma_1+\cdots+\gamma_k} \circ \mathbb{D} f, \, k = 1, \cdots, n \right\}, $$

and then it follows from [60, §3.5] that ${}^{mL}\mathbb{D}^{\mu,\gamma_1,\cdots,\gamma_m} = {}^{R}\mathbb{D}^{\mu}$ on this restricted space. A commutative diagram can also be drawn to relate the new operator ${}^{mL}\mathbb{D}^{\mu,\gamma_1,\cdots,\gamma_m}$ with the original one ${}^{mL}_{a}D^{\mu,\gamma_1,\cdots,\gamma_m}_{x}$; however, as this diagram would be very large and would not demonstrate any concepts not already shown in the existing commutative diagrams above, we omit it here.

In what follows, we shall use the convention that the new fractional integrals and derivatives of Riemann–Liouville type are the same as each other with inverted orders: ${}^{R}\mathbb{D}^{-\mu} = \mathbb{I}^{\mu}$, which enables both of these to be defined for all $\mu \in \mathbb{C}$, without restrictions on $\mathrm{Re}(\mu)$. This makes sense because it is true for the original Riemann–Liouville operators: ${}^{R}_{a}D^{-\mu}_{x} f(x)$ is the analytic continuation in the complex variable μ (from the right half-plane to the left half-plane) of ${}^{R}_{a}I^{\mu}_{x} f(x)$.

Semigroup properties in the generalised fractional differintegrals now follow immediately from the well-known semigroup properties of Riemann–Liouville differintegrals.

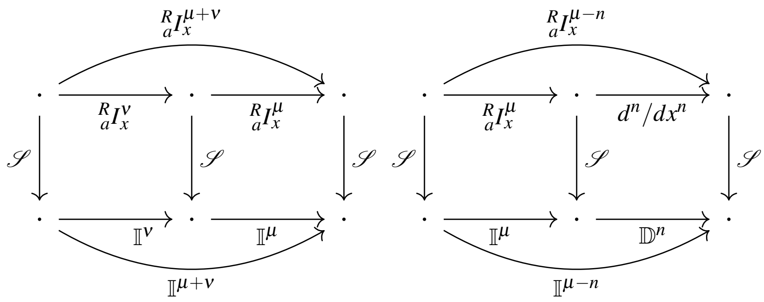
Theorem 5.1: For any $\mu, \nu \in \mathbb{C}$ with $\mathrm{Re}(\nu) > 0$ and any $n \in \mathbb{N}$, we have the following semigroup relations:

$$\mathbb{I}^{\mu}\mathbb{I}^{\nu} f = \mathbb{I}^{\mu+\nu} f,$$

$$\mathbb{D}^{n}\mathbb{I}^{\mu} f = \mathbb{I}^{\mu-n} f,$$

where in the first case $f \in \mathscr{S}\left(L^1[a,b]\right) \subset Y$ if $\mathrm{Re}(\mu) > 0$ or the appropriate subset of this space if $\mathrm{Re}(\mu) \leq 0$, and in the second case f is in the appropriate space according to whether $\mathrm{Re}(\mu) > n$ or $0 < \mathrm{Re}(\mu) \leq n$ or $\mathrm{Re}(\mu) \leq 0$.

Proof. The corresponding results in Riemann–Liouville fractional calculus are already known, so these results follow immediately from the conjugation relations. They can also be illustrated by commutative diagrams:



□

Other composition formulae for fractional differintegrals are not semigroup properties – for example, the inversion formulae for the integral of a derivative – but they can still be quite straightforwardly extended to more general fractional differintegrals via conjugation relations.

Theorem 5.2: For any $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) > 0$, and defining $n = \lfloor \mathrm{Re}(\mu) \rfloor + 1 \in \mathbb{N}$, we have:

$$\mathbb{I}^{\mu}\,{}^{R}\mathbb{D}^{\mu} f = f - \sum_{k=0}^{n-1} \mathscr{S}\left(\frac{(x-a)^{\mu-k-1}}{\Gamma(\mu-k)}\right) \cdot \left(\mathscr{S}^{-1}\,{}^{R}\mathbb{D}^{\mu-k-1} f\right)(a),$$

$$\mathbb{I}^{\mu}\,{}^{C}\mathbb{D}^{\mu} f = f - \sum_{k=0}^{n-1} \mathscr{S}\left(\frac{(x-a)^{k}}{k!}\right) \cdot \left(\mathscr{S}^{-1}\mathbb{D}^{k} f\right)(a),$$

where $f \in \mathscr{S}\left(AC^n[a,b]\right) \subset Y$.

