Non-Commutative Quantum Mechanics

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ABSTRACT

In this thesis we have summarized the background of non-commutative quantum mechanics. We have also loosely touched the related topics.

Non-commutativity in quantum mechanics completely changes our understanding of the phase space and has the potential to explain multiple problems in different branches of physics. This makes it a very significant topic to research.

As an example we have calculated the Landau Problem well-known commutative setting, and the non-commutative setting. From these results we can see that in the presence of sufficiently strong magnetic fields it is possible to measure θ and η may be experimentally accessible.

Due to the symmetry of the problem, we show that the addition of an electric field perpendicular to the magnetic field has no effect on the Landau levels for commutative and non-commutative settings.

Keywords: Networks, Oscillators, Synchronization, Desynchronization, Non-Commutativity, Quantum Field Theory, Quantum Mechanics.

ÖZ

Bu tezimizde komutatif olmayan kuantum mekaniği uzerinde çalıştık. Aynı zamanda kısaca ilgili konulara da dokunduk.

Komutatif olmayan kuantum mekaniği faz alanları hakkındakı algımızı tamamen değiştirir, Aynı zamanda fiziğin pek çok alanındakı problemlere de açiklama getirme potansiyeli vardır.

Örnek problem olarakm hem komutatif hem de komutatif olmayan durumlarda Landau problemini inceledik. Bu şekilde θ ve η'nin, deney sonuçları ve komutatif faz alanındaki beklentilerini karşılastırarak deneysel olarak ölçülebileceği sonucuna vardık.

Problemin simetrisinden dolayı, hem komutatif, hem de komutatif olmayan durumlarda manyetik alana dik olacak sekilde eklenen bir elektrik alanının, Landau seviyelerine herhangi bir katkıda bulunmayacağını gösterdik.

Anahtar Kelimeler: Ağlar, Osilatörlerler, Senkronizasyon, Desenkronizasyon, Komutatif Olmama, Kuantum Alan Teorisi, Kuantum Mekaniği.

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DEDICATION

... to my grandmother Fatma Asgar...

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LIST OF ABBREVIATIONS

CPT	Charge Parity Time
NC	Non-Commutative
NCFT	Non-Commutative Field Theory
NCG	Non-Commutative Geometry
NCQM	Non-Commutative Quantum Mechanics
QFT	Quantum Field Theory

Chapter 1

INTRODUCTION

In 1947, Hartland Snyder published his revolutionary work on non-commutative Quantum Mechanics. By the introduction of non commutativity, Snyder hoped to solve the divergence problem in quantum field theory which are explained more in detail in section 5 [1].

This idea, later developed by Connes et al [2], who worked on the appropriate mathematical background, takes a non-commutative geometry as a passive background and builds quantum phenomenon on it.

We assume the presence of a quantum gravity. This will distort the space which leads to the non commutativity of space and kinetic momentum. This will be explained more in detail later on in our discussion. As we approach Planck length, we observe that the non commutativity becomes more apparent[3].

This is also compatible with the recent findings in string theory [4, 5, 6].

The distortion of space makes the position NC, and as we cannot precisely measure the position in more than 2 dimensions which makes the position of the quantum particle ambiguous. Due to this, the non commutativity is believed to be the root cause of string theory.

Another parameter that could possible have an effect of the amount of non commutativity is the presence of a magnetic field for a charged quantum particle[7]. The argument behind is that the non commutativity of kinetic momentum leads to the modification of momentum space in the presence of a magnetic field.

Nonlocality and Lorentz invariance violation can be considered as interesting outcomes of non-commutativity.

It has been suggested that the gravity implies a minimum observable length called the Planck length [8, 9].

The scales maximum as big as the Planck scale require high energies to resolve disturb the space-time structures because of their gravitational effects. Of course this can only happen as we approach the Planck scale. If this is true, this could potentially imply that the gravity changes the quantum field theories [10]. We assume that a minimal length should be expressed as a nonzero uncertainty Δx_0 in position measurements. Arguments by string theorists also imply a minimal length in the same form [10].

It is possible to express certain non-commutative geometries as the dual space to a curved space[11, 12].

Quantum gravity causes an uncertainty principle. This uncertainty bounds the maximum accuracy of the measurement to the Planck length. This means that as we approach the Planck length, the momentum we need to make the measurement modifies the geometry [13].

It is a subject of debate weather the existence of a minimal length would imply minimal areas and minimal volumes. Similarly we can possibly come to the conclusion of the existence of maximal lengths and maximal volumes [14, 15, 16].

Chaturvedi et al[17] has presented the first work on non-relativistic non-commutative work, which has later on been worked on and developed by Gamboa [18]. Mathematicians such as Connes Douglas et al [2, 19, 20] have also very significant contributions on the construction of a non-commutative geometry which is an essential part of NCQM.

Non commutativity of space is compatible with the General Uncertainty Principle [21]. According to [22, 23], the minimum length is also consistent with the General Uncertainty Principle. As we will explain below, the existence of a limitation of distance causes a UV cut-off which in turn gives rise to a phenomenon called UV/IR Mixing [24].

According to [25], the total information in the universe is bounded which is compatible with the existence of a minimal length, or better said, the quantization of space.

It has also been proposed that non that entanglement induces non-commutativity in space [26].

In this thesis we will be discussing the effects of non-commutativity on non relativistic quantum mechanics.

In order to make this discussion easier to read, we would like to explain our notation:

Here we use $\hat{x}$, $\hat{y}$ and $\hat{z}$ to be the non-commutative position operators whereas $\hat{p}_i$ is the non-commutative momentum operator. x, y, z are the commutative position operators and p_i is the commutative momentum operator.

In the following, we will give a short overview of the sections in this thesis:

In chapter 2 we loosely discover the mathematical background on non commutativity. This includes the Heisenberg algebra, Wightman axioms, which are the mathematical representations of the quantum field theory, and we also detail the non-commutative geometry.

In Chapter 3, we talk about the current understanding of the uncertainty principles in a commutative setting.

In Chapter 4 we give a brief explication to the differences of kinetic and canonical momentum.

From Chapter 5 on we start to talk about non-commutativity in quantum physics in more detail. We define and derive the Moyal $\star$-product and give the physical meaning of its usage. After constructing NCFT, we explain its significant consequences and related research areas.

In Chapter 6, we introduce a uniform magnetic field and proceed to observe the differences of the Landau Problem in commutative and non-commutative cases.

We will end our discussion with a conclusion in chapter 7.

Chapter 2

MATHEMATICAL BACKGROUND

NCQM is worked on using the deformation quantization. Later on, this discussion has given birth to the non-commutative field theory.

