Generalized Parametric Blending-type Bernstein Polynomials

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ABSTRACT

In this thesis we focus on the blending type bivariate operators. We institute a modification of the blending type operators including four different parameters which can be considered as a new class or a generalization of the operators defined in [4]. We also study properties of these new type of operators on both the standard definition and Generalized Boolean sum case of them. More over we show the compatibility of the Korovkin type approximation theorem for these new families. In the last chapter some numerical results are given to analyze the behaviour of the proposed operators while the parameters are being changed.

Keywords: Korovkin approximation theorem , Voronovskaja approximation theorem , Mixed modulus of continuity, GBS operators.

ÖZ

Bu tezde, blending tipi iki değişkenli operatörlere odaklanılır. Yeni bir sınıf veya [4]'te tanımlanan operatörlerin bir genellemesi olarak kabul edilebilecek dört farklı parametre içeren blending tipi operatörlerin bir uyarlaması kurulur. Ayrıca bu yeni tip operatörlerin özellikleri hem standart tanımlarında hem de GBS durumlarında çalışır. İlaveten, bu yeni aileler için Korovkin tipi yaklaşım teoreminin uyumluluğu gösterilir. Son bölümde, parametreler değiştirildiğinde önerilen operatörlerin nasıl çalıştığını analiz etmek için bazı sayısal sonuçlar verilmiştir.

Anahtar Kelimeler: Krovkin yaklaşım teoremi, Voronovskaja yaklaşım teoremi, Karışık süreklilik modülü, GBS operatörleri.

... Dedication

This thesis is dedicated: to my Family.

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LIST OF SYMBOLS AND ABBREVIATIONS

$\Delta_{(t,s)}g(x,y)$	mixed difference of function g, the formula is given by $\Delta_{(t,s)}g(x,y) = g(x,y) - g(x,s) - g(t,y) + g(t,s)$,
$\Delta g(x_j)$	is the forward difference defined as $\Delta g(x_j) = g(x_{j+1}) - g(x_j) = g(x_j + T) - g(x_j)$ with the step size T, $\Delta^0 g(x_j) = g(x_j)$, $\quad \Delta^r g(x_j) = \Delta(\Delta^{r-1} g(x_j))$,
$\mathbb{N}$	the set of natural numbers,
$\mathbb{R}$	the set of real numbers,
$\Delta_T^k g(a)$	is the finite difference of order $k \in \mathbb{N}$ with the step size (non-zero) $h \in \mathbb{R}$ and $a \in X$ as starting point. Its formula is as follow $\Delta_T^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} g(a + jT)$,
$C(X)$	the set of all real-valued and continuous functions defined on X,
$C[a,b]$	the set of all real-valued and continuous functions defined on the compact interval $[a,b]$
(a,b)	an open interval,
$[a,b]$	a closed interval,
$B_b(X \times Y)$	B-bounded, Let X,Y be two intervals, $B_b(X \times Y) = \{g \mid g : X \times Y \longrightarrow \mathbb{R} :$ g is B-bounded on $X \times Y\}$,
$C_b(X \times Y)$	B-continuous, Let X,Y be two intervals, $C_b(X \times Y) = \{g \mid g : X \times Y \longrightarrow \mathbb{R} :$ g is B-continuous on $X \times Y\}$,
$D_B g(x,y)$	B-differential of g in the point (x,y), Let X,Y be two intervals, $D_B g(x,y) = \{g \mid g : X \times Y \longrightarrow \mathbb{R} :$ g is B-differential on $X \times Y\}$,
$Lip_k^{(\tau_1,\tau_2)}$	Lipchitz class with $\tau_1, \tau_2 \in (0,1]$.
GBS	Generalized Boolean Sum
iff	if and only if

Chapter 1

INTRODUCTION

The most famous theorem for convergence of linear positive operators is due to the Weierstrass , who introduced an important theorem named Weierstrass approximation theorem. This theorem is the first magnificent evolution in approximation theory of one real variable and played a basic role in the development of approximation theory. Bernstein's form began to be commonly used as a multifaceted approach for intuitively geometric shapes creation and analysis. Furthermore, Bernstein had provided further developments in mathematics. For instance, polynomials in the Bernstein basic theory have better numerical stability, and recursive algorithms with less complexity order providing a wide variety of applications in other related mathematics areas such as the combination of Bernstein with C programmer (the power of computer) for machine a shape in geometric design. The Bernstein polynomials were given in 1912, by Sergei N. S. Bernstein and it is as follow,

$$B_{n_1}(g;x_1) = \sum_{i=0}^{n_1} \binom{n_1}{i} x_1^i (1-x_1)^{n_1-i} g(\frac{i}{n_1}), \tag{1.1}$$

for any $g \in C[0,1]$, $x_1 \in [0,1]$ and $n_1 \in \mathbb{N}$. Bernstein operators constitute a powerful tool allowing one to replace many inconvenient calculations performed for continuous functions by more friendly calculations on approximating polynomials. For any operators defined on the space of all continuous functions $C[0,1]$, the following family of polynomials introduced in [20] which is called Blending-type Bernstein polynomials as,

$$L_t^{\alpha}(h;x) = \sum_{k=0}^{t} \Big[(1-\alpha)\binom{t-2}{k-2} x^{k-1}(1-x)^{t-k} + (1-\alpha)\binom{t-2}{k} x^{k}(1-x)^{t-k-1}$$
$$+ \alpha \binom{t}{k} x^{k}(1-x)^{t-k} \Big] h(\frac{k}{t}), \qquad t \geq 2. \tag{1.2}$$

where $\alpha \in [0,1]$. It is clear that for the case $\alpha = 1$ for L_t^{α}, reduces to the standard Bernstein operators (1.1). The parameter α is called shape parameter of the operators (1.2), which is defined on the interval $[0,1]$. The Blending-type operators (1.2) are a new family of linear positive operators which have shape preserving properties. Using modulus of continuity we are able to approximate the error of the operators (1.2) for any t in natural number. It should be noticed that as α increases or better saying as it approches to 1, it causes the error to be decreased. Both Bernstein operators and blending type Bernstein operators are studied by different researchers and recently are used in the papers [1], [2], [3], [4], [6], [10], [13], [17], [18] [19], [21], [25], [27], [28], [30], [31], [32], [33], [34], [35], [36], [37], [41], [42] and [43] .

The below definition comes to the bivariate blending type Bernstein operators as a bivariate type of the operators (1.2) as follows,

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(g;x_1,x_2) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} a_{n_1,n_2,i,j}^{(\alpha_1,\alpha_2)}(x_1,x_2) g(\frac{i}{n_1},\frac{j}{n_2}), \tag{1.3}$$

where

$$a_{n_1,n_2,i,j}^{(\alpha_1,\alpha_2)}(x_1,x_2) = \Big[\binom{n_1-2}{i}(1-\alpha_1)x_1 + \binom{n_1-2}{i-2}(1-\alpha_1)(1-x_1)$$
$$+ \binom{n_1}{i}\alpha_1 x_1(1-x_1) \Big] x_1^{i-1}(1-x_1)^{n_1-i-1}$$
$$\times \Big[\binom{n_2-2}{j}(1-\alpha_2)x_2 + \binom{n_2-2}{j-2}(1-\alpha_2)(1-x_2)$$
$$+ \binom{n_2}{j}\alpha_2 x_2(1-x_2) \Big] x_2^{j-1}(1-x_2)^{n_2-j-1},$$

where $\alpha_1, \alpha_2 \in [0,1]$ are introduced and discussed in [4].

In the present thesis, a new family of blending-type Bernstein operators which includes the operators (1.3) as a special case is introduced such that it is a modification of the operators (1.3) consists of four parameters α_1, α_2, s_1 and s_2 as follows,

for any bivariate continuous function h, positive integers s_1, s_2 and fixed real numbers $\alpha_1, \alpha_2 \in [0,1]$ we have the following,

$$T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2) = \begin{cases} \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} p_{n_1,n_2,i,j}^{(\alpha_1,\alpha_2,s_1,s_2)}(x_1,x_2)h(\frac{i}{n_1},\frac{j}{n_2}), & n_1,n_2 \geq max\{s_1,s_2\}, \\ \\ B_{n_1,n_2}(h;x_1,x_2), & otherwise, \end{cases} \tag{1.4}$$

where

$$\begin{aligned} p_{n_1,n_2,i,j}^{(\alpha_1,\alpha_2,s_1,s_2)}(x_1,x_2) = (1-\alpha_1)&\left[\binom{n_1-s_1}{i-s_1}x^{i-s_1+1}(1-x_1)^{n_1-i}\right. \\ &\left.+\binom{n_1-s_1}{i}x_1^i(1-x_1)^{n_1-s_1-i+1}\right] \\ &+\alpha_1\binom{n_1}{i}x_1^i(1-x_1)^{n_1-i} \\ &\times(1-\alpha_2)\left[\binom{n_2-s_2}{j-s_2}x_2^{i-s_2+1}(1-x_2)^{n_2-j}\right. \\ &\left.+\binom{n_2-s_2}{j}x_2^j(1-x_2)^{n_2-s_2-j+1}\right] \\ &+\alpha_2\binom{n_2}{j}x_2^j(1-x_2)^{n_2-j}, \end{aligned}$$

where $B_{n_1,n_2}(h;x_1,x_2)$ is the double Bernstein operators. So by different values of s_1, s_2 the approximation of our operators will be changed, which means it is more flexible and more applicable than the operators (1.3) .

In chaper 2 some definitions about positive linear operators, modulus of continuity and some nessesary propositions and theorems, which will be used through out the thesis, is given. Also Krovkin type approximation theorem about positive linear operators with both degree n_1 and (n_1, n_2) is studied.

In chapter 3 the importance of Bernstein operators and their roles to prove the Weierstrass theorem is discussed. some propositions and theorems about Bernstein operators, with both degrees n_1 and (n_1, n_2), is written and some graphs for the given functions related with both bivariate Bernstein operators and classic Bernstein operatores are illustrated. We continued some propositions, lemmas and theorems for the operators (1.2) and we demosterated some numerical results of them on the graph for different values of α and the degree n_1. Also we bring some usefull lemmas and theorems about the operators (1.3) and we finish the chapter by some numerical results for these blending type operators with the given functions and sketch them to prove that how it is working by different values of parameters α_1, α_2 and the degrees (n_1, n_2).

Through out the chapter 4 the proof of some lemmas and theorems related with the operators (1.4) are given. Also the compatibility of Korovkin type approximation theorem for the operators (1.4) can be seen.

Chapter 5 is about the approximation property of the operators (1.4). We define the GBS case of our operators, using mixed modulus of continuity. Also we discussed about the degree of approximation of the GBS case for the operators (1.4). At the end of the chapter some numerical rusults are shown on the graph for different values of the parameters α_1, α_2 and the degrees (n_1, n_2) for both GBS operators and the operators (1.4).

In chapter 6 we write our conclusion about the different type of the approximation to show the logic behind the given definition and its approximation results.

Chapter 2

POSITIVE LINEAR OPERATORS AND KOROVKIN TYPE APPROXIMATION THEOREM

In the following chapter some basic definitions and properties related to the positive linear operators, modulus of continuty and Korovkin-type approximation theorem are given. For more information see [8], [23], [29] and [39].

2.1 Positive Linear Operators

The most constructive proofs of the Weierstrass theorem concerning the approximation of continuous functions on a compact interval by polynomials use some sequences of linear positive operators. Lets begin by constructing and studying a large class of such sequences of approximation operators. An operator L, defined on a linear space of functions V, is called linear if

$$L(af_1 + bf_2) = aL(f_1) + bL(f_2),$$

where $a, b \in \mathbb{R}$ and $f_1, f_2 \in V$, and it is called positive, if

$$L(f_1) \geqslant 0,$$

for all $f_1 \geqslant 0, f_1 \in V$.

Proposition 2.1: (see [29]) Let $Q : X_1 \longrightarrow X_2$, be linear positive operators, then

1. If $f_1, f_2 \in X_1$ with $f_1 \leqslant f_2$ then $Qf_1 \leqslant Qf_2$.

2. For any $f_1 \in X_1$ we have $|Qf_1| \leqslant Q|f_1|$.

Definition 2.1: (see [29]) Consider X_1, X_2 to be two normed linear spaces of real functions such that $X_1 \subseteq X_2$ and let $Q : X_1 \to X_2$. Then we can define a norm $\|Q\|$ as the following,

$$\|Q\| := sup\{\|Qh\| : h \in X_1, \|Q\| = 1\} = sup\{\|Qh\| : h \in X_1, 0 < \|h\| \leqslant 1\}.$$

Remark 2.1: Consider the linear positive operators $Q : C[x_1, x_2] \to C[x_1, x_2]$. Then Q is continuous and also $\|Q\| = \|Q(1)\|$.

Theorem 2.1: (see [29]) Consider the linear positive operators $Q : C[x_1, x_2] \to C[x_1, x_2]$ such that $Q(1;x) = 1$. For $p_1, p_2 > 1$, $\dfrac{1}{p_1} + \dfrac{1}{p_2} = 1$, $f_1, f_2 \in C[x_1, x_2]$ and $x \in [x_1, x_2]$. Then we have,

$$Q(|f_1 f_2|, x) \leqslant (Q(|f_1|^{p_1}, x))^{\frac{1}{p_1}} (Q(|f_2|^{p_2}, x))^{\frac{1}{p_2}}. \tag{2.1}$$

The equation (2.1) is called Hölder-type inequality for positive linear operators. It should be mention that in the equation (2.1) if $p_1 = p_2 = 2$, then it is called Cauchy-Schwarz inequality.

2.2 Modulus of Continuity

The application of the modulus of continuty for the linear positive operators is to measure the degree of convergence toward the identity operator.

Definition 2.2: (see [23]) A modulus of continuity ω is called a modulus of continuity for the function $g : [0,1] \longrightarrow \mathbb{R}$ if for all $x_1, x_2 \in [0,1]$ we have,

$$|g(x_1) - g(x_2)| \leqslant \omega|(x_1 - x_2)|,$$

where ω is a function defined as $\omega : [0, \infty) \longrightarrow [0, \infty)$.

Proposition 2.2: (see [23]) By the definition of modulus of continuity the following immediately holds:

1. $\omega(0) = 0$.

2. $\omega(t)$ is non-decreasing.

3. $\omega(t)$ is semi- additive, that is

$$\omega(t_1 + t_2) \leqslant \omega(t_1) + \omega(t_2),$$

where $\omega(t) = \omega(f;t)$.

Definition 2.3: (see [23]) The following is the definition of the modulus of smoothness of order t ,

$$\omega_t(g;\eta) := sup\{|\eta_h^t g(x)| 0 \leqslant h \leqslant \eta, x, x+h \in [a,b]\}.$$

Proposition 2.3: (see [23]) For the Definition 2.3 the following are hold,

1. $\omega_t(g;0) = 0$.

2. $\omega_t(g;.)$ is a function on real numbers which is none-negative, continuous and increasing.

3. $\forall \eta \geqslant 0, \omega_{t+1}(g;\eta) \leqslant 2\omega_t(g;\eta)$.

4. $\omega_t(g;.)$ is sub-additive.

5. If $g \in C^1[x_1,x_2]$ then $\omega_{t+1}(g;\eta) \leqslant \eta\, \omega_t(g';\eta), \eta \geqslant 0$.

6. If $g \in C^t[x_1,x_2]$ then $\omega_t \leqslant \eta^t sup_{\eta \in [x_1,x_2]} |g^{(r)}(\eta)|$.

7. $\forall \eta > 0$ and $n \in \mathbb{N}$, $\omega_t(g;n\eta) \leqslant n^k \omega_t(g;\eta)$.

8. If $\eta \geqslant 0$ is fixed, then $\omega(g;.)$ is a seminorm on $C[x_1,x_2]$.

Let $g(x_1,x_2)$ be a bivariate function which is defined on the set $[0,1] \times [0,1]$, then we have the following definitions,

Definition 2.4: (see [9]) The modulus of continuity $\omega_2(g;\eta)$, for a bivariate function $g(x_1,x_2)$ can be defined as;

$$\omega_2(g;\eta) := sup_{\sqrt{(t-x_1)^2+(s-x_2)^2}\leq\eta}|f(t,s)-f(x_1,x_2)|,$$

where $\eta > 0$ and the supremum is taken on $(t,s),(x_1,x_2) \in [0,1]\times[0,1]$.

