On Langevin Type of Fractional Time - Delay Differential Equations

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ABSTRACT

The existence uniqueness of solutions of the impulsive delayed fractional differential equations(IDFDEs) was proved. The Ulam-Hyers stability of IDFDEs was demonstrated. The controllability of IDFDEs was shown via iterative learning control technique. In the sequel, the neutral fractional multi-delayed differential equations(NFMDDEs) was introduced. The existence and uniqueness of NFMDDEs was investigated in addition to its stability, and relative controllability of NFMDDEs was proved by means of fixed point technique. Lastly, new fractional integral and derivatives, i.e. ϕ-generalized Riemann Liouville k-fractional integral, ϕ-generalized Riemann Liouville k-fractional derivative, ϕ-generalized Caputo k-fractional derivative were defined and some fundamental features were discussed to build theory's basement.

Keywords: fractional derivative and integral, impulsive delayed differential equation, neutral multi-delayed differential equation, existence and uniqueness, iterative learning control, relative controllability, stability

ÖZ

İmpulsif ve gecikmeli kesirli bir diferansiyel denklemin çözümünün var ve tek olduğu ispatlandı. Bu kesirli diferansiyel denklemin Ulam-Hyers anlamında kararlı olduğu gösterildi. Yinelemeli öğrenme kontrol edilebilirlik tekniği yardımıyla da bu denklemin kontrol edilebileceği gösterildi. Hemen akabinde, çok gecikmeli nötr ve kesirli diferansiyel denklemi tanıtıldı. Bu kesirli diferansiyel denklemin kararlılığına ilaveten sistemin çözümünün var ve tek olduğu araştırıldı ve sabit nok teoremleri tekniği aracılığıyla bu kesirli diferansiyel denklemin nisbi kontrol edilebileceği ispat edildi. Son olarak, ϕ-genelleştirilmiş Riemann Liouville k-kesirli integrali, ϕ-genelleştirilmiş Riemann Liouville k-kesirli türevi, ϕ-genelleştirilmiş Caputo k-kesirli türevi olmak üzere yeni kesirli integral ve türevleri tanımlandı ve teorinin temelini inşa etmek için bazı temel özellikler tartışıldı.

Anahtar Kelimeler: kesirli türev ve integral, impulsif gecikmeli diferansiyel denklem, nötr çok gecikmeli differansiyel denklem, varlık ve teklik, yinelemeli öğrenme kontrol edilebilirlik, nisbi kontrol edilebilirlik, kararlılık

DEDICATION

To the honorable memory of my advisor Prof. Dr. Nazım Mahmudov that has lighted my academic road.

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$B(Y_1, Y_2)$	Space of All Linear Bounded Operators
$C([a,b], \mathbb{R}^n)$	Space of All Vector-valued Continuous Functions
$L^\infty(J, Y_2)$	Space of All Essentially Bounded Functions
$PC([a,b], \mathbb{R}^n)$	Space of All Piecewise Continuous Vector-valued Functions
ϕ-GC k-FD	ϕ-Generalized Caputo k-Fractional Derivative
ϕ-GRL k-FI	ϕ-Generalized Riemann Liouville k-Fractional Integral
ϕ-GRL k-FD	ϕ-Generalized Riemann Liouville k-Fractional Derivative
IDFDE	Impulsive Delayed Fractional Differential Equation
ILC	Iterative Learning Control
MDP of ML	The Multi-delayed Perturbation of the Mittag-Leffler
NFMDDE	Neutral Fractional Multi-delayed Differential Equation

Chapter 1

INTRODUCTION

When tracing the history of fractional calculus, we undoubtedly can say that the foundation of the theory of fractional-order derivatives are laid with Leibniz's note to L'Hospital in 1695 [1]. On that note, the possible meaning of the derivative of one-half order was debated. This also stimulates everyone who early encounters the differential operator d/dx, d^2/dx^2,etc to ponder what the meanings of $d^{1/2}/dx^{1/2}$, d^{-1}/dx^{-1}, and $d^{\sqrt{2}}/dx^{\sqrt{2}}$ are. As a result, these cause to appear the theory of derivatives and integrals of any order. For the last three centuries, the theory of fractional calculus was developed mostly as a pure theoretical area of mathematics. However, for a couple of decades many researchers were aware of the fact that fractional-order derivatives and integrals are more appropriate to express the real-life world problems compared to previously used integer-order derivatives and integrals. Especially, in comparison with the traditional integer-order models fractional derivatives provide the miracle argument for the description of memory and hereditary features of various processes and materials. In addition to describing rheological features of rocks, the advantages of fractional derivatives also appear to model mechanical and electrical properties of real materials and, in many other areas. A fractional differential equation is an equation that consists of fractional derivatives. Based on the features of fractional derivatives, the physical and mathematical processes and modelling gave rise to fractional-order differential system having fractional-order and oblige to solve(settle up) such equations. Fractional derivatives and integrals also become apparent in the theory of control of dynamical systems, when the controller or controlled system is expressed by fractional differential equations. Many articles are written on the subject of the history and applications of fractional calculus [9, 79–82].

Fractional calculus which regards as a generalization of the traditional calculus or integer calculus or classical calculus has been of widespread use in the scientific world. Unlike the traditional calculus, fractional calculus is exponentially exploited in many kinds of areas like mathematical physics [79], [80], electrochemistry, (Optimal) control theory [145, 146], biophysics, engineering, signal, etc; see for instance [81]- [83] to model a variety of phenomena as control theory, stability theory, viscoelasticity, existence [144], and etc [7, 8, 17, 88–92, 97, 99, 100, 111].

In the most of previous applications, a system was examined according to a principle of casualty. This means that the system's future state did not depend on the past states, that is, it is identified only with the present states. In this case, a differential equation with the state and its rate of change becomes either ordinary or partial. Of course, the future state was not logical not to depend on the past state. In the late thirties, Volterra in his works [76], [77] like viscoelasticity and predator-prey model formulated certain quite general differential equations including the past states. Again on the same days, Minorksii [78] showed the importance of the delay in the feedback mechanism in his studies of automatic steering and ship stabilization. This played a significant role in the theory of differential equations with the past state and in the control theory. The dependence of the past states can be expressed with the state variable and not derivative of the state variable. In the literature, this kind of differential equation is called retarded(delayed) differential equation. Neutral differential equation also involves derivatives of retardation(delay) in addition to the function itself depending on the past and present states.

Recently, it is studied on the expression of solutions to delay differential equations. Khusainov and Shukin [22], and Diblik and Khusainov [23, 24] managed to get the accurate representations of solutions of linear continuous and discrete delay equations by proposing the concept of delay matrix exponential function e_h^{Bt} as in (2.3). Li and Wang [110] studied the fractional analogue of the same problem in the case $A = \Theta$ by exploring delayed Mittag-Leffler type matrix function $e_{h,\alpha}^{Bt}$ as in (2.4). Motivated by Khusainov & Shuklin [23] and Li and Wang [110], Mahmudov [86] consider representation of solutions of the Caputo fractional delay differential equations by introducing delay perturbation of Mittag-Leffler function $X_{h,\alpha,\beta}^{A,B}(t)$ as in (3.3) which does not need to satisfy that A and B are permutable.

Arimoto et al. introduced the concept of iterative learning control (ILC briefly) in 1980s, which has been widely used to apply to biological systems, robotics, industrial control systems to get an awesome tracking performance in a finite interval of time. ILC has been deeply researched from practical and theoretical applications [43–51, 55–59]. But the study on this subject is still at the beginning and there are a lots of problems which are expected to solve. ILC is not restricted to the integer order differential equations. For instance, it can be used for fractional differential equations [43–45, 47, 48], for fractional impulsive equations [49, 52–54].

The paper [67] presents a second order D^{α}-type iterative learning control scheme for a class of fractional-order linear time-delay systems with fractional order $0 < \alpha < 1$. In [68] convergence conditions are derived in frequency domain via contraction mapping principle. The convergent sufficient conditions of open-loop and closed-loop iterative learning schemes are established in [52]. A robust second-order feedback PD type iterative learning control for a class of uncertain fractional-order singular systems is presented in [47]. PD^{α}-type iterative learning control for fractional-order singular time-delay system is studied in [69].

For an approximately one and half century, differential equations have been used to formulate the dynamics of changing processes. The dynamics of several developing processes depend on sudden changes like shocks, natural disasters. The phenomena have short-term deviations(perturbations) from continuous and smooth dynamics. Considering the course of the whole development, its duration is ignorable. In formulations having such deviations, these deviations treat instantaneously or in the form of ''impulses''. As a result, formulating impulsive problems have developed impulsive differential equations in population dynamics, industrial robotics, ecology, physics, optimal control and so on [93–96].

The impulsive method which is an efficient control approach is mostly exploited in today's sophisticated control systems. It is seen that perturbations can be generally described in the shape of impulsive expressions for an examined sophisticated system. In order to follow the discontinuous reference trajectory properly employing a couple of iterations in an arbitrary finite time interval, an impulsive control method is required. For an impulsive equation, there are a lot of tangible instances to describe it, e.g. the computer networking, the population control systems, the automatic control systems, and aircraft. It can be effortlessly remarked that the principle goal of the impulsive control technique is not to compare or rival with the available other control techniques. On quite the contrary, the impulsive control technique offers a novel point of view when there is at least one changeable state variable in the system. There are important theories widely used on impulsive differential systems, see [71], [72]. Also, ILC is presented to shift the state of the impulsive equations when certain conditions are hold. In fact, the ILC technique has been confirmed to be a sensible and reasonable control system to deal with the impulsive systems. In [49], in order to trace the craved discontinuous trajectory, ones discovered P-type ILC algorithm for the impulsive differential system. In [53], D and P-type ILC algorithms have been presented for fractional impulsive evolution system, and the convergency analysis of these algorithms is done in the sense of λ-norm. In [73], D and P-type ILC algorithms have been designed for a class of impulsive first order systems with distributed parameter by employing λ-norm and L^p-norm. PD^{α}-type iterative learning control for the fractional-order nonlinear time-delay systems is investigated in [70].

As you would appreciate, our efforts to find a representation of solutions to linear (fractional) differential equations provide positive results. However, we can not always say the same things about the nonlinear and partial (fractional) differential equations. Sometimes it is too difficult to solve such differential equations. In this case, it is remarkably significant to determine whether such equations have any solutions or under which conditions the solutions are existent or unique. The fixed point theorems are mostly used to identify answers of these kinds of questions. So, benefits of knowing existence and uniqueness results are beyond dispute.

Numerous researchers have debated the data dependence in the theory of differential systems. All kind of stability properties have attracted the attention of many mathematicians, see [62]- [66]. By the way, there are certain private data dependence in the theory of functional equations like Ulam-Hyers-Bourgin, Ulam-Hyers-Rassias and Ulam-Hyers. Particularly, the Ulam-Hyers stability was exploited by lots of mathematicians and the study of this area has the grown to be one of the central subjects in the mathematical analysis area. Ulam and Hyers introduced and studied the concept of the stability of systems [25–28]. This stabilities are known as the Ulam-Hyers stability. Some papers as to the Ulam-Hyers stability can be reached at [29–32] Also both the existence and the uniqueness and the stability of solutions of fractional differential systems play an important role in the fractional calculus. These studies can be found in [33–42].

This thesis consists of five principle chapters.

In Chapter 1, an introduction which expresses the available literatures and brief histories about fractional calculus and subjects of our thesis is given to enable the readers to easily understand our findings.

In Chapter 2, some special spaces endowed with their appropriate norms, the theory of some special functions(gamma function, beta function, k-gamma function, k-beta function, Mittag-Leffler (type) functions which play a wonderful role in the theory of fractional calculus), some formal definitions of stability, controllability as well as fractional derivatives and integrals, and related necessary tools are presented.

In Chapter 3, the fractional impulsive delayed system is granted. Existence uniqueness and Ulam-Hyers stability of fractional time-delay impulsive semilinear system with nonpermutable matrix coefficients are studied. Iterative learning control problem for this system and study convergence of P, D, and D^{α} type of ILC schemes is constructed. In addition to an example that satisfies all the conditions of types P, D, and D^{α}, graphs of output functions, error tables and their histograms by three different origin references trajectories for the example are offered. Possible open problems are expressed.

In Chapter 4, the qualitative concepts for neutral fractional multi-delayed differential equations with noncommutative coefficient matrices are discussed. An explicit solution to the neutral fractional linear multi-delayed differential equations with non-commutative matrices is given based on this multi-delayed perturbation function of Mittag-Leffler type matrix function. The problem of existence uniqueness and Ulam-Hyers stability of solutions to the nonlinear neutral fractional multi-delayed differential system is investigated by using the Banach Contraction Principle. The sufficient and necessary condition for relative controllability of the neutral multi-delayed homogeneous system is determined by giving the concept of the neutral fractional multi-delayed Gramian matrix. Lastly, the relative controllability result for the neutral multi-delayed semi-linear system is studied by means of Krasnoselskii's fixed point theorem. Some new problems for the readers to research are stated.

In Chapter 5, quite comprehensive ϕ-generalized Riemann-Liouville k-fractional integral and ϕ-generalized Riemann-Liouville and Caputo k-fractional derivatives which can be reduced to most of the well-known fractional integrals and derivatives depending on the choices of k, s, ϕ are presented in order to combine these conceptions into an united one and improve a theory for FDEs with an unified novel derivative due to lots of new types of fractional integral and derivatives and many of their applications of real world problems. Some fundamental features are discussed to build theory's basement.

Chapter 2

PRELIMINARY

Let $a,b \in \mathbb{R}$(or $\mathbb{R}^+$) which is the set of all real numbers(or all positive real numbers). For $-\infty < a < b < \infty$, $[a,b]$ is the interval of $\mathbb{R}$. Let $C([a,b],\mathbb{R}^n)$ be the Banach space of vector-valued continuous functions from $[a,b]$ to $\mathbb{R}^n$ endowed with the infinity norm

$$\|f\|_C := \sup_{t\in[a,b]} \|f(t)\|$$

for an arbitrary norm $\|.\|$ on $\mathbb{R}^n$. For $n \in \{0,1,2,...\}$, let $C^n([a,b],\mathbb{R})$ be the space of complex-valued functions $f(x)$ which have continuous derivatives up to order n such that $f^{(n)} \in C([a,b],\mathbb{R})$. We introduce the piecewise continuous vector-valued functions space

$$\begin{aligned} PC([0,T],\mathbb{R}^n) := \{x : [0,T] \to \mathbb{R}^n \mid x \in C((t_k,t_{k+1}],\mathbb{R}^n),\ \ k = 0,...,m \\ \text{and there exist } x\left(t_k^-\right), x\left(t_k^+\right) \text{ with } x\left(t_k^-\right) = x(t_k),\ \ k = 1,...,m\right\}, \end{aligned}$$

the jumps

$$x\left(t_i^+\right) = \lim_{\varepsilon\to 0^+} x(t_i+\varepsilon), \quad x\left(t_i^-\right) = \lim_{\varepsilon\to 0^-} x(t_i+\varepsilon)$$

represent the left and right limits of $x(t)$ at $t = t_i$, respectively, endowed with

$$\|x\|_{PC} := \sup_{t\in[0,T]} \|x(t)\|.$$

A λ-norm also is defined on $PC([0,T],\mathbb{R}^n)$ by

$$\|x\|_\lambda = \sup_{t\in[0,T]} \left\{e^{-\lambda t}\|x(t)\|\right\}, \quad \lambda > 0.$$

Remark 2.1: [143] $PC([0,T],\mathbb{R}^n)$ is the Banach space with respect to the norm $\|.\|_{PC}$.

For $A \in \mathbb{R}^{n\times m}$, $n,m \in \mathbb{N}$; the well-known maximum norm of a matrix is defined as

$$\|A\| = \max_{1\leq i\leq n} \sum_{j=1}^{m} |a_{ij}|$$

where a_{ij} are the elements of the matrix A. Let Y_1, Y_2 be two Banach spaces, $B(Y_1,Y_2)$ consists of all linear bounded operator from Y_1 to Y_2. Let J be a bounded closed interval. $L^\infty(J,Y_2)$ symbolizes the space of all essentially bounded functions which is the Banach space with $\|.\|_{L^\infty(J,Y_2)}$.

Definition 2.1: [111] The gamma function $\Gamma(z)$ is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt \quad (Re(z) > 0)$$

where $t^{z-1} = e^{(z-1)\log(t)}$. The integral is convergent for all complex $z \in \mathbb{C}$, $(Re(z) > 0)$. For this function the reduction formula

$$z\Gamma(z) = \Gamma(z+1), \quad Re(z) > 0$$

holds.

Definition 2.2: [111] The beta function $B(z,w)$ is defined by

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt \quad (Re(z) > 0 \quad Re(w) > 0)$$

The integral is convergent for all complex $z,w \in \mathbb{C}$ such that $Re(z) > 0 \quad Re(w) > 0$. The Beta function is closely related to the Gamma function; in fact, we have

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

holds.

Diaz and Pariguan define the generalisations of Gammma and Beta function as noted below.

Definition 2.3: [120] The k-gamma and the k-beta functions are defined as follows

$$\Gamma_k(\omega) = \int_0^\infty e^{-\frac{y^k}{k}} y^{\omega-1} dy, \tag{2.1}$$

and

$$B_k(\omega, \varpi) = \frac{1}{k}\int_0^1 y^{\frac{\omega}{k}-1}(1-y)^{\frac{\varpi}{k}-1} dy, \tag{2.2}$$

where $Re(\omega) > 0$ and $Re(\varpi) > 0$, respectively. Their relations with the well-known gamma and beta functions and themselves are given by

$$\Gamma(\omega) = \lim_{k\to 1}\Gamma_k(\omega), \quad \Gamma_k(\omega) = k^{\frac{\omega}{k}-1}\Gamma\left(\frac{\omega}{k}\right), \quad \Gamma_k(k) = 1, \quad \Gamma_k(\omega+k) = \omega\Gamma_k(\omega),$$

and

$$B_k(\omega, \varpi) = \frac{\Gamma_k(\omega)\Gamma_k(\varpi)}{\Gamma_k(\omega+\varpi)} = \frac{1}{k}B\left(\frac{\omega}{k}, \frac{\varpi}{k}\right).$$

Gösta Mittag-Leffler defined the following series as the Mittag-Leffler function.

Definition 2.4: [132] Mittag-Leffler function $E_\eta(\varphi)$ is defined by

$$E_\eta(\varphi) = \sum_{k=0}^{\infty}\frac{\varphi^k}{\Gamma(\eta k+1)}, \ \ \eta \in \mathbb{C}, \ \ Re(\eta) > 0.$$

Subsequently, Wiman introduced a generalized Mittag-Leffler function as follows.

Definition 2.5: [142] Mittag-Leffler function $E_{\eta,\mu}(\varphi)$ is given by

$$E_{\eta,\mu}(\varphi) = \sum_{k=0}^{\infty}\frac{\varphi^k}{\Gamma(\eta k+\mu)}, \ \ \eta, \mu \in \mathbb{C}, \ \ Re(\eta) > 0.$$

Definition 2.6: [24] Delay matrix exponential function e_h^{Bt} is defined by

$$e_h^{Bt} = \begin{cases} \Theta, & -\infty \le t < -h, \\ I, & -h \le t < 0, \\ \sum_{j=0}^{p} B^j \frac{(t-(j-1)h)^j}{j!}, & (p-1)h < t \le ph. \end{cases} \tag{2.3}$$

Definition 2.7: [110] The delay Mittag-Leffler function $e^{Bt}_{h,\alpha}$ is defined by

$$e^{Bt}_{h,\alpha} = \begin{cases} \Theta, & -\infty \le t < -h, \\ I, & -h \le t < 0, \\ \sum_{j=0}^{p} B^j \frac{(t-(j-1)h)^{\alpha j}}{\Gamma(\alpha j + 1)}, & (p-1)h < t \le ph. \end{cases} \tag{2.4}$$

Mahmudov generalizes them to that one including two independent coefficient matrices by using double summation as follows.

Definition 2.8: [86] The delayed perturbation of Mittag-Leffler type matrix function $X^{A,B}_{h,\alpha,\beta}(\cdot) : [0,\infty) \to R^n$ generated by A, B is defined by

$$X^{A,B}_{h,\alpha,\beta}(t) := \begin{cases} \Theta, & -h \le t < 0, \\ I, & t = 0, \\ \sum_{k=0}^{\infty}\sum_{j=0}^{p} Q_{k+1}(jh) \frac{(t-jh)_+^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}, & ph < t \le (p+1)h, \end{cases} \tag{2.5}$$

where $(t)_+ = \max(0,t)$, $Q_{k+1}(jh) = AQ_k(jh) + BQ_k(jh-h)$, $Q_0(s) = Q_k(-h) = \Theta$, $Q_1(0) = I$ for $k = 0,1,2,\ldots$ and $s = 0,h,2h,\ldots$ Θ and I are the zero and identity matrices.

Remark 2.2: As stated in [86, Figure 1] or will be shown with the aid of the interpretation of the first derivative, the delayed perturbation of Mittag-Leffler type matrix function $X^{A,B}_{h,\alpha,\beta}$ is increasing on $(0,\infty)$.

Remark 2.3: By determining x which makes $f'(x)$ bigger than zero, one can show that the function f is increasing for $x > 0$ If $f(x) = x^{\alpha}$, $\alpha \in (0,1)$.

Lemma 2.1: [75] In addition to the fact that the exclusive function $X^{A,B}_{h,\alpha,\beta}(\cdot)$ is continuous on $(0,\infty)$, for all $t \ge 0$, $0 < \alpha < 1$, $0 < \beta \le 1$ satisfying $\alpha + \beta \ge 1$, we have

$$\left\| X_{h,\alpha,\beta}^{A,B}(t) \right\| \leq t^{\beta-1} E_{\alpha,\beta}\left((\|A\| + \|B\|) t^{\alpha} \right), \quad \left\| X_{h,\alpha,\beta}^{A,B}(t) \right\| \leq X_{h,\alpha,\beta}^{\|A\|,\|B\|}(t). \tag{2.6}$$

Lemma 2.2: [60] Let $x \in PC([0,T], R^n)$ satisfy the following inequality

$$\|x(t)\| \leq c_1(t) + c_2 \int_0^t (t-s)^{\alpha-1} \|x(s)\| ds + \sum_{0<t_k<t} \theta_k \left\| x\left(t_k^-\right) \right\|,$$

where $c_1(t)$ is non-negative continuous and non-decreasing on $[0,T]$, c_2, $\theta_k \geq 0$ are constants. Then

$$\|x(t)\| \leq c_1(t) \left(1 + \theta E_{\alpha}\left(c_2 \Gamma(\alpha) t^{\alpha}\right)\right)^k E_{\alpha}\left(c_2 \Gamma(\alpha) t^{\alpha}\right)$$

for $t_k < t \leq t_{k+1}$, where $\theta = \max\{\theta_1, ..., \theta_m\}$ and $E_{\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}$, $z \in C$.