2.1 Heisenberg Algebra

In order to express the non commutativity mathematically, we introduce Heisenberg Algebra:

$$\left[\hat{x}_i, \hat{x}_j\right] = i\theta_{ij} \tag{2.1}$$

$$\left[\hat{p}_i, \hat{p}_j\right] = i\eta_{ij} \tag{2.2}$$

$$\left[\hat{x}_i, \hat{p}_j\right] = i\hbar\alpha^2\delta_{ij} - i\frac{\theta\eta}{4\alpha^2\hbar}\delta_{ij}, \tag{2.3}$$

where θ_{ij} and η_{ij} are anti-symmetric tensors. θ_{ij} is dependent on the gravitational field while η_{ij} is predicted to be dependent on magnetic field.

The relationship between these commutative and non-commutative versions is

$$\hat{x}_i = \alpha x_i - \frac{\theta_{ij}}{2\alpha\hbar} p_j \tag{2.4}$$

and

$$\hat{p}_i = \alpha p_i + \frac{\eta_{ij}}{2\alpha\hbar} x_j \tag{2.5}$$

We can already see that taking $\alpha = 1$ (which is the scaling factor) and $\theta = \eta = 0$ gives commutative versions of the position and momentum operators.

In [27] the antisymmetric tensor is extended which causes a change of algebra.

As θ and η are anti-symmetric tensors, it is also possible to write

$$\eta_{ij} = \eta\lambda_{ij} \tag{2.6}$$

and

$$\theta_{ij} = \theta\lambda_{ij} \tag{2.7}$$

where λ is an anti-symmetric tensor.

$$\lambda_{ij} = \begin{cases} \lambda & \text{for } ij = 12 \\ -\lambda & \text{for } ij = 21 \\ 0 & \text{else} \end{cases}$$

We will also talk about what we call the Moyal Product later in our discussion. One of the main tools to convert our understanding from commutative quantum mechanics to non-commutative quantum mechanics is done by using a Star Moyal Product. The proof, the usage and physical interpretations are detailed in 5.1.1.

The textbook definition of the time independent Schrödinger equation is

$$H\psi = E\psi \tag{2.8}$$

now becomes

$$H(x,p) \star \psi(x) = E\psi(x) \tag{2.9}$$

where $\star$ denotes the Weyl-Moyal Product

2.2 Wightman Axioms

The Wightman axioms are an attempt to mathematically model the Quantum Field Theory, therefore we will be needing the Wightman Axioms further in our discussion, in a non-commutative setting.

Wightman axioms are an attempt to mathematically represent Quantum Field Theory that have important consequences.

A common textbook representation of Wightman axioms are as follows:

Axiom 1. There exists a physical Hilbert space H in which a unitary representation $U(a,\Lambda)$ of the Poincarré spinor group, P_0 acts in.

Axiom 2. The spectrum of the energy-momentum operator P is concentrated in the closed upper light cone V^+.

Axiom 3. In H, there exists a unique unit vector $|0\rangle$. This is invariant with respect to the space-time translations $U(a,1)$.

Axiom 4. The components ϕ_i of the quantum field ϕ are operator-valued distributions $\phi_i(x)$ over the Schwartz space S(M) with domain of definition D. $|0\rangle$ is contained in D.

Axiom 5. $U(a,\Lambda)\phi_i(x)U(a,\Lambda)^{-1} = \sum_j V_{ij}(\Lambda - 1)\phi_j(\Lambda x + a)$ where $V_{ij}(\Lambda)$ is a complex or real finite-dimensional matrix representation of $SL(2,C)$.

2.3 Non-Commutative Geometry

The main tool for non-commutative geometry is a usually associative but not necessarily commutative algebra that we call A. Taking a, b as elements found in A.

For any $a, b \in A$ we can discuss three types of products. These products are ab, $a.b$ and $a \star b$

The nature of the third product will be clearly explained in chapter 5.1.1.

Assuming a space M that contains a classical complex scalar field. The addition and the multiplication $(f+g)(x) = f(x) + g(x)$ and $(f.g)(x) = f(x).g(x)$ and well defined properties of a commutative algebra A.

The first example that comes to mind when we say 'non-commutative algebra' is the Mat_n. This is the algebra of $n \times n$ matrices.

As quantum physicists have been doing for a long time know, in order to gain insight on operators in quantum mechanics, we can express them in the form of matrices. This is the first hint of non commutativity in quantum mechanics [2, 14, 20].

Chapter 3

UNCERTAINTY PRINCIPLES

3.1 Heisenberg Uncertainty Principle

This chapter is inspired by [7].

Definition 3.1: (Banach Space) A normed space E is called complete if every Cauchy sequence in E converges to an element of E. A complete normed space is called a Banach space.

Definition 3.2: (Hilbert Space) A complete inner product space is called a Hilbert space. As the standard formulations of quantum mechanics require the Hilbert spaces used to be separable, we will here give the further definition of separable Hilbert spaces as well. A Hilbert space is separable if and only if it admits a countable orthonormal basis.

Theorem 4.5. (The Uncertainty Principle) If $\hat{A}$ and $\hat{B}$ are Hermitian operators, then for any state vector we can write that for the scalar product of two wave functions, the Schwarz inequality is

$$|\langle\phi|\psi\rangle|^2 \leq \langle\phi|\phi\rangle \langle\psi|\psi\rangle \tag{3.1}$$

Proof is given below.

i For $\phi = 0$ the inequality is trivial

$$|\langle\phi|\psi\rangle|^2 = 0 \tag{3.2}$$

$$\langle\phi|\phi\rangle \langle\psi|\psi\rangle = 0 \tag{3.3}$$

ii For $\phi \neq 0$ we can write $\psi = z\phi + \xi$

where $z \in \mathbf{C}$ and $\phi, \psi \in H$ and $x\phi$ is parallel to ϕ and ξ is perpendicular to ϕ.

This means that the wavefunction ψ can be decomposed into a sum of components with inner product with zero and non-zero ϕ and this implies that $\langle\phi|\xi\rangle = 0$. and $\langle\phi|\phi\rangle = 1$

We can also write $\langle\phi|\psi\rangle = z\langle\phi|\phi\rangle$

so $z = \frac{\langle\phi|\psi\rangle}{\langle\phi|\phi\rangle}$

$$\langle\psi|\psi\rangle = \langle z\phi + \xi|z\phi + \xi\rangle = z^*z\langle\phi|\phi\rangle + \langle\xi|\xi\rangle \geq z^*z\langle\phi|\phi\rangle$$

Substitution of z gives

$$\langle\phi|\phi\rangle \geq \frac{|\langle\phi|\psi\rangle|^2}{\langle\phi|\phi\rangle} \tag{3.4}$$

Exact analogs of the Pythagorean Theorem and Parallelogram Law hold in Hilbert spaces. A Hilbert space naturally must have many restrictions. To define Hilbert spaces, we will start with an inner product space as it is simply a vector space with the additional structure of inner product.