Proposition 2.4: (see [9]) The modulus of continuity $\omega_2(g;\eta)$ satisfies the following conditions.

(i) If $0 < \eta \leq \gamma$, then $\omega_2(g;\eta) \leq \omega_2(g;\gamma)$.

(ii) The bivariate function $g(x,y)$ on the compact set $[0,1]\times[0,1]$ is uniformly continuous iff $\lim_{\eta\to 0}\omega_2(g;\eta) = 0$.

(iii) If $\eta_1 > 0$, then $\omega_2(f;\eta\eta_1) \leq (1+\eta_1)\omega_2(g;\eta)$.

In the following there is the definition of Partial modulus of continuities as;

$$\omega_{2,x}(g;\eta) := sup_{|x_1-x_2|\leq\eta}|g(x_1,y)-g(x_2,y)|. \tag{2.2}$$

$$\omega_{2,y}(g;\eta) := sup_{|y_1-y_2|\leq\eta}|g(x,y_1)-g(x,y_2)|. \tag{2.3}$$

The partial modulus of continuities satisfies the properties (i),(ii) and (iii) as well.

2.3 Korovkin-Type Approximation Theorem

A necessary and sufficient condition for the convergence of positive linear operators is provided in [39] which is initiated by P.P. Korovkin and it mostly is known as the Bohman-Korovkin theorem. After that, many other researchers has used this elegant and simple result to extend Korovkin's method to obtain some approximation results related with positive linear operators. The present section is devoted to Korovkin type approximation theorems for double sequences of functions .

Theorem 2.2: (Bohman-Korovkin Theorem)(see [39]) Let L_{n_1} be a sequence of positive linear operators such that $L_{n_1} : C([x_1,x_2]) \longrightarrow C([x_1,x_2])$ and let $\phi_j = t^j$. If $\lim_{n_1\to\infty} L_{n_1}\phi_j = \phi_j$, $j = 0,1,2$, (uniformly) on $[x_1,x_2]$, then $\lim_{n_1\to\infty} L_{n_1}g = g$ (uniformly) on $[x_1,x_2]$ for every $g \in C([x_1,x_2])$. by the Theorem 2.2 the monomials $\phi_j = t^j$, $j \in \{0,1,2\}$ have a remarkable role on the space of continuous functions in the approximation theory of linear positive operators. They are often called test functions.

Proposition 2.5: (see [22]) For the sequence of linear operators L_{n_1} and $j \in \mathbb{N}\bigcup\{0\}$ we have,

$$L_{n_1}\big((\phi_1 - x_1)^j;x_1\big) = L_{n_1}(\phi_i;x_1) - \sum_{k=0}^{j-1}\binom{j}{k}x_1^{j-k}L_{n_1}\big((\phi_1 - x_1)^k;x_1\big).$$

Proof.

$$\begin{aligned} L_{n_1}(\phi_j;x_1) &= L_{n_1}\big((\phi_1 - x_1 + x_1)^k;x_1\big) \\ &= L_{n_1}\big(\sum_{k=0}^{j}\binom{j}{k}x_1^{j-k}(\phi_1 - x_1)^k;x_1\big) \\ &= \sum_{k=0}^{j}\binom{j}{k}x_1^{j-k}L_{n_1}\big((\phi_1 - x_1)^k;x_1\big) \\ &= L_{n_1}\big((\phi_1 - x_1)^k;x_1\big) \\ &+ \sum_{k=0}^{j-1}\binom{j}{k}x_1^{j-k}L_{n_1}\big((\phi_1 - x_1)^k;x_1\big), \end{aligned}$$

□

which implies the representation of $j-th$ moment.

It should be remark that the Proposition 2.5 holds without the assumption $L_{n_1}\phi_0 = \phi_0$ and $L_{n_1}\phi_1 = \phi_1$.

Throughout the following Theorem, $C(X)$ will be the Banach space of all continuous functions of two variables on $X \subset \mathbb{R}\times\mathbb{R}$ with the usual supremum norm.

Theorem 2.3: (see [22]) Let X be a compact subset of $\mathbb{R}\times\mathbb{R}$, and let $L_{n_1,n_2} : C(X) \rightarrow$

$C(X)$ be a double sequence of positive linear operators. Then

$$L_{n_1,n_2}(g_j;x_1,x_2) \longrightarrow g_j(x_1,x_2) \quad \text{(uniformly)} \quad j=0,1,2,3 \tag{2.4}$$

iff

$$L_{n_1,n_2}(g;x_1,x_2) \longrightarrow g(x_1,x_2). \quad \text{(uniformly)} \tag{2.5}$$

Where $g_0(x_1,x_2)=1, g_1(x_1,x_2)=x_1, g_2(x_1,x_2)=x_2$ and $g_3(x_1,x_2)=x_1^2+x_2^2$.

Proof. Since, (2.5) implies (2.4), it is enough to prove that (2.4) implies (2.5). Given $g \in C(X)$, and let (x_1,x_2) be an arbitrary but fixed point in X then for every $\varepsilon > 0$ there exist a real number $\delta > 0$ such that $|g(x_1,x_2)-g(u,v)| < \varepsilon$ for all $(u,v) \in X$ satisfying $|u-x_1| < \delta$ and $|x_2-v| < \delta$ hence for all $(u,v) \in X$, we have,

$$|g(x_1,x_2)-g(u,v)| \leq \varepsilon + 2M\frac{1}{\delta^2}[(u-x_1)^2+(x_2-v)^2],$$

where $M = \|g\|_{C(X)}$. By the linearity and pozitivity of the operators L_{n_1,n_2} we have

$$\begin{aligned}
&|L_{n_1,n_2}(g;x_1,x_2)-g(x_1,x_2)| \\
&\leq L_{n_1,n_2}(|g(u,v)-g(x_1,x_2)g_0|;x_1,x_2)+|g(x_1,x_2)||L_{n_1,n_2}(g_0;x_1,x_2)-g_0(x_1,x_2)| \\
&\leq |\varepsilon L_{n_1,n_2}(g_0;x_1,x_2)+\frac{2M}{\delta^2}[L_{n_1,n_2}((u-x_1)^2+(x_2-v)^2;x_1,x_2)]| \\
&+M|L_{n_1,n_2}(g_0;x_1,x_2)-g_0(x_1,x_2)| \\
&\leq \left(\varepsilon+M+\frac{2M}{\delta^2}(E^2+F^2)\right)|L_{n_1,n_2}(g_0;x_1,x_2)-g_0(x_1,x_2)| \\
&+\frac{4M}{\delta^2}E|L_{n_1,n_2}(g_1;x_1,x_2)-g_1(x_1,x_2)|+\frac{4M}{\delta^2}F|L_{n_1,n_2}(g_2;x_1,x_2)-g_2(x_1,x_2)| \\
&+\frac{2M}{\delta^2}|L_{n_1,n_2}(g_3;x_1,x_2)-g_3(x_1,x_2)|+\varepsilon.
\end{aligned}$$

where $E := max|x_1|$, $F := max|x_2|$ and let

$$B := max\{\varepsilon+M+\frac{2M}{\delta^2}(E^2+F^2), \frac{2M}{\delta^2}, E\frac{4M}{\delta^2}, F\frac{4M}{\delta^2}\}.$$

If $n_1,n_2 \longrightarrow \infty$, then the proof is completed. □

Chapter 3

BERNSTEIN POLYNOMIALS

In this chapter we study the definition of classic Bernstein polynomials, double Bernstein polynomials for bivariate functions and Blending type Bernstein polynomials. Also some properties and theorems are given. For more information see [20] and [26].

The polynomial

$$f_n(x) = a_0 + a_1x + ... + a_nx^n,$$

is the simplest real valued functions of the variable x. Where $a_0, a_1, ...a_n$ are constant values. This function is known as algebric (ordinary) polynomial which is created by operations of multiplication and addition in the field of real numbers to x, finite number of times. The index of the highest power of x which occurs in it, is known as the degree (order) of a polynomial. The basis of the theory of approximation of functions of a real variable is a theorem discovered by Weierstrass which has a great importance in the development of the whole of mathematical analysis.

3.1 Weierstrass Approximation Theorem

The famous Weierstrass approximation theorem asserts that, if $g(x)$ is a continuous function on a closed interval $[a,b]$, and for any positive ε there is a polynomial $f(x)$ such that,

$$|g(x) - f(x)| < \varepsilon \qquad \text{for all} \qquad x \in [a,b].$$

S.N. Bernstein in 1912 introduced the following polynomials of functions, which was one of the most elegant proof for Weirestrass approximation theorem, as follows

$$B_{n_1}(g;x_1) = \sum_{i=0}^{n_1} \binom{n_1}{i} x_1^i (1-x_1)^{n_1-i} g(\frac{i}{n_1}), \tag{3.1}$$

for any $g \in C[0,1]$, $x_1 \in [0,1]$ and $n_1 \in \mathbb{N}$.

It is easily verified that the operators (3.1) are linear and monotone operators on the interval $[0,1]$ such that the conditions of the Korovkin type approximation theorem are hold on them. This justifies that $B_{n_1}(g)$ is uniformly convergent to g for all $g \in C[0,1]$.

Later Voronovskaja [49] found the asymptotic error for Bernstein polynomials, that is if $g^{''} exists$, then we have

$$\lim_{n_1 \to \infty} n_1(B_{n_1}(g;x_1) - g(x_1)) = \frac{x_1(1-x_1)}{2} g^{''}(x_1),$$

for any $g(x_1) \in [0,1]$.

Also there is another error calculation which was done by Popoviciu [48] for Bernstein polynomials, using the modulus of continuity as follow,

$$|B_{n_1}(g;x_1) - g(x_1)| \leqslant \frac{3}{2} \omega(\frac{1}{\sqrt{n_1}}).$$

There are many researchers, who work on linear operators, variation diminishing properties, convexity, the rate of convergence and Lipschitz constants as well as multivariate Bernstein polynomials.

Proposition 3.1: (see [26]) For the classic Bernstein operators (3.1) and any continuous function $g : [0,1] \longrightarrow \mathbb{R}$, the following hold,

1. $B_{n_1}(g;0) = g(0)$.

2. $B_{n_1}(g;1) = g(1)$.

3. $B_{n_1}(1;x_1) = 1$.

4. $B_{n_1}(t;x_1) = x_1$.

5. $B_{n_1}(t^2;x_1) = x_1^2 + \frac{x_1(1-x_1)}{n_1}$.

6. $B_{n_1}(t^3;x_1) = x_1^3 + \frac{3x_1^2(1-x_1)}{n_1} + \frac{(1-2x_1)x_1(1-x_1)}{n_1^2}$.

7. $B_{n_1}(t^4;x_1) = x_1^4 + \frac{6x_1^3(1-x_1)}{n_1} + \frac{x_1^2(1-x_1)^2(7-11x_1)}{n_1^2} + \frac{-6x_1^3+12x_1^3-7x_1^2+x_1}{n_1^3}$.

Proof. It is enough to prove parts 1), 2), 3) ,4) and 5). The rest Parts can be proved in the similar way.

1)
$$\begin{aligned} B_{n_1}(g;x_1) &= \sum_{i=0}^{n_1}\binom{n_1}{i}x^i(1-x)^{n-i}g(\frac{i}{n}) \\ &= \sum_{i=1}^{n_1}\binom{n_1}{i}x_1^i(1-x_1)^{n_1-i}g(\frac{i}{n_1}) + g(0) \\ &\Rightarrow B_{n_1}(g;0) = 0 + g(0) = g(0). \end{aligned}$$

2)
$$\begin{aligned} B_{n_1}(g;x) &= \sum_{i=0}^{n_1}\binom{n_1}{i}x_1^i(1-x_1)^{n_1-i}g(\frac{i}{n_1}) \\ &= \sum_{i=0}^{n_1-1}\binom{n_1}{i}x_1^i(1-x_1)^{n_1-i}g(\frac{i}{n_1}) + g(1) \\ &\Rightarrow B_{n_1}(g;1) = 0 + g(1) = g(1). \end{aligned}$$

3)
$$\begin{aligned} B_{n_1}(1;x_1) &= \sum_{i=0}^{n_1}\binom{n_1}{i}x_1^i(1-x_1)^{n_1-i} \\ &= (x_1 + (1-x_1))^{n_1} \\ &= 1. \end{aligned}$$

4)
$$\begin{aligned} B_{n_1}(t;x_1) &= \sum_{i=0}^{n_1}\binom{n_1}{i}x_1^i(1-x_1)^{n_1-i}\frac{i}{n_1} \\ &= x_1\sum_{i=1}^{n_1}\binom{n_1-1}{i-1}x_1^{i-1}(1-x_1)^{n_1-i} \\ &= x_1\sum_{i=0}^{n_1-1}\binom{n_1-1}{i}x_1^i(1-x_1)^{n_1-i-1} = x_1. \end{aligned}$$

5)
$$
\begin{aligned}
B_{n_1}(t^2;x_1) &= \sum_{i=0}^{n_1} \binom{n_1}{i} x_1^i (1-x_1)^{n_1-i} \frac{i^2}{n_1^2} \\
&= \sum_{i=1}^{n_1} \binom{n_1-1}{i-1} x_1^i (1-x_1)^{n_1-i} \frac{i}{n_1} \\
&= \sum_{i=1}^{n_1} \binom{n_1-1}{i-1} x_1^i (1-x_1)^{n_1-i} \frac{(i-1)}{n_1} \\
&+ \frac{1}{n_1} \sum_{i=1}^{n_1} \binom{n_1-1}{i-1} x_1^i (1-x_1)^{n_1-i} \\
&= \frac{(n_1-1)x_1^2}{n_1} \sum_{i=2}^{n_1} \binom{n_1-2}{i-2} x_1^{i-2} (1-x_1)^{n_1-i} + \frac{x_1}{n_1} \\
&= \frac{(n_1-1)x_1^2}{n_1} + \frac{x_1}{n_1} = x_1^2 + \frac{x_1(1-x_1)}{n_1}.
\end{aligned}
$$

□

Example 3.1: 1. Figure 3.1, Figure 3.2, Figure 3.3 and Figure 3.4 illustrate the approximation of Bernstein polynomials (3.1) to the functions $f_1(x_1) = sin(2\pi x_1^2)$, $f_2(x_1) = sin(2\pi x_1)$, $f_3(x_1) = cos(2\pi x_1^2)$ and $f_4(x_1) = x_1 cos(2\pi x_1)$ respectively for the degrees $n_1 = 10, 100$.

In Example we aim to show the approximation of the classical Bernstein polynomials to the given continuous functions on $[0,1]$which shows that by increasing the degree n_1, the approximation will be improved.

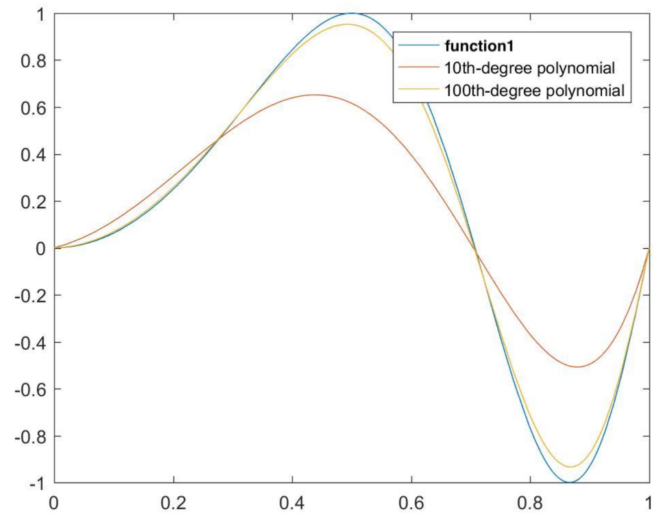

Figure 3.1: Approximation of $B_{n_1}(f_1;x_1)$ to $f_1(x_1)$ for different values of degree n_1, where $f_1(x_1)$(blue), $B_{10}(f_1;x_1)$(red) and $B_{100}(f_1;x_1)$(yellow).