Definition 2.9: [111, 112] The Riemann-Liouville fractional integrals ${}^{\mathfrak{RL}}\mathfrak{I}_{a^+}^{\alpha} f(t)$ of order $\alpha \in \mathbb{R}^+$(positive real numbers) are defined by

$${}^{\mathfrak{RL}}\mathfrak{I}_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds$$

Definition 2.10: [111, 112] The Riemann-Liouville fractional derivatives ${}^{\mathfrak{RL}}\mathfrak{D}_{a^+}^{\alpha} f(t)$ of order $0 < \alpha < 1$ are defined by

$${}^{\mathfrak{RL}}\mathfrak{D}_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} f(s) ds.$$

Definition 2.11: [111] The Caputo fractional derivatives ${}_a{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} f(t)$ of order $0 < \alpha < 1$ are defined by

$${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f'(s) ds,$$

and their relations are

$${}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\alpha} f(t) = {}^{\mathfrak{RL}}\mathfrak{D}_{a^+}^{\alpha}\left(f(t) - f(a)\right), \quad {}^{\mathfrak{RL}}\mathfrak{I}_{a^+}^{\alpha}\left({}^{\mathfrak{RL}}\mathfrak{D}_{a^+}^{\alpha} f(t)\right) = f(t), \tag{2.7}$$

and

$$^{\mathfrak{RL}}\mathfrak{D}_{a^+}^{\alpha}(x-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1} = {}^{\mathfrak{C}}\mathfrak{D}_{a^+}^{\alpha}(x-a)^{\beta-1}.$$

Theorem 2.1: [61] (Banach's Contraction Principle) Let X be a complete metric space with a contraction mapping $T : X \to X$. Then T has an unique fixed point in X.

Lemma 2.3: (Krasnoselskii's fixed point theorem, see [108]) Let $\mathscr{D}$ be a convex closed and bounded subset of Banach space Y and let $\mathscr{G} = \mathscr{G}_1 + \mathscr{G}_2$ be maps $\mathscr{D}$ into Y such that $\mathscr{G}z = \mathscr{G}_1 z + \mathscr{G}_2 z \in \mathscr{D}$ for every pair $z \in \mathscr{D}$. If $\mathscr{G}_2$ is continuous and compact and $\mathscr{G}_1$ is a contraction, then the equation $\mathscr{G}z = \mathscr{G}_1 z + \mathscr{G}_2 z = z$ is of a solution on $\mathscr{D}$.

Chapter 3

IMPULSIVE DELAYED FRACTIONAL SYSTEM

In the section of the introduction, we talked about advancements about both delayed systems and impulsive systems. It will be interesting to combine the iterative learning control and Ulam-Hyers stability issues with the semilinear fractional time-delay impulsive systems. Therefore, we consider the following semilinear impulsive fractional differential time-delay equations with noncommutative coefficients,

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} x(t) = Ax(t) + Bx(t-h) + f(t, x(t)), \ \ t \in [0,T] \setminus I, \\ x\left(t_i^+\right) = x\left(t_i^-\right) + g(x(t_i)), \ \ t_i \in I, \\ x(t) = \varphi(t), \ \ -h \le t \le 0, \end{cases} \tag{3.1}$$

where k denotes the iterative times, $I = \{t_1, t_2, ..., t_m\}$ is the impulsive times and satisfying $0 < t_1 < ... < t_m < T$, $T = lh$ for a fixed $l \in \mathbb{N}$ and $h > 0$. $A, B \in R^{n \times n}$ are constant matrices which do not have to be commutative. $f : [0,T] \times R^n \to R^n$ and $g : R^n \to R^n$ are continuous vector functions. The jumps

$$x\left(t_i^+\right) = \lim_{\varepsilon \to 0^+} x(t_i + \varepsilon), \quad x\left(t_i^-\right) = \lim_{\varepsilon \to 0^-} x(t_i + \varepsilon)$$

represent the left and right limits of $x(t)$ at $t = t_i$, respectively. We introduce the following integral equation

$$\begin{aligned} x(t) = {} & X_{h,\alpha,1}^{A,B}(t+h)\varphi(-h) + \int_{-h}^{0} X_{h,\alpha,\alpha}^{A,B}(t-s)\left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\varphi(s) - A\varphi(s)\right] ds \\ & + \int_0^t X_{h,\alpha,\alpha}^{A,B}(t-s) f(s, x(s))\, ds + \sum_{0<t_i<t} X_{h,\alpha,1}^{A,B}(t-t_i)\, g(x(t_i)), \end{aligned} \tag{3.2}$$

where delayed perturbation of Mittag-Leffler type matrix function $X_{h,\alpha,\beta}^{A,B}(\cdot) : [0,\infty) \to$

R^n generated by A,B is defined by

$$X_{h,\alpha,\beta}^{A,B}(t) := \begin{cases} \Theta, & -h \le t < 0, \\ I, & t = 0, \\ \sum\limits_{k=0}^{\infty}\sum\limits_{j=0}^{p} Q_{k+1}(jh)\dfrac{(t-jh)_{+}^{k\alpha+\beta-1}}{\Gamma(k\alpha+\beta)}, & ph < t \le (p+1)h, \end{cases} \tag{3.3}$$

where $(t)_+ = \max(0,t)$, $Q_{k+1}(jh) = AQ_k(jh) + BQ_k(jh-h)$, $Q_0(s) = Q_k(-h) = \Theta$, $Q_1(0) = I$ for $k = 0,1,2,\ldots$ and$s = 0,h,2h,\ldots$ Θ and I are the zero and identity matrices.

Remark 3.1: If we take $\alpha = \beta = 1$ and omit the impulsive part and non-commutativity of A and B, the solution in (3.2) coincides with that of the paper [22]. The solution in (3.2) without the impulse reduces to that of the work [110] provided $A = \Theta$. The solution in (3.2) without the impulse is as in the work [86].

It is well known that fractional differential equation (3.1) and integral equation (3.2) are equivalent. So far all of both impulsive and time-delay [22] [23] [24] [110] fractional systems have been considered under the condition that the coefficient matrices A and B are permutable, and also under this condition it has been investigated whether there exists a solution of the system, the solution is existent and unique, and the system is stable, controllable, etc. In the current system (3.1), the coefficient matrices A and B in the impulsive fractional order time-delay systems examined do not need to be permutable. This is the biggest difference from the peer-papers in addition to the idea of combining impulsive systems with delayed systems . Accordingly, the obtained results subsequent to principal contributions' section are different from the others and also pretty novel.

3.1 Existence and Uniqueness of Solution of the System

In this subsection, the existence and uniqueness of solution of the semilinear impulsive fractional differential time-delay equations with nonpermutable matrix coefficients is offered by the following theorem and its proof.

Lemma 3.1: Let $X^{A,B}_{h,\alpha,\beta}(t)$ be defined as in (3.3). The following equation holds.

$$\int_0^t X^{\|A\|,\|B\|}_{h,\alpha,\alpha}(t-s)\,ds = X^{\|A\|,\|B\|}_{h,\alpha,\alpha+1}(t).$$

Proof. By using definition of delay perturbation function and properties of integration, we have

$$\begin{aligned}
\int_0^t X^{\|A\|,\|B\|}_{h,\alpha,\alpha}(t-s)\,ds &= \int_0^t \sum_{i=0}^{\infty}\sum_{j=0}^{p} \|Q_{i+1}(jh)\| \frac{(t-s-jh)_+^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)} ds \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{p} \|Q_{i+1}(jh)\| \frac{1}{\Gamma((i+1)\alpha)} \int_0^{t-jh} (t-s-jh)_+^{(i+1)\alpha-1} ds \\
&+ \sum_{i=0}^{\infty}\sum_{j=0}^{p} \|Q_{i+1}(jh)\| \frac{1}{\Gamma((i+1)\alpha)} \int_{t-jh}^{t} (t-s-jh)_+^{(i+1)\alpha-1} ds \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{p} \|Q_{i+1}(jh)\| \frac{1}{\Gamma((i+1)\alpha+1)} \left[-(t-s-jh)_+^{(i+1)\alpha}\right]_0^{t-jh} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{p} \|Q_{i+1}(jh)\| \frac{(t-jh)_+^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} \\
&= X^{\|A\|,\|B\|}_{h,\alpha,\alpha+1}(t),
\end{aligned}$$

where $(t-s-jh)_+ = 0$ for $t-jh \le s \le t,\ j=0,1,\ldots,p$ due to its rule. □

Theorem 3.1: If the following assumptions are hold true,

$i)$ the function $f:[0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous;

$ii)$ $\|f(t,x)-f(t,y)\| \le L_f\|x-y\|, \quad t\in[0,T], \quad x,y\in\mathbb{R}^n, \quad L_f>0;$

$iii)$ $\|g(x)-g(y)\| \le L_g\|x-y\|, \qquad x,y\in\mathbb{R}^n, \quad L_g>0;$

$iv)$ $L_f X^{\|A\|,\|B\|}_{h,\alpha,\alpha+1}(T) + mML_g < 1$, m is the number of the impulsive times, $M := E_{\alpha,\beta}((\|A\|+\|B\|)T^\alpha) > 0$ with $\beta=\alpha,1,$ is given in (2.6) then the integral equation (3.2) has a unique solution in $[-h,T]$.

Proof. Define $F: PC([-h,T],\mathbb{R}^n) \to PC([-h,T],\mathbb{R}^n)$ by

$$\begin{aligned} Fx(t) := {} & X_{h,\alpha,1}^{A,B}(t+h)\varphi(-h) + \int_{-h}^{0} X_{h,\alpha,\alpha}^{A,B}(t-s)\left[{}^{\mathfrak{C}}\mathfrak{D}_{0+}^{\alpha}\varphi(s) - A\varphi(s)\right]ds \\ & + \int_0^t X_{h,\alpha,\alpha}^{A,B}(t-s) f(s,x(s))\,ds + \sum_{0<t_i<t} X_{h,\alpha,1}^{A,B}(t-t_i)\, g(x(t_i)) \end{aligned} \tag{3.4}$$

By taking arbitrary $x,y \in PC([-h,T],\mathbb{R}^n)$ and employing (3.4), we consider

$$\begin{aligned} \|Fx(t) - Fy(t)\| &\le \left\| \int_0^t X_{h,\alpha,\alpha}^{A,B}(t-s)\left[f(s,x(s)) - f(s,y(s))\right]ds \right\| \\ &+ \left\| \sum_{0<t_i<t} X_{h,\alpha,1}^{A,B}(t-t_i)\left[g(x(t_i)) - g(y(t_i))\right] \right\| \\ &\le \int_0^t X_{h,\alpha,\alpha}^{\|A\|\|,B\|}(t-s)\,\|[f(s,x(s)) - f(s,y(s))]\|\,ds \\ &+ \sum_{0<t_i<t} \left\| X_{h,\alpha,1}^{A,B}(t-t_i) \right\| \|g(x(t_i)) - g(y(t_i))\| \\ &\le L_f \int_0^t X_{h,\alpha,\alpha}^{\|A\|,\|B\|}(t-s)\,\|x(s) - y(s)\|\,ds + mML_g\|x-y\|_{PC} \\ &\le L_f \int_0^t X_{h,\alpha,\alpha}^{\|A\|,\|B\|}(t-s)\,ds\,\|x-y\|_{PC} + mML_g\|x-y\|_{PC} \end{aligned} \tag{3.5}$$

By combining inequality 3.5 with Lemma 3.1, we get

$$\|Fx(t) - Fy(t)\| \le \left[L_f X_{h,\alpha,\alpha+1}^{\|A\|,\|B\|}(T) + mML_g\right] \|x-y\|_{PC}.$$

The fourth condition (iv) of this theorem guarantees that F is a contraction. By Banach's Contraction Principle, F has a unique fixed point on $[0,T]$. This shows us the existence and uniqueness of solution of (3.1). □

3.2 Ulam-Hyers Stability of the System

We present a theorem which proves that the nonlinear impulsive fractional differential time-delay equation with Caputo fractional derivative is Ulam-Hyers stable.

Definition 3.1: Let $\eta > 0$. The system (3.1) is said to be Ulam - Hyers stable if for every solution $z \in PC([0,T], \mathbb{R}^n)$ of inequality,

$$\left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} z(t) - Az(t) - Bz(t-h) - f(t, z(t)) \right\| \leq \eta, \tag{3.6}$$

there exists a solution $x \in PC([0,T], \mathbb{R}^n)$ of the system (3.1), and $u_h > 0$ such that

$$\|z(t) - x(t)\| \leq u_h.\eta \;\; t \in [0,T]. \tag{3.7}$$

Remark 3.2: A function $z \in PC^1([0,T], \mathbb{R}^n)$ is a solution of the inequality equation (3.6) if and only if there exists a function $h \in PC([0,T], \mathbb{R}^n)$, such that

- $\|h(t)\| < \eta$,
- ${}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} z(t) = Az(t) + Bz(t-h) + f(t, z(t)) + h(t)$.

Theorem 3.2: Assume that all statements of Theorem 3.1 are satisfied. Then system (3.1) is Ulam-Hyers stable.

Proof. Let $z \in PC([0,T], \mathbb{R}^n)$ be a solution of the inequality (3.6), i.e.

$$\left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} z(t) - Az(t) - Bz(t-h) - f(t, z(t)) \right\| \leq \eta. \tag{3.8}$$

Let $x \in PC([0,T], \mathbb{R}^n)$ be the unique solution of the system (3.1), so that

$${}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} x(t) = Ax(t) + Bx(t-h) + f(t, x(t))$$

for each $t \in [0,T]$ and $0 < \alpha < 1$; $x(t) = z(t) = \varphi(t)$, $t \in [-h, 0]$ and $g(x(t_i)) = g(z(t_i))$, $t \in I$. By Remark 3.2 and inequality (3.8), there exists a function $h \in PC([0,T], \mathbb{R}^n)$, such that $\|h(t)\| < \eta$ and

$${}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} z(t) = Az(t) + Bz(t-h) + f(t, z(t)) + h(t). \tag{3.9}$$

So we get the solution $z(t)$ from (3.9):

$$\begin{aligned} z(t) = {} & X_{h,\alpha,1}^{A,B}(t+h)\varphi(-h) + \int_{-h}^{0} X_{h,\alpha,\alpha}^{A,B}(t-s)\left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\varphi(s) - A\varphi(s)\right] ds \\ & + \int_{0}^{t} X_{h,\alpha,\alpha}^{A,B}(t-s)\left[f(s, z(s)) + h(s)\right] ds + \sum_{0<t_i<t} X_{h,\alpha,1}^{A,B}(t-t_i)\, g(z(t_i)), \end{aligned}$$

$$
\begin{aligned}
&= X^{A,B}_{h,\alpha,1}(t+h)\varphi(-h) + \int_{-h}^{0} X^{A,B}_{h,\alpha,\alpha}(t-s)\left[{}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\varphi(s) - A\varphi(s)\right]ds \\
&\quad + \int_0^t X^{A,B}_{h,\alpha,\alpha}(t-s) f(s,z(s))\,ds + \sum_{0<t_i<t} X^{A,B}_{h,\alpha,1}(t-t_i)\,g(z(t_i)) \\
&\quad + \int_0^t X^{A,B}_{h,\alpha,\alpha}(t-s)\,h(s)\,ds \\
&= Fz(t) + \int_0^t X^{A,B}_{h,\alpha,\alpha}(t-s)\,h(s)\,ds
\end{aligned}
$$

By using Lemma 3.1, we have the following estimation

$$
\|Fz(t) - z(t)\| \le \int_0^t \left\| X^{A,B}_{h,\alpha,\alpha}(t-s) \right\| \|h(s)\|\,ds \le X^{\|A\|,\|B\|}_{h,\alpha,\alpha+1}(T)\,\eta. \qquad (3.10)
$$

Therefore, we deduce by the fixed point property of the operator F given in the proof of Theorem 3.1,

$$
\begin{aligned}
\|x(t) - z(t)\| &\le \|x(t) - Fz(t)\| + \|Fz(t) - z(t)\| \\
&\le \|Fx(t) - Fz(t)\| + \|Fz(t) - z(t)\| \\
&\le \left[L_f X^{\|A\|,\|B\|}_{h,\alpha,\alpha+1}(T) + mML_g\right] \|x-z\|_{PC} + X^{\|A\|,\|B\|}_{h,\alpha,\alpha+1}(T)\,\eta
\end{aligned}
$$

Rearranging the inequality,

$$
\left(1 - \left[L_f X^{\|A\|,\|B\|}_{h,\alpha,\alpha+1}(T) + mML_g\right]\right)\|x-z\|_{PC} \le X^{\|A\|,\|B\|}_{h,\alpha,\alpha+1}(T)\,\eta
$$

then we get

$$
\|x-z\|_{PC} \le u_h.\eta,
$$

where

$$
u_h = \frac{X^{\|A\|,\|B\|}_{h,\alpha,\alpha+1}(T)}{1 - \left[L_f X^{\|A\|,\|B\|}_{h,\alpha,\alpha+1}(T) + mML_g\right]} > 0.
$$

This completes the proof. □

Note that so far, we have investigated that the novel system (3.1) with nonpermutable constant coefficient matrices is of an unique solution and stable. We wonder if the system (3.1) is controllable. According to one of the traditional approaches to check whether any system is controllable, a control problem we faced is made into a fixed point problem. Unlike this approach, we would like to use the new iterative learning method for showing that the system (3.1) is controllable. Now we can pass the following subsection.

3.3 Iterative Learning Control

In this subsection, we express the ILC method with the aid of its diagram. In the sequel, we introduce our ILC problem and investigate it according to P, D, and D^{α}-types updating laws, respectively.

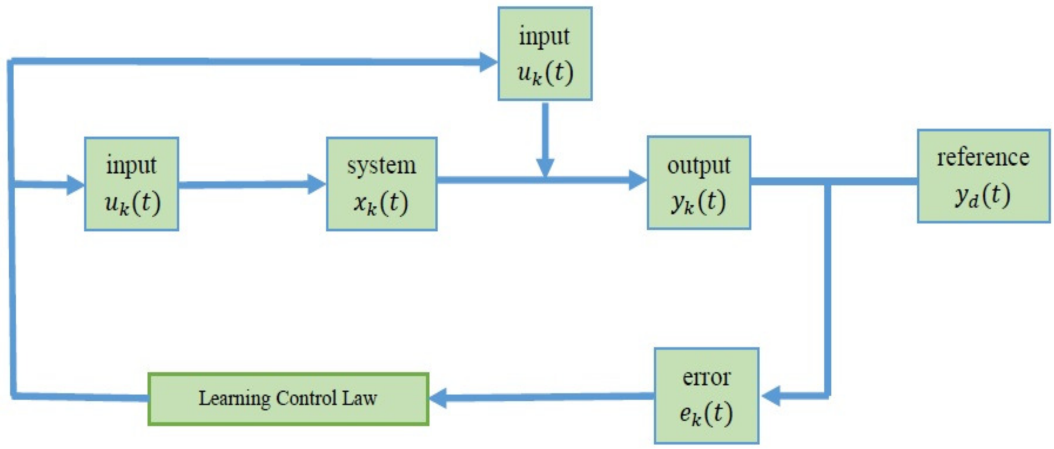

Figure 3.1: Block diagram of the iterative learning control method

In any control system, the most significant stuff is to determine the control input. The ILC system is solved repetitively in order to obtain the desired control input. ILC is a method for enhancing the provisional signal and tracking the performance of any tangible system that is needed to employ a certain operation repeatedly. By tracking the error in the output signal subsequent to each of operations and employing this error to adjust the input signal to the system, ILC tries to develop the system performance.

We consider the following ILC problem

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} x_k(t) = Ax_k(t) + Bx_k(t-h) + f(t, x_k(t)) + Cu_k(t), \ \ t \in [0,T] \backslash I, \\ x_k\left(t_i^+\right) = x_k\left(t_i^-\right) + g\left(x_k(t_i)\right), \ \ t_i \in I, \\ x_k(t) = \varphi(t), \ \ -h \leq t \leq 0, \end{cases} \tag{3.11}$$

and

$$y_k(t) = Dx_k(t) + Eu_k(t), \ \ t \in [0,T] \backslash I, \tag{3.12}$$

where k denotes the iterative times, $I = \{t_1, t_2, ..., t_m\}$ is the impulsive times and satisfying $0 < t_1 < ... < t_m < T$. $A, B \in R^{n\times n}, C \in R^{n\times r}, D \in R^{n\times s}, E \in R^{s\times r}$ are constant matrices. $f : [0,T] \times R^n \to R^n$ and $g : R^n \to R^n$ are continuous vector functions. $x_k \in R^n$, $u_k \in R^r$, and $y_k \in R^s$ symbolize state, input, and output, respectively. Define $\Delta x_k(t) := x_{k+1}(t) - x_k(t)$, $\Delta u_k(t) := u_{k+1}(t) - u_k(t)$, $e_k(t) := y_d(t) - y_k(t)$. Recall that the jumps

$$x_k\left(t_i^+\right) = \lim_{\varepsilon \to 0^+} x_k(t_i + \varepsilon), \quad x_k\left(t_i^-\right) = x_k(t_i) = \lim_{\varepsilon \to 0^-} x_k(t_i + \varepsilon)$$

represent the left and right limits of $x_k(t)$ at $t = t_i$, respectively.

It is well known that if $f(t, x_k)$, $g(x_k)$ satisfy Lipschitz conditions with respect to x_k, then the system (3.11) has a unique solution which is represented by the following integral equation:

$$\begin{aligned} x_k(t) = {} & X_{h,\alpha,1}^{A,B}(t+h)\varphi(-h) + \int_{-h}^{0} X_{h,\alpha,\alpha}^{A,B}(t-s)\left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\varphi(s) - A\varphi(s)\right] ds \\ & + \int_0^t X_{h,\alpha,\alpha}^{A,B}(t-s)\left[f(s, x_k(s)) + Cu_k(s)\right] ds + \sum_{0<t_i<t} X_{h,\alpha,1}^{A,B}(t-t_i)\, g(x_k(t_i)). \end{aligned}$$

3.3.1 *P*-Type Iterative Learning Control

In this subsection, we share some details about the concept of *P*-type and, investigate its convergency with *P*-type under some conditions. For (3.11), we employ open-loop *P*-type updating law with non-initial state learning

$$\Delta u_k(t) = \Gamma_1 e_k(t) \tag{3.13}$$

where $t \in [0,T]$ and $\Gamma_1 \in R^{s\times r}$ is unknowable element which can be obtained. Here the primary aim is to use delayed perturbation of Mittag-Leffler type matrix function to obtain the control input u_k such that the time-delay system output y_k is tracking the iteratively varying reference trajectories y_d as far as possible when $k \to \infty$ uniformly on $[0,T]$ in the sense of the λ-norm by adopting P-type updating ILC.

Theorem 3.3: Let $y_d : [0,T] \to \mathbb{R}^s$ be a desired trajectory from the system (3.11). The vector functions $f(t,x)$ and $g(x)$ satisfy Lipschitz condition, which means that for any $x_1, x_2 \in R^n$ that there exist constants $L_f > 0$ and $L_g > 0$, such that

$$\|f(t,x_1) - f(t,x_2)\| \le L_f \|x_1 - x_2\|, \|g(x_1) - g(x_2)\| \le L_g \|x_1 - x_2\|.$$

If $\|I - E\Gamma_1\| < 1$, then for any initial input u_0, the P-type updating ILC law (3.13) guarantees that y_k tends to y_d as $k \to \infty$ in the sense of λ-norm.