3.2 Uncertainty Principle

Theorem 1. If $\hat{A}$ and $\hat{B}$ are Hermitian operators, then for any state vector

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}\left\|\left\langle[\hat{A},\hat{B}]\right\rangle\right\| \tag{3.5}$$

Proof:

$$\Delta\hat{A} = \|\psi_1\| \tag{3.6}$$

where $\psi_1 = (\hat{A} - <\hat{A}>)\psi$

and

$$\Delta\hat{B} = \|\psi_2\| \tag{3.7}$$

where $\psi_2 = (\hat{B} - <\hat{B}>)\psi$

From the Schwartz inequality we can write

$$\begin{aligned}
\|\psi_1\|\|\psi_2\| &\geq \|\langle\psi_1|\psi_2\rangle\| \\
&\geq \|\mathrm{Im}\,\langle\psi_1|\psi_2\rangle\| \\
&\geq \left\|\frac{1}{2i}(\langle\psi_1|\psi_2\rangle - \langle\psi_1|\psi_2\rangle^*)\right\| \\
&\geq \left\|\frac{1}{2i} < \psi_1|\psi_2> - < \psi_2|\psi_1>\right\|
\end{aligned}$$

Since $\hat{A}$ and $\hat{B}$ are Hermitian operators

$$\begin{aligned}
\Delta\hat{A}\Delta\hat{B} \geq \left\|\frac{1}{2i}\left(\left\langle(\hat{A}-\langle\hat{A}\rangle)\psi\middle|(\hat{B}-\langle\hat{B}\rangle)\psi\right\rangle - \left\langle(\hat{B}-\langle\hat{B}\rangle)\psi\middle|(\hat{A}-\langle\hat{A}\rangle)\psi\right\rangle\right)\right\| \\
\geq \left\|\frac{1}{2i}\left\langle\psi\middle|\left(\hat{A}\hat{B}-\hat{B}\hat{A}\right)\psi\right\rangle\right\|
\end{aligned}$$

Therefore we end up with

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{1}{2}\left\|\left\langle[\hat{A},\hat{B}]\right\rangle\right\|$$

So now, taking $\hat{A} = x_j$ and $\hat{B} = p_j$, we get

$$\Delta\hat{A}\Delta\hat{B} \geq \frac{\hbar}{2} \tag{3.8}$$

3.3 Energy Time Uncertainty

Up until now, we have been interested in the observables at a certain instant and their commutation relations. More interestingly there are uncertainty principles of energy and time relations. The proof of this uncertainty is way more difficult so it cannot be given in a simple manner.

In this chapter, we will refer to a time interval and the difference of energies E_1 and E_2 which are two different values at two different instants separated by Δt [7].

$$\Delta E \Delta t \geq \frac{\hbar}{2} \tag{3.9}$$

For Hamiltonians permitted by the laws of physics we can write

$$[H, t] = i\hbar \tag{3.10}$$

which brings us to the conclusion that the time observable does not exist [28, 29].

Chapter 4

KINETIC AND CANONICAL MOMENTUM

The definition of kinetic momentum is

$$m\vec{\dot{x}} = \vec{p} - \frac{e}{c}A \tag{4.1}$$

We call $\vec{p}$ to be the canonical momentum.

They satisfy the following commutation relations

$$[x_i, p_j] = i\hbar\ \delta_{ij} \tag{4.2}$$

$$[x_i, m\dot{x}_j] = i\hbar\delta_{ij} \tag{4.3}$$

$$[\pi_i, \pi_j] = i\hbar\varepsilon_{ijk}B_k \tag{4.4}$$

By applying the Maxwell equation $\vec{\nabla}.\vec{B} = 0 \rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$ so $B_i = \varepsilon_{ijk}\partial_j A_k$

where, ε_{ijk} is the anti-symmetric Levi-Civita tensor

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\} \text{ even permutation of } (i,j,k) \\ -1 & \text{if } (i,j,k) \in \{(2,1,3),(3,2,1),(1,3,2)\} \text{ odd permutation of } (i,j,k) \\ 0 & \text{else} \end{cases}$$

and with the vector potential $\vec{A}$

$$\nabla \times \vec{A} = \vec{B} \Longleftrightarrow \varepsilon i jk \frac{\partial A_k}{\partial x_j} = B_i$$

As shown above, the kinetic momentum does not commutate with itself for more than one dimension and this has important consequences, particularly in the presence of a magnetic field [7].

Quantum field theory provides equal terms for particles and fields. This is done to accurately explain the interaction of radiation and matter, and is an important point of view to get a better understanding of particle physics events involving collisions, and a change in the total number of particles.

The change of total number of particles involves relativistic explications.

However, there are divergence problems in QFT coming from the large number of virtual states summing over for high energies.

There are three main parts of QFT [30]. These are Causality, Positivity of Energy and Unitarity.

In quantum field theory we talk about bosonic and fermionic fields. The bosonic fields obey canonical commutatuion relations. Fermionic fields, on the other hand obey canonical anticommutation relations. In this part we will be interested in the bosonic fields.

In quantum field theory we define the vanishing of field commutators for bosonic fields as causality.It can be mathematically expressed as

$$[\phi(x), \phi(y)] = 0 \tag{4.5}$$

for $(x-y)^2 \leq 0$

Here, x, y are two points and are space-like separated. It has been argued that it is possible to interpret Quantum Field Theory as a description of causal interactions [31].

The positivity of energy implies the positivity of the Hamiltonian in the representation associated to the quantum states.

Unitarity is written as

$$\psi(a^* a) \geq 0 \tag{4.6}$$

for all $a \in A$.

The causality and the positivity of energy are incompatible in quantum mechanics however the positivity of energy implies the propagation of particles outside the light cone. This is the reason for the existence of antiparticles [32].

Chapter 5

NON-COMMUTATIVE FIELD THEORY

The first ever indication of non commutativity in quantum mechanics is the usage of matrix theory to represent the operators [14, 33, 34].

Inclusion of non-commutativity in quantum field theory can be achieved in two different ways [30]. This can be done by either using a Moyal $\star$-product on the space of ordinary functions, or by defining the field theory on a coordinate operator space which is intrinsically non-commutative.

From now on we will use the example of $\mathbf{R}^d_\theta$ where $\mathbf{R}^d_\theta$ is an algebra of all complex linear combinations of products of d variables. The reason we particularly choose $\mathbf{R}^d_\theta$ is because it is one of the most commonly studied by theoretical physicist. It is of course, possible to use other examples to define non-commutative field theories, however the physics behind them is less understood.