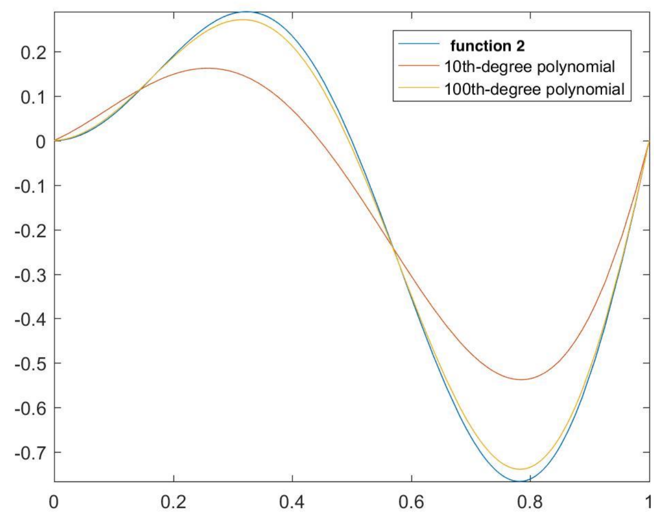

Figure 3.2: Approximation of $B_{n_1}(f_2;x_1)$ to $f_2(x_1)$ for different values of degree n_1, where $f_2(x_1)$(blue), $B_{10}(f_2;x_1)$(red) and $B_{100}(f_2;x_1)$(yellow).

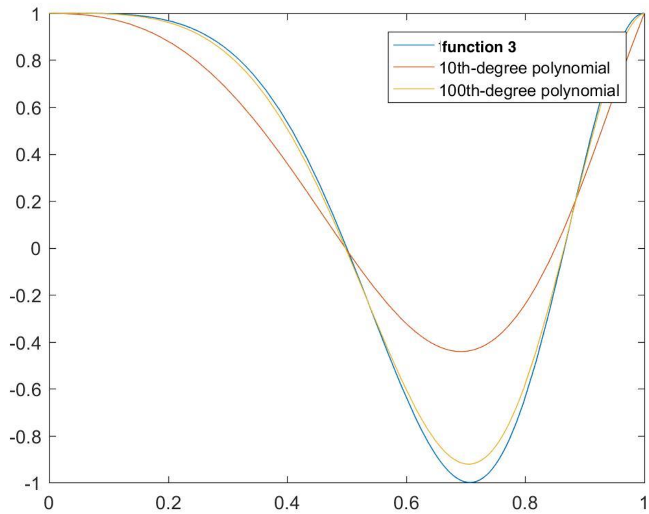

Figure 3.3: Approximation of $B_n(f_3;x_1)$ to $f_3(x_1)$ for different values of degree n_1, where $f_3(x_1)$(blue), $B_{10}(f_3;x_1)$(red) and $B_{100}(f_3;x_1)$(yellow).

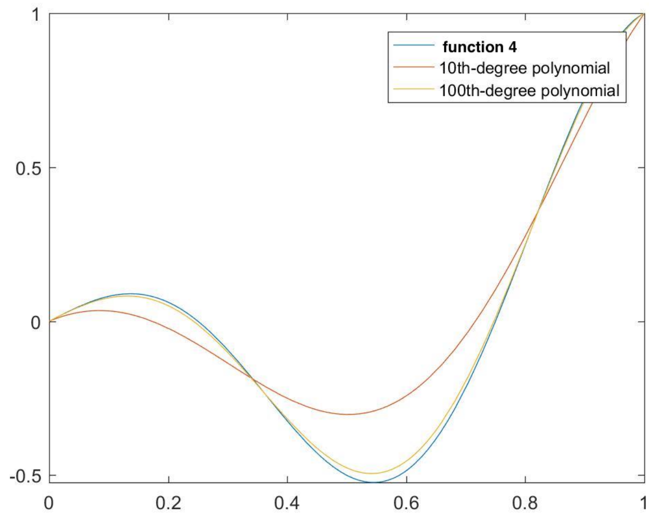

Figure 3.4: Approximation of $B_{n_1}(f_4;x_1)$ to $f_4(x_1)$ for different values of degree n_1, where $f_4(x_1)$(blue), $B_{10}(f_4;x_1)$(red) and $B_{100}(f_4;x_1)$(yellow).

There is another expression for the Bernstein polynomials as follows ,

$$B_{n_1}(g;x_1) = \sum_{i=0}^{n_1} \binom{n_1}{i} \Delta^i g_{n_1}(0) x_1^i,$$

where $g_{n_1}(x_1) = g(\frac{x_1}{n_1})$ and Δ is the forward difference operator with step size $\frac{1}{n_1}$ defined as follows,

$$\Delta g(x_i) = g(x_{i+1}) - g(x_i)$$

$$\Delta^{t+1} g(x_i) = \Delta(\Delta^t g(x_i)) = \Delta^t g(x_{i+1}) - \Delta^t g(x_i) \qquad t \geq 1,$$

where Δ^t is called the t^{th} difference.

Theorem 3.1: (see [46]) For the Bernstein polynomials B_{n_1+1} it's derivative can be defined as follows,

$$B'_{n_1+1}(g;x_1) = (n_1+1) \sum_{j=0}^{n_1} \Delta g(\frac{n_1+1}{j}) \binom{n_1}{j} x_1^j (1-x_1)^{n_1-j},$$

where the size of Δ has the step size $\frac{1}{n_1+1}$, for $n_1 \geq 0$.

Theorem 3.2: (see [46]) The derivative of $B_{n_1+j}(g;x_1)$ for any $j \geq 0$ can be written in terms of j-th difference of g as follows,

$$B'_{n_1+j}(g;x_1) = \frac{(n_1+j)!}{n_1!} \sum_{i=0}^{n_1} \Delta^j g(\frac{j}{n_1+i}) \binom{n_1}{i} x_1^i (1-x_1)^{n_1-i},$$

where the size of Δ has the step size $\frac{1}{n_1+j}$, for $n_1 \geq 0$.

Definition 3.1: (see [46]) Let $f \in C([0,1] \times [0,1])$. Then two-dimensional Bernstein operators is as follows,

$$B_{n_1,n_2}(g;x_1,x_2) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \binom{n_1}{i} \binom{n_2}{j} x_1^i (1-x_1)^{n_1-i} x_2^j (1-x_2)^{n_2-j} g(\frac{i}{n_1}, \frac{j}{n_2}). \quad (3.2)$$

It is easy to verify that the operators (3.2) are linear operators which also are monotone on the compact interval $[0,1] \times [0,1]$.

Proposition 3.2: For the Bernstein operators (3.2) the following hold,

1. $B_{n_1,n_2}(g_0;x_1,x_2) = 1$.

2. $B_{n_1,n_2}(g_1;x_1,x_2) = x_1$.

3. $B_{n_1,n_2}(g_2;x_1,x_2) = x_2$.

4. $B_{n_1,n_2}(g_3;x_1,x_2) = x_1^2 + \frac{x_1(x_1-1)}{n_1} + x_2^2 + \frac{x_2(x_2-1)}{n_2}$,

where $g_0(x_1,x_2) = 1, g_1(x_1,x_2) = x_1, g_2(x_1,x_2) = x_2$ and $g_3(x_1,x_2) = x_1^2 + x_2^2$.

Proof. By the Proposition 3.1 one can write,

$$1) \qquad B_{n_1,n_2}(g_0;x_1,x_2) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\binom{n_1}{i}\binom{n_2}{j}x_1^i(1-x_1)^{n_1-i}x_2^j(1-x_2)^{n_2-j}$$

$$= \sum_{i=0}^{n_1}\binom{n_1}{i}x_1^i(1-x_1)^{n_1-i}\sum_{j=0}^{n_2}\binom{n_2}{j}x_2^j(1-x_2)^{n_2-j} = 1$$

$$2) \qquad B_{n_1,n_2}(g_1;x_1,x_2) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\binom{n_1}{i}\binom{n_2}{j}x_1^i(1-x_1)^{n_1-i}x_2^j(1-x_2)^{n_2-j}\frac{i}{n_1}$$

$$= \sum_{i=0}^{n_1}\binom{n_1}{i}x_1^i(1-x_1)^{n_1-i}\frac{i}{n_1}\sum_{j=0}^{n_2}\binom{n_2}{j}x_2^j(1-x_2)^{n_2-j}$$

$$= x_1(x_2+1-x_2)^{n_2} = x_1.$$

$$3) \qquad B_{n,m}(g_2;x_1,x_2) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\binom{n_1}{i}\binom{n_2}{j}x_1^i(1-x_1)^{n_1-i}x_2^j(1-x_2)^{n_2-j}\frac{j}{n_2}$$

$$= \sum_{i=0}^{n_1}\binom{n_1}{i}x_1^i(1-x_1)^{n_1-i}\sum_{j=0}^{n_2}\binom{n_2}{j}x_2^j(1-x_2)^{n_2-j}\frac{j}{n_2}$$

$$= x_2(x_1+1-x_1)^{n} = x_2.$$

$$4) \qquad B_{n_1,n_2}(g_3;x_1,x_2) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\binom{n_1}{i}\binom{n_2}{j}x_1^i(1-x_1)^{n_1-i}x_2^j(1-x_2)^{n_2-j}\frac{i^2}{n_1^2}$$

$$+\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\binom{n_1}{i}\binom{n_2}{j}x_1^i(1-x_1)^{n_1-i}x_2^j(1-x_2)^{n_2-j}\frac{j^2}{n_2^2}$$

$$= \sum_{i=0}^{n_1}\binom{n_1}{i}\frac{i^2}{n_1^2}x_1^i(1-x_1)^{n_1-i}\sum_{j=0}^{n_2}\binom{n_2}{j}x_2^j(1-x_2)^{n_2-j}$$

$$+\sum_{i=0}^{n_1}\binom{n_1}{i}x_1^i(1-x_1)^{n_1-i}\sum_{j=0}^{n_2}\frac{j^2}{n_2^2}\binom{n_2}{j}x_2^j(1-x_2)^{n_2-j}$$

$$= x_1^2 + \frac{x_1(1-x_1)}{n_1} + x_2^2 + \frac{x_2(1-x_2)}{n_2}.$$

□

Proposition 3.3: For the Bernstein operators (3.2) the following hold,

1. $B_{n_1,n_2}((t-x_1);x_1,x_2)=0.$

2. $B_{n_1,n_2}((s-x_2);x_1,x_2)=0.$

3. $B_{n_1,n_2}((t-x_1)^2;x_1,x_2)=\frac{x_1(x_1-1)}{n_1}.$

4. $B_{n_1,n_2}((s-x_2)^2;x_1,x_2)=\frac{x_2(x_2-1)}{n_2}.$

Proof. By the Proposition 3.2 the result comes out. □

Theorem 3.3: (see [40]) Let $g\in[0,1]\times[0,1]$ and $B_{n_1,n_2}g$ be the two-dimensional Bernstein polynomials of g, then

$$|B_{n_1,n_2}(g;x,y)-g(x,y)|\leqslant\frac{3}{2}[\omega_{2,x}(g;\sqrt{n_1})+\omega_{2,y}(g;\sqrt{n_2})]$$

Where $\omega_{2,x}$ and $\omega_{2,y}$ are partial modulus of continuity given by 2.2 and 2.3.

Example 3.2: 1. Figure 3.5 shows the approximation of Bernstein polynomials (3.2) to $g_1(x_1,x_2)=cos(\pi x_1)x_1x_2$, for the degrees $n_1=n_2=10$ and $n_1=n_2=50$.

2. Figure 3.6 shows the approximation of Bernstein polinomials (3.2) to $g_2(x_1,x_2)=x_2sin(\pi x_1^2)$, for the degrees $n_1=n_2=10$ and $n_1=n_2=50$.

In this example we are trying to show the approximation of Bernstein polynomials (3.2) to the given bivariate continuous functions which shows that by increasing n_1 and n_2, its approximation will be improved.

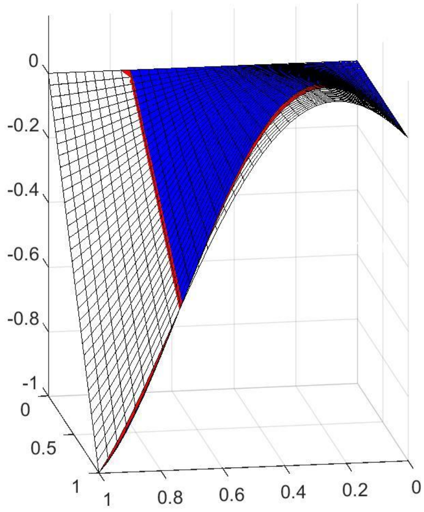

Figure 3.5: Approximation of $B_{n_1,n_2}(g_1;x_1,x_2)$ to $g_1(x_1,x_2)$ for different degrees (n_1,n_2), where $g_1(x_1,x_2)$ is denoted by blue, $B_{(10,10)}(g_1;x_1,x_2)$ by white and $B_{(50,50)}(g_1;x_1,x_2)$ by red.

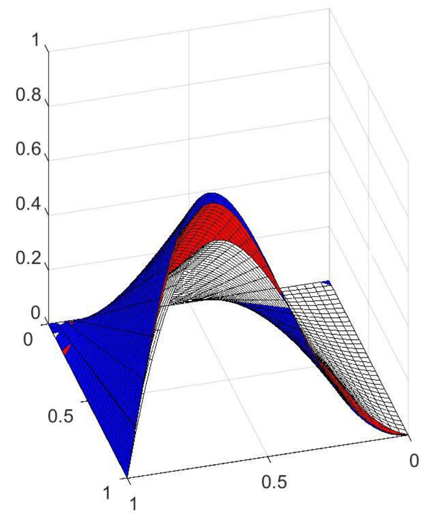

Figure 3.6: Approximation of $B_{n_1,n_2}(g_2;x_1,x_2)$ to $g_2(x_1,x_2)$ for different degrees (n_1,n_2), where $g_2(x_1,x_2)$ is denoted by blue, $B_{(10,10)}(g_2;x_1,x_2)$ by white and $B_{(50,50)}(g_2;x_1,x_2)$ by red.

The Two variable Bernstein polynomials are also written in the following form,

$$B_{n_1,n_2}(g;x_1,x_2) = \sum_{i_2=0}^{n_2}\sum_{i_1=0}^{n_1}\binom{n_1}{i_1}\binom{n_2}{i_2}\Delta_1^{i_1}\Delta_2^{i_2}g_{n_1,n_2}(0,0)x_1^{i_1}x_2^{i_2},$$

where $g_{n_1.n_2}(i_1,i_2) = g(\frac{i_1}{n_1},\frac{i_2}{n_2})$ and $\Delta_t^{i_t}$, $t \in 1,2$ is the i_t^{th} differences related to t^{th} variable x_t, and applied with the step size $\frac{1}{n_t}$.

Theorem 3.4: ([46]) For $b_1,b_2 \geq 0$ integers with $0 \leq b_i \leq n_i$, $i \in \{1,2\}$ the derivative of g is as follow,

$$B^{b_1,b_2}_{n_1+b_1,n_2+b_2} = \frac{(n_1+b_1)!}{n_1!}\frac{(n_2+b_2)!}{n_2!}\sum_{i_2=0}^{n_2}\sum_{i_1=0}^{n_1}\binom{n_1}{i_1}\binom{n_2}{i_2}\Delta_1^{b_1}\Delta_2^{b_2}f(\frac{n_1}{n_1+b_1},\frac{n_2}{n_2+b_2}) \times x_1^{i_1}(1-x_1)^{n_1-i_1}x_2^{i_2}(1-x_2)^{n_2-i_2}.$$

3.2 Blendig-Type Bernstein Operators

In this part we survey on the Blending-type Bernstein operators and some propositions about the class of such polynomials. For more information see [20] and [5].

The following defenition about Blending-type Bernstein operators, is done by Chen in [20], as a new family of generalized Bernstein operators.