Proof. Consider $e_{k+1}(t) - e_k(t)$:

$$\begin{aligned} e_{k+1}(t) - e_k(t) &= y_k(t) - y_{k+1}(t) \\ &= Dx_k(t) + Eu_k(t) - Dx_{k+1}(t) - Eu_{k+1}(t) \\ &= -D\Delta x_k(t) - E\Delta u_k(t). \end{aligned}$$

It follows that

$$\begin{aligned} e_{k+1}(t) &= e_k(t) - D\Delta x_k(t) - E\Delta u_k(t) \\ &= e_k(t) - E\Gamma_1 e_k(t) - D\Delta x_k(t) \\ &= (I - E\Gamma_1) e_k(t) - D\Delta x_k(t). \end{aligned} \tag{3.14}$$

For an arbitrary $t \in (t_i, t_{i+1}]$, $i = 0, 1, ..., m$, we have

$$\begin{aligned}\Delta x_k(t) &= x_{k+1}(t) - x_k(t) \\ &= \int_0^t X_{h,\alpha,\alpha}^{A,B}(t-s)\left[f(s,x_{k+1}(s)) - f(s,x_k(s)) + C\Delta u_k(s)\right] ds \\ &+ \sum_{0<t_i<t} X_{h,\alpha,1}^{A,B}(t-t_i)\left(g(x_{k+1}(t_i)) - g(x_k(t_i))\right).\end{aligned}$$

Next we estimate $\Delta x_k(t)$ as follows by keeping Lemma 2.1 in our mind,

$$\begin{aligned}\|\Delta x_k(t)\| &\le \int_0^t \left\|X_{h,\alpha,\alpha}^{A,B}(t-s)\right\| \left(L_f\|\Delta x_k(s)\| + \|C\|\|\Gamma_1\|\|e_k(s)\|\right) ds \\ &+ L_g \sum_{0<t_i<t} \left\|X_{h,\alpha,1}^{A,B}(t-t_i)\right\| \|\Delta x_k(t_i)\| \\ &\le L_f \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left((\|A\|+\|B\|)(t-s)^{\alpha}\right)\|\Delta x_k(s)\| ds \\ &+ \|C\|\|\Gamma_1\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\left((\|A\|+\|B\|)(t-s)^{\alpha}\right)\|e_k(s)\| ds \\ &+ L_g \sum_{0<t_i<t} E_{\alpha,1}\left((\|A\|+\|B\|)(t-t_i)^{\alpha}\right)\|\Delta x_k(t_i)\|. \end{aligned} \tag{3.15}$$

Introduce the following notations:

$$\begin{aligned} M &:= E_{\alpha,\beta}\left((\|A\|+\|B\|)T^{\alpha}\right), \ (\beta = \alpha \ \text{ or } \ 1), \\ c(t) &:= M\|C\|\|\Gamma_1\| \int_0^t (t-s)^{\alpha-1}\|e_k(s)\| ds. \end{aligned}$$

Inequality (3.15) can be written as follows:

$$\|\Delta x_k(t)\| \le c(t) + L_f M \int_0^t (t-s)^{\alpha-1}\|\Delta x_k(s)\| ds + L_g M \sum_{0<t_i<t} \|\Delta x_k(t_i)\|.$$

By the Gronwall lemma, we have

$$\|\Delta x_k(t)\| \le c(t)\left(1 + L_g M E_{\alpha}\left(L_f M \Gamma(\alpha) T^{\alpha}\right)\right)^m E_{\alpha}\left(L_f M \Gamma(\alpha) T^{\alpha}\right) \tag{3.16}$$

for $t_k < t \le t_{k+1}$. To take λ-norm, we multiply both sides of the above inequality by $e^{-\lambda t}$:

$$\begin{aligned}\|\Delta x_k\|_{\lambda} &\le \left(M\|C\|\|\Gamma_1\| \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} ds \|e_k\|_{\lambda}\right) \\ &\times \left(1 + L_g M E_{\alpha}\left(L_f M \Gamma(\alpha) T^{\alpha}\right)\right)^m E_{\alpha}\left(L_f M \Gamma(\alpha) T^{\alpha}\right).\end{aligned}$$

Let's compute the inner integral in the above inequality

$$\int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} ds = \int_{\lambda t}^0 \left(\frac{u}{\lambda}\right)^{\alpha-1} e^{-u} \left(-\frac{du}{\lambda}\right)$$
$$= \frac{1}{\lambda^\alpha} \int_0^{\lambda t} u^{\alpha-1} e^{-u} du$$
$$\leq \frac{1}{\lambda^\alpha} \int_0^{\infty} u^{\alpha-1} e^{-u} du = \frac{\Gamma(\alpha)}{\lambda^\alpha} \tag{3.17}$$

By use of estimation of (3.14), we have

$$\|e_{k+1}\|_\lambda \leq \|I - E\Gamma_1\| \|e_k\|_\lambda + \|D\| \|\Delta x_k\|_\lambda \leq \rho_1 \|e_k\|_\lambda,$$

where

$$\rho_1 := \|I - E\Gamma_1\| + \left(1 + L_g M E_\alpha \left(L_f M \Gamma(\alpha) T^\alpha\right)\right)^m E_\alpha \left(L_f M \Gamma(\alpha) T^\alpha\right)$$
$$\times \|D\| M \|C\| \|\Gamma_1\| \frac{\Gamma(\alpha)}{\lambda^\alpha}.$$

Having in mind the condition $\|I - E\Gamma_1\| < 1$, it is possible to make $\rho_1 < 1$ for sufficiently large λ. Therefore,

$$\|e_{k+1}\|_\lambda \leq \rho_1^k \|e_1\|_\lambda,$$

implies that

$$\lim_{k\to\infty} \|e_{k+1}\|_\lambda = 0,$$

which in fact says that $\lim_{k\to\infty} y_k = y_d$. The proof is completed. □

Remark 3.3: Theorem 3.3 with $\alpha = \beta = 1$ without the impulse and with commutative matrices reduces to Theorem 3.1 in the study [74].

3.3.2 *D*-Type Iterative Learning Control

In this section, we hand in brief details about the concept of *D*-type and, investigate its convergency with *D*-type for (3.11) under some conditions. For (3.11), we employ open-loop *D*-type updating law with non-initial state learning

$$\Delta u_k(t) = \Gamma_2 \left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(t)\right] \tag{3.18}$$

where $t \in [0,T]$ and $\Gamma_2 \in R^{s\times r}$ is unknowable element which can be obtained. Here the primary aim is to use delayed perturbation of Mittag-Leffler type matrix function to obtain the control input u_k such that the time-delay system output y_k is tracking the iteratively varying reference trajectories y_d as far as possible when $k \to \infty$ uniformly on $[0,T]$ in the sense of the λ-norm by adopting D-type updating ILC.

Theorem 3.4: Let $y_d : [0,T] \to \mathbb{R}^s$ be a desired trajectory from the system (3.11) with $E = \Theta$. The vector functions $f(t,x)$ and $g(x)$ satisfy Lipschitz condition, which means that for any $x_1, x_2 \in R^n$ that there exist constants $L_f > 0$ and $L_g > 0$, such that

$$\|f(t,x_1) - f(t,x_2)\| \leq L_f \|x_1 - x_2\|, \|g(x_1) - g(x_2)\| \leq L_g \|x_1 - x_2\|.$$

If $\|I - DC\Gamma_2\| < 1$, and $e_k(0) = 0, k = 1,2,...$, then for any initial input u_0, the D-type updating ILC law (3.18) guarantees that y_k tends to y_d as $k \to \infty$ in the sense of λ-norm.

Proof. By

$$\begin{aligned}
{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1}(t) - {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(t) &= {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} y_k(t) - {}^{C}D_t^{\alpha} y_{k+1}(t) \\
&= D\,{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} x_k(t) - D\,{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} x_{k+1}(t) \\
&= DA[x_k(t) - x_{k+1}(t)] + DB[x_k(t-h) - x_{k+1}(t-h)] \\
&\quad + DC[u_k(t) - u_{k+1}(t)] + [f(t,x_k(t)) - f(t,x_{k+1}(t))],
\end{aligned}$$

$$\begin{aligned}
&= -DA[\Delta x_k(t)] - DB[\Delta x_k(t-h)] - DC[\Delta u_k(t)] \\
&\quad - [f(t,x_{k+1}(t)) - f(t,x_k(t))]
\end{aligned}$$

Therefore we have

$$\begin{aligned}
{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1}(t) &= (I - DC\Gamma_2)\left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(t)\right] - DA[\Delta x_k(t)] - DB[\Delta x_k(t-h)] \\
&\quad - [f(t,x_{k+1}(t)) - f(t,x_k(t))]
\end{aligned}$$

It follows that

$$\left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1}(t) \right\| \leq \|(I - DC\Gamma_2)\| \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(t) \right\| + \|D\| \|A\| \|\Delta x_k(t)\|$$
$$+ \|D\| \|B\| \|\Delta x_k(t-h)\| + L_f \|\Delta x_k(t)\| \tag{3.19}$$

On taking λ-norm, we multiply both sides of the above inequality by $e^{-\lambda t}$

$$\left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1} \right\|_\lambda \leq \|(I - DC\Gamma_2)\| \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k \right\|_\lambda + \|D\| \|A\| \|\Delta x_k\|_\lambda$$
$$+ \|D\| \|B\| \|\Delta x_k\|_\lambda + L_f \|\Delta x_k\|_\lambda$$

then

$$\left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1} \right\|_\lambda \leq \|(I - DC\Gamma_2)\| \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k \right\|_\lambda$$
$$+ \left(L_f + \|D\| \|A\| + \|D\| \|B\|\right) \|\Delta x_k\|_\lambda \tag{3.20}$$

We know that by property (2.7)

$${}^{\mathfrak{RL}}\mathfrak{J}_{0^+}^{\alpha} \left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(t) \right] = e_k(t) - e_k(0)$$

Keeping the statement $e_k(0) = 0, k = 1, 2, ...,$ in mind, we get

$${}^{\mathfrak{RL}}\mathfrak{J}_{0^+}^{\alpha} \left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(t) \right] = e_k(t)$$

By using the definition of Riemann-Liouville fractional integral,

$$\begin{aligned} e_k(t) &= {}^{\mathfrak{RL}}\mathfrak{J}_{0^+}^{\alpha} \left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(t) \right] \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(s) \right] ds. \end{aligned}$$

By taking the norm on the both sides, we have

$$\|e_k(t)\| \leq \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(s) \right\| ds.$$

In order to take λ-norm, we multiply the both sides of the above inequality by $e^{-\lambda t}$,

$$\|e_k(t)\| e^{-\lambda t} \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda t} e^{\lambda s} ds \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k \right\|_\lambda .$$

We rewrite the inequality and take λ-norm on the left-hand side to obtain the following inequality

$$\|e_k(t)\|_\lambda \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} ds \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k \right\|_\lambda$$

with the similar calculation in the proof of Theorem 3.3, we get

$$\|e_k(t)\|_\lambda \leq \frac{1}{\lambda^\alpha} \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k \right\|_\lambda \tag{3.21}$$

In the proof of Theorem 3.3, we have by using the equation (3.16) with (3.18)

$$\begin{aligned} \|\Delta x_k(t)\| \leq & \left(M \|C\| \|\Gamma_2\| \int_0^t (t-s)^{\alpha-1} \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(s) \right\| ds \right) \\ & \times \left(1 + L_g M E_\alpha \left(L_f M \Gamma(\alpha) T^\alpha\right)\right)^m E_\alpha \left(L_f M \Gamma(\alpha) T^\alpha\right) \end{aligned}$$

To take λ-norm, we multiply both sides of the above inequality by $e^{-\lambda t}$:

$$\begin{aligned} \|\Delta x_k\|_\lambda \leq & \left(M \|C\| \|\Gamma_2\| \int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} ds \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k \right\|_\lambda \right) \\ & \times \left(1 + L_g M E_\alpha \left(L_f M \Gamma(\alpha) T^\alpha\right)\right)^m E_\alpha \left(L_f M \Gamma(\alpha) T^\alpha\right). \end{aligned}$$

By making the integral calculation in the above inequality

$$\begin{aligned} \|\Delta x_k\|_\lambda \leq & M \|C\| \|\Gamma_2\| \frac{\Gamma(\alpha)}{\lambda^\alpha} \left(1 + L_g M E_\alpha \left(L_f M \Gamma(\alpha) T^\alpha\right)\right)^m \\ & \times E_\alpha \left(L_f M \Gamma(\alpha) T^\alpha\right) \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k \right\|_\lambda . \end{aligned}$$

By combining the last inequality with (3.20)

$$\left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1} \right\|_\lambda \leq \rho_2 \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k \right\|_\lambda$$

where

$$\begin{aligned} \rho_2 := & \left(L_f + \|D\| \|A\| + \|D\| \|B\|\right) \\ & \times M \|C\| \|\Gamma_2\| \frac{\Gamma(\alpha)}{\lambda^\alpha} \left(1 + L_g M E_\alpha \left(L_f M \Gamma(\alpha) T^\alpha\right)\right)^m E_\alpha \left(L_f M \Gamma(\alpha) T^\alpha\right) \\ & + \|(I - DC\Gamma_2)\| \end{aligned}$$

Keeping the condition $\|I - DC\Gamma_2\| < 1$ in mind, it is possible to make $\rho_2 < 1$ for sufficiently large λ. Therefore,

$$\left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1} \right\|_\lambda \leq \rho_2^k \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_1 \right\|_\lambda ,$$

implies that

$$\lim_{k\to\infty}\left\|{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}e_{k+1}\right\|_{\lambda}=0.$$

By employing the inequality (3.21) with the last information, we get

$$\lim_{k\to\infty}\|e_{k+1}\|_{\lambda}=0,$$

which in fact says that $\lim_{k\to\infty}y_k=y_d$. The proof is completed. $\square$

Remark 3.4: Theorem 3.4 with $\alpha=\beta=1$ without the impulse and non-commutativity reduces to Theorem 4.1 in the study [74].

3.3.3 D^{α}-Type Iterative Learning Control

In this section, we talk about some details about the concept of D^{α}-type and, investigate its convergency with D^{α}-type for (3.11) under some conditions. Define the output equation

$$y_k(t)=Dx_k(t)+E\,{}^{\mathfrak{RL}}\mathfrak{I}_{0^+}^{\alpha}u_k(t). \tag{3.22}$$

where $t\in[0,T]\backslash I$. For (3.11), we employ open-loop D^{α}-type updating law with non-initial state learning

$$\Delta u_k(t)=\Gamma_3\left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}e_k(t)\right] \tag{3.23}$$

where $t\in[0,T]$ and $\Gamma_3\in R^{s\times r}$ is unknowable element which can be obtained. Here the primary aim is to use delayed perturbation of Mittag-Leffler type matrix function to obtain the control input u_k such that the time-delay system output y_k is tracking the iteratively varying reference trajectories y_d as far as possible when $k\to\infty$ uniformly on $[0,T]$ in the sense of the λ-norm by adopting D^{α}-type updating ILC.

Theorem 3.5: Let $y_d:[0,T]\to\mathbb{R}^s$ be a desired trajectory from the system (3.11). The vector functions $f(t,x)$ and $g(x)$ satisfy Lipschitz condition, which means that for any

$x_1, x_2 \in R^n$ that there exist constants $L_f > 0$ and $L_g > 0$, such that

$$\|f(t,x_1) - f(t,x_2)\| \le L_f \|x_1 - x_2\|, \|g(x_1) - g(x_2)\| \le L_g \|x_1 - x_2\|.$$

If $\|I - (DC+E)\Gamma_3\| < 1$, then for any initial input satisfying $u_0(t)$ the D^{α}-type updating ILC law (3.23) guarantees that y_k tends to y_d as $k \to \infty$ in the sense of λ-norm.

Proof. Consider $e_{k+1}(t) - e_k(t)$:

$$\begin{aligned} e_{k+1}(t) - e_k(t) &= y_d(t) - y_{k+1}(t) - (y_d(t) - y_k(t)) \\ &= -(y_{k+1}(t) - y_k(t)) \\ &= \left[Dx_{k+1}(t) + E^{\mathfrak{RL}}\mathfrak{I}_{0^+}^{\alpha} u_{k+1}(t) - Dx_k(t) - E^{\mathfrak{RL}}\mathfrak{I}_{0^+}^{\alpha} u_k(t)\right] \\ &= -D\Delta x_k(t) - E^{\mathfrak{RL}}\mathfrak{I}_{0^+}^{\alpha}\Delta u_k(t) \end{aligned}$$

We apply the operator ${}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}$ to the last equation,

$$\begin{aligned} &{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1}(t) - {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(t) \\ &= -D\left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\Delta x_k(t)\right] - E\left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\left(D_t^{-\alpha}\Delta u_k(t)\right)\right] \\ &= -D\left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\Delta x_k(t)\right] - E\Delta u_k(t) \end{aligned}$$

$$\begin{aligned} &= -D\left[A\Delta x_k(t) + B\Delta x_k(t-h) + C\Delta u_k(t) + \left[f(t,x_{k+1}(t)) - f(t,x_k(t))\right]\right] \\ &- E\Delta u_k(t) \\ &= -DC\Delta u_k(t) - E\Delta u_k(t) - DA\Delta x_k(t) - DB\Delta x_k(t-h) \\ &- D\left[f(t,x_{k+1}(t)) - f(t,x_k(t))\right] \\ &= -(DC+E)\Delta u_k(t) - DA\Delta x_k(t) - DB\Delta x_k(t-h) \\ &- D\left[f(t,x_{k+1}(t)) - f(t,x_k(t))\right] \\ &= -(DC+E)\Gamma_3\left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(t)\right] - DA\Delta x_k(t) - DB\Delta x_k(t-h) \\ &- D\left[f(t,x_{k+1}(t)) - f(t,x_k(t))\right] \end{aligned}$$

One can rewrite the above inequality again,

$$
\begin{aligned}
{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1}(t) = (I-(DC+E)\Gamma_3)\left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(t)\right] - DA\Delta x_k(t) \\
- DB\Delta x_k(t-h) - D\left[f(t,x_{k+1}(t)) - f(t,x_k(t))\right]
\end{aligned}
$$

By taking the norm on the both sides, we reach to

$$
\begin{aligned}
\left\|{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1}(t)\right\| \leq \|I-(DC+E)\Gamma_3\| \left\|{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(t)\right\| + \|D\|\|A\|\|\Delta x_k(t)\| \\
+ \|D\|\|B\|\|\Delta x_k(t-h)\| + L_f\|D\|\|\Delta x_k(t)\|
\end{aligned}
$$

To take λ-norm, we multiply both sides of the above inequality by $e^{-\lambda t}$,

$$
\begin{aligned}
\left\|{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1}(t)\right\| e^{-\lambda t} \leq \|I-(DC+E)\Gamma_3\| \left\|{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(t)\right\| e^{-\lambda t} + \|D\|\|A\|\|\Delta x_k(t)\| e^{-\lambda t} \\
+ \|D\|\|B\|\|\Delta x_k(t-h)\| e^{-\lambda t} + L_f\|D\|\|\Delta x_k(t)\| e^{-\lambda t}
\end{aligned}
$$

and, so we get

$$
\begin{aligned}
\left\|{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1}\right\|_\lambda &\leq \|I-(DC+E)\Gamma_3\| \left\|{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k\right\|_\lambda + \|D\|\|A\|\|\Delta x_k\|_\lambda \\
&\quad + \|D\|\|B\|\|\Delta x_k\|_\lambda + L_f\|D\|\|\Delta x_k\|_\lambda \\
&\leq \|I-(DC+E)\Gamma_3\| \left\|{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k\right\|_\lambda \\
&\quad + \left(L_f\|D\| + \|D\|\|A\| + \|D\|\|B\|\right)\|\Delta x_k\|_\lambda \\
&\leq \sigma_1 \left\|{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k\right\|_\lambda + \sigma_2\|\Delta x_k\|_\lambda \qquad (3.24)
\end{aligned}
$$

where $\sigma_1 = \|I-(DC+E)\Gamma_3\|$, $\sigma_2 = \left(L_f\|D\| + \|D\|\|A\| + \|D\|\|B\|\right)$. In the proof of Theorem (3.3), we have by using the equation (3.16) with (3.23)

$$
\begin{aligned}
\|\Delta x_k(t)\| \leq \left(M\|C\|\|\Gamma_3\| \int_0^t (t-s)^{\alpha-1} \left\|{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k(s)\right\| ds\right) \\
\times \left(1 + L_g M E_\alpha\left(L_f M\Gamma(\alpha) T^\alpha\right)\right)^m E_\alpha\left(L_f M\Gamma(\alpha) T^\alpha\right)
\end{aligned}
$$

To take λ-norm, we multiply both sides of the above inequality by $e^{-\lambda t}$:

$$\|\Delta x_k\|_\lambda \leq \left(M\|C\|\|\Gamma_3\|\int_0^t (t-s)^{\alpha-1} e^{-\lambda(t-s)} ds \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k \right\|_\lambda\right)$$
$$\times \left(1+L_g M E_\alpha \left(L_f M \Gamma(\alpha) T^\alpha\right)\right)^m E_\alpha \left(L_f M \Gamma(\alpha) T^\alpha\right).$$

by making the integral calculation in the above inequality

$$\|\Delta x_k\|_\lambda \leq \sigma_3 \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k \right\|_\lambda$$

where

$$\sigma_3 = M\|C\|\|\Gamma_3\| \frac{\Gamma(\alpha)}{\lambda^\alpha} \left(1+L_g M E_\alpha \left(L_f M \Gamma(\alpha) T^\alpha\right)\right)^m E_\alpha \left(L_f M \Gamma(\alpha) T^\alpha\right).$$

By combining the last inequality with (3.24), we get

$$\left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1} \right\|_\lambda \leq (\sigma_1 + \sigma_2\sigma_3) \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_k \right\|_\lambda.$$

Since $\sigma_1 < 1$, it is possible to make $\rho_3 := \sigma_1 + \sigma_2\sigma_3 < 1$ for sufficiently large λ. Therefore,

$$\left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1} \right\|_\lambda \leq \rho_3^k \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_1 \right\|_\lambda,$$

implies that

$$\lim_{k\to\infty} \left\| {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} e_{k+1} \right\|_\lambda = 0,$$

and employing the inequality (3.21) with the last information, we get

$$\lim_{k\to\infty} \|e_{k+1}\|_\lambda = 0,$$

which in fact says that $\lim_{k\to\infty} y_k = y_d$. This completes the proof. □

Remark 3.5: If we remove the impulsive and non-linear parts, Theorem 3.5 in this paper reduces to Theorem 1 with $\gamma = 1$ in the study [67] on ignoring the bounded external disturbance since $\rho\left(I-(DC+E)\Gamma_3\right) \leq \|I-(DC+E)\Gamma_3\| < 1$, where ρ is the spectral radius.