Proof. These results follow from the corresponding well-known inversion formulae for the original Riemann–Liouville and Caputo derivatives of a function $f \in AC^n[a,b]$, namely:

$${}^{R}_{a}I^{\mu}_{x}\,{}^{R}_{a}D^{\mu}_{x} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\mu-k-1}}{\Gamma(\mu-k)} \cdot \left({}^{R}_{a}D^{\mu-k-1}_{x} f\right)(a),$$

$${}^{R}_{a}I^{\mu}_{x}\,{}^{C}_{a}D^{\mu}_{x} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{k}}{k!} \cdot \left(D^{k}_{x} f\right)(a).$$

□

The above results have made use of the $\mathscr{S}$-image of power functions. We can also quickly establish a result on the generalised fractional differintegrals of these $\mathscr{S}$-images, as follows.

Theorem 5.3: Define $h_\mu = \mathscr{S}\left(\frac{(x-a)^\mu}{\Gamma(\mu+1)}\right) \in Y$ for all $\mu \in \mathbb{C}$ and $e_{\mu;\omega} = \mathscr{S}\left(E_\mu\left(\omega(x-a)^\mu\right)\right) \in Y$ for all $\mu, \omega \in \mathbb{C}$ with $\mathrm{Re}(\mu) > 0$. Then we have the following relations:

$$\mathbb{I}^{\mu}\left(h_\nu\right) = h_{\nu+\mu}, \qquad \mu, \nu \in \mathbb{C},\ \mathrm{Re}(\nu) > -1;$$

$${}^{C}\mathbb{D}^{\mu}\left(e_{\mu;\omega}\right) = \omega e_{\mu;\omega}, \qquad \mu, \omega \in \mathbb{C},\ \mathrm{Re}(\mu) > 0.$$

Proof. These follow immediately from the corresponding relations for Riemann–Liouville and Caputo differintegrals of power functions and Mittag-Leffler functions:

$$ {}^{R}_{a}I^{\mu}_{x}\left(\frac{(x-a)^{\nu}}{\Gamma(\nu+1)}\right)=\frac{(x-a)^{\nu+\mu}}{\Gamma(\nu+\mu+1)}, \qquad \mu,\nu\in\mathbb{C},\ \mathrm{Re}(\nu)>-1; $$

$$ {}^{C}_{a}D^{\mu}_{x}\left(E_{\mu}\left(\omega(x-a)^{\mu}\right)\right)=\omega E_{\mu}\left(\omega(x-a)^{\mu}\right), \qquad \mu,\omega\in\mathbb{C},\ \mathrm{Re}(\mu)>0. $$

□

5.2 Specific cases

The above work can be seen as creating a theory of fractional powers ($\mathbb{I}^{\mu}$ and $\mathbb{D}^{\mu}$) of a modified first-order derivative operator $\mathbb{D}$ which is defined by conjugation of the usual derivative $\frac{\mathrm{d}}{\mathrm{d}x}$. Then the question arises: what particular cases of such modified operators $\mathbb{D}$ are actually useful in practice? As it turns out, some of the basic operations of calculus, used every day in differential equations, can be written in this way of conjugations, and therefore their fractional powers can be defined using the theory outlined above. We investigate some examples in the following subsections.

5.2.1 Left-sided and right-sided fractional calculus

Let us consider the operator $\mathbb{D}=-\frac{\mathrm{d}}{\mathrm{d}x}$, simply the negation of the original derivative operator. This can be written as a conjugation when we define $\mathscr{S}$ by $(\mathscr{S}f)(x)=f(-x)$, or indeed by $(\mathscr{S}f)(x)=f(c-x)$ for any constant c. In particular, defining $(\mathscr{S}f)(x)=f(a+b-x)$ is a natural choice, because then conjugation by $\mathscr{S}$ precisely swaps the left-sided and right-sided fractional integral operators on the interval $[a,b]$, as noted in [4, Eq. (2.19)].