Definition 3.2: (see [20]) For any function $g \in C([0,1])$, positive integer n_1 and a fixed real number $\alpha \in [0,1]$, the α- Bernstein operators can be defined as follows,

$$L^{\alpha}_{n_1}(g;x) = \sum_{i=0}^{n_1} a^{\alpha}_{n_1,i}(x_1)g(\frac{i}{n_1}), \qquad n_1 \geq 2 \tag{3.3}$$

where

$$a^{\alpha}_{n_1,i}(x_1) = \left(\binom{n_1-2}{i}(1-\alpha)x_1 + \binom{n_1-2}{i-2}(1-\alpha)(1-x_1) + \binom{n_1}{i}\alpha x_1(1-x_1) \right) x_1^{i-1}(1-x_1)^{n_1-i-1},$$

with

$$\binom{n_1}{i} = \begin{cases} \frac{n_1!}{i!(n_1-i)!}, & 0 \le i \le n_1, \\ 0, & otherwise. \end{cases}$$

.

Example 3.3: For some values $n_1 = 2,3$, $i = 0,1,2,3$ and any $x_1, \alpha \in [0,1]$, $a^{\alpha}_{n_1,i}$ will be as follows,

$$a^{\alpha}_{2,0}(x_1) = (1-\alpha x_1)(1-x_1).$$

$$a^{\alpha}_{2,1}(x_1) = 2\alpha x_1(1-x_1).$$

$$a^{\alpha}_{3,0}(x_1) = (1-\alpha x_1)(1-x_1)^2.$$

$$a^{\alpha}_{3,1}(x_1) = (1-2\alpha-3\alpha x_1)x_1(1-x_1).$$

$$a^{\alpha}_{3,2}(x_1) = (1-\alpha+3\alpha_1)x_1(1-x_1).$$

Theorem 3.5: (see [20]) For the operators (3.3) we have,

$$L^{\alpha}_{n_1}(1;x_1) = 1.$$

$$L^{\alpha}_{n_1}(x_1;x_1) = x_1.$$

$$L^{\alpha}_{n_1}(x_1^2;x_1) = x_1^2 + \frac{x_1(1-x_1)[n_1+2(1-\alpha)]}{n_1^2}.$$

$$L^{\alpha}_{n_1}(x_1^3;x_1) = x_1^3 + \frac{3x_1^2(1-x_1)[n_1+2(1-\alpha)]}{n_1^2} + \frac{x_1(1-x_1)(1-2x_1)[n_1+6(1-\alpha)]}{n_1^3}.$$

$$
\begin{aligned}
L_{n_1}^{\alpha}(x_1^4;x_1) = x_1^4 &+ \frac{6x_1^3(1-x_1)[n_1+2(1-\alpha)]}{n_1^2} \\
&+ \frac{4x_1^2(1-x_1)(1-2x_1)[n_1+6(1-\alpha)]}{n_1^3} \\
&+ \frac{[3n_1(n_1-2)+12(n_1-6)(1-\alpha)]x_1^2(1-x_1)^2}{n_1^4}. \\
&+ \frac{[n_1+14(1-\alpha)]x_1(1-x_1)}{n_1^4}
\end{aligned}
$$

Proposition 3.4: (see [20]) For the operators (3.3) the following hold,

1. $L_{n_1}^{\alpha}(a_1x_1+a_2;x_1) = a_1x_1+a_2$.

2. If for any $g_1(x_1), g_2(x_1) \in C([0,1])$ and $\alpha \in [0,1]$, $g_1(x_1) \leqslant g_2(x_1)$, then $L_{n_1}^{\alpha}(g_1;x_1) \leqslant L_{n_1}^{\alpha}(g_2;x_1)$.

3. If $g_1(x_1) \in C([0,1])$ is a non-negative function, so is the operators $L_{n_1}^{\alpha}(g_1;x_1)$ for any $\alpha \in [0,1]$.

Theorem 3.6: (see [20]) If $g(x_1)$ is a continuous function on the compact interval $[0,1]$ and for any $\alpha \in [0,1]$, then $L_{n_1}^{\alpha}(g;x_1)$ uniformly convergent to g on $[0,1]$, which can be concluded from Bohman–Korovkin theorem.

For the function $f(x_1) = 2x_1 sin(x_1^2)$ and some different values of n_1 and α the approximation of the operators (3.3) will be as follows,

1. In Figure 3.7 the approximation of the operators $L_{n_1}^{\alpha}(f;x_1)$ to function $f(x_1)$, with the degrees $n_1 = 10, 15$ and 25 with fixed $\alpha = 0.5$, is considered. It is numerically shown that incresing the degree n_1 gives better approximation.

2. In Figure 3.8 the approximation of the operators $L_{n_1}^{\alpha}(f;x_1)$ to function $f(x_1)$, with fixed degree $n_1 = 10$ and with different values of $\alpha = 0.1, 0.5$ and 0.9, is considered. It is observed that incresing the value α gives better approximation.

For the function $g(x_1) = 2x_1 cos(x_1)$ and some different values for n_1 and α the approximation of the operators (3.3) will be as follows,

3. In Figure 3.9 the approximation of the operators $L_{n_1}^{\alpha}(g;x_1)$ to function $g(x_1)$, with the degrees $n_1 = 10, 15$ and 25 with fixed $\alpha = 0.5$, is considered. It is achieved that incresing the degree n_1 gives better approximation.

4. In Figure 3.10 the approximation of the operators $L_{n_1}^{\alpha}(g;x_1)$ to function $g(x_1)$, with fixed degree $n_1 = 10$ and with different values of $\alpha = 0.1, 0.5$ and 0.9, is considered. It is clear that incresing the value of α gives better approximation.

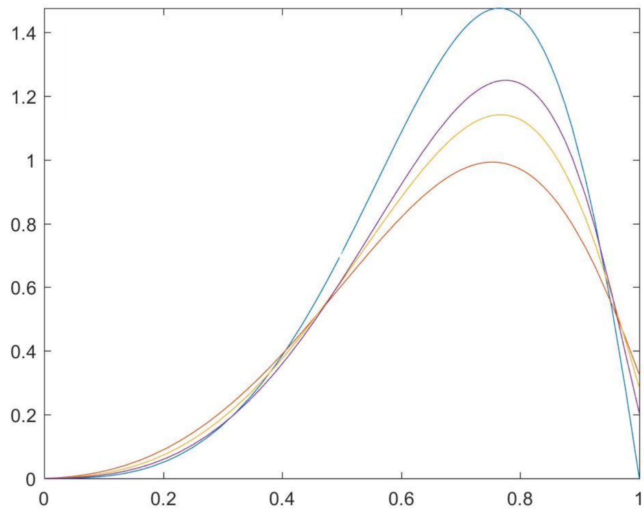

Figure 3.7: Approximation of $L_{n_1}^{\alpha}(f;x_1)$ to $f(x_1)(blue)$ for different degree $n_1 = 10(red), 15(yellow), 25(violet)$ and fixed $\alpha = 0.5$.

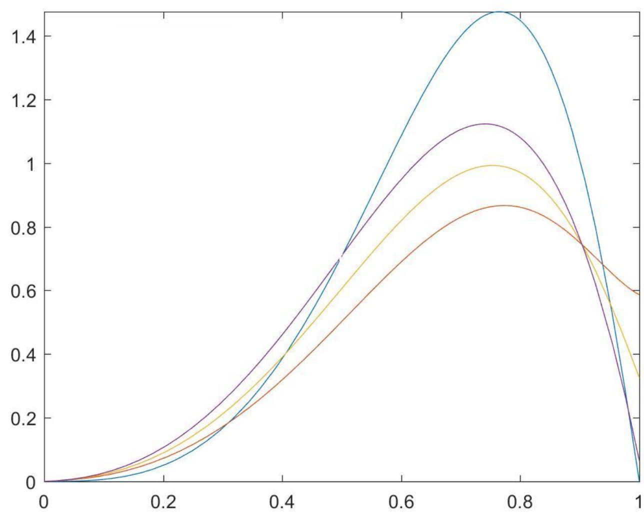

Figure 3.8: Approximation of $L_{n_1}^{\alpha}(f;x_1)$ to $f(x_1)(blue)$ for different values of $\alpha = 0.1(red), 0.5(yellow), 0.9(violet)$ and fixed $n_1 = 10$.

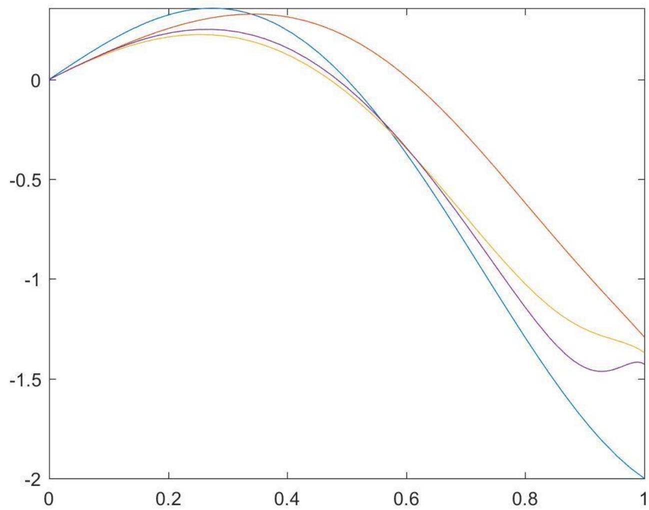

Figure 3.9: Approximation of $L_{n_1}^{\alpha}(g;x_1)$ to $g(x_1)(blue)$ for different degree $n_1 = 10(red), 15(yellow), 25(violet)$ and fixed $\alpha = 0.5$.

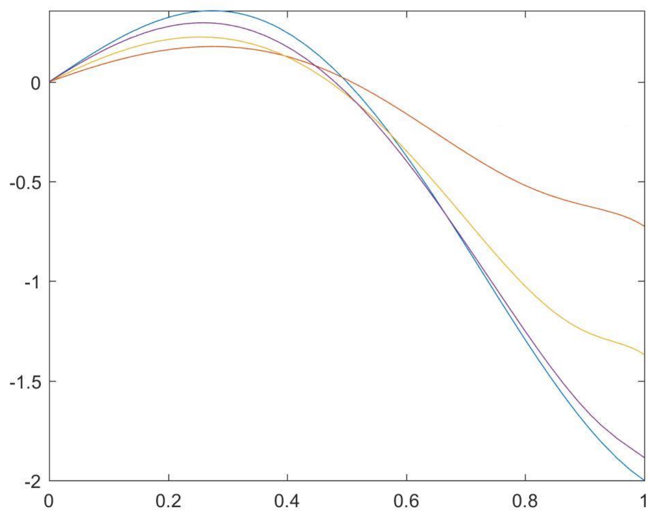

Figure 3.10: Approximation of $L_{n_1}^{\alpha}(g;x_1)$ to $g(x_1)(blue)$ for different values of $\alpha = 0.1(red), 0.5(yellow), 0.9(violet)$ and fixed $n_1 = 10$.

The larger the value of $\alpha \in [0,1]$ and n_1 the approximation is going to be better with less error.

Later Acar T. and Kajla A. in [5] introduced the two variables Blending-type Bernstein operators of degree (n_1, n_2) for bivariate functions $f(x_1,x_2) \in C([0,1]\times[0,1])$. They already extended the α-Bernstein operators (3.3) to two dimentional (α_1, α_2)- Bernstein operators for any α_1, α_2 belong to $[0,1]$.

Definition 3.3: (see [5]) For any $f(x_1,x_2) \in C([0,1]\times[0,1])$, positive integers n_1, n_2 and any fixed real numbers $\alpha_1, \alpha_2 \in [0,1]$, the bivariate extention of the operators (3.3) can be defined as follow, for any $n_1, n_2 \geq 2$,

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(g;x_1,x_2) = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} a_{n_1,n_2,i,j}^{(\alpha_1,\alpha_2)}(x_1,x_2)g(\frac{i}{n_1},\frac{j}{n_2}), \tag{3.4}$$

where

$$\begin{aligned} a_{n_1,n_2,i,j}^{(\alpha_1,\alpha_2)}(x_1,x_2) = &\left[\binom{n_1-2}{i}(1-\alpha_1)x_1 + \binom{n_1-2}{i-2}(1-\alpha_1)(1-x_1)\right. \\ &\left.+\binom{n_1}{i}\alpha_1 x_1(1-x_1)\right] x_1^{i-1}(1-x_1)^{n-i-1} \\ &\times\left[\binom{n_2-2}{j}(1-\alpha_2)x_2 + \binom{n_2-2}{j-2}(1-\alpha_2)(1-x_2)\right. \\ &\left.+\binom{n_2}{j}\alpha_2 x_2(1-x_2)\right] x_2^{j-1}(1-x_2)^{n_2-j-1}. \end{aligned}$$

Lemma 3.1: (see [5]) For the operators (3.4) the following hold,

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(1;x_1,x_2) = 1.$$

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(t;x_1,x_2) = x_1.$$

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(s;x_1,x_2) = x_2.$$

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(t^2;x_1,x_2) = x_1^2 + \frac{x_1(1-x_1)[n+2(1-\alpha_1)]}{n^2}.$$

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(s^2;x_1,x_2) = x_2^2 + \frac{x_2(1-x_2)[n_2+2(1-\alpha_2)]}{n_2^2}.$$

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(t^3;x_1,x_2) = x_1^3 + \frac{3x_1^2(1-x_1)[n_1+2(1-\alpha_1)]}{n_1^2} + \frac{x_1(1-x_1)(1-2x_1)[n_1+6(1-\alpha_1)]}{n_1^3}.$$

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(s^3;x_1,x_2) = x_2^3 + \frac{3x_2^2(1-x_2)[n_2+2(1-\alpha_2)]}{n_2^2} + \frac{x_2(1-x_2)(1-2x_2)[n_2+6(1-\alpha_2)]}{n_2^3}.$$

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(t^4;x_1,x_2) = x_2^4 + \frac{6x^3(1-x)[n+2(1-\alpha_1)]}{n^2} + \frac{4x_1^2(1-x_1)(1-2x_1)[n_1+6(1-\alpha_1)]}{n_1^3} + \frac{[3n_1(n_1-2)+12(n_1-6)(1-\alpha_1)]x_1^2(1-x_1)^2}{n_1^4} + \frac{[n_1+14(1-\alpha_1)]x_1(1-x_1)}{n_1^4}.$$

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(s^4;x_1,x_2) = x_2^4 + \frac{6x_2^3(1-x_2)[n_2+2(1-\alpha_2)]}{n_2^2} + \frac{4x_2^2(1-x_2)(1-2x_2)[n_2+6(1-\alpha_2)]}{n_2^3} + \frac{[3n_2(n_2-2)+12(n_2-6)(1-\alpha_2)]x_2^2(1-x_2)^2}{n_2^4} + \frac{[n_2+14(1-\alpha_2)]x_2(1-x_2)}{n_2^4}.$$

Corollary 3.1: (see [5]) For the operators (3.4) the following hold,

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}((t-x_1);x_1,x_2) = 0.$$

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}((s-x_2);x_1,x_2) = 0.$$

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}((t-x_1)^2;x_1,x_2) = \frac{x_1(1-x_1)[n_1+2(1-\alpha_1)]}{n_1^2}.$$

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}((s-x_2)^2;x_1,x_2) = \frac{x_2(1-x_2)[n_2+2(1-\alpha_2)]}{n_2^2}.$$

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}((t-x_1)^4;x_1,x_2) = \frac{x_2(1-x_2)[n_2+2(1-\alpha_2)]}{n_2^2}.$$

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}((t-x_2)^4;x_1,x_2) = \frac{x_1^4[3n_1^2+6n_1(1-2\alpha_1)+72(\alpha_1-1)]}{n_1^4}$$
$$+\frac{x_1^3[-6n_1^2-12n_1(1-2\alpha_1)-144(\alpha_1-1)]}{n_1^4}$$
$$+\frac{x^2[3n_1^2-n_1+6n_1(1-2\alpha_1)+86(\alpha_1-1)]}{n_1^4}$$
$$+\frac{x_1(n_1-14(\alpha_1-1))}{n_1^4}.$$

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}((s-x_2)^4;x_1,x_2) = \frac{x_2^4[3n_2^2+6n_2(1-2\alpha_2)+72(\alpha_2-1)]}{n_2^4}$$
$$+\frac{x_2^3[-6n_2^2-12n_2(1-2\alpha_2)-144(\alpha_2-1)]}{n_2^4}$$
$$+\frac{x_2^2[3n_1^2-n_1+6n(1-2\alpha_2)+86(\alpha_2-1)]}{n_2^4}$$
$$+\frac{x_2(n_2-14(\alpha_2-1))}{n_2^4}.$$

Lemma 3.2: (see [5]) Using the Corollary 3.1 we have the following ,

1. $L_{n_1,n_2}^{(\alpha_1,\alpha_2)}((t-x_1)^2;x_1,x_2) \leq \frac{3x_1(1-x_1)}{n_1}$.
2. $L_{n_1,n_2}^{(\alpha_2,\alpha_2)}((s-x_2)^2;x_1,x_2) \leq \frac{3x_2(1-x_2)}{n_2}$.