3.4 An Illustrative Example

We consider the iterative learning control problem of the impulsive fractional differential time-delay equation

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{0.5}x_k(t) = x_k(t) + x_k(t-0.3) + u_k(t), \ t \in \left[0, \frac{3}{5}\right] \backslash \{0.4\}, \\ x_k(0,4^+) = x_k(0.4^-) \\ x_k(t) = t, \ -0.3 \le t \le 0, \\ y_k(t) = x_k(t) + 1.2u_k(t) \ \text{ or } \ y_k(t) = x_k(t) + 1.2{}^{\mathfrak{RL}}\mathfrak{J}_{0^+}^{0.5}u_k(t), \ t \in \left[0, \frac{3}{5}\right] \backslash \{0.4\}, \end{cases} \tag{3.25}$$

and P-type, D-type and D^{α}-type updating laws are

$$\Delta u_k(t) = 0.3e_k(t), \Delta u_k(t) = 0.3\left[{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}e_k(t)\right]$$

It is clear that the vector functions $f(t,x)$ and $g(x)$ satisfy Lipschitz condition. $\|I - E\Gamma_1\| = 0.64 < 1$, $\|I - DC\Gamma_2\| = 0.7 < 1$, $\|I - (DC+E)\Gamma_3\| = 0.34 < 1$, so all of conditions of Theorem 3.3, 3.4 and 3.5 are satisfied. Especially, we give extra information like graphs and tables about P-type for each of three different original reference trajectories. The first original reference trajectory is a continuous function

$$y_d(t) = 12t(1-t),$$

the second original reference trajectory is a piecewise continuous function

$$y_d(t) = \begin{cases} 2t^3, & 0 \le x \le 0.4; \\ 3t^3 + 2, & 0.4 < x \le 0.6, \end{cases}$$

the third original reference trajectory is a piecewise trigonometric continuous function

$$y_d(t) = \begin{cases} 2\sin 4\pi t - 1, & 0 \le x \le 0.4; \\ 2\sin 4\pi t + \cos 4\pi t + 3, & 0.4 < x \le 0.6. \end{cases}$$

So, we have three cases. For case i, $i = 1,2,3$, we use the ith original reference trajectory in the P-type iterative learning control problem (3.25) and share one figure which includes one histogram graph as well as one chart with graphs of $y_d(t)$ and

$y_k(t)$, $i = 1, 2, \ldots, 10$ and one table about the obtained results for each of the original reference trajectories. The left-hand side of Figure i, $i = 1, 2, 3$ presents the output $y_k(t)$ of the P-type iterative learning control problem (3.25) of the first 10th iterations and the ith original reference trajectory $y_d(t)$. The right-hand side of Figure i, $i = 1, 2, 3$ shows the infinity norm of the tracking error (see also Table i, $i = 1, 2, 3$) in each iteration.

Table 3.1: Error $e_k(t)$ for Figure 3.2 in Case 1.

k	1	2	3	4	5	6	7	8	9	10
error	1.307	0.467	0.187	0.088	0.047	0.030	0.019	0.012	0.008	0.005

The tracking error of each iteration for the first original reference trajectory

Table 3.2: Error $e_k(t)$ for Figure 3.3 in Case 2.

k	1	2	3	4	5	6	7	8	9	10
error	1.702	0.360	0.118	0.074	0.047	0.030	0.019	0.012	0.008	0.005

The tracking error of each iteration for the second original reference trajectory

Table 3.3: Error $e_k(t)$ for Figure 3.4 in Case 3.

k	1	2	3	4	5	6	7	8	9	10
error	4.670	1.062	0.333	0.190	0.121	0.077	0.049	0.031	0.020	0.010

The tracking error of each iteration for the third original reference trajectory

Remark 3.6: We demonstrated that system (3.1) is controllable via iterative learning control technique in spite of three different sorts of P, D, and D^{α}-types updating laws under the certain conditions of Theorems 3.3, 3.4, and 3.5 which express how to select the design parameters. With the help of example's section, we verified the theoretical control results. We also showed that by using P-type updating law the output signal

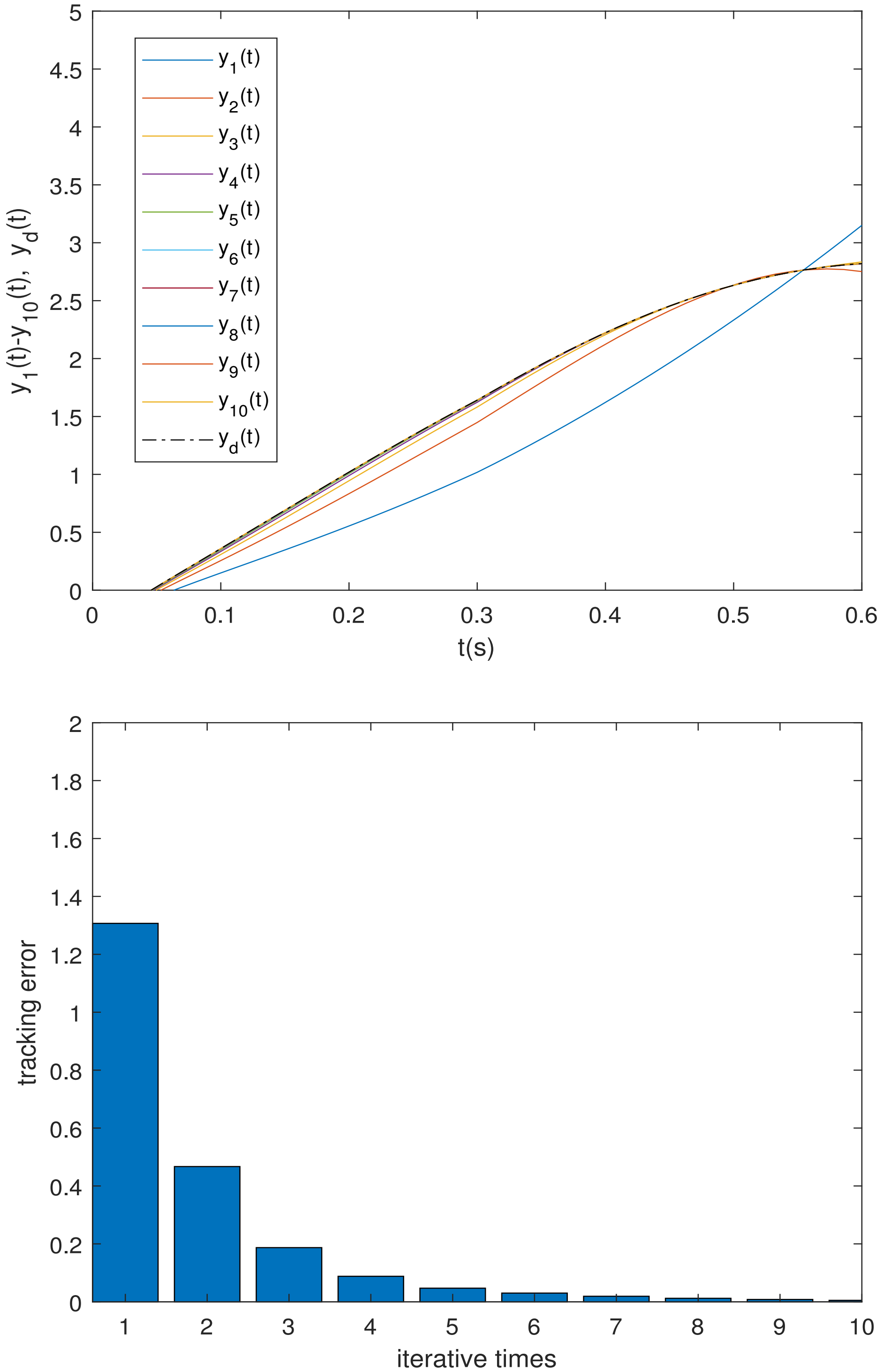

Figure 3.2: Trajectory $y_k(t)$ and error $e_k(t)$ for Case 1.

y_k tends to the original reference trajectory y_d which is continuous or discontinuous or trigonometric discontinuous. These results can be extended for D and D^{α}-types updating laws.

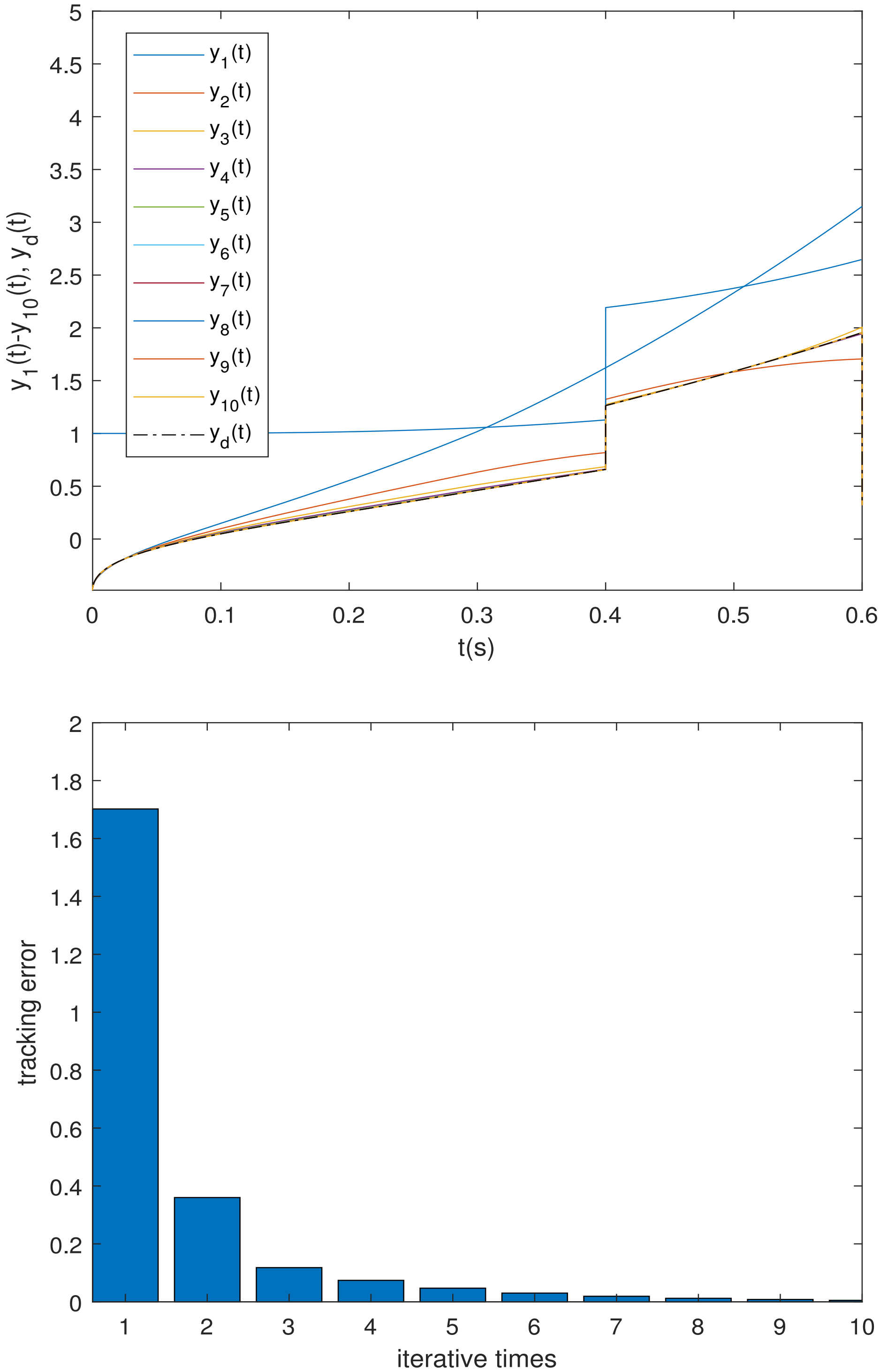

Figure 3.3: Trajectory $y_k(t)$ and error $e_k(t)$ for Case 2.

Remark 3.7: For $h = 1$ and $T = 2$, we reconsider (3.25) to show how the parameter α effects the control performance. As one can easily observe from Tables 3.4 and 3.5, the speed of convergence is faster when $\alpha \in (0,1)$ approaches to 1 and the other design

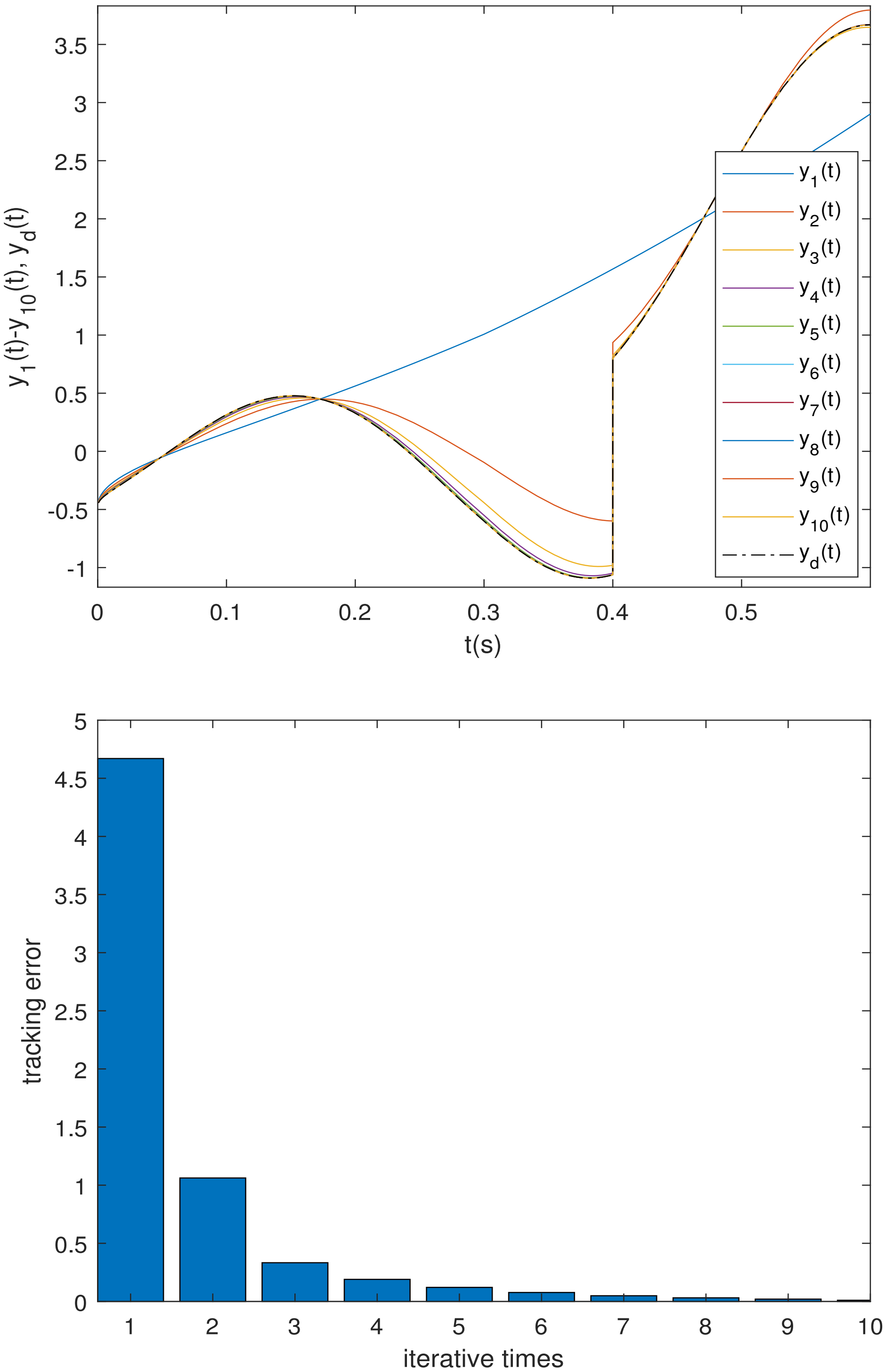

Figure 3.4: Trajectory $y_k(t)$ and error $e_k(t)$ for Case 3.

Table 3.4: $\|e_k\|_C$ for choosing α.

	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.3$	$\alpha=0.4$	$\alpha=0.5$
$\|e_1\|_C$	3.1983	2.7349	2.7798	2.6463	2.5412
$\|e_2\|_C$	2.0469	1.7504	1.5416	1.3975	1.2927
$\|e_3\|_C$	1.3100	1.1202	0.9866	0.8944	0.8273
$\|e_4\|_C$	0.8384	0.7169	0.6314	0.5724	0.5295
$\|e_5\|_C$	0.5366	0.4588	0.4041	0.3664	0.3389

Table 3.5: $\|e_k\|_C$ for choosing α.

	$\alpha=0.6$	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$
$\|e_1\|_C$	2.4310	2.3536	2.3749	2.4149
$\|e_2\|_C$	1.2085	1.1345	1.0835	1.1269
$\|e_3\|_C$	0.7734	0.7261	0.6824	0.6417
$\|e_4\|_C$	0.4950	0.4647	0.4367	0.4107
$\|e_5\|_C$	0.3168	0.2974	0.2795	0.2628

parameters connected with P-type updating law which are given in (3.12) and (3.13) are chosen to satisfy the statements of Theorem 3.3.

3.5 Open Problems

We are sure that this paper will become a source of inspiration for the works which will be conducted in this subject. A possible duty is to investigate approximate controllability, exponential stability, finite time stability, asymptotic stability, and also Lyapunov type stability of the semilinear impulsive fractional differential time-delay equations with noncommutative coefficients. Another possible duty is to extend our system (3.1) to the μ-semilinear impulsive fractional differential time-delay equations with noncommutative coefficients which means that system (3.1) is reconsidered via Caputo fractional derivative with respect to another function μ, or the semilinear neutral impulsive fractional differential time-delay equations with noncommutative coefficients, or the semilinear neutral impulsive fractional multi-delayed differential equations with noncommutative coefficients, or the semilinear neutral impulsive fractional multi-delayed differential evolution equations with noncommutative coefficients, or the μ-semilinear neutral impulsive fractional multi-delayed differential evolution equations with noncommutative coefficients. All possibilities as noted above can be questioned once again for these new systems.

Chapter 4

THE NEUTRAL FRACTIONAL MULTI-DELAYED SYSTEM

Just as the most important notion in the traditional calculus is a derivative of order $n \in \mathbb{Z}^+$, fractional derivative of order $\alpha \in \mathbb{R}^+$ and a fractional differential equation related to it are the heart of the fractional calculus. For this and similar reasons, many differential equations of integer order have been converted into fractional differential equations and their solutions, controllability and stability, etc have been examined. Two of them are delayed fractional differential equations and neutral fractional differential equations.

When we have a look at the studies conducted in this subject more specifically, Khusainov and Shuklin in pioneering work [84] managed to obtain a representation of a solution of the following delayed homogeneous linear system by defining the delayed exponential matrix.

$$\begin{cases} z^{'}(x) = \mathscr{T} z(x) + \mathscr{U} z(x-r), \;\; x > 0 \;\; r > 0 \;\; \text{(delay)}, \\ \quad z(x) = \phi(x), \;\; -r \leq x \leq 0. \end{cases}$$

The first result for pure delay fractional differential equations is solved by Li and Wang [85]. The first result for delay fractional differential equations with nonpermutable case is solved by Mahmudov [86]. He was able to get a solution by introducing delay perturbation of Mittag-Leffler type matrix function with two parameters. You et al. [75] investigated the relative controllability of fractional delay system whose solution is as in the work [86]. In the sequel, Mahmudov [87] extended the fractional delay differential equations [86] to the fractional multi-delay differential equation with nonpermutable matrices by defining multi-delayed perturbation of Mittag-Leffler type matrix function. The research [84] yielded plenty of novel results on the representation of solutions [88]- [96], which are applied in the stability analysis [97], [98], and control problems [99], [100] of time-delay systems.

On the other hand, Pospíšil and Škripková [101] considered the following neutral linear differential equations

$$\begin{cases} z'(x) - \mathscr{I}z'(x-r) = \mathscr{T}z(x) + \mathscr{U}z(x-r) + f(x), \;\; x > 0 \;\; r > 0 \;\; , \\ z(x) = \phi(x), \;\; -r \leq x \leq 0. \end{cases} \tag{4.1}$$

where r is a retardation, ϕ is continuously differentiable from $[-r,0]$ to $\mathbb{R}^n$ and f is continuous from $[0,\infty)$ to $\mathbb{R}^n$. The coefficient matrices $\mathscr{I}, \mathscr{T}, \mathscr{U}$ are permutable, that is $\mathscr{I}\mathscr{T} = \mathscr{T}\mathscr{I}, \mathscr{U}\mathscr{T} = \mathscr{T}\mathscr{U}, \mathscr{I}\mathscr{U} = \mathscr{U}\mathscr{I}$. As a special case, $\mathscr{T} = \Theta$, Pospíšil [102] made a study of relative controllability of the below neutral system with permutable matrices

$$\begin{cases} z'(x) - \mathscr{I}z'(x-r) = \mathscr{U}z(x-r) + f(x), \;\; x > 0 \;\; r > 0 \;\; , \\ z(x) = \phi(x), \;\; -r \leq x \leq 0. \end{cases} \tag{4.2}$$

Pospíšil in this work [102] achieved to supply a description of all control functions for system (4.2) with the aid of the shifted Legendre polynomial and granted an equal condition of Kalman type for the relative controllability of system (4.2). You et al. [105] proved the relative controllability for system (4.1) by Krasnoselskii's fixed point theorem. Zhang et al. [103] looked into the representation of the solution to the neutral fractional linear differential system having a constant delay

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0+}\left(z(x) - \mathscr{I}z(x-r)\right) = \mathscr{T}z(x) + \mathscr{U}z(x-r) + f(x), \;\; x > 0 \\ z(x) = \phi(x), \;\; -r \leq x \leq 0, \quad r > 0. \end{cases} \tag{4.3}$$

where ${}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}$ is Caputo fractional derivative of order α, $0 < \alpha < 1$, $\mathscr{I}, \mathscr{T}, \mathscr{U} \in \mathbb{R}^{n\times n}$. ϕ is continuously differentiable from $[-r,0]$ to $\mathbb{R}^n$ and f is continuous from $[0,\infty)$ to $\mathbb{R}^n$. In an attempt to solve system (4.3), Zhang et al. [103] exploited Laplace integral transform. This produced some drawback and mistakes because the representation of power series of the fundamental solution is unknowable. Huseynov and Mahmudov in the study [104] took the following system into consideration

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} z(x) = \mathscr{T} z(x) + \mathscr{U} z(x-r_1) + \mathscr{I}\, {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} z(x-r_2) + f(x), \ \ x > 0, \\ z(x) = \phi(x), \ \ -r \le x \le 0, \quad r = \max\{r_1, r_2\}, \ r_1, r_2 > 0. \end{cases} \tag{4.4}$$

By proposing delayed Mittag-Leffler type matrix function, they [104] gave the analytic representation of solutions to linear and semilinear neutral fractional differential difference system with time delay. The existence and uniqueness of the solutions was demonstrated by the Banach contraction principle along with a weighted space of continuous functions with respect to classical Mittag-Leffler functions besides showing that it is Ulam-Hyers stable based on fixed-point approach.