5.1 Construction of Non-commutative Field Theory

5.1.1 Construction of Star Product

We consider a non-commutative plane

$$[\hat{x}, \hat{y}] = i\theta \tag{5.1}$$

$\hat{x}$ and $\hat{y}$ are phase space variables on Hilbert Space H_θ which is a part of the specification of the classical field theory. This already implies the non locality in a quantum scale.

We have a wave function on a square integrable domain $L^2(R)$. We need a square integrable domain to make sure to have a normalisation term.

Writing down the following relationships

$$\hat{x}\psi = x\psi \tag{5.2}$$

and

$$\hat{y}\psi = -i\theta\partial_x\psi(x) \tag{5.3}$$

The standard operation identity is

$$e^{ip\hat{y}} f(\hat{x}) = f(\hat{x} - p\theta)e^{ip\hat{y}} \tag{5.4}$$

$\hat{y}$ is the conjugated momentum.

As we can clearly deduct from (5.4), $\hat{y}$ generates translations of $\hat{x}$ eigenvalues.

Composition rule for plane waves are as follows

$$e^{ip_\mu \hat{x}^\mu} \times e^{iq_\mu \hat{x}^\mu} = e^{-\frac{i}{2}pq} e^{i(p+q)_\mu \hat{x}^\mu} \tag{5.5}$$

with x_μ, p_μ being the position and momentum operators.

From Baker-Cambel-Haussdorf equation.

Baker-Cambel-Haussdorf equation is the solution Z of

$$\exp\{Z\} = \exp\{X\}\exp\{Y\} \tag{5.6}$$

where X and Y may be non-commutative and are the elements of the Lie algebra, of a Lie Group.

The above representation is the first order approximation and is written in the form

$$Z = X + Y + \frac{1}{2}[X,Y] + \dots \tag{5.7}$$

As stated above, we can use the operator algebra to map it to some deformed function algebra.

Using components in a suitable basis we can map the operator algebra to a deformed function algebra.

The basic idea is to use the infinite dimensional generalisation of the standard choice of a basis in a finite dimensional $U(N)$ lie algebra is stated below.

A Lie algebra is a vector space G together with a certain bi-linear map, also called a Lie Bracket, that satisfies a Jacobi identity.

Let T^a be the basis of generators.

Let A be a Hermitian matrix with vector components A^a in T^a

$$A = \Sigma A^a T^a \tag{5.8}$$

For an operator $\hat{O}$ acting on H_θ the vector components in a given basis is a function of a continuous label $f_{\hat{O}}(\hat{x}^\mu)$

The map will be expressed as

$$\hat{O} = \int d^d x f_{\hat{O}(x^\mu)} \hat{T}_x^\mu \tag{5.9}$$

Definition 5.1: A representation of the operator product in components in the space of a component functions is called a star product and is written as below

$$f_{\hat{O}\hat{O}'} = f_{\hat{O}}(x) \star f_{\hat{O}'}(x) \tag{5.10}$$

We will be using the following basis

$$T^a T^b = \sum_c C_c^{ab} T^c$$

where C_c^{ab} is a constant.

Let's say A and B are two hermitian matrices expressed as

$$A = \sum_a A_a T_a \tag{5.11}$$

$$B = \sum_a B_a T_a \tag{5.12}$$

Using equation (5.11) and (5.12) we can express AB as

$$AB = \sum_{a,b} A_a B_b C_c^{ab} T^c = \Sigma(AB)_c T^c \tag{5.13}$$

As explained above, we are using a Weyl map $\mathbf{R}^d_\theta$.

A Weyl Map is an invertable mapping between functions in phase space formulation and Hilbert space operators.

The mathematical expression will be

$$\hat{T}_{x^\mu} = \int \frac{d^d k}{(2\pi)^d} \exp\{ik(\hat{x} - x)\} \tag{5.14}$$

The inverse Weyl map can be written as

$$f_{\hat{O}}(x^\mu) = \int \frac{d^d k}{(2\pi)^d} \mathrm{Tr} \exp\left\{ik_\mu(x-\hat{x})^\mu\right\} \hat{O}(\hat{x}^\mu) \tag{5.15}$$

It is useful to keep in mind that the plane operator $\exp\{ip\hat{x}\}$ is associated to the plane wave function $\exp\{ipx\}$.

Using (5.10) and (5.5) we can write

$$\exp\{ipx\} \star \exp\{ipx\} = \exp\{ipx\} \times \exp\{iqx\} = \tag{5.16}$$

$$= \exp\left\{-\frac{i}{2}pq\right\} \exp\{i(p+q)x\} \tag{5.17}$$

Therefore

$$\exp\{ipx\} \star \exp\{iqx\} = \exp\left\{-\frac{i}{2}pq\right\}\exp\{i(p+q)x\} \tag{5.18}$$

which, by superposition gives

$$f(x) \star g(x) = f(x)\exp\left\{\frac{i}{2}\overleftarrow{\partial}_\alpha \theta^{\alpha\beta}\overrightarrow{\partial}_\beta\right\} \tag{5.19}$$

This expression is called a Moyal Product. It is associative but non-commutative.

Therefore we can say that the NCG amounts to a smooth deformation of the classical algebra of functions is equivalent to the changing of composition rules, but the elements of algebra stay the same. Since A_θ can be viewed as a deformation of the ordinary algebra of functions on $\mathbf{R^d}$ we can construct NCFT by deforming action functionals in a straightforward way[14, 20, 35].

An important property of the Moyal product is that it is cyclic

$$\int d^dx f(x) \star g(x) \star h(x) = \int d^dx g(x) \star h(x) \star f(x) \tag{5.20}$$

Under the same conditions we can write

$$\int d^dx f(x) \star g(x) = \int d^dx f(x) g(x) \tag{5.21}$$

5.1.2 Physical Interpretation of the Moyal Product

We assume the existence of a non-commutative field $\phi(x)$and a particle in it.

This particle interacts with a potential $V(x)$ by a term

$$\int d^d x(V(x) \star \phi(x) - \phi(x) \star V(x)) \tag{5.22}$$

For a plane wave we have

$$V(x) \star \exp\{ip.x\} - \exp\{ip.x\} \star V(x) \tag{5.23}$$

We can see that the non-commutative interaction is reproduced by a rigid dipole oriented along the vector:

$$L^\mu = \theta^{\mu\nu} p_\nu \tag{5.24}$$

interacting through the end-points, like a rigid open string.

This analogy is actually rather literal, as we will see in the next section.