Proof. It is enough to prove 1. By the Corollary 3.1 we have,

$$L_{n_1,n_2}^{(\alpha_1,\alpha_2)}((t-x_1)^2;x_1,x_2) = \frac{x_1(1-x_1)}{n_1} + \frac{x_1(1-x_1)[2(1-\alpha_1)]}{n^2}$$
$$+\frac{x_1(1-x_1)}{n_1} + \frac{2x_1(1-x_1)}{n_1^2} \leq \frac{3x_1(1-x_1)}{n_1}.$$

□

Theorem 3.7: (see [5]) for all $g(x_1,x_2) \in C([0,1]\times[0,1])$, it follows $L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(g;x_1,x_2)$ converges to $g(x_1,x_2)$ uniformly, $\forall \alpha_1,\alpha_2 \in [0,1]$ which are real fixed numbers and $(x_1,x_2) \in [0,1]\times[0,1]$.

Proof. By the Korovkin type approximation Theorem 2.3 and Lemma 3.1 the result holds. □

Lets us consider the function $f(x_1,x_2) = 2x_1 cos(x_1)x_2^3$ and some different values for n_1,n_2 and α_1,α_2 the approximation of the operators (3.4) will be as follows,

1. In Figure 3.11 the approximation of the operators $L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(f;x_1,x_2)$ to the function $f(x_1,x_2)$, with the degrees $n_1 = n_2 = 10,15,25$ and fixed $\alpha_1 = \alpha_2 = 0.5$, is shown. It can be viewed that incresing the degrees (n_1,n_2) give better approximation.

2. In Figure 3.12 the approximation of the operators $L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(f;x_1,x_2)$ to the function $f(x_1,x_2)$, with fixed degree $n_1 = n_2 = 10$ and different values of $\alpha_1 = \alpha_2 = 0.1,0.5$ and 0.9, is given. It can be observed that incresing the values of α_1,α_2 give better approximation.

For the function $g(x_1,x_2) = 2x_1 sin(x_1^2)x_2^3$ and some different values for n_1,n_2 and α_1,α_2 the approximation of the operators (3.3) will be as follows,

1. In Figure 3.13 the approximation of the operators $L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(g;x_1,x_2)$ to the function $g(x_1,x_2)$, with the degrees $n_1 = n_2 = 10,15,25$ and fixed values $\alpha_1 = \alpha_2 = 0.5$, is considered. It is clear that incresing the degrees (n_1,n_2) give better approximation.

2. In Figure 3.14 the approximation of the operators $L_{n_1,n_2}^{(\alpha_1,\alpha_2)}(g;x_1,x_2)$ to function $g(x_1,x_2)$, with fixed degree $n_1 = n_2 = 10$ and different values of $\alpha_1 = \alpha_2 = 0.1,0.5$ and 0.9, is considered. It is easy to see that incresing the values α_1,α_2 give better approximation.

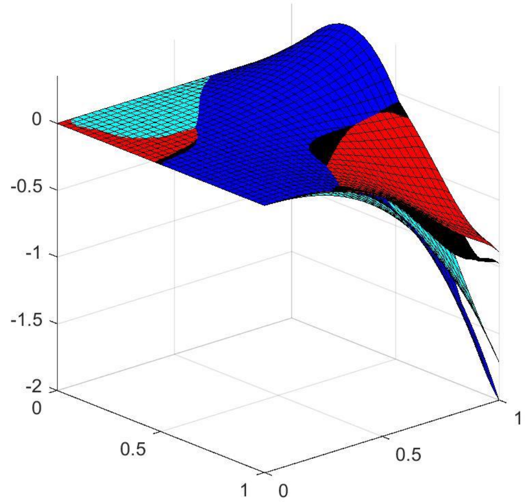

Figure 3.11: Approximation of $L_{n_1,n_2}^{\alpha_1,\alpha_2}(f;x_1,x_2)$ to $f(x_1,x_2)$ (blue) for different degrees (n_1,n_2) and fixed value (α_1,α_2), $(n_1 = n_2 = 10)$(red), $(n_1 = n_2 = 15)$(black),$(n_1 = n_2 = 25)$(green).

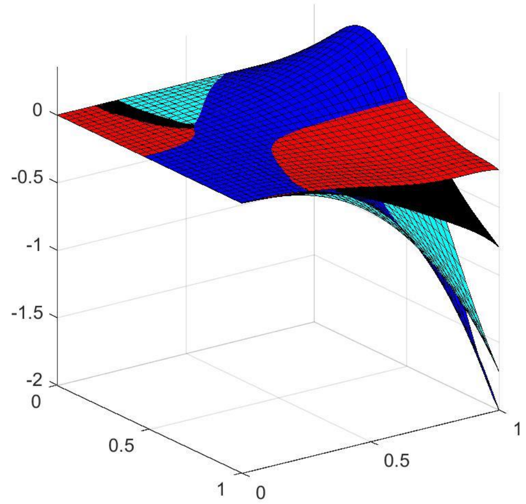

Figure 3.12: Approximation of $L_{n_1,n_2}^{\alpha_1,\alpha_2}(f;x_1,x_2)$ to $f(x_1,x_2)$(blue) for different values (α_1,α_2) and fixed degree (n_1,n_2), $\alpha_1 = \alpha_2 = 0.1$ (red), $\alpha_1 = \alpha_2 = 0.5$ (black), $\alpha_1 = \alpha_2 = 0.9$ (green).

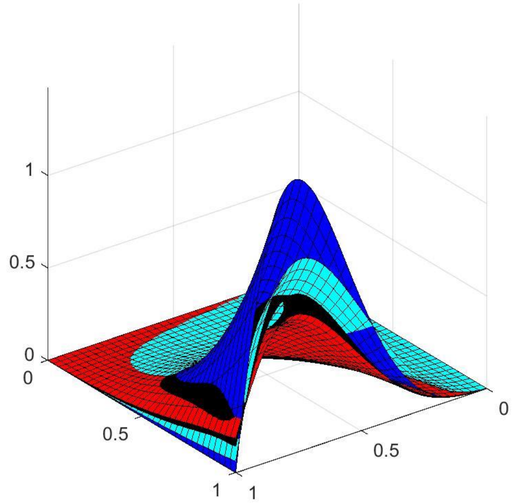

Figure 3.13: Approximation of $L_{n_1,n_2}^{\alpha_1,\alpha_2}(g;x_1,x_2)$ to $g(x_1,x_2)$ (blue) for different degrees (n_1,n_2) and fixed value (α_1,α_2) , $(n_1 = n_2 = 10)$ (red), $(n_1 = n_2 = 15)$ (black), $(n_1 = n_2 = 25)$ (green).

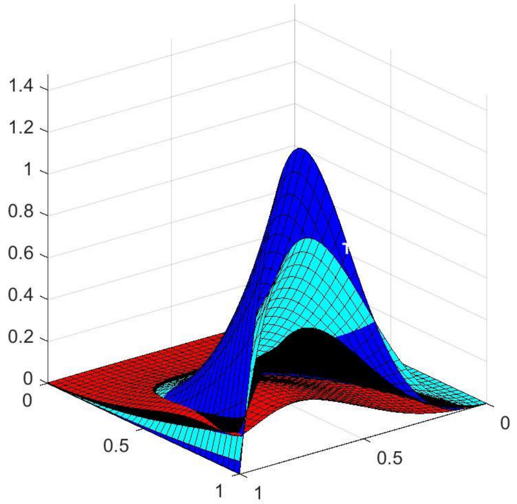

Figure 3.14: Approximation of $L_{n_1,n_2}^{\alpha_1,\alpha_2}(g;x_1,x_2)$ to $g(x_1,x_2)$(blue) for different values (α_1,α_2) and fixed degree (n_1,n_2) , $\alpha_1 = \alpha_2 = 0.1$ (red), $\alpha_1 = \alpha_2 = 0.5$ (black), $\alpha_1 = \alpha_2 = 0.9$ (green).

Chapter 4

PARAMETRIC BLENDING-TYPE OPERATORS

This chapter includes a new family of parametric Blending-type operators, considering four different parameters α_1, α_2, s_1 and s_2. Then we prove some theorems and lemmas which will be used to show the convergence of our operators.

Now lets define a new generalization of the Blending-tpye operators given by (3.4).

Definition 4.1: For all $h \in C([0,1]\times[0,1])$, positive integers s_1, s_2 and fixed real numbers $\alpha_1, \alpha_2 \in [0,1]$ we have the following,

$$T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2) = \begin{cases} \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} p_{n_1,n_2,i,j}^{(\alpha_1,\alpha_2,s_1,s_2)}(x_1,x_2)h(\frac{i}{n_1},\frac{j}{n_2}), & n_1,n_2 \geq max\{s_1,s_2\}, \\ \\ B_{n_1,n_2}(h;x_1,x_2), & otherwise, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} p_{n_1,n_2,i,j}^{(\alpha_1,\alpha_2,s_1,s_2)}(x_1,x_2) = (1-\alpha_1)\Bigg[&\binom{n_1-s_1}{i-s_1}x_1^{i-s_1+1}(1-x_1)^{n_1-i} \\ &+\binom{n_1-s_1}{i}x_1^{i}(1-x_1)^{n_1-s_1-i+1}\Bigg] \\ &+\alpha_1\binom{n_1}{i}x_1^{i}(1-x_1)^{n_1-i} \\ &\times(1-\alpha_2)\Bigg[\binom{n_2-s_2}{j-s_2}x_2^{j-s_2+1}(1-x_2)^{n_2-j} \\ &+\binom{n_2-s_2}{j}x_2^{j}(1-x_2)^{n_2-s_2-j+1}\Bigg] \\ &+\alpha_2\binom{n_2}{j}x_2^{j}(1-x_2)^{n_2-j}, \end{aligned}$$

and $B_{n_1,n_2}(h;x_1,x_2)$ is the double Bernstein operators given in (3.2).

Lemma 4.1: For any $\alpha_1, \alpha_2 \in [0,1]$ and $n_1, n_2 \geqslant max\{s_1,s_2\} \geqslant 2$, the following hold for the operators 4.1,

$$1) T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(1;x_1,x_2) = 1.$$

$$2) T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(t;x_1,x_2) = x_1.$$

$$3) T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(s;x_1,x_2) = x_2.$$

$$4) T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(t^2;x_1,x_2) = x_1^2 + \frac{x_1(1-x_1)\left[n_1+(1-\alpha_1)s_1(s_1-1)\right]}{n^2}.$$

$$5) T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(s^2;x_1,x_2) = x_2^2 + \frac{x_2(1-x_2)\left[n_2+(1-\alpha_2)s_2(s_2-1)\right]}{n_2^2}.$$

$$6) T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(t^3;x_1,x_2) = x_1^3 + x_1^2(1-x_1)\left[\frac{3n_1-2}{n_1^2} + (1-\alpha_1)\frac{s_1(s_1-1)(3n_1-2s_1-2)}{n_1^3}\right] + x_1(1-x_1)\left[\frac{1}{n_1^2} + (1-\alpha_1)\frac{s_1(s_1-1)(s_1+1)}{n_1^3}\right].$$

$$7) T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(s^3;x_1,x_2) = x_2^3 + x_2^2(1-x_2)\left[\frac{3n_2-2}{n_2^2} + (1-\alpha_2)\frac{s_2(s_2-1)(3n_2-2s_2-2)}{n_2^3}\right] + x_2(1-x_2)\left[\frac{1}{n_2^2} + (1-\alpha_2)\frac{s_2(s_2-1)(s_2+1)}{n_2^3}\right].$$

$$8) T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(t^4;x_1,x_2) = x_1^4 + x_1^3(1-x_1)\left[\frac{6n_1^2-11n_1+6}{n_1^3} + (1-\alpha_1)\frac{s_1(1-s_1)\left[3(s_1+1)(s_1+2)+2n_1(3n_1-4s_1-7)\right]}{n_1^4}\right] + x_1^2(1-x_1)\left[\frac{7(n_1-1)}{n_1^3} + (1-\alpha_1)\frac{s_1(s_1-1)[(n_1-s_1)(4s_1+10)-7]}{n_1^4}\right] + x_1(1-x_1^2)\left[\frac{1}{n_1^3} + (1-\alpha_1)\frac{s_1(s_1-1)(s_1^2+s_1+1)}{n_1^4}\right].$$

$$9) T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(s^4;x_1,x_2) = x_2^4 + x_2^3(1-x_2)\left[\frac{6n_2^2-11n_2+6}{n_2^3} + (1-\alpha_2)\frac{s_2(1-s_2)\left[3(s_2+1)(s_2+2)+2n_2(3n_2-4s_2-7)\right]}{n_2^4}\right] + x_2^2(1-x_2)\left[\frac{7(n_2-1)}{n_2^3} + (1-\alpha_2)\frac{s_2(s_2-1)[(n_2-s_2)(4s_2+10)-7]}{n_2^4}\right] + x_2(1-x_2^2)\left[\frac{1}{n_2^3} + (1-\alpha_2)\frac{s_2(s_2-1)(s_2^2+s_2+1)}{n_2^4}\right].$$