Therefore, the model of fractional calculus we obtain by starting from the original (left-sided) operators (2.1)–(2.3) and conjugating by the operator $\mathscr{S}$ defined by $(\mathscr{S}f)(x)=f(a+b-x)$ is precisely the corresponding right-sided operators:

$$
\begin{aligned}
{}^{R}_{x}I^{\mu}_{b} f(x) &= \frac{1}{\Gamma(\mu)} \int_{x}^{b} (t-x)^{\mu-1} f(t)\,\mathrm{d}t, \\
{}^{R}_{x}D^{\mu}_{b} f(x) &= (-1)^{n} \frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}} \left({}^{R}_{x}I^{n-\mu}_{b} f(x) \right), \\
{}^{C}_{x}D^{\mu}_{b} f(x) &= (-1)^{n}\, {}^{R}_{x}I^{n-\mu}_{b} \left(\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}} f(x) \right), \\
{}^{H}_{x}D^{\mu,\nu}_{b} f(x) &= (-1)^{n}\, {}^{R}_{x}I^{\nu(n-\mu)}_{b} \left(\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}} \left({}^{R}_{x}I^{(1-\nu)(n-\mu)}_{b} f(x) \right) \right),
\end{aligned}
$$

where $x \in (a,b)$ in every case and $n = \lfloor \mathrm{Re}(\mu) \rfloor + 1 \in \mathbb{N}$ for the fractional derivatives.

This is a very simple example of a conjugation relation, but a useful one, as it means we do not need to waste time proving the same results twice for left-sided and right-sided fractional calculus: it is usually enough to prove them once for left-sided operators and then the corresponding results for right-sided operators will follow automatically.

5.2.2 Fractional calculus with respect to functions

Let us consider the operator $\mathbb{D} = A(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}$, where A is a positive function. This type of operator would be frequently used in any setting of differential equations with non-constant coefficients, and it can be written as a conjugation as follows.

If we let $\mathscr{S} = Q_{\psi}$ be an operator of right composition with a bijective differentiable function ψ, namely $\mathscr{S}(f) = f \circ \psi$, then the chain rule gives

$$
\mathbb{D} = \mathscr{S} \circ \frac{\mathrm{d}}{\mathrm{d}x} \circ \mathscr{S}^{-1} = Q_{\psi} \circ \frac{\mathrm{d}}{\mathrm{d}x} \circ Q_{\psi}^{-1} = \frac{1}{\psi'(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}.
$$

Therefore, putting $\psi = \int \frac{1}{A}$ (where the constant of integration can be chosen freely, e.g. in order to ensure $\psi(a) = 0$ if desired), we can obtain all fractional powers of the operator $\mathbb{D} = A(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}$, in a natural way that preserves properties such as semigroup and composition relations.

The model of fractional calculus thus obtained is called fractional calculus with respect to functions, and it is a general class of operators which includes, according to specific choices of ψ, the Hadamard and Katugampola fractional calculi. Studies of this class started from the work of Erdélyi [69] and Osler [18], with more detailed overviews in the textbooks of Samko et al [4, §18.2] and Kilbas et al [3, §2.5]. The operators of fractional differintegration in this setting are given explicitly as follows:

$$
\begin{aligned}
{}^{R}_{c}I^{\mu}_{\psi(x)}f(x) &= \frac{1}{\Gamma(\mu)}\int_c^x \big(\psi(x)-\psi(t)\big)^{\mu-1} f(t)\psi'(t)\,\mathrm{d}t, \\
{}^{R}_{c}D^{\mu}_{\psi(x)}f(x) &= \left(\frac{1}{\psi'(x)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n} {}^{R}_{c}I^{n-\mu}_{\psi(x)}f(x), \\
{}^{C}_{c}D^{\mu}_{\psi(x)}f(x) &= {}^{R}_{c}I^{n-\mu}_{\psi(x)}\left(\frac{1}{\psi'(x)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n} f(x), \\
{}^{H}_{c}D^{\mu,\nu}_{\psi(x)}f(x) &= (-1)^{n}\,{}^{R}_{c}I^{\nu(n-\mu)}_{\psi(x)}\left(\frac{1}{\psi'(x)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\left({}^{R}_{c}I^{(1-\nu)(n-\mu)}_{\psi(x)}f(x)\right),
\end{aligned}
$$

where $n=\lfloor\mathrm{Re}(\mu)\rfloor+1\in\mathbb{N}$ for the fractional derivatives, and these operators are the conjugations of the Riemann–Liouville operators with constant of differintegration $a=\psi(c)$. Note that the Caputo and Hilfer type derivatives in this setting were defined long after the Riemann–Liouville type operators [20, 21], but it is clear (at least from the conjugation viewpoint) that they are a natural and obvious modification after the Riemann–Liouville type operators have already been defined.