The proof of this phenomenon is given by [6]. When the propagator is calculated using boundary conditions. For the boundary

$$g^{ij}(\partial - \bar{\partial})x_j + 2\pi\alpha' B_{ij}(\partial + \bar{\partial})x_j = 0 \tag{5.25}$$

at $z = \bar{z}$

By doing the calculation, we end up with

$$\langle x_i(t), x_j(t') \rangle = -\alpha' G_{ij} \log\left(t - t'\right)^2 + \frac{i}{2}\theta_{ij}\varepsilon(t - t') \tag{5.26}$$

where t is time and

$$\varepsilon(t) = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \end{cases}$$

G_{ij} is interpreted as the effective metric seen by the open string and is called the open string metric.

Fields interacting in the "fundamental representation" as

$$\int d^d x V(x) \star \phi(x) \tag{5.27}$$

behave as half-dipoles of length $L^\mu/2$.

Therefore, the non-locality implied by the non commutativity constructed using the Moyal products is equavalent to the reinterpretation of the elementary excitations as extended rigid objects [35, 36].

This makes us to recconsider the Heisenberg's principle The effective size of a particle grows with the increase of momentum at high velocities in a linear manner in a non-commutative setting [35, 37, 38, 39].

$$L_{eff} = \max(\frac{1}{|p|}, |\theta.p|) \tag{5.28}$$

5.1.3 Non-commutative Uncertainty

As discussed above, in a non-commutative framework, the uncertainty principles discussed above may be modified [10, 11, 40, 41, 42, 43, 44].

We continue our discussion in a dense domain $D \in H$ where H denotes the Hilbert space.

The uncertainty relations should be in the form

$$\Delta A \Delta B \geq \frac{1}{2} |< \langle [A,B] \rangle >| \tag{5.29}$$

to make sure that for $[x_i, x_j] \neq 0$ we have $\Delta x_i \Delta x_j \geq 0$

For one dimension, we may rewrite the general uncertainty relation [10]

$$[x_i, p_j] = i\hbar(1 + \beta(\Delta p)^2 + \gamma) \tag{5.30}$$

and for more than one dimensions it can be generalised to

$$[x_i, p_i] = i\hbar(\delta_{ij} + \alpha_{ijkl} x_k x_l + \beta_{ijkl} p_k p_l) \tag{5.31}$$

Here when we talk about α and β are appropriate matrices with complex variables and are obliged to be consistent with the *-convolution so they have to obey $\alpha^*_{ijkl} = \alpha_{ijkl}$ and $\beta^*_{ijkl} = \beta_{ijkl}$.

In Chapter 6, we explain the consequences of non-commutativity and introduce the relating subjects.

5.2 Important Consequences of Non-commutativity

5.2.1 Nonlocality

Quantum nonlocality is an important problem of the understanding of quantum concepts [45].

It has been suggested that the quantum phase space may have a connection to hidden variables but as of now, no direct connection has been found [46].

At this point, space and time stop appearing in the fundamental formulation of the quantum theory. Especially the lack of a time operator have been proposed as the missing link between quantum mechanics and general relativity [47, 48].

The smaller the scale, the more the geometry of the space becomes ambiguous and this effect pushes us to reconsider the validity of locality in quantum scales.

Of course, introduction of non-locality means the point interactions should also be reconsidered.

The string theory, along with non commutativity support this idea. It is argued that the ambiguities of the space is the result of the nature of the strings.

Another interesting argument relies the non-locality to the cosmological constant. In the past it was proposed that the UV/IR Mixing that we will detail below relates to the cosmological constant. Recently it has been argued that the cosmological constant is not dependent on any other parameters. More research is needed to see weather non-locality has any relationship to the cosmological constant.

Increased energies do not mean an increase in the resolution. Actually, the resolution reduces. This causes significant ambiguities in the space. It is highly unlikely to find locality both in a space and its dual. Moreover, there is absolutely no reason for locality to exist in either of them[11].

An example of non-locality is given by

$$\delta(x) \star \delta(x) = \frac{1}{|\det(\pi\theta)|} \tag{5.32}$$

This means that the behaviour of the fields in short distances its long wavelength properties [2, 49].

5.2.2 Breakdown of Lorentz Invariance

We remember the space (non) commutation relation

$$[x_i, x_j] = i\theta_{ij}$$

The presence of a non zero θ_{ij} automatically causes the violation of the Lorentz symmetry.

It is very interesting to observe the Lorentz indices on θ_{ij}. There are two main types of Lorentz invariance [50].

Very basically, we can express them as

a) Where the motion leaves the physical laws independent on the observer. This is because the field operators and θ_{ij} transform covariantly.

b) Transformations that leave θ_{ij} unaffected Therefore the physics principles stay the same.

This sounds very similar to the result of spontaneous Lorentz invariance violation [51].

It is also interesting to note that the θ_{ij} acts like an expectation value tensor.

A non-commutative theory violates particle Lorentz symmetry because θ_{ij} forces a direction in space-time [52].

It is useful to note that the θ_{ij} found in non-commutative quantum mechanics carries Lorentz indices. The physics is modified to keep the θ_{ij} invariant therefore forces a direction in the space-time. This is analogous to the violation of Lorentz symmetry

[52].

On the other hand, it has been suggested [53] to bound η such that

$$\eta \geq (1.7 \times 10^{-25})E^2 \tag{5.33}$$

in order to conserve the validity of the Lorentz invariant.

5.2.2.1 Non-Commutative Standard Model

It has been proposed by Calmet et al [54] to restructure the standard model on a non-commutative setting.

They argue that the zeroeth order of θ_{ij}'s expansion gives the commutative electroweak model and during the study of a minimal non-commutative standard model, there happens to be a significant difference. In the non-commutative model, all interactions should be considered together because a master field had been introduced, which is nothing other than the superposition of different gauge fields.

The violation of Lorentz invariance could explain the unexpected cancellation of CPT-even perturbations [55] so it is interesting to also research high energy cosmic rays in a non-commutative setting.

The standard model building in a non-commutative setting has also been researched by [56].

The non-commutativity may introduce new research topics for the standard model and can potentially provide a better understanding on problems such as naturalness and the hierarchy problem[54].

5.2.3 UV/IR Mixing

One of the most interesting properties of the Non-Commutative Field Theory is UV/IR mixing.

We introduce the idea of loop momenta in this chapter as a unique integrand does not make sense in field theory. Loop momenta are introduced by the Feynman diagrams and is the straightforward tools to the proof of UV/IR Mixing.

By definition, UV/IR Mixing is a lack of Wilsonian decoupling between UV and IR scales and is rooted in the position uncertainty.

In general the non-commutative quantum field theory is not a smooth deformation of the ordinary $\theta = 0$ theory, even if it was so in the classical approximation. We also learn that, at fixed non-zero θ, the NCFT is IR singular as a result of divergences that originally had an UV interpretation, which is the origin of the name UV/IR mixing [35, 57].