Proof. Let us give the proof of parts 1) - 7) so the rest will be satisfied in the same way,

$$1)T_{n_1,n_2}^{(\alpha_1,\alpha_2),s_1,s_2}(1;x_1,x_2) = (1-\alpha_1)\Big[x_1(x_1+(1-x_1))^{n_1-s_1}+(1-x_1)(x_1+(1-x_1))^{n_1-s_1}\Big]$$

$$+\alpha_1(x_1+(1-x_1))^n = (1-\alpha_1)+\alpha_1 = 1$$

$$2)T_{n_1,n_2}^{(\alpha_1,\alpha_2),s_1,s_2}(t;x_1,x_2), = \frac{1}{n_1}\Bigg((1-\alpha_1)\Big[s_1x_1+(n_1-s_1)x_1\Big]+\alpha_1n_1x_1\Bigg)$$

$$\times\Bigg((1-\alpha_2)(x_2+(1-x_2))^{n_2-s_2}+\alpha_2(x_2+(1-x_2))^{n_2}\Bigg)$$

$$= x_1.$$

$$3)T_{n_1,n_2}^{(\alpha_1,\alpha_2),s_1,s_2}(s;x_1,x_2) = \frac{1}{n_2}\Bigg((1-\alpha_2)\Big[s_2x_2+(n_2-s_2)x_2\Big]+\alpha_2n_2x_2\Bigg)$$

$$\times\Bigg((1-\alpha_1)(x_1+(1-x_1))^{n_1-s_1}+\alpha_1(x_1+(1-x_1))^{n_1}\Bigg)$$

$$= x_2.$$

$$4)T_{n_1,n_2}^{(\alpha_1,\alpha_2),s_1,s_2}(t^2;x_1,x_2) = \frac{1}{n_1^2}\Bigg((1-\alpha_1)\Big[s_1(s_1-1)x_1+2s_1x_1(n_1-s_1)$$

$$-(n_1-s_1)x_1^2+(n_1-s_1)^2x_1^2\Big]-\alpha_1(n_1-s_1)x_1^2$$

$$+\alpha_1(n_1-s_1)^2x_1^2\Bigg)\times\Bigg((1-\alpha_2)(x_2+(1-x_2))^{n_2-s_2}+\alpha_2(x_2+(1-x_2))^{n_2}\Bigg)$$

$$= x_1^2+\frac{x_1(1-x_1)\big[n_1+(1-\alpha_1)s_1(s_1-1)\big]}{n_1^2}.$$

$$5)T_{n_1,n_2}^{(\alpha_1,\alpha_2),s_1,s_2}(s^2;x_1,x_2) = \frac{1}{n_2^2}\Bigg((1-\alpha_2)\Big[s_2(s_2-1)y+2s_2x_2(n_2-s_2)$$

$$-(n_2-s_2)x_2^2+(n_2-s_2)^2x_2^2\Big]-\alpha_2(n_2-s_2)x_2^2$$

$$+\alpha_2(n_2-s_2)^2x_2^2\Bigg)\times\Bigg((1-\alpha_1)(x_1+(1-x_1))^{n_1-s_1}+\alpha_1(x_1+(1-x_1))^{n_1}\Bigg)$$

$$= x_2^2+\frac{x_2(1-x_2)\big[n_2+(1-\alpha_2)s_2(s_2-1)\big]}{n_2^2}.$$

$$6)T_{n_1,n_2}^{(\alpha_1,\alpha_2),s_1,s_2}(t^3;x_1,x_2) = \Bigg(\frac{(1-\alpha_1)}{n_1^3}\Big[s_1^3x_1+s_1^2(n_1-s_1)x_1^2+(n_1-s_1)s_1(s_1+1)x_1^2$$

$$+s_1(n_1-s_1)(n_1-s_1)x_1^3+(n_1-s_1)(n_1-s_1-1)(n_1-s_1-2)x_1^3$$

$$+(n_1-s_1)(n_1-s_1-1)x_1^3+2(s_1+1)(n_1-s_1)(n_1-s_1-1)x_1^3$$

$$+2(s_1+1)^2x_1^2-(n_1-s_1)(s_1-1)^2x_1^2+x_1(1-x_1)(n_1-s_1)$$

$$+2x_1^2(1-x_1)+x_1^3(n_1-s_1)(n_1-s_1-1)(n_1-s_1-2)$$

$$
\begin{aligned}
&+x_1^2(1-x_1)(n_1-s_1)(n_1-s_1-1)\Big]\\
&+\alpha_1\big[x_1^3+\frac{3x_1^2(1-x_1)}{n_1}+\frac{x_1(1-2x_1)(1-x_1)}{n_1^2}\big]\Big)\\
&\times\Big((1-\alpha_2)(x_2+(1-x_2))^{n_2-s_2}+\alpha_2(x_2+(1-x_2))^{n_2}\Big)\\
&=x_1^3+x_1^2(1-x_1)\left[\frac{3n_1-2}{n_1^2}+(1-\alpha_1)\frac{s_1(s_1-1)(3n_1-2s_1-2)}{n_1^3}\right]\\
&+x_1(1-x_1)\left[\frac{1}{n_1^2}+(1-\alpha_1)\frac{s_1(s_1-1)(s_1+1)}{n_1^3}\right]
\end{aligned}
$$

$$
\begin{aligned}
7)T_{n_1,n_2}^{(\alpha_1,\alpha_2),s_1,s_2}(s^3;x_1,x_2)=&\Big(\frac{(1-\alpha_2)}{n_2^3}\Big[s_2^3y+s_2^2(m-s_2)y^2+(n_2-s_2)s_2(s_2+1)x_2^2\\
&+s_2(n_2-s_2)(n_2-s_2)x_2^3+(n_2-s_2)(n_2-s_2-1)(n_2-s_2-2)x_2^3\\
&+(n_2-s_2)(n_2-s_2-1)x_2^3+2(s_2+1)(n_2-s_2)(n_2-s_2-1)x_2^3\\
&+2(s_2+1)^2x_2^2-(n_2-s_2)(s_2-1)^2x_2^2+x_2(1-x_2)(n_2-s_2)\\
&+2x_2^2(1-x_2)+x_2^3(n_2-s_2)(n_2-s_2-1)(n_2-s_2-2)\\
&+x_2^2(1-x_2)(n_2-s_2)(n_2-s_2-1)\Big]\\
&+\alpha_2\big[x_2^3+\frac{3x_2^2(1-x_2)}{n_2}+\frac{x_2(1-2x_2)(1-x_2)}{n_2^2}\big]\Big)\\
&\times\Big((1-\alpha_1)(x_1+(1-x_1))^{n_1-s_1}+\alpha_1(x_1+(1-x_1))^{n_1}\Big)\\
&=x_2^3+x_2^2(1-x_2)\left[\frac{3n_2-2}{n_2^2}+(1-\alpha_2)\frac{s_2(s_2-1)(3n_2-2s_2-2)}{n_2^3}\right]\\
&+x_2(1-x_2)\left[\frac{1}{n_2^2}+(1-\alpha_2)\frac{s_2(s_2-1)(s_2+1)}{n_2^3}\right].
\end{aligned}
$$

□

For the operators (4.1), using the Lemma 4.1, the following lemma can be written.

Lemma 4.2: For any $\alpha_1,\alpha_2\in[0,1]$ we have,

$$
T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)^2;x_1,x_2)=\begin{cases}\frac{x_1(1-x_1)\left[n_1+(1-\alpha_1)s_1(s_1-1)\right]}{n_1^2}, & n_1,n_2\geq max\{s_1,s_2\}\geq 2\\[2ex] \frac{x_1(1-x_1)}{n_1}, & otherwise.\end{cases}
$$

$$
T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((s-x_2)^2;x_1,x_2)=\begin{cases}\frac{x_2(1-x_2)\left[n_2+(1-\alpha_2)s_2(s_2-1)\right]}{n_2^2}, & n_1,n_2\geq max\{s_1,s_2\}\geq 2\\[2ex] \frac{x_2(1-x_2)}{n_2}, & otherwise.\end{cases}
$$

$$T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)^3;x_1,x_2)=\begin{cases} x_1(1-x_1)(1-2x_1)\frac{\left[n_1+(1-\alpha_1)s_1(s_1-1)(s_1+1)\right]}{n_1^3}, & n_1,n_2\geq max\{s_1,s_2\}\geq 2\\ \\ B_{n_1,n_2}((t-x_1)^3;x_1,x_2),\ otherwise. \end{cases}$$

$$T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((s-x_2)^3;x_1,x_2)=\begin{cases} x_2(1-x_2)(1-2x_2)\frac{\left[n_2+(1-\alpha_2)s_2(s_2-1)(s_2+1)\right]}{n_2^3}, & n_1,n_2\geq max\{s_1,s_2\}\geq 2\\ \\ B_{n_1,n_2}((s-x_1)^3;x_1,x_2),\ otherwise. \end{cases}$$

$$T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)^4;x_1,x_2)=\begin{cases} 3x_1^2(1-x_1)^2\left[\frac{n_1(n_1-2)+(1-\alpha_1)s_1(s_1-1)[2n_1-(s_1+1)(s_1+2)]}{n_1^4}\right] & \\ +x_1(1-x_1)\left[\frac{n_1+(1-\alpha_1)s_1(s_1-1)(s_1^2+s_1+1)}{n_1^4}\right], & n_1,n_2\geq max\{s_1,s_2\}\geq 2\\ \\ B_{n,m}((t-x_1)^4;x_1,x_2),\ otherwise. \end{cases}$$

$$T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((s-x_2)^4;x_1,x_2)=\begin{cases} 3x_2^2(1-x_2)^2\left[\frac{n_2(n_2-2)+(1-\alpha_2)s_2(s_2-1)[2n-(s_2+1)(s_2+2)}{n_2^4}]\right] & \\ +x_2(1-x_2)\left[\frac{n_2+(1-\alpha_2)s_2(s_2-1)(s_2^2+s_2+1)}{n_2^4}\right], & n_1,n_2\geq max\{s_1,s_2\}\geq 2\\ \\ B_{n_1,n_2}((s-x_2)^4;x_1,x_2),\ otherwise. \end{cases}$$

Proof. Using the Lemma 4.1 proof can be completed. □

Lemma 4.3: $\forall\alpha_1,\alpha_2\in[0,1]$ and positive integers s_1,s_2, we have,

$$T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)^2;x_1,x_2)\leq\frac{s_1^2-s_1+1}{4n_1}. \tag{4.2}$$

$$T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((s-x_2)^2;x_1,x_2)\leq\frac{s_2^2-s_2+1}{4n_2}. \tag{4.3}$$

Proof. If $n_1,n_2\geq max\{s_1,s_2\}$ then by the used idea in the Lemma 3.2 and Lemma 4.2, we have

$$
\begin{aligned}
T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)^2;x_1,x_2) &= \frac{x_1(1-x_1)\left[n_1+(1-\alpha_1)s_1(s_1-1)\right]}{n_1^2}\\
&\leq \frac{x_1(1-x_1)}{n_1}+\frac{x_1(1-x_1)[s_1(s_1-1)]}{n_1}\\
&\leq \frac{x_1(1-x_1)[s_1^2-s_1+1]}{n_1}\leq \frac{s_1^2-s_1+1}{4n_1},
\end{aligned}
$$

which gives equation (4.2). If $n_1,n_2 \geq max\{s_1,s_2\}$ does not hold then by Proposition 3.3 we have,

$$
\begin{aligned}
T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)^2;x_1,x_2 = B_{n_1,n_2}((t-x_1)^2;x_1,x_1) &\leq B_{n_1,n_2}(t^2;x_1,x_2)\\
&-2x_1B_{n_1,n_2}(t;x_1,x_2)+B_{n_1,n_2}(x_1^2;x_1,x_2)\\
&+x_1^2+\frac{x_1(1-x_1)}{n_1}-2x_1^2+x_1^2\\
&=\frac{x_1(1-x_1)}{n_1}\leq\frac{s_1^2-s_1+1}{4n_1}.
\end{aligned}
$$

If $n_1,n_2 \geq max\{s_1,s_2\}$ then by same method we have,

$$
\begin{aligned}
T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((s-x_2)^2;x_1,x_2) &= \frac{x_2(1-x_2)\left[n_2+(1-\alpha_2)s_2(s_2-1)\right]}{n_2^2}\\
&\leq \frac{x_2(1-x_2)}{n_2}+\frac{x_2(1-x_2)[s_2(s_2-1)]}{n_2}\\
&\leq \frac{x_2(1-x_2)[s_2^2-s_2+1]}{n_2}\leq \frac{s_2^2-s_2+1}{4n_2},
\end{aligned}
$$

which gives equation (4.3). If $n_1,n_2 \geq max\{s_1,s_2\}$ does not hold then by Proposition 3.3 we have,

$$
\begin{aligned}
T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((s-x_2)^2;x_1,x_2) = B_{n_1,n_2}((s-x_2)^2;x_1,x_2) &\leq B_{n_1,n_2}(s^2;x_1,x_2)\\
&-2x_2B_{n_1,n_2}(s;x_1,x_2)+B_{n_1,n_2}(x_2^2;x_1,x_2)\\
&+x_2^2+\frac{x_2(1-x_2)}{n_2}-2x_2^2+x_2^2\\
&=\frac{x_2(1-x_2)}{n_2}\leq\frac{s_2^2-s_2+1}{4n_2}.
\end{aligned}
$$

□

Chapter 5

APPROXIMATION PROPERTIES OF $T^{\alpha_1,\alpha_2,S_1,S_2}_{N_1,N_2}$

In the present chapter we consider some approximation properties of the operators (4.1) on the space of $C([0,1]\times[0,1])$.

Now using the given preliminaries in the pervious chapters the following Korovkin type approximation theorem can be proved for the operators $T^{\alpha_1,\alpha_2,s_1,s_2}_{n_1,n_2}$. For more information see [9].

Theorem 5.1: Let consider two real fixed numbers $\alpha_1,\alpha_2\in[0,1]$ and let $s_1,s_2\in\mathbb{N}$ then for all $h\in C([0,1]\times[0,1])$,

$$\lim_{n_1,n_2\to\infty} T^{\alpha_1,\alpha_2,s_1,s_2}_{n_1,n_2}(h;x_1,x_2)=h(x_1,x_2),$$

uniformly for any $(x_1,x_2)\in[0,1]\times[0,1]$.

Proof. By the definition of the operators $T^{\alpha_1,\alpha_2,s_1,s_2}_{n_1,n_2}$ and Lemma 4.1, we have,

$$T^{\alpha_1,\alpha_2,s_1,s_2}_{n_1,n_2}(1;x_1,x_2)=1.$$

$$T^{\alpha_1,\alpha_2,s_1,s_2}_{n_1,n_2}(t;x_1,x_2)=x_1.$$

$$T^{\alpha_1,\alpha_2,s_1,s_2}_{n_1,n_2}(s;x_1,x_2)=x_2.$$

$$T^{\alpha_1,\alpha_2,s_1,s_2}_{n_1,n_2}(t^2;x_1,x_2)=\begin{cases} x_1^2+\frac{x_1(1-x_1)\left[n_1+(1-\alpha_1)s_1(s_1-1)\right]}{n_1^2}, & n_1,n_2\geq max\{s_1,s_2\}\geq 2\\ \\ x_1^2+\frac{x_1(1-x_1)}{n_1},\ otherwise. \end{cases}$$

$$T^{\alpha_1,\alpha_2,s_1,s_2}_{n_1,n_2}(s^2;x_1,x_2)=\begin{cases} x_2^2+\frac{x_2(1-x_2)\left[n_2+(1-\alpha_2)s_2(s_2-1)\right]}{n_2^2}, & n_1,n_2\geq max\{s_1,s_2\}\geq 2\\ \\ x_2^2+\frac{x_2(1-x_2)}{n_2},\ otherwise. \end{cases}$$

Therefore,
$$\lim_{n,m\to\infty} T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(1,x_1,x_2) = 1,$$
$$\lim_{n_1,n_2\to\infty} T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(t;x_1,x_2) = x_1,$$
$$\lim_{n_1,n_2\to\infty} T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(s;x_1,x_2) = x_2,$$
$$\lim_{n_1,n_2\to\infty} T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(t^2+s^2;x_1,x_2) = (x_1^2+x_2^2),$$

(uniformly) for any $(x_1,x_2)\in[0,1]\times[0,1]$, by which the proof is completed. □

Corollary 5.1: Let $h(t,s,x_1,x_2)\in C([0,1]\times[0,1])$, such that $h(t,s,x_1,x_2)\to 0$ as $(t,s)\to(x_1,x_2)$. For any $\alpha_1,\alpha_2\in[0,1]$ and positive integers s_1,s_2 with $n_1\geq max\{s_1,s_2\}$ we have,

$$\lim_{n_1\to\infty} n_1 T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}(h(t,s,x_1,x_2)\sqrt{(t-x_1)^4+(s-x_2)^4};x_1,x_2) = 0.$$

Proof. By the Cauchy-Schwarz inequality, we can write the following,

$$\begin{aligned}
& n_1 T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}(h(t,s,x_1,x_2)\sqrt{(t-x_1)^4+(s-x_2)^4};x_1,x_2) \\
& \leq \left(T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}(h^2(t,s,x_1,x_2)\right)^{\frac{1}{2}}\left(n_1^2 T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)^4+(s-x_2)^4;x_1,x_2)\right)^{\frac{1}{2}} \\
& \leq \left(T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}(h^2(t,s,x_1,x_2)\right)^{\frac{1}{2}}\left(n_1^2 T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)^4;x_1,x_2)\right. \\
& \left. + T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}((s-x_2)^4;x_1,x_2)\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.1}$$

On the other hand,

$$\lim_{n_1\to\infty} n_1^2 T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)^4;x_1,x_2) = 3x_1^2(1-x_1^2), \tag{5.2}$$

$$\lim_{n_1\to\infty} n_1^2 T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}((s-x_2)^4;x_1,x_2) = 3x_2^2(1-x_2^2), \tag{5.3}$$

and as a consequence of Lemma 4.2, we have,

$$\lim_{n\to\infty} T_{n,n}^{\alpha_1,\alpha_2,s_1,s_2} h(t,s,x,y) = 0. \tag{5.4}$$

By considering (5.2),(5.3) and (5.4) in (5.1) the proof will be completed. □

Theorem 5.2: Let $\alpha_1, \alpha_2 \in [0,1]$ and let $n_1, n_2 \geq max\{s_1, s_2\} \geq 2$, then for any $g \in C([0,1]\times[0,1])$,

$$\lim_{n_1\to\infty} n_1\left(T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}(g;x_1,x_2) - g(x_1,x_2)\right) = \frac{1}{2}x_1(1-x_1)\frac{\partial^2 g}{\partial x_1^2}(x_1,x_2) + \frac{1}{2}x_2(1-x_2)\frac{\partial^2 g}{\partial x_2^2}(x_1,x_2), \tag{5.5}$$

(uniformly) on $[0,1]\times[0,1]$.