Fractional differential equations with respect to functions can therefore be seen as the fractional version of differential equations with variable coefficients that are solvable using substitution methods. Such fractional differential equations have attracted interest in recent years [45, 49, 70], and various methods for their solution have been used. We emphasise here that all problems involving these operators can be greatly simplified by using conjugation relations, which enable the solutions to be deduced directly from those for classical fractional differential equations [23, 71].

It is also worth noting that, by using the ideas of this subsection together with the previous one, we are able to construct meaningful fractional powers of the modified derivative operator $\mathbb{D} = A(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}$ for any measurable function A, either positive (from this subsection) or negative (from combining this subsection with the previous one).

5.2.3 Weighted fractional calculus

Let us consider the operator $\mathbb{D} = B(x) + \frac{\mathrm{d}}{\mathrm{d}x}$, where B is a function. This type of operator is frequently useful in differential equations, e.g. in integrating factor methods, and it can be written as a conjugation as follows.

If we let $\mathscr{S} = M_{w(x)}^{-1}$ be an operator of division by a weight function w, namely $(\mathscr{S}f)(x) = \frac{f(x)}{w(x)}$, then the product rule gives

$$\mathbb{D} = \mathscr{S} \circ \frac{\mathrm{d}}{\mathrm{d}x} \circ \mathscr{S}^{-1} = M_{w(x)}^{-1} \circ \frac{\mathrm{d}}{\mathrm{d}x} \circ M_{w(x)} = \frac{w'(x)}{w(x)} + \frac{\mathrm{d}}{\mathrm{d}x}.$$

Therefore, putting $w = \exp\left(\int B\right)$ (where the constant of integration can be chosen freely, e.g. in order to ensure $w(a) = 1$ if desired), we can obtain all fractional powers of the operator $\mathbb{D} = B(x) + \frac{\mathrm{d}}{\mathrm{d}x}$, in a natural way that preserves properties such as semigroup and composition relations.

The model of fractional calculus thus obtained is called weighted (or scaled) fractional calculus, and it is a general class of operators which includes, according to specific choices of w, the tempered and Kober–Erdélyi fractional calculi. This is a subclass of weighted fractional calculus with respect to functions, discussed in the next subsection below. Again, its conjugation relations have been noted in the literature [72] but have not been used in some recent studies [73, 74], so we feel it is important to emphasise the power and usefulness of these conjugation relations. The operators of fractional differintegration in this setting are given explicitly as follows:

$$
\begin{aligned}
{}_{a}I^{\mu}_{x,w(x)}f(x) &= \frac{1}{\Gamma(\mu)w(x)}\int_a^x (x-t)^{\mu-1}w(t)f(t)\,\mathrm{d}t,\\
{}^{R}_{a}D^{\mu}_{x,w(x)}f(x) &= \left(\frac{\mathrm{d}}{\mathrm{d}x}+\frac{w'(x)}{w(x)}\right)^n {}_{a}I^{n-\mu}_{x,w(x)}f(x),\\
{}^{C}_{a}D^{\mu}_{x,w(x)}f(x) &= {}_{a}I^{n-\mu}_{x,w(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x}+\frac{w'(x)}{w(x)}\right)^n f(x),\\
{}^{H}_{a}D^{\mu,\nu}_{x,w(x)}f(x) &= {}_{a}I^{\nu(n-\mu)}_{x,w(x)}\left(\frac{\mathrm{d}}{\mathrm{d}x}+\frac{w'(x)}{w(x)}\right)^n {}_{a}I^{(1-\nu)(n-\mu)}_{x,w(x)}f(x),
\end{aligned}
$$

where $n = \lfloor \mathrm{Re}(\mu) \rfloor + 1 \in \mathbb{N}$ for the fractional derivatives.

Weighted fractional differential equations can therefore be seen as the fractional version of differential equations with derivatives modified by addition. Such fractional differential equations have attracted interest in recent years [75], with applications in variational calculus and probabilistic processes [72, 76], and their solution is greatly simplified by using conjugation relations [55].