5.3 CPT for NCFT

According to CPT Symmetry, if we reverse the charge, momentum and time, the physical laws governing the system will stay the same. This is particularly true for Lorentz invariant systems.

CPT Symmetry is a direct result of Wightman axioms.

However, for the construction of a non-commutative field theory we have to modify the Wightman axioms. This work is done by [10].

We assume the existence of a separable Hilbert space carrying the group $\beta = [O(1,1)SO(2)]\tau_4$. Also there is a vacuum state $|\Omega\rangle$ which is unique and invariant under β.

The momentum operator P is represented as

$$\mathrm{Spec}(P) = (p^0)^2 - (p^1)^2 \geq 0 \; where \; p^0 \geq 0 \tag{5.34}$$

Fields $\phi(x)$ are operator-valued distributions in the non-commutative space $\mathbf{R}^{1,3}$ transforming under β as

$$U(\Lambda,a)\phi(x)U(\Lambda,a)^{-1} = U_\Lambda \phi(\Lambda x + a) \tag{5.35}$$

In this equation, U_Λ is the matrix acting on the indices of $\phi(x)$

Wightman functions:

$$W(x_1,\ldots,x_n) = \langle\Omega|\phi(x_1),\ldots,\phi(x_n)|\Omega|\Omega|\phi(x_1),\ldots,\phi(x_n)|\Omega\rangle \tag{5.36}$$

are fine tempered distributions on the Schwartz space $S(\mathbf{R}^{1,3})$ of smooth test functions where a tempered distribution is a continuous linear functional on that Schwartz space.

In a non-commutative framework, UV/IR mixing may spoil the character of the tempered distributions.

We are forced to replace the light cone by the light wedge to relax the condition that the commutators vanish outside the light cone.

$$[\phi(x), \phi(y)]_{\pm} = f(x^0 - y^0)^2 - (x^0 - y^0)^2 \geq 0 \tag{5.37}$$

The other axioms stay the same.

One of the most important conclusions of previous Wightman axioms was the CPT symmetry. After studies on the matter,it was concluded the CPT Symmetry is still valid. However the UV/IR mixing may prove the CPT symmetry untrue for non-commutative field theory.

Chapter 6

NON COMMUTATIVITY IN MAGNETIC FIELD

Non-commutativity is detailed in [58, 59].

6.1 Charged Particles in Electromagnetic Fields

We assume the existence of a charge q mass m and an electromagnetic field $\vec{E},\vec{B}$.

$\vec{E}$ and $\vec{B}$ are physical fields.

There is a concept called "potential" that helps us to rewrite $\vec{E}$ and $\vec{B}$ in terms of mathematical quantities.

Quantum mechanics couples to the potentials so we can say that the potentials are more important than fields.

Most of the time, we consider an open Minkowski space, it doesn't really make a difference weather we describe an fundamental field or the potentials.

However, when we consider a topological content, we have to consider potentials.

$$\vec{B} \to \nabla \cdot \vec{B} = 0, \vec{B} = \nabla \times \vec{A} \tag{6.1}$$

$$\vec{E} \to \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{6.2}$$

therefore we can write

$$\nabla^2 \times (E + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}) = 0 \tag{6.3}$$

with

$$\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\nabla\phi \tag{6.4}$$

So we can write

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{6.5}$$

But now there is a freedom with this potentials which are called gauge transformations.

$$\nabla \times (\nabla u) = 0 \tag{6.6}$$

$$\vec{A} \to \vec{A}' = \vec{A} + \nabla\Lambda \tag{6.7}$$

$$\vec{B}' = \nabla \times \vec{A}' = \nabla \times \vec{A} = \vec{B} \tag{6.8}$$

We can see that vector potential $\vec{A}$ has changed by the addition of $\nabla\Lambda$, but $\vec{B}$ remains unchanged.

On the other hand, if you change the vector potential, if you change the vector potential you can affect $\vec{E}$ but if simultaneously you change ϕ $\vec{E}'$ will be left unchanged.

We want to keep $\vec{E}$ unchanged, therefore we define ϕ' as

$$\phi' = \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t} \tag{6.9}$$

By changing A and ϕ in this way, both B and E are unchanged.

We summarise the gauge invariance by saying that the new electric field due to ϕ' and A' is equal to the

$$\vec{E}'(\phi',A') = \vec{E}(\phi,A), \vec{B}'(\phi',A') = \vec{B}(\phi,A) \tag{6.10}$$

So from now on we will be describing electric and magnetic fields with $\vec{E}$ and $\vec{B}$ and in terms of their corresponding potentials.

And we assume that $(\phi',\vec{A}')$ is physically equivalent to $(\phi,\vec{A})$ it there is a Λ such that they are gauge transforms of each other.

6.2 Quantum Harmonic Oscillator

A quantum harmonic oscillator is never at rest as the ground state energy is necessarily greater than zero. This shows us the property of quantum fluctuations. An important distinction from classical harmonic oscillators is that stationary quantum states can exist on classically prohibited regions [60].

For a particle in the context of quantum mechanics, there are two main states;

1. Bound States

   This is the state that the wavefunction $\psi(x)$ approaches 0 as the position x goes to infinity

2. Unbound States

   We can consider bound and unbound states to model particles in quantum mechanics.

A particle in a bound state will have a discrete energy spectrum while a particle in an unbound state has a continuous one. In a for a continuous potential V(x) ψ_n has $n-1$ nodes for a quantum harmonic oscillator (see node theorem). There are a lot of symmetries in the quantum harmonic oscillator system and discrete energy spectrum is evenly spaced [7], with difference of hf between each consecutive n.

Unlike the Hamiltonian in classical mechanics, the quantum Hamiltonian always represents the total energy of the system.