Proof. By the aid of Taylor's expansion we get,

$$\begin{aligned} g(t,s) &= g(x_1,x_2) + (t-x_1)\frac{\partial g}{\partial x_1}(x_1,x_2) + (s-x_2)\frac{\partial g}{\partial x_2}(x_1,x_2) + \frac{1}{2}(t-x_1)^2\frac{\partial^2 g}{\partial x_1^2}(x_1,x_2) \\ &+ (t-x_1)(s-x_2)\frac{\partial^2 g}{\partial x_1 x_2}(x_1,x_2) \\ &+ \frac{1}{2}(s-x_2)^2\frac{\partial^2 g}{\partial x_2^2}(x_1,x_2) + h(t,s,x_1,x_2)\sqrt{(t-x_1)^4+(s-x_2)^4}, \end{aligned} \tag{5.6}$$

where $h(t,s,x_1,x_2) \in C([0,1]\times[0,1])$ and $\lim_{(t,s)\to(x_1,x_2)} h(t,s,x_1,x_2) = 0$.

Using (5.6) and the Lemma 4.2, we have,

$$\begin{aligned} T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}(g;x_1,x_2) &= g(x_1,x_2) + \frac{1}{2}\frac{\partial^2 g}{\partial x_1^2}(x_1,x_2)T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)^2;x_1,x_2) \\ &+ \frac{1}{2}\frac{\partial^2 g}{\partial x_2^2}(x_1,x_2)T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}((s-x_2)^2;x_1,x_2) \\ &+ f\frac{\partial^2 g}{\partial x_1 x_2}(x_1,x_2)T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)(s-x_2);x_1,x_2) \\ &+ T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}(h(t,s,x_1,x_2)\sqrt{(t-x_1)^4+(s-x_2)^4};x_1,x_2). \end{aligned}$$

By the proved inequality in Lemma 4.3, we have,

$$\begin{aligned} &T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}(g;x_1,x_2) \leq g(x_1,x_2) \\ &+ \frac{1}{2}\frac{\partial^2 g}{\partial x_1^2}(x_1,x_2)\frac{x_1(1-x_1)[s_1^2-s_1+1]}{n_1} \\ &+ \frac{1}{2}\frac{\partial^2 g}{\partial x_2^2}(x_1,x_2)\frac{x_2(1-x_2)[s_2^2-s_2+1]}{n_1} \\ &+ \frac{\partial^2 g}{\partial x_1 x_2}(x_1,x_2)T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)(s-x_2);x_1,x_2) \\ &+ T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}(h(t,s,x_1,x_2)\sqrt{(t-x_1)^4+(s-x_2)^4};x_1,x_2). \end{aligned} \tag{5.7}$$

Thus by Corollary 5.1 and the following equation

$$T_{n_1,n_1}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)(s-x_2);x_1,x_2) = T_{n_1}^{\alpha_1,s_1}((t-x_1);x_1)T_{n_1}^{\alpha_2,s_2}((s-x_2);x_2) = 0,$$

in (5.7) we get (5.5). □

Theorem 5.3: If $h \in C([0,1]\times[0,1])$ then for any $\alpha_1,\alpha_2 \in [0,1]$, positive integers s_1,s_2 and for all $n_1,n_2 \geq max\{s_1,s_2\}$ we have,

$$|T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2) - h(x_1,x_2)| \leq 2\omega_2(h;\delta_{n_1,n_2}),$$

where $\delta_{n_1,n_2}^2 = \frac{n_2\left(s_1^2-s_1+1\right)+n_1\left(s_2^2-s_2+1\right)}{4n_1n_2}$.

Proof. By the properties of positive linear operators and the modulus of continuity we have,

$$\begin{aligned}
&|T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2) - h(x_1,x_2)| = T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(|h(t,s)-h(x_1,x_2)|;x_1,x_2) \\
&\leq T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}\left(\omega_2\big(h;\sqrt{(t-x_1)^2+(s-x_2)^2}\big);x_1,x_2\right) \\
&\leq \omega_2\big(h;\delta_{n_1,n_2};x_1,x_2\big) \\
&\times\left\{1+\frac{1}{\delta_{n_1,n_2}}T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}\left(\sqrt{(t-x_1)^2+(s-x_2)^2};x_1,x_2\right)\right\}, \qquad (5.8)
\end{aligned}$$

where $\delta_{n_1,n_2} > 0$. Appliying Cauchy-Schwarz inequality to (5.8) and Lemma 4.3, we get;

$$\begin{aligned}
&|T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2) - h(x_1,x_2)| \\
&\leq \omega_2\big(h;\delta_{n_1,n_2};x_1,x_2\big)\left[1+\frac{1}{\delta_{n_1,n_2}}\sqrt{L_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}\left((t-x_1)^2+(s-x_2)^2;x_1,x_2\right)}\right] \\
&\leq \omega_2\big(h;\delta_{n_1,n_2};x_1,x_2\big)\left[1+\frac{1}{\delta_{n_1,n_2}}\sqrt{L_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}\big((t-x_1)^2;x_1,x_2\big)+T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}\big((s-x_2)^2;x_1,x_2\big)}\right] \\
&\leq \omega_2\big(h;\delta_{n_1,n_2};x_1,x_2\big)\left[1+\frac{1}{\delta_{n_1,n_2}}\sqrt{\frac{s_1^2-s_1+1}{4n_1}+\frac{s_2^2-s_2+1}{4n_2}}\right],
\end{aligned}$$

taking $\delta_{n_1,n_2} = \sqrt{\frac{s_1^2-s_1+1}{4n_1}+\frac{s_2^2-s_2+1}{4n_2}}$, the proof is completed. □

Theorem 5.4: Let $h \in C([0,1]\times[0,1])$. For any $\alpha_1,\alpha_2 \in [0,1]$, positive integers s_1,s_2 and for all $n_1,n_2 \geq max\{s_1,s_2\}$ we have,

$$T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2)-h(x_1,x_2)|\leq 2\bigg(\omega_{2,x_1}(h;\delta_{n_1})+\omega_{2,x_2}(h;\delta_{n_2})\bigg),$$

taking $\delta_{n_1}=\sqrt{\frac{s_1^2-s_1+1}{4n_1}}$ and $\delta_{n_2}=\sqrt{\frac{s_2^2-s_2+1}{4n_2}}$.

Proof. As the operators $T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}$ is linear positive and by the properties of modulus of continuity we have,

$$\begin{aligned}
&|T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2)-h(x_1,x_2)| = T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(|h(t,s)-h(x_1,x_2)|;x_1,x_2)\\
&\leq T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(|h(t,s)-h(x_1,s)|;x_1,x_2)+T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(|h(x_1,s)-h(x_1,x_2)|;x_1,x_2)\\
&\leq T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(\omega_{2,x_1}(h,|t-x_1|)+T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(\omega_{2,x_2}(h;|s-x_2|)\\
&\leq \omega_{2,x_1}(h;\delta_{n_1})\left[1+\frac{1}{\delta_{n_1}}T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(|t-x_1|;x_1,x_2)\right]\\
&+\omega_{2,x_2}(h;\delta_{n_2})\left[1+\frac{1}{\delta_{n_2}}T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(|s-x_2|;x_1,x_2)\right].
\end{aligned}$$

Using Lemma 4.3 and Cauchy-Schwarz inequality we get,

$$\begin{aligned}
&|T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2)-h(x_1,x_2)|\\
&\leq \omega_{2,x_1}(h;\delta_{n_1})\left[1+\frac{1}{\delta_n}\sqrt{T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)^2;x_1,x_2)}\right]\\
&+\omega_{2,x_2}(h;\delta_{n_2})\left[1+\frac{1}{\delta_{n_2}}\sqrt{T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((s-x_2)^2;x_1,x_2)}\right]\\
&\leq \omega_{2,x_1}(h;\delta_{n_1})\left[1+\frac{1}{\delta_{n_1}}\sqrt{\frac{s_1^2-s_1+1}{4n_1}}\right]+\omega_{2,x_2}(h;\delta_{n_2})\left[1+\frac{1}{\delta_{n_2}}\sqrt{\frac{s_2^2-s_2+1}{4n_2}}\right],
\end{aligned}$$

where $\delta_{n_1}=\sqrt{\frac{s_1^2-s_1+1}{4n_1}}$ and $\delta_{n_2}=\sqrt{\frac{s_2^2-s_2+1}{4n_2}}$, this completes the proof. □

Consider real numbers $\tau_1,\tau_2\in(0,1]$ and the following Lipshitz class $Lip_K^{(\tau_1,\tau_2)}$ for $h(x,y)$ such that;

$$|h(t,s)-h(x_1,x_2)|\leq K|t-x_1|^{\tau_1}|s-x_2|^{\tau_2}.$$

Theorem 5.5: For any $\alpha_1,\alpha_2\in[0,1]$, positive integers s_1,s_2 , $h\in Lip_K^{(\tau_1,\tau_2)}$ and all $n_1,n_2\geq max\{s_1,s_2\}$ we have,

$$\|T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h)-h\|_\infty$$
$$\leq K\left(\frac{n_1+(1-\alpha_1)\left(s_1^2-s_1+1\right)}{4n_1^2}\right)^{\frac{\tau_1}{2}}\left(\frac{n_2+(1-\alpha_2)\left(s_2^2-s_2+1\right)}{4n_2^2}\right)^{\frac{\tau_2}{2}}. \tag{5.9}$$

Proof. Let $h\in Lip_K^{(\tau_1,\tau_2)}$, then

$$|T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2)-h(x_1,x_2)|$$
$$\leq T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(|h(t,s)-h(x_1,x_2)|;x_1,x_2)$$
$$\leq T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(K|t-x_1|^{\tau_1}|s-x_2|^{\tau_2};x_1,x_2)$$
$$\leq KT_{n_1}^{\alpha_1,s_1}(|t-x_1|^{\tau_1};x_1)T_{n_2}^{\alpha_2,s_2}(|s-x_2|^{\tau_2};x_1,x_2). \tag{5.10}$$

Applying Hölder's inequality to (5.10) for $p_1=\frac{2}{\tau_1}$ $q_1=\frac{2}{2-\tau_1}$ and $p_2=\frac{2}{\tau_2}$ $q_2=\frac{2}{2-\tau_2}$ we get,

$$|T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2)-h(x_1,x_2)|$$
$$\leq K\left(T_{n_1}^{\alpha_1,s_1}((t-x_1)^2;x_1)\right)^{\frac{\tau_1}{2}}\left(T_{n_1}^{\alpha_1,s_1}(1;x_1)\right)^{\frac{2-\tau_1}{2}}\left(T_{n_2}^{\alpha_2,s_2}((s-x_2)^2;x_1)\right)^{\frac{\tau_2}{2}}\left(T_{n_2}^{\alpha_2,s_2}(1;x_1)\right)^{\frac{2-\tau_2}{2}}$$
$$\leq K\left(T_{n_1}^{\alpha_1,s_1}((t-x_1)^2;x_1)\right)^{\frac{\tau_1}{2}}\left(T_{n_2}^{\alpha_2,s_2}((s-x_2)^2;x_1)\right)^{\frac{\tau_2}{2}}$$
$$\leq K\left(\frac{x_1(1-x_1)(n_1+(1-\alpha)s_1(s_1-1)}{n_1^2}\right)^{\frac{\tau_1}{2}}\left(\frac{x_2(1-x_2)(n_2+(1-\alpha)s_2(s_2-1)}{n_2^2}\right)^{\frac{\tau_2}{2}}.$$

Taking supremum from both sides gives (5.9) so it completes the proof. □

Theorem 5.6: For any $\alpha_1,\alpha_2\in[0,1]$, positive integers s_1,s_2, $\forall n_1,n_2\geq max\{s_1,s_2\}$ and $h\in C^1([0,1]\times[0,1])$, we have,

$$|T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2)-h(x_1,x_2)|$$
$$\leq\|\frac{\partial h}{\partial x_1}\|_\infty\left(\frac{x_1(1-x_1)\left[n_1+(1-\alpha_1)\left(s_1^2-s_1+1\right)\right]}{n_1^2}\right)^{\frac{1}{2}}$$
$$+\|\frac{\partial h}{\partial x_2}\|_\infty\left(\frac{x_2(1-x_2)\left[n_2+(1-\alpha_2)\left(s_2^2-s_2+1\right)\right]}{n_2^2}\right)^{\frac{1}{2}}.$$

Proof. Given $h\in C^1([0,1]\times[0,1])$ then,

$$h(t,s)-h(x_1,x_2)=\int_{x_1}^{t}\frac{\partial h}{\partial x_1}(\tau,s)d\tau+\int_{x_2}^{s}\frac{\partial h}{\partial x_2}(x_1,\theta)d\theta.$$

If we apply $T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}$ to both sides we get;

$$|T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2)-h(x_1,x_2)| \leq T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}\left(\left|\int_{x_1}^{t}|\frac{\partial h}{\partial x_1}(\tau,s)|d\tau\right|;x_1,x_2\right) + T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}\left(\left|\int_{x_2}^{s}|\frac{\partial h}{\partial x_2}(x_1,\theta)|d\theta\right|;x_1,x_2\right),$$

which gives,

$$|T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2)-h(x_1,x_2)| \leq \|\frac{\partial h}{\partial x_1}\|_\infty T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(|t-x_1|;x_1,x_2) + \|\frac{\partial h}{\partial x_2}\|_\infty T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(|s-y_2|;x_1,x_2).$$

Then by applying Cauchy-Schwarz inequality to the above inequality we have,

$$|T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2)-h(x_1,x_2)| \leq \|\frac{\partial h}{\partial x_1}\|_\infty \left(T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)^2;x_1,x_2)\right)^{\frac{1}{2}} + \|\frac{\partial h}{\partial x_2}\|_\infty \left(T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((s-x_2)^2;x_1,x_2)\right)^{\frac{1}{2}}.$$

Using Lemma 4.2, in the above inequality completes the proof. □

5.1 The GBS Case of Generalized Belending Type-Bernstein Operators

In this section we consider the definition of B-continuity, B-differentiability, B-bounded, mixed modulus of continuity and the GBS case of our operators.

Definition 5.1: A function $h:[0,1]\times[0,1] \longrightarrow \mathbb{R}$ is called a B-continuous function in $(x_1,x_2)\in[0,1]\times[0,1]$ if and only if

$$\lim_{(t,s)\to(x_1,x_2)} \triangle_{(t,s)}h(x_1,x_2) = 0,$$

, where

$$\triangle_{(t,s)}h(x_1,x_2) = h(x_1,x_2)-h(x_1,s)-h(t,x_2)+h(t,s),$$

which is called the mixed difference of h.

Definition 5.2: A function $h: X_1\times X_2 \longrightarrow \mathbb{R}$ is called a B-continuous function on

$[0,1]\times[0,1]$ if and only if it is B-continuous on any points $[0,1]\times[0,1]$.

Definition 5.3: A function $h:[0,1]\times[0,1]\longrightarrow\mathbb{R}$ is called a B-differentiable function in $(t,s)\in[0,1]\times[0,1]$ iff

$$\lim_{(x_1,x_2)\to(t,s)}\frac{\triangle_{(t,s)}h(x_1,x_2)}{(x_1-t)(x_2-s)},$$

exists. This limit is called B-differential of h at the point (t,s) and it is denoted as $D_Bh(t,s)$.