Motivated by studies [84], [85], [86], [101], we consider the following neutral Caputo fractional multi-delayed differential equations with noncommutative coefficient matrices

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} \left[z(x) - \sum_{i=1}^{d} E_i z(x-r_i)\right] = M z(x) + \sum_{i=1}^{d} U_i z(x-r_i) + \urcorner(x), x > 0, \\ z(x) = \phi(x), \ \ -r \le x \le 0, \end{cases} \tag{4.5}$$

where ${}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$. For each of $i = 1,2,3,\ldots,d$, $r_i > 0$ is a retardation and E_i, M, U_i are n-by-n coefficient matrices which do not have to be commutative. $\phi(x)$ is an arbitrary continuously differentiable vector function and $\urcorner \in C([0,T], \mathbb{R}^n)$ with $T = lr$ for a fixed $l \in \mathbb{N}$, $r = \max\{r_i : i = 1,2,3,\ldots,d\}$. Subsequent to investigating the explicit solutions of

(4.5), we reach to the explicit solutions to the below equations

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\left[z(x)-\sum_{i=1}^{d}E_i z(x-r_i)\right]=Mz(x)+\sum_{i=1}^{d}U_i z(x-r_i)+\daleth(x,z(x)), \\ \quad z(x)=\phi(x),\ \ -r\leq x\leq 0, \end{cases} \tag{4.6}$$

where the function $\daleth$ is continuous from $[0,T]\times\mathbb{R}^n$ to $\mathbb{R}^n$ and the rest of terms are the same as (4.5).

Remark 4.1: It is clear that neutral Caputo fractional multi-delayed differential equations with noncommutative coefficient matrices reduces to fractional linear multi-delay differential equations in the reference [9] when for each of $i=1,2,3,\ldots,d$, $E_i=\Theta$ which is the suitable dimensional zero matrix.

4.1 The Multi-Delayed Perturbation of Mittag-Leffler Type Matrix Function

In this section, we share main findings like introducing the neutral multi-delayed perturbation of Mittag-Leffler type matrix function, an explicit solution of system (4.5), the existence and uniqueness of solutions and Ulam-Hyers stability of system (4.5).

In the rest of this chapter, we use the abbreviation the MDP of ML function for the multi-delayed perturbation of Mittag-Leffler type matrix function.

It is a fact that the ML function is known as a generalization of the exponential function. Delayed version which is called delayed Mittag-Leffler type matrix function is presented in the reference [110]. Delayed perturbation version which is named as delayed perturbation of Mittag-Leffler type matrix function is introduced in the reference [86] and multi-delayed perturbation version which is recently given by multi-delayed perturbation of Mittag-Leffler type matrix function is introduced in the reference [87] . In the current study, the MDP of ML function is given through determining matrix equation for $Q_j(s)$ for $j = 0, 1, 2, \ldots$

$$Q_{j+1}(s_1, s_2, \ldots, s_d) = MQ_j(s_1, s_2, \ldots, s_d) + \sum_{i=1}^{d} U_i Q_j(s_1, s_2, \ldots, s_i - r_i, \ldots, s_d)$$
$$+ \sum_{i=1}^{d} E_i Q_{j+1}(s_1, s_2, \ldots, s_i - r_i, \ldots, s_d) \tag{4.7}$$

$$Q_0(s_1, s_2, \ldots, s_d) = Q_j(-r_1, \ldots, s_d) = Q_j(s_1, \ldots, -r_d) = \Theta, \quad Q_1(0, \ldots, 0) = I,$$

$$Q_1(s_1, s_2, \ldots, s_d) = \Theta,\ s_i \neq 0$$

where $s_i = 0, r_i, 2r_i, \ldots$, Θ is the zero matrix, and I is the unit matrix.

If $d = 1$ for the problem with single delay determining equation has the following simple form.

$$Q_{j+1}(s) = MQ_j(s) + UQ_j(s-r) + EQ_{j+1}(s-r)$$

$$Q_0(s) = Q_j(-r) = \Theta, \quad Q_1(0) = I.$$

In order to calculate $Q_j(s)$ we may use the following table.

	$s=0$	$s=r$	$s=2r$	$\cdots$
$Q_1(s)$	I	E	E^2	$\cdots$
$Q_2(s)$	M	$ME+U+EM$	$ME^2+UE+E(ME+U+EM)$	$\cdots$
$Q_3(s)$	M^2	$M(ME+U+EM)+UM+EM^2$	$MQ_2(2r)+UQ_2(r)+EQ_3(r)$	$\cdots$
$Q_4(s)$	M^3	$MQ_3(r)+UQ_3(0)+EQ_4(0)$	$MQ_3(2r)+UQ_3(r)+EQ_4(r)$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

If $M = \Theta$ then the above table becomes simpler:

	$s=0$	$s=r$	$s=2r$	$s=3r$	$\cdots$	$s=pr$
$Q_1(s)$	I	E	E^2	E^3	$\cdots$	E^p
$Q_2(s)$	Θ	U	$UE+EU$	$UE^2+E(UE+EU)$	$\cdots$	
$Q_3(s)$	Θ	Θ	U^2	$U(UE+EU)+EU^2$	$\cdots$	Θ
$Q_4(s)$	Θ	Θ	Θ	U^3	$\cdots$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	Θ
$Q_{p+1}(s)$	Θ	Θ	Θ	Θ	$\cdots$	U^p

It is a high time to define the MDP of ML type matrix function by employing the multivariate function $Q_{k+1}(s_1,s_2,\dots,s_d)$ in the below definition.

Definition 4.1: The multi-delayed perturbation of the Mittag-Leffler type matrix function $\mathscr{P}_{\alpha,\beta}(x)$ is given by

$$\mathscr{P}_{\alpha,\beta}(x)=\begin{cases}\Theta, & x\in[-r,0),\\ I, & x=0,\\ \sum\limits_{m=0}^{\infty}\sum\limits_{n_1,n_2,\dots,n_d=0}^{\infty}Q_{m+1}(n_1r_1,\dots,n_dr_d)\frac{\left(x-\sum_{i=1}^{d}n_ir_i\right)_+^{m\alpha+\beta-1}}{\Gamma(m\alpha+\beta)}, & x\in(0,\infty)\end{cases}\tag{4.8}$$

where Θ is the zero matrix, I is the identity matrix, and $(t)_+=\max(0,t)$.

Remark 4.2: Here are some special cases depending on selections of the coefficient matrices.

1. The MDP of ML type matrix function coincides with the Mittag-Leffler matrix function [111], [112] provided that $E_i=U_i=\Theta$, $i=1,2,\dots,d$ i.e. $\mathscr{P}_{\alpha,\beta}(x)=x^{\beta-1}E_{\alpha,\beta}(Mx^{\alpha})$.
2. If $E_i=\Theta$, $i=1,2,\dots,d$ and $U_i=\Theta$, $i=2,\dots,d$, then the MDP of ML type

matrix function matches up with delay perturbation of two-parameter Mittag-Leffler matrix function [86].

3. $\mathscr{P}_{\alpha,\beta}(x)$ reduces to delayed Mittag-Leffler type matrix function [110] providing that $E_i = \Theta$, $i = 1,2,\ldots,d$ and $U_i = \Theta$, $i = 2,\ldots,d$, $M = \Theta$.

4. Since the determining matrix function $Q_k(s)$ for $k = 0,1,2,\ldots$ in (4.7) is accurately different from that one of the reference [87], the definition of the MDP of ML function is not equal to that one of [87, Def. 3.3]. But they overlap under the condition $E_i = \Theta$, $i = 1,2,\ldots,d$. As it is remarked in [87] multivariate determining matrix equation (4.7) is a delayed analogue of the multinomial formula (theorem) for non-commutative matrices.

5. If $d = 1$, $M = \Theta$, $\alpha = \beta = 1$ and the coefficient matrices are permutable, in addition to appropriate selections, $\mathscr{P}_{\alpha,\beta}(t)$ reduces to $X(t)$ in (2.4) in the paper [101]:

$$X(t) = \begin{cases} \Theta, & t < 0, \\ \sum\limits_{j=0}^{k}\sum\limits_{i=0}^{k-j} U^i E^j \begin{pmatrix} i+j \\ i \end{pmatrix} \frac{[t-(i+j)r]^j}{j!}, & kr \leq t < (k+1)r,\ k = 0,1,\ldots \end{cases}$$

6. $\varepsilon^{r_1,r_2}_{\alpha,\beta}(\mathscr{T},\mathscr{U},\mathscr{I};x)$ in Definition 3.1 in the work [104] can be obtained from $\mathscr{P}_{\alpha,\beta}(x)$ depending on appropriate selections r_i, E_i, U_i, T, $i = 1,2,\ldots,d$.

4.2 The Explicit Solutions of Neutral Fractional Multi-Delayed System

Prior to investigating an explicit solution of system (4.5), we present some useful lemma and theorem to be used in the coming proofs.

Theorem 4.1: Let $\mathscr{P}_{\alpha,\beta}(x)$ be as defined in (4.8).

$$^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\left[\mathscr{P}_{\alpha,1}(x) - \sum_{i=1}^{d} E_i \mathscr{P}_{\alpha,1}(x-r_i)\right] = M\mathscr{P}_{\alpha,1}(x) + \sum_{i=1}^{d} U_i \mathscr{P}_{\alpha,1}(x-r_i)$$

hold true.

Proof. To begin with the first item, we need to separately calculate

$$^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\mathscr{P}_{\alpha,1}(x) = ? \quad \text{and} \quad ^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\mathscr{P}_{\alpha,1}(x-r_i) = ?.$$

The latter is easily obtained from the former, so it is enough to compute the first one.

$$\begin{aligned}
&^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\mathscr{P}_{\alpha,1}(x)\\
&= \sum_{m=1}^{\infty}\sum_{n_1,n_2,\dots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\dots,n_dr_d)\,^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\left(\frac{\left(x-\sum_{i=1}^{d} n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)}\right)\\
&= \sum_{m=1}^{\infty}\sum_{n_1,n_2,\dots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\dots,n_dr_d)\left(\frac{\Gamma(m\alpha+1)}{\Gamma(m\alpha-\alpha+1)}\frac{\left(x-\sum_{i=1}^{d} n_ir_i\right)_+^{m\alpha-\alpha}}{\Gamma(m\alpha+1)}\right)\\
&= \sum_{m=0}^{\infty}\sum_{n_1,n_2,\dots,n_d=0}^{\infty} Q_{m+2}(n_1r_1,\dots,n_dr_d)\frac{\left(x-\sum_{i=1}^{d} n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)}. \qquad (4.9)
\end{aligned}$$

One can easily obtain

$$^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\mathscr{P}_{\alpha,1}(x-r_k) := \sum_{m=0}^{\infty}\sum_{n_1,n_2,\dots,n_d=0}^{\infty} Q_{m+2}(n_1r_1,\dots,n_dr_d)\frac{\left(x-r_k-\sum_{i=1}^{d} n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)}.$$

If we use (4.7) in (4.9), we carry on

$$\begin{aligned}
&^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\mathscr{P}_{\alpha,1}(x)\\
&= M\sum_{m=0}^{\infty}\sum_{n_1,n_2,\dots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\dots,n_dr_d)\frac{\left(x-\sum_{i=1}^{d} n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)}\\
&+\sum_{k=1}^{d}U_k\sum_{m=0}^{\infty}\sum_{n_1,n_2,\dots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\dots,n_kr_k-r_k,\dots,n_dr_d)\frac{\left(x-\sum_{i=1}^{d} n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)}\\
&+\sum_{k=1}^{d}E_k\sum_{m=0}^{\infty}\sum_{n_1,n_2,\dots,n_d=0}^{\infty} Q_{m+2}(n_1r_1,\dots,n_kr_k-r_k,\dots,n_dr_d)\frac{\left(x-\sum_{i=1}^{d} n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)}\\
&= M\sum_{m=0}^{\infty}\sum_{n_1,n_2,\dots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\dots,n_dr_d)\frac{\left(x-\sum_{i=1}^{d} n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)}\\
&+\sum_{k=1}^{d}U_k\sum_{m=0}^{\infty}\sum_{n_1,n_2,\dots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\dots,n_dr_d)\frac{\left(x-r_k-\sum_{i=1}^{d} n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)}\\
&+\sum_{k=1}^{d}E_k\sum_{m=0}^{\infty}\sum_{n_1,n_2,\dots,n_d=0}^{\infty} Q_{m+2}(n_1r_1,\dots,n_dr_d)\frac{\left(x-r_k-\sum_{i=1}^{d} n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)}\\
&= M\mathscr{P}_{\alpha,1}(x)+\sum_{k=1}^{d}U_k\mathscr{P}_{\alpha,1}(x-r_k)+\sum_{k=1}^{d}E_k\,^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\mathscr{P}_{\alpha,1}(x-r_k),
\end{aligned}$$

which is the desired result concluding the proof of this lemma. $\square$

Lemma 4.1: Let $\mathscr{P}_{\alpha,\beta}(x)$ be as in (4.8). The following mathematical equation is true:

$$\int_0^x (x-s)^{-\alpha} \int_0^s \mathscr{P}_{\alpha,\alpha}(s-t) \urcorner(t)\, dtds$$
$$= \sum_{m=0}^{\infty} \sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d) \int_0^x \frac{\Gamma(1-\alpha)\left(x-t-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)} \urcorner(t)\, dt.$$

Proof. With a simple substitution as $v = \frac{s-\sum_{i=1}^d n_ir_i - t}{x-\sum_{i=1}^d n_ir_i - t}$, we get

$$\int_0^x (x-s)^{-\alpha} \int_0^s \mathscr{P}_{\alpha,\alpha}(s-t) \urcorner(t)\, dtds$$
$$= \sum_{m=0}^{\infty} \sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d)$$
$$\times \int_0^x \int_t^x (x-s)^{-\alpha} \frac{\left(s-t-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)} ds \urcorner(t)\, dt$$
$$= \sum_{m=0}^{\infty} \sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d)$$
$$\times \int_0^x \int_{t+\sum_{i=1}^d n_ir_i}^x (x-s)^{-\alpha} \frac{\left(s-t-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)} ds \urcorner(t)\, dt$$
$$= \sum_{m=0}^{\infty} \sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d) \int_0^x \frac{\Gamma(1-\alpha)\left(x-t-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)} \urcorner(t)\, dt.$$

$\square$

Now, the coming theorem is one of main theorems as to the desired solutions. It gives a part of the solution under the zero initial condition .

Theorem 4.2: The following function

$$z(x) = \int_0^x \mathscr{P}_{\alpha,\alpha}(x-t) \urcorner(t)\, dt, \quad x \geq 0,$$

is a solution of inhomogeneous system (4.5) under the condition $z(x) = 0$ with $-r \leq x \leq 0$.

Proof. To see this, we consider the following expression by keeping Lemma 4.1 in mind

$$
\begin{aligned}
&{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\left(\int_0^x \mathscr{P}_{\alpha,\alpha}(x-t)\,\daleth(t)\,dt\right)\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-s)^{-\alpha}\int_0^s \mathscr{P}_{\alpha,\alpha}(s-t)\,\daleth(t)\,dtds\\
&=\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d)\frac{d}{dx}\int_0^x \frac{\left(x-t-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)}\daleth(t)\,dt\\
&=\sum_{m=1}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d)\frac{d}{dx}\int_0^x \frac{\left(x-t-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)}\daleth(t)\,dt\\
&+\daleth(x)\\
&=\sum_{m=1}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d)\frac{d}{dx}\int_0^{x-\sum_{i=1}^d n_ir_i} \frac{\left(x-t-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)}\daleth(t)\,dt\\
&+\daleth(x)\\
&=\sum_{m=1}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d)\frac{d}{dx}\int_0^{x-\sum_{i=1}^d n_ir_i} \frac{\left(x-t-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha}}{\Gamma(m\alpha+1)}\daleth(t)\,dt\\
&+\daleth(x)\\
&=\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+2}(n_1r_1,\ldots,n_dr_d)\int_0^x \frac{\left(x-t-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)}\daleth(t)\,dt+\daleth(x).
\end{aligned}
\tag{4.10}
$$

One can easily obtain

$$
\begin{aligned}
&{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\left(\int_0^x \mathscr{P}_{\alpha,\alpha}(x-t-r_k)\,\daleth(t)\,dt\right)\\
&=\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+2}(n_1r_1,\ldots,n_dr_d)\int_0^x \frac{\left(x-t-r_k-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)}\daleth(t)\,dt
\end{aligned}
$$

We put (4.7) and Lemma 4.1 into (4.10) to obtain the following

$$
\begin{aligned}
&{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\left(\int_0^x \mathscr{P}_{\alpha,\alpha}(x-t)\urcorner(t)\,dt\right)\\
&= M\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d)\int_0^x \frac{\left(x-t-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)}\urcorner(t)\,dt+\urcorner(x)\\
&+\sum_{k=1}^d U_k\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_kr_k-r_k,\ldots,n_dr_d)\\
&\times\int_0^x \frac{\left(x-t-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)}\urcorner(t)\,dt\\
&+\sum_{k=1}^d E_k\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+2}(n_1r_1,\ldots,n_kr_k-r_k,\ldots,n_dr_d)\\
&\times\int_0^x \frac{\left(x-t-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)}\urcorner(t)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= M\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d)\int_0^x \frac{\left(x-t-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)}\urcorner(t)\,dt+\urcorner(x)\\
&+\sum_{k=1}^d U_k\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d)\int_0^x \frac{\left(x-t-r_k-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)}\urcorner(t)\,dt\\
&+\sum_{k=1}^d E_k\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+2}(n_1r_1,\ldots,n_dr_d)\int_0^x \frac{\left(x-t-r_k-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)}\urcorner(t)\,dt\\
&= M\int_0^x \mathscr{P}_{\alpha,\alpha}(x-t)\urcorner(t)\,dt+\sum_{k=1}^d U_k\int_0^x \mathscr{P}_{\alpha,\alpha}(x-t-r_k)\urcorner(t)\,dt\\
&+\sum_{k=1}^d E_k{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\left(\int_0^x \mathscr{P}_{\alpha,\alpha}(x-t-r_k)\urcorner(t)\,dt\right)+\urcorner(x)
\end{aligned}
$$

which gives the inevitable result. □

The next theorem is the last one of main theorems as to a solution of the homogeneous part of system (4.5).

Theorem 4.3: The following $\mathbb{R}^n$-valued continuous function

$$
\begin{aligned}
z(x) &= \left[\mathscr{P}_{\alpha,1}(x)-\sum_{m=1}^d \mathscr{P}_{\alpha,1}(x-r_m)E_m\right]\phi(0)\\
&+\sum_{j=1}^d\int_{-r_j}^0 \mathscr{P}_{\alpha,\alpha}\left(x-r_j-t\right)\left[U_j\phi(t)+E_j{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi(t)\right]dt
\end{aligned}
$$

is a solution of homogeneous system (4.5) with $z(x)=\phi(x)$, $-r\le x\le 0$ and $\urcorner=0$.

Proof. Now we consider

$$
\begin{aligned}
&{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\left(\sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}\left(x-r_j-t\right)U_j\phi\left(t\right)dt\right)\\
&=\sum_{j=1}^{d}\int_{-r_j}^{0}\sum_{m=1}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty}Q_{m+1}(n_1r_1,\ldots,n_dr_d)^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\\
&\times\left(\frac{\left(x-r_j-t-\sum_{i=1}^{d}n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)}\right)U_j\phi\left(t\right)dt\\
&=\sum_{j=1}^{d}\int_{-r_j}^{0}\sum_{m=1}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty}Q_{m+1}(n_1r_1,\ldots,n_dr_d)\\
&\times\frac{\left(x-r_j-t-\sum_{i=1}^{d}n_ir_i\right)_+^{m\alpha-1}}{\Gamma(m\alpha)}U_j\phi\left(t\right)dt\\
&=\sum_{j=1}^{d}\int_{-r_j}^{0}\sum_{m=1}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty}Q_{m+1}(n_1r_1,\ldots,n_dr_d)\\
&\times\frac{\left(x-r_j-t-\sum_{i=1}^{d}n_ir_i\right)_+^{m\alpha-1}}{\Gamma(m\alpha)}U_j\phi\left(t\right)dt\\
&=\sum_{j=1}^{d}\int_{-r_j}^{0}\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty}Q_{m+2}(n_1r_1,\ldots,n_dr_d)\\
&\times\frac{\left(x-r_j-t-\sum_{i=1}^{d}n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)}U_j\phi\left(t\right)dt
\end{aligned}
\tag{4.11}
$$

By applying (4.7) to (4.11), we get

$$
\begin{aligned}
&{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\left(\sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}\left(x-r_j-t\right)U_j\phi\left(t\right)dt\right)\\
&=M\sum_{j=1}^{d}\int_{-r_j}^{0}\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty}Q_{m+1}(n_1r_1,\ldots,n_dr_d)\\
&\times\frac{\left(x-r_j-t-\sum_{i=1}^{d}n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)}U_j\phi\left(t\right)dt\\
&+\sum_{k=1}^{d}U_k\sum_{j=1}^{d}\int_{-r_j}^{0}\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty}Q_{m+1}(n_1r_1,\ldots,n_kr_k-r_k,\ldots,n_dr_d)\\
&\times\frac{\left(x-r_j-t-\sum_{i=1}^{d}n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)}U_j\phi\left(t\right)dt\\
&+\sum_{k=1}^{d}E_k\Bigg(\sum_{j=1}^{d}\int_{-r_j}^{0}\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty}Q_{m+2}(n_1r_1,\ldots,n_kr_k-r_k,\ldots,n_dr_d)\\
&\times\frac{\left(x-r_j-t-\sum_{i=1}^{d}n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)}U_j\phi\left(t\right)dt\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
&= M\sum_{j=1}^{d}\int_{-r_j}^{0}\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d)\\
&\times \frac{\left(x-r_j-t-\sum_{i=1}^{d} n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)} U_j\phi\left(t\right)dt\\
&+\sum_{k=1}^{d} U_k\sum_{j=1}^{d}\int_{-r_j}^{0}\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d)\\
&\times \frac{\left(x-r_j-r_k-t-\sum_{i=1}^{d} n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)} U_j\phi\left(t\right)dt\\
&+\sum_{k=1}^{d} E_k\Bigg(\sum_{j=1}^{d}\int_{-r_j}^{0}\sum_{m=0}^{\infty}\sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+2}(n_1r_1,\ldots,n_dr_d)\\
&\times \frac{\left(x-r_j-r_k-t-\sum_{i=1}^{d} n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)} U_j\phi\left(t\right)dt\Bigg)\\
&= M\left(\sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}\left(x-r_j-t\right)U_j\phi\left(t\right)dt\right)\\
&+\sum_{k=1}^{d} U_k\left(\sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}\left(x-r_j-r_k-t\right)U_j\phi\left(t\right)dt\right)\\
&+\sum_{k=1}^{d} E_k\,{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\left(\sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}\left(x-r_j-r_k-t\right)U_j\phi\left(t\right)dt\right).
\end{aligned}
$$

In a similar way, one can easily obtain

$$
\begin{aligned}
&{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\left(\sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}\left(x-r_j-t\right)E_j\,{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi\left(t\right)dt\right)\\
&= M\left(\sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}\left(x-r_j-t\right)E_j\,{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi\left(t\right)dt\right)\\
&+\sum_{k=1}^{d} U_k\left(\sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}\left(x-r_j-r_k-t\right)E_j\,{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi\left(t\right)dt\right)\\
&+\sum_{k=1}^{d} E_k\,{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\left(\sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}\left(x-r_j-r_k-t\right)E_j\,{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi\left(t\right)dt\right).
\end{aligned}
$$

Now we use variation of constant technique to determine the coefficient of the first square bracket term in the statement theorem. With the help of Theorem 4.1 and the obtained results of just above proof, the solution which satisfies the initial condition $z\left(x\right)=\phi\left(x\right)$, $-r\leq x\leq 0$, has in the following pattern formula,

$$z(x)=\left[\mathscr{P}_{\alpha,1}(x)-\sum_{m=1}^{d}\mathscr{P}_{\alpha,1}(x-r_m)E_m\right]\kappa$$
$$+\sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}\left(x-r_j-t\right)\left[U_j\phi(t)+E_j{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi(t)\right]dt:=\phi(x)$$

where κ is an unknown constants. For $x=0$, we get $\mathscr{P}_{\alpha,1}(0)=I$, $\mathscr{P}_{\alpha,1}(-r_m)=\Theta$, and $\mathscr{P}_{\alpha,\alpha}\left(-r_j-t\right)=\Theta$ from the definition of $\mathscr{P}_{\alpha,\beta}$ in (4.8). So, $\kappa=\phi(0)$. This completes the proof. □

So far we have found the parts of the step-by-step solution, now let's put the parts together in the below corollary which stands for a whole solution of system (4.5).