6.3 Non-Commutative Landau Problem

This problem is previously worked on by [61, 62] In this section, two dimensional Landau problem is considered for a uniform magnetic field in z-direction, i.e. $\vec{B} = B\vec{e}_z$. The vector potential for a uniform magnetic field can be easily determined as

$$\vec{A}(\vec{r}) = \frac{1}{2}\vec{r} \times \vec{B} \tag{6.11}$$

As the magnetic field is only in z-direction, the vector potential becomes

$$\vec{A} = \frac{1}{2}\left(-yB\vec{e}_x + xB\vec{e}_y\right) \tag{6.12}$$

And the Hamiltonian has the following form

$$H = \frac{1}{2m}[(p_x^2 + \frac{qB}{2c}y)^2 + (p_y^2 - \frac{qB}{2c}x)^2 + p_z^2] \tag{6.13}$$

Developing this equation, we end up with

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega_L^2(x^2 + y^2) - \omega_L L_z + \frac{1}{2m}p_z^2 \tag{6.14}$$

where $L_z = xp_y - yp_x$ and $\omega_L = \frac{qB}{2mc}$

The non-commutativity in the problem forces us to express the Schrodinger's equation as

$$H(x,p) \star \psi = E\psi \tag{6.15}$$

We can replace the $\star$-product with the Bopp's Shift therefore the Schrödinger's equation will now be written as

$$H(\hat{x}_i, p_i) = H(x_i - \frac{1}{2\hbar}\theta_{ij}p_j, p_i)\psi = E\psi \tag{6.16}$$

The Hamiltonian is

$$H = \frac{1}{2m}(p_x^2 + (p_y^2) + \frac{1}{2}m\omega_L^2(x^2+y^2) - \omega_L L_z + \frac{1}{2m}p_z^2 =$$
$$\frac{1}{2\mu'}(p_x^2 + p_y^2) + \frac{1}{2}\tilde{\mu}\tilde{\omega}_L(x^2+y^2) - \omega_L L_z + \frac{1}{2m}p_z^2 = H_{xy} + H_{L_z} + H_{||} \quad (6.17)$$

where
$$\tilde{\mu} = \frac{\mu}{(1+\frac{qB}{4\hbar c}\theta)^2} \quad (6.18)$$

$$\tilde{\omega}_L = \frac{qB}{2\tilde{\mu}c(1+\frac{qB}{4\hbar c}\theta})} \quad (6.19)$$

The eigenvalues are in the form $E_{||} = \frac{\hbar^2 k^2}{2m}$ and $E_{L_z} = -n\hbar\tilde{\omega}_L$

The solution is written as

$$\psi = R(\rho)\exp\{in\phi\}\exp\{ikz\} \quad (6.20)$$

with $n = 0, \pm 1, \ldots$

The radial equation is

$$\left[-\frac{\hbar^2}{2\tilde{\mu}}(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho - \frac{n^2}{\rho^2}) + \frac{1}{2}\mu\tilde{\omega}_L\rho^2\right]R(\rho) = E_{xy}R(\rho) \quad (6.21)$$

The eigenvalues are

$E_{xy} = (N+1)\hbar\tilde{\omega}_L$

with $N = 2n_\rho + |m|$ where $n_\rho = 0, 1, 2, \ldots$ The corresponding eigenfunctions are

$$R(\rho) = \rho^{|m|}F(-n_\rho, |m|+1, \zeta\rho^2)\exp\left\{-\zeta^2\frac{\rho^2}{2}\right\} \quad (6.22)$$

where $\zeta^2 = \frac{\tilde{\mu}\tilde{\omega}_L}{\hbar}$

Therefore energy eigenfunctions become

$$\psi_{n_\rho mk} = \rho^{|m|} F(-n_\rho, |m|+1, \zeta\rho^2) \exp\{im\phi + ikz\} \tag{6.23}$$

For $\theta \to 0$ we return back to the commutative case.

6.4 Harmonic Oscillator in a Non-Commutative Background

The Schrödinger equation in a non-commutative case is

$$H(\hat{x}, y) \star \psi = E\psi \tag{6.24}$$

However we can write, using the Bopp's shift

$$H(\hat{\hat{x}}, \hat{y})\psi = E\psi \tag{6.25}$$

The Hamiltonian for the non-commutative case is the same as the Hamiltonian for the commutative case except that x_i, p_i becomes $\hat{x}_i, \hat{p}_i$.

We remember that the main difference of NC version of the harmonic oscillator is the replacement of space and momentum operators with the following expressions

$$\hat{x} = \alpha x - \frac{\theta}{2\alpha\hbar} p_y$$

$$\hat{y} = \alpha y + \frac{\theta}{2\alpha\hbar} p_x \tag{6.26}$$

$$\hat{p}_x = \alpha p_x + \frac{\eta}{2\alpha\hbar} y \tag{6.27}$$

$$\hat{p}_y = \alpha p_y - \frac{\eta}{2\alpha\hbar} x \tag{6.28}$$

So the new angular momentum is expressed as

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \alpha^2 L_z + \frac{\theta}{2\hbar}(-p_x^2 + p_y^2) + \frac{\eta}{2\hbar}(-x^2 + y^2) + \frac{\theta\eta}{4\alpha^2\hbar^2}(L_z) \tag{6.29}$$

$$\hat{x}^2 + \hat{y}^2 = \alpha^2(x^2 + y^2) + \frac{\theta}{\hbar}(-L_z) + \frac{\theta^2}{4\alpha^2\hbar^2}(p_x^2 + p_y^2) \tag{6.30}$$

We know that the Weyl-Moyal product can be replaced by the standard product by changing x and p with the non-commutative operators $\hat{x}$ and $\hat{p}$ and therefore for the Hamiltonian, we get

$$\hat{H}(\hat{x},\hat{p})\psi = \hat{H}(\alpha x_i - \frac{\theta_{ij}}{2\alpha\hbar}p_j, \alpha p_i + \frac{\eta_{ij}}{2\alpha\hbar}x_j) \tag{6.31}$$

By replacing $\hat{x}$ and $\hat{p}$ with the (6.4), (6.26) and (6.27), (6.28) we get

$$\begin{aligned} H = \frac{1}{2m}\left[(\alpha p_x + \frac{\eta}{2\alpha\hbar}y)^2 + (\alpha p_y - \frac{\eta}{2\alpha\hbar}x)^2\right] + \\ + \frac{q^2B^2 + 4m^2\omega^2c^2}{8mc^2}\left[(\alpha x - \frac{\theta}{2\alpha\hbar}p_y)^2 + (\alpha y + \frac{\theta}{2\alpha\hbar}p_x)^2\right] \\ - \frac{qB}{2mc}\left[(\alpha x - \frac{\theta}{2\alpha\hbar}p_y)(\alpha p_y - \frac{\eta}{2\alpha\hbar}x) - (\alpha y + \frac{\theta}{2\alpha\hbar}p_x)(\alpha p_x + \frac{\eta}{2\alpha\hbar}y)\right] \end{aligned} \tag{6.32}$$

So by expanding and regrouping this equation, we can write

$$\begin{aligned} H = \frac{1}{2\tilde{m}}(p_x^2 + p_y^2) + \frac{1}{2}\tilde{m}\tilde{\omega}(x^2 + y^2) \\ - \left[\frac{qB}{2mc} + \frac{\eta}{\hbar} + \frac{(q^2B^2)\theta}{8mc\hbar c^2}\right](xp_y - yp_x) \end{aligned} \tag{6.33}$$

where

$$\tilde{m} = \left[\frac{\alpha^2}{m} + \frac{qB\theta}{2mc\hbar}\right]^{-1} \tag{6.34}$$

and

$$\tilde{\omega}^2 = \left[\frac{\alpha^2(q^2B^2)}{4mc^2} + \frac{qB\eta}{2mc\hbar} + \frac{\eta^2}{4m\alpha^2\hbar^2}\right]/\tilde{m} \tag{6.35}$$

We can clearly see that for $\alpha = 1$ and $\theta = \eta = 0$ we return back to the known case of commutative case of a harmonic oscillator in a magnetic field.