Definition 5.4: A function $h:[0,1]\times[0,1]\longrightarrow\mathbb{R}$ is called a B-bounded on $[0,1]\times[0,1]$ iff there exists $M_1>0$ such that

$$|\triangle_{(t,s)}h(x_1,x_2)|\leq M_1,$$

for any $(x_1,x_2),(t,s)\in[0,1]\times[0,1]$.

The sets $C_b([0,1]\times[0,1])$ and $B_b([0,1]\times[0,1])$ are denoted as B-continuous and B-bounded function respectively.

The above definitions are introduced by Bögel in [14], [15] and [16]. Later, approximating properties of GBS operators of bivariate Bernstein polynomials was shown by Dobrescu and Matei in [24], using B-continuity and B-differentiability. Nowadays, one of the popular and hot topics among researchers is Bögel space. Some of those research can be mentioned as follow, [1], [7], [11], [12], [36], [38], [45] and [47].

Definition 5.5: The mixed modulus of smoothness of a real valud B-continuous function $h(x_1,x_2)$ is denoted by $\omega_{mixed}(h,\delta_1,\delta_2)$ and defined as,

$$\omega_{mixed}(h,\delta_1,\delta_2) := sup\{|\triangle_{(t,s)} h(x_1,x_2)| : |t-x_1| < \delta_1, |s-x_2| < \delta_2\},$$

where $(x_1,x_2),(t,s) \in [0,1]\times[0,1]$ and $\delta_1,\delta_2 > 0$. It should be mentioned that the properties of $\omega_{mixed}(h,\delta_1,\delta_2)$ is similar with usual modulus of continuity.

Definition 5.6: A bivariate function $h(x_1,x_2)$, $(x_1,x_2) \in [0,1]\times[0,1]$ is called uniformly B-continuous function iff

$$\lim_{\delta_1,\delta_2\to 0} \omega_{mixed}(h,\delta_1,\delta_2) = 0.$$

The GBS case of the operators $T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}$ can be written as follow,

$$G_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h(t,s);x_1,x_2) = T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h(x_1,s)+h(t,x_2)-h(t,s);x_1,x_2),$$

if $n_1,n_2 \geq max\{s_1,s_2\}$.

That is

$$\begin{aligned}
&G_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h(t,s);x_1,x_2)\\
&=\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}(1-\alpha_1)\left[\binom{n_1-s_1}{i-s_1}x_1^{i-s_1+1}(1-x_1)^{n_1-i}+\binom{n_1-s_1}{i}x_1^{i}(1-x_1)^{n_1-s_1-i+1}\right]\\
&\quad+\alpha_1\binom{n_1}{i}x_1^{i}(1-x_1)^{n_1-i}\Big]\\
&\times(1-\alpha_2)\left[\binom{n_2-s_2}{j-s_2}x_2^{i-s_2+1}(1-x_2)^{n_2-j}+)\binom{n_2-s_2}{j}x_2^{j}(1-x_2)^{n_2-s_2-j+1}\right.\\
&\quad\left.+\alpha_2\binom{n_2}{j}x_2^{j}(1-x_2)^{n_2-j}\right]\left(h(\frac{i}{n_1},x_2)+h(x_1,\frac{j}{n_2})-h(\frac{i}{n_1},\frac{j}{n_2})\right),
\end{aligned}$$

or

$$G_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h(t,s);x_1,x_2) = B_{n_1,n_2}(h(x_1,s)+h(t,x_2)-h(t,s);x_1,x_2),$$

otherwise.

5.2 Degree of Approximation for The Operators $G_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}$

In the present section we study the degree of approximation of the GBS case of our operators and also we construct some numerical results of our operators to show that how does it work.

Theorem 5.7: Let $(x_1,x_2) \in [0,1]\times[0,1]$ and let f be a B-continuous function, for real values $\alpha_1,\alpha_2 \in [0,1]$ and positive integers s_1,s_2 and for all $n_1,n_2 \geq max\{s_1,s_2\}$, we have

$$|G_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h(t,s);x_1,x_2)-h(x_1,x_2)| \leq \left(1+\sqrt{M_{s_1,s_2}}\right)^2 \omega_{mixed}(h,\frac{1}{\sqrt{n_1}},\frac{1}{\sqrt{n_2}}),$$

where $M_{s_1,s_2} > 0$ and it depends on s_1 and s_2.

Proof. For any positive real numbers τ_1 and τ_2 we have,

$$\omega_{mixed}(h,\tau_1\delta_1,\tau_2\delta_2) \leq (1+\tau_1)(1+\tau_2)\omega_{mixed}(f,\delta_1,\delta_2).$$

For all $\delta_1 > 0$, $\delta_2 > 0$ and for any $(x_1,x_2),(t,s) \in [0,1]\times[0,1]$ we have,

$$\begin{aligned}\triangle_{(t,s)}h(x_1,x_2) &\leq \omega_{mixed}(h,|t-x_1|,|s-x_2|)\\ &\leq (1+\frac{|t-x_1|}{\delta_1})(1+\frac{|s-x_2|}{\delta_2})\omega_{mixed}(h,\delta_1,\delta_2).\end{aligned}$$

By the definition of $\triangle_{(t,s)}h(x_1,x_2)$ we get,

$$h(t,x_2)+h(x_1,s)-h(t,s) = h(x_1,x_2)-\triangle_{(t,s)}h(x_1,x_2),$$

and

$$\begin{aligned}&|G_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h(t,s);x_1,x_2)-h(x_1,x_2)|\\ &\leq T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(|\triangle_{(t,s)}h(x_1,x_2)|;x_1,x_2)\\ &\leq \omega_{mixed}(h,\delta_1,\delta_2)T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((1+\frac{|t-x_1|}{\delta_1})(1+\frac{|s-x_2|}{\delta_2});x_1,x_2)\\ &\leq \omega_{mixed}(h,\delta_1,\delta_2)\Bigg(1+\frac{1}{\delta_1}\sqrt{T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)^2;x_1,x_2)}\\ &+\frac{1}{\delta_2}\sqrt{L_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((s-x_2)^2;x_1,x_2)}\\ &+\frac{1}{\delta_1\delta_2}\sqrt{T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((t-x_1)^2;x_1,x_2)T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}((s-x_2)^2;x_1,x_2)}\Bigg).\end{aligned}$$

By Lemma 4.3, we get,

$$|G_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h(t,s);x_1,x_2)-h(x_1,x_2)|$$
$$\leq \omega_{mixed}(h,\delta_1,\delta_2)\Bigg(1+\frac{1}{\delta_1}\sqrt{\frac{s_1^2-s_1+1}{4n_1}}+\frac{1}{\delta_2}\sqrt{\frac{s_2^2-s_2+1}{4n_2}}$$
$$+\frac{1}{\delta_1\delta_2}\sqrt{(\frac{s_1^2-s_1+1}{4n_1})(\frac{s_2^2-s_2+1}{4n_2})}\Bigg).$$

Now, assume that $\delta_1=n_1^{-\frac{1}{2}}$ and $\delta_2=n_2^{-\frac{1}{2}}$ we get

$$|G_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h(t,s);x_1,x_2)-h(x_1,x_2)|$$
$$\leq \omega_{mixed}(h,n_1^{-\frac{1}{2}},n_2^{-\frac{1}{2}})\Bigg(1+\sqrt{\frac{s_1^2-s_1+1}{4}}+\sqrt{\frac{s_2^2-s_2+1}{4}}$$
$$+\sqrt{(\frac{s_1^2-s_1+1}{4})(\frac{s_2^2-s_2+1}{4})}\Bigg)$$
$$\leq (1+M_{s_1,s_2})^2\omega_{mixed}(h,n_1^{-\frac{1}{2}},n_2^{-\frac{1}{2}}),$$

where $M_{s_1,s_2}:=max\left\{\frac{s_1^2-s_1+1}{4},\frac{s_2^2-s_2+1}{4}\right\}$. □

5.3 Graphical Analysis

In this part I am going to show the approximation of the operators $T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}$ and $G_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}$ to a function $h(x_1,x_2)$ where $h(x_1,x_2)=2x_1 sin(\pi x_1)x_2^3$ to study their approach for different values of $\alpha_1,\alpha_2,s_1,s_2,n_1$ and n_2. Then immediately the following can be observed,

1. The small values of s_1 and s_2, while the other parameters are fixed the better approximation of $T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h)$ to the function h (see Figure 5.1).

2. The large values of α_1 and α_2, while the other parameters are fixed , the better approximation of $T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h)$ to the function h (see Figure 5.2).

3. In Figure 5.3 The approximation of the operator $T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h)$ to the function h is shown, which is also proved theoretically in the paper that by increasing the degrees (n_1,n_2) will be better.

4. Approximation of the GBS operators $G_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h)$ is better than approximation of the $T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h)$ to the function h (see Figure 5.4).

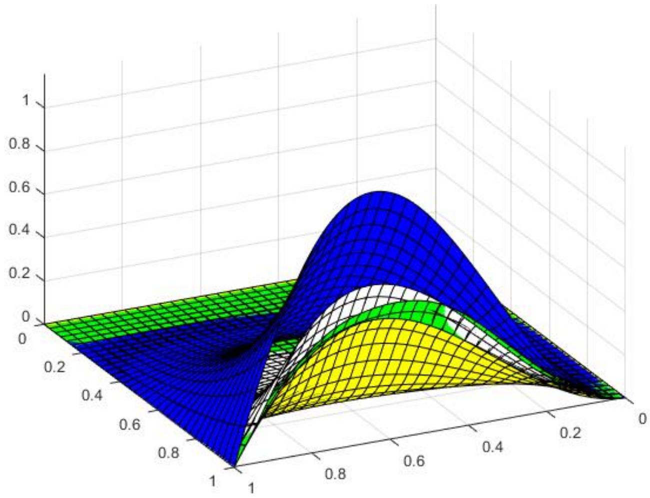

Figure 5.1: Approximation of $T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2)$ to $h(x_1,x_2)$ (blue) for $s_1,s_2 = 5(white)$, $15(green)$, $30(yellow)$, $\alpha_1,\alpha_2 = 0.7$ and $n_1,n_2 = 45$

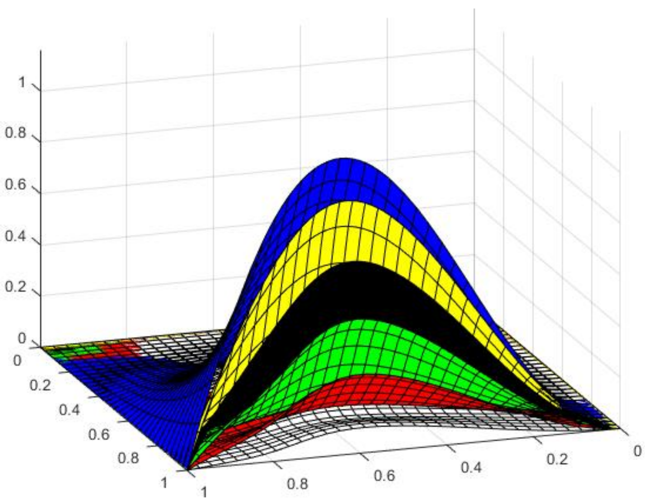

Figure 5.2: Approximation of $T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(h;x_1,x_2)$ to $h(x_1,x_2)$ (blue) for $s_1,s_2 = 10$, $\alpha_1 = \alpha_2 = 0.1(white)$, $0.3(red)$, $0.5(green)$, $0.7(black)$, $0.9(yellow)$ and $n_1,n_2 = 45$

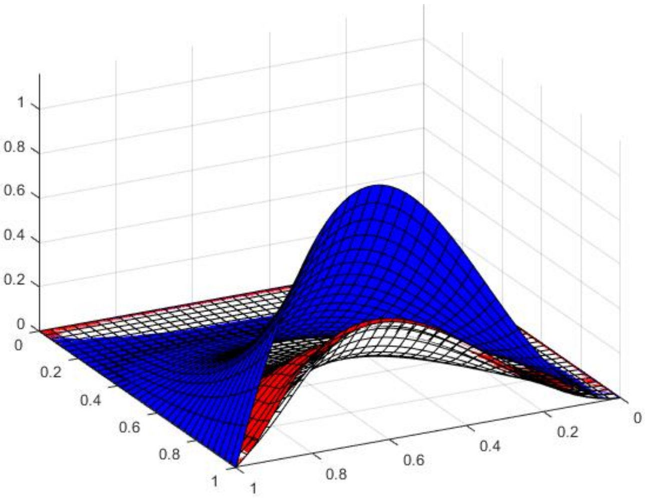

Figure 5.3: Approximation of $T^{\alpha_1,\alpha_2,s_1,s_2}_{n_1,n_2}$ to $h(x_1,x_2)$ (blue) for $s_1 = s_2 = 6$, $\alpha_1, \alpha_2 = 0.4$ and $n_1, n_2 = 35(white)$, $n_1, n_2 = 50(red)$

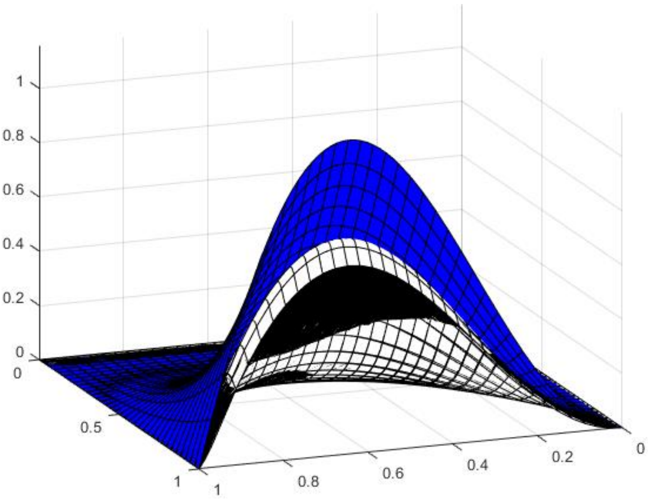

Figure 5.4: Approximation to $h(x_1,x_2)$ (blue) with $s_1 = s_2 = 2$, $\alpha_1 = \alpha_2 = 0.7$ and $n_1 = n_2 = 60$, $G^{\alpha_1,\alpha_2,s_1,s_2}_{n_1,n_2}(h;x_1,x_2)(white)$, $T^{\alpha_1,\alpha_2,s_1,s_2}_{n_1,n_2}(h;x_1,x_2)$(black)
.

Chapter 6

CONCLUSION

In this thesis, a new class of non-paltry extention of bivariate blending type Bernstein operators are introduced. The blending type Bernstein operators which have been studied by other researchers includes parameters α_1 and α_2 but the proved operators here contain parameters α_1, α_2, s_1 and s_2. There are some handouts achieved in this thesis as follow:

1. Proposed operators in this thesis depend on four parametes α_1, α_2, s_1 and s_2. So there are two more variables s_1 and s_2 by which investigating of approximation of proposed operators are more advantageous.

2. The parameters s_1 and s_2 effect the sum and cause more flexible than the defined blending type Bernstein operators in [5].

3. The Korovkin and Voronoskaja type theorems are proved for the given operators.

4. The GBS case for the defined operators, with four parameters, are introduced as $G_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}$ where approximation is defined through different values of s_1 and s_2.

5. The associate GBS operators $G_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}$ give better approximation than $T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}$.

6. Some approximation results for $T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}$ are also obtained.

7. The operators $T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}$ satisfy the condition $T_{n_1,n_2}^{\alpha_1,\alpha_2,s_1,s_2}(1) = 1$ for some values of n_1 and n_2. It is while the suggested pervious operators by other researchers do not satisfy in the mentioned condition. So it is easy to see that the Korovkin type approximation theorem still hold and the same results can be obtained. It should be mentioned that defining the suggested operators as a piecewise function have solved

this issue easily.

8. It should be mentioned that Blending-type Bernstein operators can be used in the control theory, modeling theory and finding numerical solution of integral equations. For example see Maleknejad et al 2011 ([44]) for the approximate solution of Volterrra integral equations using Bernstein operators .

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