Corollary 4.1: The following $\mathbb{R}^n$-valued continuous function

$$z(x)=\left[\mathscr{P}_{\alpha,1}(x)-\sum_{m=1}^{d}\mathscr{P}_{\alpha,1}(x-r_m)E_m\right]\phi(0)$$
$$+\sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}\left(x-r_j-t\right)\left[U_j\phi(t)+E_j{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi(t)\right]dt$$
$$+\int_{0}^{x}\mathscr{P}_{\alpha,\alpha}(x-t)\urcorner(t)\,dt$$

is a solution of system (4.5).

Now we are ready to share an equivalent definition of solution of system (4.6). The following corollary expresses it.

Corollary 4.2: The solution of the following integral equation

$$z(x)=\left[\mathscr{P}_{\alpha,1}(x)-\sum_{m=1}^{d}\mathscr{P}_{\alpha,1}(x-r_m)E_m\right]\phi(0)$$
$$+\sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}\left(x-r_j-t\right)\left[U_j\phi(t)+E_j{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi(t)\right]dt$$
$$+\int_{0}^{x}\mathscr{P}_{\alpha,\alpha}(x-t)\urcorner(t,z(t))\,dt$$

is a solution of nonlinear system (4.6) and vice versa.

Remark 4.3: Here are some special cases depending on selections of the coefficient matrices.

1. If $E_i = \Theta$, $i = 1,2,\dots,d$ and $U_i = \Theta$, $i = 2,\dots,d$, then Corollary 4.1 matches up with Corollary 1 in the reference [80].
2. Corollary 4.1 with $\urcorner = 0$ reduces to Theorem 3.2 in the work [79] providing that $E_i = \Theta$, $i = 1,2,\dots,d$ and $U_i = \Theta$, $i = 2,\dots,d$, $M = \Theta$.
3. Corollary 4.1 overlaps with Theorem 4.2 in the study [9] under the condition $E_i = \Theta$, $i = 1,2,\dots,d$.
4. Even if the constant coefficient matrices are commutative, our findings also are valid. If the coefficient matrices are permutable in addition to appropriate selections, Corollary 4.1 reduces to Theorem 6 in the paper [7].
5. Corollary 4.1 reduces to Theorem 3.5 in the paper [104] on taking $d = 2$ and without loss of generality $E_1 = U_2 = \Theta$.

4.3 Existence and Uniqueness, and Ulam-Hyers Stability

In this subsection, we look for answers to three kinds of questions : is there a solution for system (4.6)?, is the solution unique? Subsequent to given answers, we finis discussing. When we look at features of each term in system (4.6) like $\urcorner(x,z(x))$ is continuous, we find an explicit solution in corollary 4.1. Unfortunately, these features or conditions are not enough to make the explicit solution unique. So, we add one more feature to the continuous function $\urcorner(x,z(x))$ in order to make the explicit solution satisfy the uniqueness. This feature is : the continuous function $\urcorner(x,z(x))$ satisfies the Lipschitz condition in the second component with the Lipschitz constant $L_\urcorner$, that is

$$\|\urcorner(x,z) - \urcorner(x,y)\| \leq L_\urcorner \|z-y\|, \quad z,y \in \mathbb{R}^n.$$

Prior to carrying on, we discuss an equality about $\mathscr{P}_{\alpha,\alpha}$ in the following lemma.

Lemma 4.2: Let $\mathscr{P}_{\alpha,\alpha}(x)$ be as in (4.8).

$$\int_0^x \|\mathscr{P}_{\alpha,\alpha}(x-s)\|\,ds = \left\|\mathscr{P}_{\alpha,\alpha+1}(x)\right\|$$

holds true.

Proof. It is easy to see that

$$\begin{aligned}
&\int_0^x \|\mathscr{P}_{\alpha,\alpha}(x-s)\|\,ds \\
&= \int_0^x \left\| \sum_{m=0}^{\infty} \sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d) \frac{\left(x-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)} \right\| ds \\
&= \left\| \sum_{m=0}^{\infty} \sum_{n_1,n_2,\ldots,n_d=0}^{\infty} Q_{m+1}(n_1r_1,\ldots,n_dr_d) \right\| \int_0^x \frac{\left(x-\sum_{i=1}^d n_ir_i\right)_+^{m\alpha+\alpha-1}}{\Gamma(m\alpha+\alpha)} ds \\
&= \left\|\mathscr{P}_{\alpha,\alpha+1}(x)\right\|.
\end{aligned}$$

□

Here is the following existence and uniqueness' theorem.

Theorem 4.4: If the jointly continuous function $\daleth(x,z)$ satisfies the Lipschitz condition in the second component with the Lipschitz constant $L_\daleth$ with $L_\daleth \left\|\mathscr{P}_{\alpha,\alpha+1}(T)\right\| < 1$, then the integral equation in the corollary 4.1 is of a unique solution in $[-r,T]$.

Proof. Define $\mathscr{G} : C([-r,T],\mathbb{R}^n) \to C([-r,T],\mathbb{R}^n)$ by

$$\begin{aligned}
\mathscr{G}z(x) = &\left[\mathscr{P}_{\alpha,1}(x) - \sum_{m=1}^{d} \mathscr{P}_{\alpha,1}(x-r_m)E_m\right]\phi(0) \\
&+ \sum_{j=1}^{d} \int_{-r_j}^{0} \mathscr{P}_{\alpha,\alpha}\left(x-r_j-t\right)\left[U_j\phi(t) + E_j{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi(t)\right] dt \\
&+ \int_0^x \mathscr{P}_{\alpha,\alpha}(x-t)\daleth(t,z(t))\,dt
\end{aligned}$$

For arbitrary $z,y \in C([-r,T],\mathbb{R}^n)$, we consider by using Lemma 4.2

$$\|\mathscr{G}z(x)-\mathscr{G}y(x)\| \leq \int_0^x \|\mathscr{P}_{\alpha,\alpha}(x-s)\| \|\daleth(s,z(s))-\daleth(s,y(s))\| ds$$
$$= L_{\daleth} \left\|\mathscr{P}_{\alpha,\alpha+1}(T)\right\| \|z-y\|_C.$$

The statements of this theorem ensure that $\mathscr{G}$ is a contraction. By the Banach Contraction principle, $\mathscr{G}$ is of a unique fixed point on $[-r,T]$, that is $\exists! z_0 \in C([-r,T],\mathbb{R}^n)$, $z_0(x) = \mathscr{G}z_0(x)$. □

As the last theoretical result, we investigate the stability of system (4.6).

Definition 4.2: Let $\varepsilon > 0$. The system (4.6) is said to be Ulam-Hyers stable if for every solution $z \in C([0,T],\mathbb{R}^n)$ of inequality,

$$\left\| {}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\left[z(x)-\sum_{i=1}^{d} E_i z(x-r_i)\right] - Mz(x) - \sum_{i=1}^{d} U_i z(x-r_i) - \daleth(x,z(x)) \right\| \leq \varepsilon, \quad (4.12)$$

there exists a solution $z_0 \in C([0,T],\mathbb{R}^n)$ of system (4.6), and $u_h > 0$ such that

$$\|z(x)-z_0(x)\| \leq u_h.\varepsilon, \;\; x \in [0,T].$$

Remark 4.4: A function $z \in C^1([0,T],\mathbb{R}^n)$ is a solution of the inequality equation (4.12) if and only if there exists a function $u \in C([0,T],\mathbb{R}^n)$, such that

i. $\|u(x)\| < \varepsilon$,

ii. ${}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\left[z(x)-\sum_{i=1}^{d} E_i z(x-r_i)\right] = Mz(x) + \sum_{i=1}^{d} U_i z(x-r_i) + \daleth(x,z(x)) + u(x)$.

Theorem 4.5: Suppose that all of statements of Theorem 4.4 are hold. Then system (4.6) is Ulam-Hyers stable.

Proof. Let $z \in C([0,T],\mathbb{R}^n)$ which satisfies the inequality (4.12), and let $z_0 \in C([0,T],\mathbb{R}^n)$ which is the unique solution of system (4.6) with the initial condition $z_0(x) = z(x)$ for each $t \in [-r,0]$. By keeping the definition of $\mathscr{G}$ and Remark 4.4 in mind, we can acquire

$$\|u(x)\| < \varepsilon, \quad z(x) = \mathscr{G}z(x) + \int_0^x \mathscr{P}_{\alpha,\alpha}(x-t)\,u(t)\,dt,$$

and also $z_0(t) = (\mathscr{G}z_0)(t)$ for each $t \in [0,T]$. One can easily make the following estimation

$$\|\mathscr{G}z(t) - z(t)\| \leq \int_0^t \|\mathscr{P}_{\alpha,\alpha}(t-s)\| \, \|u(s)\| \, ds \leq t \left\|\mathscr{P}_{\alpha,\alpha+1}(T)\right\| \varepsilon.$$

We are all set to estimate $\|z_0(t) - z(t)\|$:

$$\begin{aligned} \|z_0(t) - z(t)\| &\leq \|\mathscr{G}z_0(t) - \mathscr{G}z(t)\| + \|\mathscr{G}z(t) - z(t)\| \\ &\leq L_\neg \left\|\mathscr{P}_{\alpha,\alpha+1}(T)\right\| \|z_0 - z\|_C + t \left\|\mathscr{P}_{\alpha,\alpha+1}(T)\right\| \varepsilon, \end{aligned}$$

which provides

$$\left(1 - L_\neg \left\|\mathscr{P}_{\alpha,\alpha+1}(T)\right\|\right) \|z - y\|_C \leq t \left\|\mathscr{P}_{\alpha,\alpha+1}(T)\right\| \varepsilon,$$

from this just above inequality, we obtain the desired result

$$\|z - y\|_C \leq u_h \varepsilon, \qquad u_h = \frac{T \left\|\mathscr{P}_{\alpha,\alpha+1}(T)\right\|}{\left(1 - L_\neg \left\|\mathscr{P}_{\alpha,\alpha+1}(T)\right\|\right)} > 0.$$

□

Remark 4.5: The results of existence and uniqueness and stability match up with these ones of the study [104].

4.4 Relative Controllability of the Neutral Fractional Multi-Delayed System.

In this subsection, we deal with the neutral Caputo type fractional multi-delayed differential system while it is not only linear but also semilinear.

It is clear that $\mathscr{P}_{\alpha,\beta}$ depends on the coefficient matrices E_i, M, U_i for $i = 1, 2, \ldots, d$. In order to make them visible, we use $\mathscr{P}_{\alpha,\beta}^{\mathfrak{E},M,\mathfrak{U}}$ instead of $\mathscr{P}_{\alpha,\beta}$ where $\mathfrak{E} = \sum_{i=1}^{d} E_i$ and $\mathfrak{U} = \sum_{i=1}^{d} U_i$.

Definition 4.3: [109, Definition 4] System (4.5) is said to be relatively controllable, if for the final state $z_\tau \in \mathbb{R}^n$, and time τ, any initial function $\phi \in C^1([-r,0],\mathbb{R}^n)$, there is a control $u \in L^\infty(J,\mathbb{R}^n)$ such that system (4.5) is of a solution $z \in C^1([-r,\tau],\mathbb{R}^n)$ that holds the initial ϕ and $z(\tau) = z_\tau$.

Lemma 4.3: Let $\mathscr{P}_{\alpha,\beta}^{\mathfrak{E},M,\mathfrak{U}}$ be defined as in (4.8)

- $\left(\mathscr{P}_{\alpha,\beta}^{\mathfrak{E},M,\mathfrak{U}}\right)^T(x) = \mathscr{P}_{\alpha,\beta}^{\mathfrak{E}^T,M^T,\mathfrak{U}^T}(x)$,
- $\int_0^x \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(x-s)\,ds = \mathscr{P}_{\alpha,\alpha+1}^{\mathfrak{E},M,\mathfrak{U}}(x)$,
- $\left\|\mathscr{P}_{\alpha,\beta}^{\mathfrak{E},M,\mathfrak{U}}(x)\right\| \leq \mathscr{P}_{\alpha,\beta}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(x)$, for $0<\alpha<1$, $0<\beta\leq 1$, and $\alpha+\beta\geq 1$.

Proof. Their proofs are similar to Lemma 4.2 by employing fundamental definition $\mathscr{P}_{\alpha,\beta}^{\mathfrak{E},M,\mathfrak{U}}$ and Lemma 2.1. □

We firstly deal with the case $\urcorner(x,z(x)) = 0 \in \mathbb{R}^n$, $x \in J = [0,\tau]$, i.e. the following linear neutral fractional multi-delayed control system

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\left[z(x) - \sum_{i=1}^d E_i z(x-r_i)\right] = Mz(x) + \sum_{i=1}^d U_i z(x-r_i) + Su(x), \quad x \in J, \\ z(x) = \phi(x), \quad -r \leq x \leq 0, \end{cases} \tag{4.13}$$

whose solution is

$$\begin{aligned} z(x) = &\left[\mathscr{P}_{\alpha,1}^{\mathfrak{E},M,\mathfrak{U}}(x) - \sum_{m=1}^{d}\mathscr{P}_{\alpha,1}^{\mathfrak{E},M,\mathfrak{U}}(x-r_m)E_m\right]\phi(0) \\ &+ \sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}\left(x-r_j-t\right)\left[U_j\phi(t) + E_j\,{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi(t)\right]dt \\ &+ \int_0^x \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(x-t)Su(t)\,dt. \end{aligned}$$

Now, there is no barrier to present a representation of the neutral fractional multi-delayed Gramian matrix as noted below:

$$W_{r,\alpha}[0,\tau] = \int_0^\tau \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(\tau-s)SS^T\mathscr{P}_{\alpha,\alpha}^{\mathfrak{E}^T,M^T,\mathfrak{U}^T}(\tau-s)\,ds \tag{4.14}$$

where $\mathfrak{E}^T = \sum_{i=1}^d E_i^T$ and $\mathfrak{U}^T = \sum_{i=1}^d U_i^T$.

Theorem 4.6: System (4.13) is relatively controllable if and only if $W_{r,\alpha}[0,\tau]$ is nonsingular.

Proof. *Necessity:* Assume that $W_{r,\alpha}[0,\tau]$ is singular, i.e., there exists at least nonzero $h \in \mathbb{R}^n$ such that

$$W_{r,\alpha}[0,\tau]h = 0.$$

One obtains

$$\begin{aligned}
0 = h^{\intercal} W_{r,\alpha}[0,\tau]h &= h^{\intercal} \int_0^{\tau} \mathscr{P}_{\alpha,\alpha}^{\mathfrak{C},M,\mathfrak{U}}(\tau - s)\, SS^T\, \mathscr{P}_{\alpha,\alpha}^{\mathfrak{C}^T,M^T,\mathfrak{U}^T}(\tau - s)\, dsh \\
&= \int_0^{\tau} \left\| S^T\, \mathscr{P}_{\alpha,\alpha}^{\mathfrak{C}^T,M^T,\mathfrak{U}^T}(\tau - s)\, h \right\|^2 ds,
\end{aligned}$$

which implies that

$$S^T\, \mathscr{P}_{\alpha,\alpha}^{\mathfrak{C}^T,M^T,\mathfrak{U}^T}(\tau - s)\, h = 0, \quad 0 \leq s \leq \tau,$$

or

$$h^T\, \mathscr{P}_{\alpha,\alpha}^{\mathfrak{C},M,\mathfrak{U}}(\tau - s)\, S = 0, \quad 0 \leq s \leq \tau.$$

Since system (4.13) is relatively exact controllable, according to definition, there exists a control u_1 that drives the initial state to zero at time T, i.e.,

$$\begin{aligned}
z(\tau) = &\left[\mathscr{P}_{\alpha,1}^{\mathfrak{C},M,\mathfrak{U}}(\tau) - \sum_{m=1}^{d} \mathscr{P}_{\alpha,1}^{\mathfrak{C},M,\mathfrak{U}}(\tau - r_m)\, E_m \right] \phi(0) \\
&+ \sum_{j=1}^{d} \int_{-r_j}^{0} \mathscr{P}_{\alpha,\alpha}^{\mathfrak{C},M,\mathfrak{U}}\left(\tau - r_j - t\right) \left[U_j \phi(t) + E_j\, {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} \phi(t) \right] dt \\
&+ \int_0^{\tau} \mathscr{P}_{\alpha,\alpha}^{\mathfrak{C},M,\mathfrak{U}}(\tau - t)\, Su_1(t)\, dt = 0.
\end{aligned}$$

Similarly, there also exists a control u_2 that drives the initial state to nonzero h at time τ

$$
\begin{aligned}
z(\tau) = & \left[\mathscr{P}_{\alpha,1}^{\mathfrak{E},M,\mathfrak{U}}(\tau) - \sum_{m=1}^{d} \mathscr{P}_{\alpha,1}^{\mathfrak{E},M,\mathfrak{U}}(\tau - r_m) E_m \right] \phi(0) \\
& + \sum_{j=1}^{d} \int_{-r_j}^{0} \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}\left(\tau - r_j - t\right) \left[U_j \phi(t) + E_j \, {}^{\mathfrak{C}}\mathfrak{D}_{0+}^{\alpha} \phi(t) \right] dt \\
& + \int_{0}^{\tau} \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(\tau - t) S u_2(t) \, dt = h.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
h &= \int_{0}^{\tau} \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(\tau - t) S [u_2(t) - u_1(t)] \, dt, \\
h^{\mathsf{T}} h &= \int_{0}^{\tau} h^{\mathsf{T}} \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(\tau - t) S [u_2(t) - u_1(t)] \, dt = 0.
\end{aligned}
$$

Thus $h = 0$, which contradicts with h being nonzero.

Sufficiency: Since $W_{r,\alpha}[0,\tau]$ is non-singular, its inverse $(W_{r,\alpha}[0,\tau])^{-1}$ is well-defined. For any final state h, the following control functions can be selected:

$$
u(s) := S^T \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E}^T,M^T,\mathfrak{U}^T}(\tau - s) (W_{r,\alpha}[0,\tau])^{-1} \eta,
$$

where

$$
\begin{aligned}
\eta = h - & \left[\mathscr{P}_{\alpha,1}^{\mathfrak{E},M,\mathfrak{U}}(\tau) - \sum_{m=1}^{d} \mathscr{P}_{\alpha,1}^{\mathfrak{E},M,\mathfrak{U}}(\tau - r_m) E_m \right] \phi(0) \\
& - \sum_{j=1}^{d} \int_{-r_j}^{0} \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}\left(\tau - r_j - t\right) \left[U_j \phi(t) + E_j \, {}^{\mathfrak{C}}\mathfrak{D}_{0+}^{\alpha} \phi(t) \right] dt.
\end{aligned}
$$

Then

$$
\begin{aligned}
z(\tau) = & \left[\mathscr{P}_{\alpha,1}^{\mathfrak{E},M,\mathfrak{U}}(\tau) - \sum_{m=1}^{d} \mathscr{P}_{\alpha,1}^{\mathfrak{E},M,\mathfrak{U}}(\tau - r_m) E_m \right] \phi(0) \\
& + \sum_{j=1}^{d} \int_{-r_j}^{0} \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}\left(\tau - r_j - t\right) \left[U_j \phi(t) + E_j \, {}^{\mathfrak{C}}\mathfrak{D}_{0+}^{\alpha} \phi(t) \right] dt \\
& + \int_{0}^{\tau} \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(\tau - t) S S^T \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E}^T,M^T,\mathfrak{U}^T}(\tau - t) (W_{r,\alpha}[0,\tau])^{-1} \eta \, dt = h.
\end{aligned}
$$

□

Remark 4.6: It should be stressed out that the rank condition for the relative controllability for single delayed neutral non-fractional linear systems was considered in [102]. Rank condition for the relative controllability for single delayed neutral Caputo fractional linear systems was considered in [113], [114]. Rank condition for the relative controllability of fractional multi-delayed neutral linear system will be considered in forthcoming papers.

Secondly we consider the case $\daleth(x,z(x)) \neq 0 \in \mathbb{R}^n$, $x \in J = [0,\tau]$, i.e. the following semilinear neutral fractional multi-delayed control system

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0+}\left[z(x) - \sum_{i=1}^{d} E_i z(x-r_i)\right] = Mz(x) + \sum_{i=1}^{d} U_i z(x-r_i) + Su(x) + \daleth(x,z(x)), x \in J, \\ z(x) = \phi(x), \quad -r \leq x \leq 0, \end{cases} \tag{4.15}$$

with the solution of a form

$$\begin{aligned} z(x) = &\left[\mathscr{P}^{\mathfrak{E},M,\mathfrak{U}}_{\alpha,1}(x) - \sum_{m=1}^{d} \mathscr{P}^{\mathfrak{E},M,\mathfrak{U}}_{\alpha,1}(x-r_m) E_m\right]\phi(0) \\ &+ \sum_{j=1}^{d} \int_{-r_j}^{0} \mathscr{P}^{\mathfrak{E},M,\mathfrak{U}}_{\alpha,\alpha}\left(x-r_j-t\right)\left[U_j \phi(t) + E_j \, {}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0+}\phi(t)\right] dt \\ &+ \int_0^x \mathscr{P}^{\mathfrak{E},M,\mathfrak{U}}_{\alpha,\alpha}(x-t) Su(t)\, dt + \int_0^x \mathscr{P}_{\alpha,\alpha}(x-t) \daleth(t,z(t))\, dt. \end{aligned} \tag{4.16}$$

Prior to giving the pioneer theorem, let's make some assumptions:

$(\mathbf{R_1})$ The operator $W_c : L^\infty(J,\mathbb{R}^n) \to \mathbb{R}^n$ given by

$$W_c u = \int_0^\tau \mathscr{P}^{\mathfrak{E},M,\mathfrak{U}}_{\alpha,\alpha}(\tau - s) Su(s) ds,$$

has an inverse operator W_c^{-1} which take values in $L^2(J,\mathbb{R}^n)/kerW_c$.