Again we take

$$\psi(\rho,\phi) = \xi(\rho)\exp\{im'\phi\} \tag{6.36}$$

where $m' = 0, \pm 1, \pm 2, \ldots$

The non-commutative version of the Schrodinger in radial coordinates is

$$\left[-\frac{\hbar^2}{2\tilde{m}(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho - \frac{m^2}{\rho^2}}) + \frac{1}{2}\tilde{m}\tilde{\omega}^2\rho^2 \right] - \frac{qB}{2mc} + \frac{\eta}{\hbar} + \frac{(q^2B^2)\theta}{8m\hbar c^2} \tag{6.37}$$

So the energy of the system is

$$E_{n_\rho m'} = \hbar\tilde{\omega}(2n_\rho + |m'| + 1) + m'\hbar \left[\frac{qB}{2mc} + \frac{\eta}{\hbar} + \frac{(q^2B^2 + 4m^2\omega^2c^2)\theta}{8m\hbar c^2} \right] \tag{6.38}$$

with

$$\xi(\rho) = \rho^{|m'|} F(-n_\rho, |m'| + 1, \bar{\beta}^2\rho^2) \exp\left\{ -\bar{\beta}\frac{\rho^2}{2} \right\} \tag{6.39}$$

with $\beta = \frac{qB}{2mc} + \frac{\bar{\theta}}{\hbar} + \frac{(q^2B^2)\theta}{8m\hbar c^2}$

Therefore the eigenfunction is

$$\psi(\rho, \phi) = \tilde{N}\rho^{|m_1|} F(-n_\rho, |m_1| + 1, \rho^2\tilde{\beta}^2) \exp\left\{ -\tilde{\beta}\rho^2/2 \right\} \tag{6.40}$$

with where $m' = 0, \pm 1, \pm 2, \ldots$ and $\tilde{N}$ being the normalisation constant. with $\tilde{\beta} = \frac{qB}{2mc} + \frac{\eta}{\hbar} + \frac{(q^2B^2)\theta}{8m\hbar c^2}$

6.5 Addition of Electric Field and First and Second Order Differentiation

The Hamiltonian is written as

$$H = \frac{1}{2m}((\vec{p}_x + \frac{q}{2c}yB\vec{e}_x) + (\vec{p}_y - \frac{q}{2c}yB\vec{e}_y)) + \frac{1}{2}m\omega^2(x^2+y^2) \tag{6.41}$$

With the Coulomb Gauge

$$A_x = -\frac{1}{2}Bx \tag{6.42}$$

$$A_y = \frac{1}{2}By \tag{6.43}$$

So the Hamiltonian becomes

$$H = \frac{1}{2m}[(\hat{p}_x^2 + \frac{q}{c}B\hat{y})^2 + (\hat{p}_y^2 - \frac{q}{c}B\hat{x})^2] + \frac{1}{2}\omega^2(x^2+y^2) - qFx \tag{6.44}$$

where F is the magnetic field strength.

For $\omega = 0$ this becomes

$$H = \frac{1}{2m}[(\hat{p}_x^2 + \frac{q}{c}B\hat{y})^2 + (\hat{p}_y^2 - \frac{q}{c}B\hat{x})^2] + \frac{1}{2}\omega^2(x^2+y^2) - qFx \tag{6.45}$$

By regrouping the equation we get

$$H = \frac{1}{2m}[(\hat{p}_x^2 + \hat{p}_y^2) + \frac{q^2B^2}{4c^2}(\hat{x}^2+\hat{y}^2) + \frac{qB}{c}L_z] - qFx \tag{6.46}$$

Using (6.4), we end up with

$$\hat{p}_x^2 + \hat{p}_y^2 = \alpha^2(p_x+p_y) + \frac{\eta^2}{4\alpha^2\hbar^2}(x^2+y^2) + \frac{\eta}{\hbar}(p_xy - p_yx) \tag{6.47}$$

and

$$x^2+y^2=\alpha^2(x^2+y^2)+\frac{\theta^2}{4\alpha^2\hbar^2}(p_x^2+p_y^2)+\frac{\theta}{\hbar}(p_xy-p_yx) \tag{6.48}$$

Passing to polar forms we end up with

$$\left\{\left[-\frac{\hbar}{2\tilde{m}}\left(\partial_r^2+\frac{1}{r}\partial_r-\frac{n}{r}\right)+\frac{1}{2}\tilde{m}\tilde{\omega}^2r^2\right]+\right.$$
$$\left.+\left[\frac{qB}{2mc}+\frac{\eta}{\hbar}+\frac{(q^2B^2+4m^2\omega^2c^2)\theta}{8m\hbar c^2}\right]n_1\hbar-qFr\cos\theta\right\}\psi=E\psi \tag{6.49}$$

The non-perturbed solution is written in the form

$$\psi=r^{|n|}\exp\{-\omega_Lr^2/2\}L^{|n|}_{(n'-|n|-1)/2} \tag{6.50}$$

This is also the perturbation in zeroth order, and the corresponding energies are

$$E=\hbar\tilde{\omega}(2n_r|m_1|+1)+m_1\hbar[\frac{qB}{2mc}+\frac{\eta}{\hbar}+\frac{(q^2B^2+4m^2\omega^2c^2)\theta}{8m\hbar c^2}] \tag{6.51}$$

The first order correction to energy due to the perturbation of the electric field is 0 due to parity.

The same goes for the second order correction due to electric field

$$E_2^{(n)}=0 \tag{6.52}$$

Corrections of higher orders are also 0 which can be seen from the eigenvalues of energy, $E_{||}$ and E_{L_z}.

Chapter 7

CONCLUSION

In conclusion, introduction of non-commutative geometry into quantum physics is a ground braking idea that has specifically mathematical justifications in string theory etc.

We have touched the mathematical background and related topics that the non commutativity could potentially have an effect on.

In this thesis we have worked on the differences between commutative and non-commutative cases of harmonic oscillator in a uniform magnetic field.

We saw that for a constant magnetic field, commutative and non-commutative harmonic oscillator acts significantly differently. It is clear that the eigenfunctions, mass and Larmor frequency changes in a non-commutative setting.

This could potentially be measured experimentally.

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