5.2.4 Weighted fractional calculus with respect to functions

Let us consider the operator $\mathbb{D} = B(x) + A(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}$, where A and B are functions. This type of operator can be seen as the general first-order differential operator, and it can be written as a conjugation as follows.

If we let $\mathscr{S} = M^{-1}_{w(x)} \circ Q_\psi$ be an operator of right composition with a bijective differentiable function ψ followed by division by a weight function w, namely $(\mathscr{S}f)(x) = \frac{f\circ\psi(x)}{w(x)}$, then the product rule and chain rule give

$$
\mathbb{D} = \mathscr{S}\circ\frac{\mathrm{d}}{\mathrm{d}x}\circ\mathscr{S}^{-1} = M^{-1}_{w(x)}\circ Q_\psi\circ\frac{\mathrm{d}}{\mathrm{d}x}\circ Q^{-1}_\psi\circ M_{w(x)} = \frac{1}{\psi'(x)}\left(\frac{w'(x)}{w(x)}+\frac{\mathrm{d}}{\mathrm{d}x}\right).
$$

Therefore, putting $\psi = \int \frac{1}{A}$ and $w = \exp\left(\int B/A\right)$ (where the constants of integration can both be chosen freely, according to desired initial conditions on ψ and w), we can obtain all fractional powers of the operator $\mathbb{D} = B(x) + A(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}$, in a natural way that preserves properties such as semigroup and composition relations.

The model of fractional calculus thus obtained is called weighted (or scaled) fractional calculus with respect to functions, and it is a general class of operators which includes, according to specific choices of ψ and w, the Hadamard-type and Erdélyi–Kober fractional calculi. The definition was first introduced by Agrawal in 2012 [77], and there have been just a few further studies of this class in its full generality [55, 72, 74, 76]. The operators of fractional differintegration in this setting are given explicitly as follows:

$$
\begin{aligned}
{}^{R}_{c}I^{\mu}_{\psi(x),w(x)}f(x) &= \frac{1}{\Gamma(\mu)w(x)}\int_c^x \big(\psi(x)-\psi(t)\big)^{\mu-1}w(t)f(t)\psi'(t)\,\mathrm{d}t, \\
{}^{R}_{c}D^{\mu}_{\psi(x),w(x)}f(x) &= \left(\frac{1}{\psi'(x)}\left[\frac{\mathrm{d}}{\mathrm{d}x}+\frac{w'(x)}{w(x)}\right]\right)^{n} {}^{R}_{c}I^{n-\mu}_{\psi(x),w(x)}f(x), \\
{}^{C}_{c}D^{\mu}_{\psi(x),w(x)}f(x) &= {}^{R}_{c}I^{n-\mu}_{\psi(x),w(x)}\left(\frac{1}{\psi'(x)}\left[\frac{\mathrm{d}}{\mathrm{d}x}+\frac{w'(x)}{w(x)}\right]\right)^{n} f(x), \\
{}^{H}_{c}D^{\mu,\nu}_{\psi(x),w(x)}f(x) &= {}^{R}_{c}I^{\nu(n-\mu)}_{\psi(x),w(x)}\left(\frac{1}{\psi'(x)}\left[\frac{\mathrm{d}}{\mathrm{d}x}+\frac{w'(x)}{w(x)}\right]\right)^{n} {}^{R}_{c}I^{(1-\nu)(n-\mu)}_{\psi(x),w(x)}f(x),
\end{aligned}
$$

where $n = \lfloor \mathrm{Re}(\mu) \rfloor + 1 \in \mathbb{N}$ for the fractional derivatives, and these operators are the conjugations of the Riemann–Liouville operators with constant of differintegration $a = \psi(c)$.

We note in passing that, although Hilfer fractional derivatives with respect to functions have been intensively studied, this current work is (to the best of our knowledge) the first time that a weighted Hilfer fractional derivative with respect to a function has been presented in the literature. So our operator ${}^{H}_{c}D^{\mu,\nu}_{\psi(x),w(x)}$ is new, albeit a natural definition when the Riemann–Liouville and Caputo type derivatives are already defined [76, 77].

Weighted fractional differential equations with respect to functions can therefore be seen as the fractional version of differential equations with variable coefficients modified by addition: the combination of both of the previous two subsections into one even more general class. These operators have been studied from the viewpoint of variational calculus [76] and probability theory [72], but some recent work on them [74] has failed to take account of the power of the conjugation relations, which has been further discussed recently [55] to emphasise the approach.