$(\mathbf{R_2})$ The function $\daleth : J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $L_\daleth(.) \in L^\infty(J,\mathbb{R}^+)$ such that for arbitrary $z, y \in \mathbb{R}^n$

$$\|\daleth(x,z(x)) - \daleth(x,y(x))\| \leq L_\daleth(x)\|z(x) - y(x)\|, \quad x \in J.$$

now, let's introduce the following notations:

$$H = \|W_c\|^{-1}_{B\left(\mathbb{R}^n, L^2(J,\mathbb{R}^n)/kerW_c\right)},$$

$$H_1 = \mathscr{P}_{\alpha,1}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\|\phi(0)\| + \sum_{m=1}^{d}\|E_m\|\,\mathscr{P}_{\alpha,1}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}(\tau - r_m)\|\phi(0)\|$$
$$+ \sum_{m=1}^{d}\int_{-r_m}^{0}\mathscr{P}_{\alpha,\alpha}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}(\tau - r_m - t)\left\|U_m\phi(t) + E_m{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi(t)\right\|dt$$
$$+ N_\neg\mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}(\tau),$$

$$H_2 = \mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\|L_\neg\|_{L^\infty(J,\mathbb{R}^n)},$$

where $N_\neg = \max_{[0,\tau]}\|\urcorner(x,0)\|$. One can obtain the following information from Remark 3.3 [106]

$$H = \sqrt{\left\|W_{r,\alpha}^{-1}[0,\tau]\right\|}.$$

Theorem 4.7: Suppose that $1 > \alpha \geq 0.5$, $(\mathbf{R_1})$ and $(\mathbf{R_2})$ are hold. Then system (4.15) is relatively controllable if

$$H_2\left(1 + \mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\|S\|H\right) < 1. \tag{4.17}$$

Proof. With the aid of $(\mathbf{R_1})$ for any $z \in C = C(J,\mathbb{R}^n)$, we define the below control function $u_z(x)$:

$$u_z(x) = W_c^{-1}\left[z_\tau - \mathscr{P}_{\alpha,1}^{\mathfrak{C},M,\mathfrak{U}}(\tau)\phi(0) - \sum_{m=1}^{d}\mathscr{P}_{\alpha,1}^{\mathfrak{C},M,\mathfrak{U}}(\tau - r_m)E_m\phi(0)\right.$$
$$- \sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}^{\mathfrak{C},M,\mathfrak{U}}(\tau - r_j - t)\left[U_j\phi(t) + E_j{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi(t)\right]dt$$
$$\left. - \int_{0}^{\tau}\mathscr{P}_{\alpha,\alpha}^{\mathfrak{C},M,\mathfrak{U}}(\tau - t)\urcorner(t,z(t))\,dt\right](x). \tag{4.18}$$

By employing this control function, we define $\mathscr{K}: C \to C$ by

$$\mathscr{K}z(x) = \left[\mathscr{P}_{\alpha,1}^{\mathfrak{C},M,\mathfrak{U}}(x) + \sum_{m=1}^{d}\mathscr{P}_{\alpha,1}^{\mathfrak{C},M,\mathfrak{U}}(x - r_m)E_m\right]\phi(0)$$
$$+ \sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}_{\alpha,\alpha}^{\mathfrak{C},M,\mathfrak{U}}(x - r_j - t)\left[U_j\phi(t) + E_j{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi(t)\right]dt$$
$$+ \int_{0}^{x}\mathscr{P}_{\alpha,\alpha}^{\mathfrak{C},M,\mathfrak{U}}(x - t)Su(t)\,dt + \int_{0}^{x}\mathscr{P}_{\alpha,\alpha}(x - t)\urcorner(t,z(t))\,dt. \tag{4.19}$$

which is of a fixed point z being the mild solution of system (4.15).

If we take the definition of relative controllability into consideration, system (4.15) with (4.18) is relatively controllable if and only if (4.19) is of a solution $z \in C([-r,\tau],\mathbb{R}^n)$ with $z(\tau) = z_\tau$ and $z(x) = \phi(x)$, $x \in [-r,\tau]$.

It is well-known that for each $\varepsilon > 0$

$$\mathscr{D}_\varepsilon = \{z \in C : \|z\|_C \leq \varepsilon\}$$

is a convex, bounded and closed set of C. In an attempt to make the rest of this proof more understandable, it is divided into three steps.

Step 1 : One can find at least a positive real number $\varepsilon > 0$ such that

$$\mathscr{K}(\mathscr{D}_\varepsilon) \subseteq \mathscr{D}_\varepsilon.$$

It is time to compute the norm of the control function $u_z(x)$ by using $(\mathbf{R_1})$ and $(\mathbf{R_2})$ and Lemma 4.3 and Hölder's inequality, we get

$$\begin{aligned}
\|u_z(x)\| \leq{}& \left\|W_c^{-1}\right\|_{B(\mathbb{R}^n,L^\infty(J,\mathbb{R}^n)/kerW_c)} \Bigg[\|z_\tau\| + \left\|\mathscr{P}_{\alpha,1}^{\mathfrak{C},M,\mathfrak{U}}(\tau)\right\| \|\phi(0)\| \\
&+ \left\|\sum_{m=1}^{d} E_m \mathscr{P}_{\alpha,1}^{\mathfrak{C},M,\mathfrak{U}}(\tau - r_m)\right\| \|\phi(0)\| \\
&+ \sum_{j=1}^{d} \int_{-r_j}^{0} \left\|\mathscr{P}_{\alpha,\alpha}^{\mathfrak{C},M,\mathfrak{U}}(\tau - r_j - t)\right\| \left\|U_j \phi(t) + E_j {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} \phi(t)\right\| dt \\
&+ \int_0^\tau \|\mathscr{P}_{\alpha,\alpha}(\tau - t)\| \|\daleth(t,z(t))\| \, dt\Bigg] \\
\leq{}& H \Bigg[\|z_\tau\| + \mathscr{P}_{\alpha,1}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}(\tau) \|\phi(0)\| + \sum_{m=1}^{d} \|E_m\| \mathscr{P}_{\alpha,1}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}(\tau - r_m) \|\phi(0)\| \\
&+ \sum_{m=1}^{d} \int_{-r_m}^{0} \mathscr{P}_{\alpha,\alpha}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}(\tau - r_m - t) \left\|U_m \phi(t) + E_m {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha} \phi(t)\right\| dt \\
&+ \int_0^x \mathscr{P}_{\alpha,\alpha}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}(\tau - t) \left(\|\daleth(t,z(t)) - \daleth(t,0)\| + \|\daleth(t,0)\|\right) dt\Bigg],
\end{aligned}$$

$$\leq H\left[\|z_\tau\| + \mathscr{P}_{\alpha,1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\|\phi(0)\| + \sum_{m=1}^{d}\|E_m\|\mathscr{P}_{\alpha,1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau-r_m)\|\phi(0)\|\right.$$
$$+\sum_{m=1}^{d}\int_{-r_m}^{0}\mathscr{P}_{\alpha,\alpha}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau-r_m-t)\left\|U_m\phi(t)+E_m{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi(t)\right\|dt$$
$$\left.+\int_{0}^{x}\mathscr{P}_{\alpha,\alpha}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau-t)\,dt\,\|L_\neg\|_{L^\infty(J,\mathbb{R}^+)}\|z\|_C + N_\neg\int_{0}^{x}\mathscr{P}_{\alpha,\alpha}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau-t)\,dt\right]$$
$$\leq H\|z_\tau\| + HH_1 + HH_2\|z\|_C.$$

To determine $\varepsilon > 0$ such that $\mathscr{K}z(x)\in\mathscr{D}_\varepsilon$, we consider by using $(\mathbf{R_1})$ and $(\mathbf{R_2})$ and Lemma 4.3,

$$\|\mathscr{K}z(x)\| \leq \mathscr{P}_{\alpha,1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\|\phi(0)\| + \sum_{m=1}^{d}\|E_m\|\mathscr{P}_{\alpha,1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau-r_m)\|\phi(0)\|$$
$$+\sum_{m=1}^{d}\int_{-r_m}^{0}\mathscr{P}_{\alpha,\alpha}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau-r_m-t)\left\|U_m\phi(t)+E_m{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{\alpha}\phi(t)\right\|dt$$
$$+\int_{0}^{x}\mathscr{P}_{\alpha,\alpha}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau-t)\|\neg(t,z(t))\|\,dt$$
$$+\int_{0}^{x}\mathscr{P}_{\alpha,\alpha}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau-t)\|S\|\,\|u_z(t)\|\,dt.$$

If we use control estimation in the last inequality, we get

$$\|\mathscr{K}z(x)\| \leq \left(1+\mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\|S\|H\right)H_1 + \left(\mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\|S\|H\right)\|z_\tau\|$$
$$+\left(1+\mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\|S\|H\right)H_2\|z\|_C$$
$$\leq \left(1+\mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\|S\|H\right)H_1 + \left(\mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\|S\|H\right)\|z_\tau\|$$
$$+\left(1+\mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\|S\|H\right)H_2\varepsilon = \varepsilon$$

One can easily obtain
$$\varepsilon := \frac{\left(1+\mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\|S\|H\right)H_1 + \left(\mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\|S\|H\right)\|z_\tau\|}{1-\left(1+\mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\|S\|H\right)H_2} > 0$$
which provides $\mathscr{K}(\mathscr{D}_\varepsilon)\subseteq\mathscr{D}_\varepsilon$ Now we split the operator $\mathscr{K}$ into two operators $\mathscr{K}_1$ and $\mathscr{K}_2$ on $\mathscr{D}_\varepsilon$ as follows:

$$\begin{aligned}\mathscr{K}_1 z(x) &= \left[\mathscr{P}^{\mathfrak{C},M,\mathfrak{U}}_{\alpha,1}(x) - \sum_{m=1}^{d}\mathscr{P}^{\mathfrak{C},M,\mathfrak{U}}_{\alpha,1}(x-r_m)E_m\right]\phi(0)\\ &+\sum_{j=1}^{d}\int_{-r_j}^{0}\mathscr{P}^{\mathfrak{C},M,\mathfrak{U}}_{\alpha,\alpha}\left(x-r_j-t\right)\left[U_j\phi(t)+E_j{}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\phi(t)\right]dt\\ &+\int_0^x\mathscr{P}^{\mathfrak{C},M,\mathfrak{U}}_{\alpha,\alpha}(x-t)Su(t)\,dt, \qquad (4.20)\end{aligned}$$

and

$$\mathscr{K}_2 z(x) = \int_0^x \mathscr{P}^{\mathfrak{C},M,\mathfrak{U}}_{\alpha,\alpha}(x-t)\urcorner(t,z(t))\,dt,$$

for $x\in J$, respectively.

Step 2 : we will prove that $\mathscr{K}_1$ is a contraction. Let $z,y\in\mathscr{D}_\varepsilon$. Keeping $(\mathbf{R_1})$ and $(\mathbf{R_2})$ in mind, we get

$$\begin{aligned}\left\|u_z(x)-u_y(x)\right\| &\le H\int_0^\tau \mathscr{P}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}_{\alpha,\alpha}(\tau-t)\left(\|\urcorner(t,z(t))-\urcorner(t,y(t))\|\right)dt\\ &\le H\int_0^\tau \mathscr{P}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}_{\alpha,\alpha}(\tau-t)\,dt\,\|L_\urcorner\|_{L^\infty(J,\mathbb{R}^+)}\|z-y\|_C\\ &\le HH_2\|z-y\|_C.\end{aligned}$$

So,

$$\begin{aligned}\|\mathscr{K}_1 z(x)-\mathscr{K}_1 y(x)\| &\le \int_0^x \mathscr{P}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}_{\alpha,\alpha}(x-t)\|S\|\left\|u_z(t)-u_y(t)\right\|dt\\ &\le \mathscr{P}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}_{\alpha,\alpha+1}(\tau)\|S\|HH_2\|z-y\|_C.\end{aligned}$$

Since (4.17), $\mathscr{P}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}_{\alpha,\alpha+1}(\tau)\|S\|HH_2<1$, this grants us that $\mathscr{K}_1$ is a contraction.

Step 3 : We will demonstrate that $\mathscr{K}_2$ is compact and continuous. Let $z_n\in\mathscr{D}_\varepsilon$ with $z_n\to z$ in $\mathscr{D}_\varepsilon$. $(\mathbf{R_2})$ ensures that $\urcorner(x,z_n(x))\to\urcorner(x,z(x))$ in C. By using dominated convergence theorem

$$\|\mathscr{K}_2 z_n(x)-\mathscr{K}_2 z(x)\|\le\int_0^x\mathscr{P}^{\|\mathfrak{C}\|,\|M\|,\|\mathfrak{U}\|}_{\alpha,\alpha}(x-t)\|\urcorner(t,z_n(t))-\urcorner(t,z(t))\|\,dt\to 0$$

as $n\to\infty$. So $\mathscr{K}_2$ is continuous on $\mathscr{D}_\varepsilon$. In an attempt to be able to confirm that $\mathscr{K}_2$ is compact, we must show that $\mathscr{K}_2(\mathscr{D}_\varepsilon) \subseteq C$ is uniformly bounded and equicontinuous. For any $z \in \mathscr{D}_\varepsilon$, $0 < x < x+h < \tau$

$$\begin{aligned}\mathscr{K}_2 z(x+h) - \mathscr{K}_2 z(x) = &\int_x^{x+h} \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(x+h-t)\,\daleth(t,z(t))\,dt\\ &+\int_0^x \left(\mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(x+h-t) - \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(x-t)\right)\daleth(t,z(t))\,dt.\end{aligned}$$

Set the following notations:

$$\eta_1 := \int_x^{x+h} \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(x+h-t)\,\daleth(t,z(t))\,dt,$$

$$\eta_2 := \int_0^x \left(\mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(x+h-t) - \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(x-t)\right)\daleth(t,z(t))\,dt.$$

Since

$$\|\mathscr{K}_2 z(x+h) - \mathscr{K}_2 z(x)\| \leq \|\eta_1\| + \|\eta_2\|,$$

it is necessary to show that $\|\eta_j\| \to 0$ as $h \to 0$, $j1,2$. With a simple calculation

$$\|\eta_1\| \leq \int_x^{x+h} \|\mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(x+h-t)\|dt\,\|L\daleth\|_{L^\infty(J,\mathbb{R}^+)}\|z\|_C + N\daleth\int_x^{x+h}\|\mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(x+h-t)\|dt,$$

and

$$\begin{aligned}\|\eta_2\| \leq &\int_0^x \left\|\mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(x+h-t) - \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(x-t)\right\|dt\,\|L\daleth\|_{L^\infty(J,\mathbb{R}^+)}\|z\|_C\\ &+N\daleth\int_0^x \left\|\mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(x+h-t) - \mathscr{P}_{\alpha,\alpha}^{\mathfrak{E},M,\mathfrak{U}}(x-t)\right\|dt\end{aligned}$$

So $\|\eta_j\| \to 0$ for $j = 1,2$ as $h \to 0$. As a consequence, one can immediately obtain that for $z \in \mathscr{D}_\varepsilon$,

$$\|\mathscr{K}_2 z(x+h) - \mathscr{K}_2 z(x)\| \to 0, \quad h \to 0.$$

$\mathscr{K}_2(\mathscr{D}_\varepsilon)$ is bounded because one can easily reach to the following inequality with the similar computation

$$\|\mathscr{K}_2 z(x)\| \leq \mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\,\|L\daleth\|_{L^\infty(J,\mathbb{R}^+)}\,\varepsilon + N\daleth\mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau).$$

So by Arzelà-Ascoli theorem $\mathscr{K}_2(\mathscr{D}_\varepsilon)$ is relatively compact in C. Therefore, $\mathscr{K}_2$ is compact and continuous. By Lemma 2.3, $(\mathbf{R_1})$ and $(\mathbf{R_2})$, $\mathscr{K}$ is of a fixed point $z \in \mathscr{D}_\varepsilon$. Obviously, z is such a solution of system (4.15) that it satisfies $z(\tau) = z_\tau$ and $z(x) = \phi(x)$ with $-r \leq x \leq 0$ is satisfied by (4.15). This completes the proof. $\square$

4.5 Illustrated Examples

Here are some examples to illustrate theoretical results. We now consider the following neutral Caputo fractional multi-delayed differential equations with distinct kinds of parameters.

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}^{0.5}_{0^+}\left[z(x) - E_1 z(x-0.2) - E_2 z(x-0.1)\right] = Mz(x) \\ +Uz(x-0.2) + \frac{e^x}{1+e^x} x^2 \sin(z(x)), \ \ x \in (0, 0.4], \\ \quad z(x) = \phi(x), \ \ -0.2 \leq x \leq 0, \end{cases} \tag{4.21}$$

where $E_1 = \begin{pmatrix} 0.870 & 0.130 \\ 0 & 0.650 \end{pmatrix}$, $M = \begin{pmatrix} 0.33 & 0 \\ 0.03 & 0.125 \end{pmatrix}$, $U = \begin{pmatrix} 0.66 & 0.34 \\ 0.17 & 0.01 \end{pmatrix}$, and $E_2 = I$ which are pairwise noncommutative matrices, e.g., $E_1 U \neq U E_1$ and $MU \neq UM$. The initial function is given by nonlinear functions $\phi(x) = \begin{pmatrix} x^3 \\ 2x+1 \end{pmatrix}$. From Corollary 4.2, one could easily obtain the closed-form formula of the solution $z(x) \in C\left([-0.2, 0.4], \mathbb{R}^2\right)$ of system (4.21) is of the integral representation as noted below

$$\begin{aligned} z(x) &= \left[\mathscr{P}_{0.5,1}(x) + \mathscr{P}_{0.5,1}(x-0.2)E_1 + \mathscr{P}_{0.5,1}(x-0.1)\right]\phi(0) \\ &\quad + \int_{-0.2}^{0} \mathscr{P}_{0.5,0.5}(x-0.2-t)\left(U\phi(t) + E_1\, {}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\phi(t)\right)dt \\ &\quad + \int_{-0.1}^{0} \mathscr{P}_{0.5,0.5}(x-0.10-t)\, {}^{\mathfrak{C}}\mathfrak{D}^{\alpha}_{0^+}\phi(t)\,dt \\ &\quad + \int_{0}^{x} \mathscr{P}_{0.5,0.5}(x-t)\frac{e^t}{1+e^t} t^2 \sin(z(t))\,dt, \end{aligned}$$

where $\mathscr{P}_{0.5,0.5}$ and $\mathscr{P}_{0.5,1}$ is as in (4.8). It is clear that $\daleth(x, z(x)) = \frac{e^x}{1+e^x} x^2 \sin(z(x))$ is continuous as well as being the Lipschitz function with the Lipschitz constant $L_\daleth =$

0.16 and $L_{\neg}\left\|\mathscr{P}_{0.5,1.5}(0.4)\right\| \cong 0.186 < 1$. Hence, all of conditions of Theorem 4.4 and 4.5 holds, so system (4.21) is of an unique solution in addition to being Ulam-Hyers stable.

In order to illustrate relative controllability of the neutral fractional differential linear multi-delayed homogeneous system , we examine the following system

$$\begin{cases} {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{0.9}\left[z(x) - E_1 z(x-0.5) - E_2 z(x-0.2)\right] = Mz(x) \\ +U_1 z(x-0.5) + Su(x), x \in (0, 1.5] \\ \quad z(x) = \phi(x), -0.5 \leq x \leq 0, \end{cases} \tag{4.22}$$

where

$$M = \begin{bmatrix} 0.2 & 0.36 & 0.45 \\ 0.96 & 0 & 0.12 \\ 0.16 & 0.3 & 0.45 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 0.1 & 0.7 & 0.4 \\ 0.36 & 0.52 & 0.2 \\ 0.6 & 0.56 & 0.2 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.3 & 0.5 & 0.18 \\ 0.21 & 0.41 & 0 \\ 1.01 & 0.8 & 0.43 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.4 & 0.98 & 0.4 \\ 0.3 & 0.81 & 0.87 \\ 0.2 & 0.41 & 0.87 \end{bmatrix},$$

$$S = \begin{bmatrix} 0.7 & 0.44 & 0.9 \\ 2 & 0.91 & 0.56 \\ 0.1 & 0.3 & 0.4 \end{bmatrix},$$

and $\phi(x) = [x^2+5 \;\; 2x+4 \;\; 5x+7]^T \in \mathbb{R}^3$. The closed form of the solution of system (4.22) is

$$
\begin{aligned}
z(x) = & \left[\mathscr{P}_{0.9,1}^{\sum_{i=1}^{2} E_i, M, U_1}(x) - E_1 \mathscr{P}_{0.9,1}^{\sum_{i=1}^{2} E_i, M, U_1}(x-0.5) - E_2 \mathscr{P}_{0.9,1}^{\sum_{i=1}^{2} E_i, M, U_1}(x-0.2)\right] \phi(0) \\
& + \int_{-0.5}^{0} \mathscr{P}_{0.9,0.9}^{\mathfrak{E}, M, \mathfrak{U}}(x-0.5-t) \left[U_1 \phi(t) + E_1 {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{0.9} \phi(t)\right] dt \\
& + \int_{-0.2}^{0} \mathscr{P}_{0.9,0.9}^{\sum_{i=1}^{2} E_i, M, U_1}(x-0.2-t) E_2 {}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{0.9} \phi(t) \, dt \\
& + \int_{0}^{x} \mathscr{P}_{0.9,0.9}^{\sum_{i=1}^{2} E_i, M, U_1}(x-t) S u(t) \, dt.
\end{aligned}
$$

On the other hand, a representation of the neutral fractional multi-delayed Gramian matrix as follows:

$$
\begin{aligned}
W_{0.5,0.9}[0,1] &= \int_{0}^{1} \mathscr{P}_{0.9,0.9}^{\sum_{i=1}^{2} E_i, M, U_1}(1-s) S S^T \mathscr{P}_{0.9,0.9}^{\sum_{i=1}^{2} E_i^T, M^T, U_1^T}(1-s) \, ds \\
&= \begin{bmatrix} 0.1171 & 1.9438 & 0.0338 \\ 1.9438 & 0.3307 & 0.0353 \\ 0.0338 & 0.0353 & 0.5392 \end{bmatrix}.
\end{aligned}
$$

We calculate the determinant of Gramian matrix $W_{0.5,0.9}[0,1]$ which is -2.0123, so $W_{0.5,0.9}[0,1]$ is nonsingular. By Theorem 4.6, system (4.22) is relatively controllable.