Chapter 6

CONCLUSION

This thesis contains the results of three publications [45, 46, 59], divided into three chapters following the introduction and preliminaries. In Chapter 3, we have established a new extension of the concept of Mikusiński's operational calculus, already well-known for classical derivatives and integrals, and also in the last two decades for fractional derivatives and integrals of various types. Our work is devoted to the extension of this mathematical formalism to the class of fractional derivatives and integrals of one function with respect to another function. This is a class of operators which covers, for example, the Hadamard and Katugampola models of fractional calculus, which have various applications in modelling.

Mikusiński's operational calculus is a useful method for solving differential equations, formally similar to the method of Laplace transforms, but easier to justify rigorously, and applicable in some problems where Laplace transforms cannot be used. Therefore, the new extension defined in Chapter 3 can be used similarly to solve differential equations using fractional derivative operators with respect to functions. In this work, we have demonstrated the application of this operational calculus to solve some linear fractional differential equations with constant coefficients and Riemann–Liouville derivative operators with respect to functions.

In Chapter 4, we have studied the theory and practice of Mikusiński's operational calculus as it applies to Caputo fractional derivatives of a function with respect to another function. This is a continuation of our Chapter 3 in which we applied Mikusiński's operational calculus to Riemann–Liouville fractional derivatives of a function with respect to another function; however, this new setting requires different function spaces and different sets of results both for the functional relations and for the solutions of fractional differential equations.

Additionally, we have elucidated some of the general theory of Mikusiński's operational calculus in fractional calculus, e.g. in Theorem 4.5 above where we have described clearly the group structure generated by the field elements corresponding to the operations of fractional integration and differentiation. We have also related this theory to several types of Mittag-Leffler functions, including some recently defined ones which have emerged naturally from solving differential equations.

In order to demonstrate the usefulness of the formalism constructed here, we have used it to solve some fractional differential equations posed using Caputo derivatives with respect to functions. We compared our results with those of another recent work which studied such differential equations using the method of successive approximations, and found that their results are consistent with ours.

Finally in Chapter 5, we have provided a brief glimpse at the power of conjugation relations in fractional calculus. With an abstract linear map $\mathscr{S}$, conjugation relations allow the notion of fractional integrals and fractional derivatives to be extended to a much more general setting, while keeping many of their fundamental properties such as semigroup and composition relations and analogues of power and Mittag-Leffler functions. As concrete applications of this abstract framework, we have considered the general classes of fractional calculus with respect to functions and weighted fractional calculus. These settings and their combination allow us to define fractional powers of any first-order differential operator $B(x)+A(x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}$ in a natural way, and also to include many useful fractional calculi, including Hadamard, tempered, and Erdélyi–Kober, as special cases.

The use of ideas from abstract algebra in understanding fractional calculus has already been promoted in the operational calculus of Mikusiński [33], which has recently been applied to more general operators, such as Hilfer derivatives [37], fractional calculus with respect to functions [45, 46], or Sonine kernels and their generalisations [78, 79]. The author of reference [80] investigated the use of Mikusiński's operational calculus in the general conjugated fractional calculus and found that the structures and results established by Luchko in classical fractional calculus can be transferred to the more general setting via the conjugating bijection with minimal modifications necessary. Here we see how a little algebraic understanding can be very helpful in finding deep results and connections in fractional calculus. In future work, we hope to extend and enrich the connections between abstract algebra and fractional calculus.

It will also be possible to combine the work done here with other directions of generalisation in fractional calculus. Some general classes of operators are related to classical fractional calculus via conjugation relations, but others are related via series formulae [15] or other kernel generalisations such as Sonine kernels [81]. Some work has already begun on combining general analytic kernels with conjugation relations [19], and this sort of combination can be extended further by using the more general setting for conjugations proposed herein [82].

As the moral of this work, we would like to emphasise the necessity of taking into account connections between different operators when doing any mathematical work with new types of fractional calculus. Newly invented operators should be critically examined to understand how they fit into the existing structure, and such connections can enable many of their properties to be immediately seen without need for detailed proofs. This is not to say that operators with connections to old ones are useless – on the contrary, even a minor modification of an existing operator may discover some real applications to make itself useful – but, from the mathematical point of view, these connections and overall structure should always be borne in mind, for more clean and efficient mathematical work.

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