To exemplify the neutral fractional differential semilinear multi-delayed system , we investigate the following system

$$
\begin{cases}
{}^{\mathfrak{C}}\mathfrak{D}_{0^+}^{0.75}\left[z(x) - E_1 z(x-2) - E_2 z(x-1)\right] = M z(x) + U_1 z(x-2) \\
+ S u(x) + \daleth(x, z(x)), \ \ 0 \le x \le 6, \\
\quad z(x) = \phi(x), \ \ -2 \le x \le 0,
\end{cases}
\tag{4.23}
$$

where

$$
M = \begin{bmatrix} 0 & 0.36 \\ 0.96 & 0.81 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 0.1 & 0.7 \\ 0.3 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.3 & 0 \\ 0.21 & 0.41 \end{bmatrix},
$$

$$
E_2 = \begin{bmatrix} 0.4 & 0.98 \\ 0.3 & 0.81 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0.44 \\ 2 & 0.91 \end{bmatrix},
$$

and $\phi(x) = [1 \ \ 5]^T$ and $\daleth(x, z(x)) = \left[\frac{\tan^{-1} z(x)}{(\pi^2 x)^2} \ \ \frac{\sin z(x)}{\pi^6 x^2}\right]^T$ with the solution of a closed-form obtained from (4.16). On having a look at the assumptions for system (4.23),

$$W_{2,0.75}[0,3] = \begin{bmatrix} 5.7296 - 0.000i & -15.4580 + 5.4391i \\ -15.4580 + 5.4391i & -23.8019 + 33.8270i \end{bmatrix}$$

and

$$W_{2,0.75}^{-1}[0,3] = \begin{bmatrix} 0.0817 - 0.0123i & -0.0292 - 0.0148i \\ -0.0292 - 0.0148i & -0.0079 - 0.0083i \end{bmatrix}$$

From the reference [106, Remark 3.3], we know

$$H = \|W_c^{-1}\|_{B\left(\mathbb{R}^n, L^2(J,\mathbb{R}^n)/kerW_c\right)} = \sqrt{\|W_{r,\alpha}^{-1}[0,\tau]\|},$$

and so, we acquire

$$H = \sqrt{\left\|W_{2,0.75}^{-1}[0,3]\right\|} = 0.3397,$$

which ensures that the inverse operator W_c^{-1} exists, so the operator $W_c : L^\infty\left([0,3],\mathbb{R}^2\right) \to \mathbb{R}^2$ satisfies $(\mathbf{R_1})$. The function $\daleth : [0,3] \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and $L_\daleth(.) \in L^\infty(J, \mathbb{R}^+)$ such that for arbitrary $z, y \in \mathbb{R}^n$

$$\left\| \left[\frac{\tan^{-1} z(x)}{(\pi^2 x)^2} \ \ \frac{\sin z(x)}{\pi^6 x^2}\right]^T - \left[\frac{\tan^{-1} y(x)}{(\pi^2 x)^2} \ \ \frac{\sin y(x)}{\pi^6 x^2}\right]^T \right\| \leq L_\daleth(x)\|z(x) - y(x)\|, \ \ x \in [0,3]$$

where $L_\daleth(x) = \frac{1}{\pi^2 x} \in \mathbb{R}^+$. So $(\mathbf{R_2})$ is hold for system (4.23). It is time to verify whether the inequality (4.17) is satisfied

$$H_2\left(1 + \mathscr{P}_{\alpha,\alpha+1}^{\|\mathfrak{E}\|,\|M\|,\|\mathfrak{U}\|}(\tau)\,\|S\|\,H\right) = 0.2765 < 1.$$

As a result, each of the conditions of Theorem 4.7 is verified. Theorem 4.7 provides us that system (4.23) is relatively controllable under the control function

$$\begin{aligned} u_z(x) = W_c^{-1}\Big[& z_\tau - \left(\mathscr{P}_{\alpha,1}^{E_1+E_2,M,U_1}(\tau) + E_1\mathscr{P}_{\alpha,1}^{E_1+E_2,M,U_1}(\tau-2)\right)[1 \ \ 5]^T \\ & - \left(E_2\mathscr{P}_{\alpha,1}^{E_1+E_2,M,U_1}(\tau-1) + \int_{-2}^{0}\mathscr{P}_{\alpha,\alpha}^{E_1+E_2,M,U_1}(\tau-2-t)\,U_1 dt\right)[1 \ \ 5]^T \\ & - \int_0^\tau \mathscr{P}_{\alpha,\alpha}^{E_1+E_2,M,U_1}(\tau-t)\left[\frac{\tan^{-1} z(t)}{(\pi^2 t)^2} \ \ \frac{\sin z(t)}{\pi^6 t^2}\right]^T dt\Big](x). \end{aligned}$$

Remark 4.7: We want to note that if we reduce relative controllability of the semilinear neutral fractional multi-delayed control system (4.15) with permutable matrices and one single delay and $\alpha = \beta = 1$ to the relative controllability of neutral delay differential equations [105], their results overlap.

4.6 New Problems

We are sure that this paper will become a source of inspiration for the works which will be conducted in this subject. A possible duty is to verify the explicit solution result of the nonlinear neutral fractional multi-delayed differential system with the aid of the Laplace transform which is the most powerful tool for differential equations. Another possible duty is to investigate approximate controllability, exponential stability, finite time stability, asymptotic stability, and also Lyapunov type stability of the neutral fractional multi-delayed differential equations with noncommutative coefficient matrices. Another possible duty is to extend our system (4.5) to the nonlinear neutral fractional multi-delayed differential evolution equation or μ-the nonlinear neutral fractional multi-delayed differential system which means that (4.5) is reconsidered via Caputo fractional derivative with respect to another function μ. All possibilities as noted above can be questioned once again for these new systems.

Chapter 5

NEW FRACTIONAL INTEGRAL AND DERIVATIVES

In order to make this section more understandable, we talk about the history of fractional derivatives and integrals and their related notions again in addition to information given in the section of introduction. It was understood that n was one of the non-negative integers when one talked about derivative of order n or n-fold integrals. Because the former was in need of knowing instantaneous rates of change, areas under or between curves, the slopes of curves, and accumulation of quantities. These needs produced the well-known traditional calculus. Unlike traditional calculus, although fractional calculus at that time was a production of only innocent curiosity which is in Leibnitz's letter to L' Hospital in 1695, it has been widely improved along with the extension of the needs in the recent decades. Many researchers not only in the past like Euler, Fourier, Abel, Liouville, Riemann, Grünwald, Hadamard, Weyl, Erdélyi-Kober, Caputo have tried to understand and define fractional derivatives and integrals [131] [130] [134] [141] [135] [136] [118] [133], ones but also in the present make an attempt to define a new derivative or integral of fractional order depending generalizing available concepts like gamma function and appearing new ones and needs. For instance, Katugampola [126] introduced a novel fractional operator generalising the well-known Hadamard fractional and the Riemann-Liouville derivatives to a individual form. Romero [137] et al. presented a novel fractional derivative named by k-Riemann-Liouville fractional derivative by utilizing the

k-gamma function and relationships with the k-Riemann-Liouville integral and some features employing Laplace and Fourier transforms. Sarikaya [140] et al. gave a new version of fractional integral called (k,s)-Riemann-Liouville fractional integral generalising the Riemann-Liouville fractional integral and presented some features for this one as well as new integral inequalities employing the novel version of fractional integral. Subsequently, Azam [117] et al. developed the generalized k-fractional derivative in the sense of Riemann-Liouville and generalized Caputo type k-fractional derivative which are the generalized forms of some existing fractional derivatives. Almeida [115] studied a Caputo type fractional derivative with respect to another function and investigated some features, like the inverse law and the semigroup law, Fermat's and Taylor's Theorems, etc.

Fractional calculus has a prevailing usage in the scientific world. Nowadays, it has been employed in the areas of mathematical physics, statistical mechanics, electrochemistry, electric conductance of biological systems, astrophysics, computed tomography, control theory, the mathematical modelling of viscoelastic material, thermodynamics, the modelling of diffusion, biophysics, electric conductance of biological systems, fractional order models of neurons, hydrology, geological surveying, signal and image possessing, engineering, finance, etc. Almeida [115] et al. took a Population Growth Model into consideration and demonstrated that the process utilizing a Caputo FD with respect to different functions(kernels) can be more accurately modelled. With the help of the generalized fractional derivatives, mathematically the variant of post-Newtonian mechanics and the relativistic-covariant generalization of the traditional equations in the gravitational field are studied by Kobelev [128]

As a source of inspiration, the main works [140] [117] [115] and papers mentioned above encourage us to describe a new fractional integral including sorts of fractional-order integrals and two fractional derivatives which are novel and can include many available fractional derivatives. We looked for certain their properties and found their relations, and coped with the fundamental Cauchy problem.

5.1 The ϕ-Generalized R-L k-Fractional Integral and Derivative

In this section, we introduce both the ϕ-generalized Riemann Liouville k-fractional integral(ϕ-GRL k-FI) of order $\alpha > 0$ and the ϕ-generalized Riemann Liouville k-fractional derivative(ϕ-GRL k-FD) of order $\alpha > 0.$ we examine some properties and relations between them. Now, let's start with the definition of (ϕ-GRL k-FI).

Definition 5.1: Let f be a continuous function on the real interval $[a,b]$ and let $\phi \in C^1[a,b]$ be an increasing function with $\phi'(x) \neq 0,\ \forall x \in [a,b]$. Then the ϕ-generalized Riemann Liouville k-fractional integral of $\alpha > 0$ is given by

$$\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi} f\right)(x) = \frac{s^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x \left(\phi^s(x) - \phi^s(t)\right)^{\frac{\alpha}{k}-1} \phi'(t)\, \phi^{s-1}(t)\, f(t)\, dt,$$

where $k > 0$ and $s \in \mathbb{R}\backslash\{-1\}$. For the sake of simplicity, we denote ϕ-GRL k-FI using the differential concept by

$$\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi} f\right)(x) = \frac{s^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x \left(\phi^s(x) - \phi^s(t)\right)^{\frac{\alpha}{k}-1} f(t)\, d\phi^s(t).$$

Depending on the selections of s,k,ϕ, we acquire distinct kinds of the available definitions of fractional integrals, e.g. ϕ-GRL k-FI coincides with (k,s)-Riemann-Liouville fractional integral [140] if $\phi(x) = x$. If$s = 1,\ k \to 1$, it reduces to the ϕ-Riemann-Liouville fractional integrals [127] [139] [116] [115]. ϕ-GRL k-FI under the choices of $s = 1,\ \phi(x) = x,\ k \to 1$ reduces to the traditional

Riemann-Liouville fractional integrals. Selecting $\phi(x) = x,\ k \to 1,\ s \to 0^+$ turns ϕ-GRL k-FI into the Hadamard fractional integral [124], etc.

The following theorem expresses semi-group and commutative property of ϕ-GRL k-FI.

Theorem 5.1: Let f be a continuous function on the real interval $[a,b]$ and let $\phi \in C^1[a,b]$ be an increasing function with $\phi'(x) \neq 0,\ \forall x \in [a,b]$. Then, $\forall \alpha, \beta > 0$

$$ {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s}\left[{}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\beta,\phi}_{k,s} f(x)\right] = {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha+\beta,\phi}_{k,s} f(x) = {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\beta,\phi}_{k,s}\left[{}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s} f(x)\right]. $$

Proof. Assume that given conditions are satisfied. By using Fubini's theorem, consider

$$ \begin{aligned} &{}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s}\left[{}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\beta,\phi}_{k,s} f(x)\right] \\ &= \frac{s^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (\phi^s(x) - \phi^s(y))^{\frac{\alpha}{k}-1} \left[\frac{s^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_a^y (\phi^s(y) - \phi^s(t))^{\frac{\beta}{k}-1} d\phi^s(t) f(t)\right] d\phi^s(y) \\ &= \frac{s^{-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)} \int_a^x f(t) \left[\int_t^x (\phi^s(x) - \phi^s(y))^{\frac{\alpha}{k}-1} (\phi^s(y) - \phi^s(t))^{\frac{\beta}{k}-1} d\phi^s(y)\right] d\phi^s(t) \end{aligned} $$

By substituting $z = \frac{\phi^s(y)-\phi^s(t)}{\phi^s(x)-\phi^s(t)}$, we obtain $z = 0$, $z = 1$, $[\phi^s(x) - \phi^s(t)]z = d\phi^s(y)$, and

$$ \begin{aligned} &{}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s}\left[{}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\beta,\phi}_{k,s} f(x)\right] \\ &= \frac{s^{-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)} \int_a^x (\phi^s(x) - \phi^s(t))^{\frac{\alpha+\beta}{k}-1} f(t) \left[\int_0^1 (1-z)^{\frac{\alpha}{k}-1} z^{\frac{\beta}{k}-1} dz\right] d\phi^s(t) \\ &= \frac{s^{-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)} \int_a^x (\phi^s(x) - \phi^s(t))^{\frac{\alpha+\beta}{k}-1} f(t)\, d\phi^s(t)\, B_k\left(\frac{\alpha}{k}, \frac{\beta}{k}\right) \\ &= \frac{s^{-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)} \int_a^x (\phi^s(x) - \phi^s(t))^{\frac{\alpha+\beta}{k}-1} f(t)\, d\phi^s(t)\, k \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} \\ &= {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha+\beta,\phi}_{k,s} f(x). \end{aligned} $$

By changing places of α and β, commutativity of ϕ-GRL k-FI can be easily followed.

□

The following corollary says that ϕ-GRL k-FI is linear.

Corollary 5.1: Let g and h be a continuous function on the real interval $[a,b]$ and let

$\phi \in C^1[a,b]$ be an increasing function with $\phi'(x) \neq 0, \ \forall x \in [a,b], \ \alpha \in \mathbb{R}^+, \ \mu \in \mathbb{R}$. Then

$$ {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s}\left[g(x) + \mu h(x)\right] = {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s}\, g(x) + \mu\, {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s}\, h(x). $$

Lemma 5.1: Let an increasing function $\phi \in C^1[a,b]$ have the property of $\phi'(x) \neq 0, \ \forall x \in [a,b]$ and let $\alpha, \beta, k > 0$ and $s \in \mathbb{R}\backslash\{-1\}$. Then we have

$$ {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s}\left(\phi^s(x) - \phi^s(a)\right)^{\frac{\beta}{k}-1} = \frac{\Gamma_k(\beta)}{s^{\frac{\alpha}{k}}\Gamma_k(\alpha+\beta)}\left(\phi^s(x) - \phi^s(a)\right)^{\frac{\alpha+\beta}{k}-1}. $$

Proof. In the light of the definition of ${}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s}$

$$ {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s}\left(\phi^s(x) - \phi^s(a)\right)^{\frac{\beta}{k}-1} = \frac{s^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x \left(\phi^s(x) - \phi^s(t)\right)^{\frac{\alpha}{k}-1}\left(\phi^s(t) - \phi^s(a)\right)^{\frac{\beta}{k}-1} d\phi^s(t) $$

By substituting $z = \frac{\phi^s(t)-\phi^s(a)}{\phi^s(x)-\phi^s(a)}$, we obtain $z = 0$, $z = 1$, $[\phi^s(x) - \phi^s(a)]z = d\phi^s(t)$, and

$$ \begin{aligned} {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s}\left(\phi^s(x) - \phi^s(a)\right)^{\frac{\beta}{k}-1} &= \frac{s^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \left(\phi^s(x) - \phi^s(a)\right)^{\frac{\alpha+\beta}{k}-1}(1-z)^{\frac{\alpha}{k}-1} z^{\frac{\beta}{k}-1} dz \\ &= \frac{s^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\left(\phi^s(x) - \phi^s(a)\right)^{\frac{\alpha+\beta}{k}-1} kB_k(\alpha,\beta). \end{aligned} $$

which provides the desired result. □

Definition 5.2: Let f be a continuous function on $[0,\infty)$ and let $\phi \in C^1[0,\infty)$ be an increasing function with $\phi'(x) \neq 0, \ \forall x \in [0,\infty)$. $s, \alpha \in \mathbb{R}^+$, and $n, k \in \mathbb{N}$ with $n = [\alpha]+1$. Then the ϕ-generalized Riemann Liouville k-fractional derivative of $\alpha > 0$ is given by

$$ \begin{aligned} \left({}^{\mathfrak{R}}_{a^+}\mathfrak{D}^{\alpha,\phi}_{k,s} f\right)(x) &= \frac{s^{\frac{\alpha-nk+k}{k}}}{k\Gamma_k(nk-\alpha)}\left(\phi^{1-s}(x)\frac{1}{\phi'(x)}\frac{d}{dx}\right)^n \\ &\times \int_a^x \left(\phi^s(x) - \phi^s(t)\right)^{\frac{nk-\alpha}{k}-1}\phi'(t)\,\phi^{s-1}(t)\, f(t)\, dt, \end{aligned} $$

where $\forall 0 < a < x$. For the sake of simplicity and making calculations easy, we denote ϕ-GRL k-FD using the differential concept by

$$ \left({}^{\mathfrak{R}}_{a^+}\mathfrak{D}^{\alpha,\phi}_{k,s} f\right)(x) = \frac{s^{\frac{\alpha}{k}}}{k\Gamma_k(nk-\alpha)}\left(\frac{d}{d\phi^s(x)}\right)^n \int_a^x \left(\phi^s(x) - \phi^s(t)\right)^{\frac{nk-\alpha}{k}-1} f(t)\, d\phi^s(t). $$

It can be expressed as follows

$$\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}f\right)(x)=\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^n\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{nk-\alpha,\phi}f\right)(x).$$

Depending on the selections of ϕ,s,k, we can reach to many of fractional derivatives, e.g. ϕ-GRL k-FD coincides with the generalized k-fractional derivative [117] if $\phi(x)=x$. If $s=1$, $k\to 1$, it corresponds to the ϕ-Riemann-Liouville fractional derivative [127] [139] [116] [115]. ϕ-GRL k-FD under the special choices of $s=1$, $\phi(x)=x$, $k\to 1$, reduces to the traditional Riemann-Liouville fractional derivative. Depending on selecting suitable choices of ϕ,s,k from ϕ-GRL k-FD, one can easily obtain the generalized fractional derivative [126], the k-Riemann-Liouville fractional derivative [137], the k-Weyl fractional derivative [138], the k-Hadamard fractional derivative [121] as well as classical Riemann-Liouville fractional derivative, Weyl fractional derivative, Hadamard fractional derivative, etc. One can find more details in the references [127] [139] [116] [115].

Now, we discuss the inverse property of the ϕ-GRL k-FD.

Theorem 5.2: Let f be a continuous function on $[0,\infty)$ and let $\phi\in C^1[0,\infty)$ be an increasing function with $\phi^{'}(x)\neq 0,\quad \forall x\in[0,\infty)$. $s,\alpha\in\mathbb{R}^+$, and $n,k\in\mathbb{N}$ with $n=[\alpha]+1$. Then $\forall 0<a<x$,

$${}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)(x)=\frac{1}{k^n}f(x).$$

Proof. With the help of both their definitions, we get

$$\begin{aligned}
&{}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)(x)\\
&=\frac{s^{\frac{\alpha-nk}{k}}}{k\Gamma_k(nk-\alpha)}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^n\int_a^x(\phi^s(x)-\phi^s(y))^{\frac{nk-\alpha}{k}-1}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\varphi}f\right)(y)\,d\phi^s(y)\\
&=\frac{s^{\frac{\alpha}{k}-n-\frac{\alpha}{k}}}{k^2\Gamma_k(nk-\alpha)\Gamma_k(\alpha)}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^n\int_a^x(\phi^s(x)-\phi^s(y))^{\frac{nk-\alpha}{k}-1}\\
&\times\left[\int_a^y(\phi^s(y)-\phi^s(t))^{\frac{\alpha}{k}-1}f(t)\,d\phi^s(t)\right]d\phi^s(y)
\end{aligned}$$

By utilizing Fubini's theorem, we have

$$
\begin{aligned}
&{}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)(x)\\
&=\frac{s^{\frac{\alpha}{k}-n-\frac{\alpha}{k}}}{k^2\Gamma_k(nk-\alpha)\Gamma_k(\alpha)}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^n\int_a^x f(t)\\
&\times\left[\int_t^x(\phi^s(x)-\phi^s(y))^{\frac{nk-\alpha}{k}-1}(\phi^s(y)-\phi^s(t))^{\frac{\alpha}{k}-1}d\phi^s(t)\right]d\phi^s(y)
\end{aligned}
$$

By substituting $z=\frac{\phi^s(y)-\phi^s(t)}{\phi^s(x)-\phi^s(t)}$, we obtain $z=0$, $z=1$, $[\phi^s(x)-\phi^s(t)]z=d\phi^s(y)$, and

$$
\begin{aligned}
&{}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)(x)\\
&=\frac{s^{-n}}{k^2\Gamma_k(nk-\alpha)\Gamma_k(\alpha)}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^n\\
&\times\int_a^x(\phi^s(x)-\phi^s(t))^{n-1}f(t)\int_0^1(1-z)^{\frac{nk-\alpha}{k}-1}z^{\frac{\alpha}{k}-1}dzd\phi^s(t)
\end{aligned}
$$

In the light of the definition and properties of beta function,

$$
\begin{aligned}
&{}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)(x)\\
&=\frac{1}{k^2\Gamma_k(nk-\alpha)\Gamma_k(\alpha)}\left(\frac{d}{d\phi^s(x)}\right)^n\int_a^x(\phi^s(x)-\phi^s(t))^{n-1}f(t)d\phi^s(t)kB_k(nk-\alpha,\alpha)\\
&=\frac{s^{-n}}{k^n\Gamma(n)}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^n\int_a^x(\phi^s(x)-\phi^s(t))^{n-1}f(t)d\phi^s(t)\\
&=\frac{1}{k^n\Gamma(n)}\left(\frac{d}{d\phi^s(x)}\right)^n\int_a^x(\phi^s(x)-\phi^s(t))^{n-1}f(t)d\phi^s(t).
\end{aligned}
$$

By applying derivative of an integral of a two-variable function by n-times to the above equality, we get the required result as follows,

$$
\begin{aligned}
&= \frac{1}{k^n\Gamma(n)}\left(\frac{d}{d\phi^s(x)}\right)^{n-1}\int_a^x \frac{d}{d\phi^s(x)}\left(\phi^s(x)-\phi^s(t)\right)^{n-1} f(t)\,d\phi^s(t)\\
&= \frac{(n-1)}{k^n\Gamma(n)}\left(\frac{d}{d\phi^s(x)}\right)^{n-1}\int_a^x \left(\phi^s(x)-\phi^s(t)\right)^{n-2} f(t)\,d\phi^s(t)\\
&= \frac{(n-1)(n-2)}{k^n\Gamma(n)}\left(\frac{d}{d\phi^s(x)}\right)^{n-2}\int_a^x \left(\phi^s(x)-\phi^s(t)\right)^{n-3} f(t)\,d\phi^s(t)\\
&\vdots\\
&= \frac{(n-1)!}{k^n\Gamma(n)}\frac{d}{d\phi^s(x)}\int_a^x f(t)\,d\phi^s(t)\\
&= \frac{\Gamma(n)}{k^n\Gamma(n)}\frac{1}{s\phi^{s-1}(x)\,\phi'(x)}\frac{d}{dx}\int_a^x f(t)\,s\phi^{s-1}(t)\,\phi^{'}(t)\,dt\\
&= \frac{1}{k^n}f(x).
\end{aligned}
$$

□

Corollary 5.2: Let f be a continuous function on $[0,\infty)$ and let $\phi \in C^1[0,\infty)$ be an increasing function with $\phi^{'}(x) \neq 0,\ \ \forall x \in [0,\infty)$. $s,\alpha,\beta \in \mathbb{R}^+$, and $n,k \in \mathbb{N}$ with $n = [\alpha]+1$. Then $\forall 0 < a < x$,

$$
{}^{\mathfrak{R}}_{a^+}\mathfrak{D}^{\alpha,\varphi}_{k,s}\left({}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\beta,\varphi}_{k,s}f\right)(x) = \frac{1}{k^n}\left({}^{\mathfrak{R}}_{a^+}\mathfrak{D}^{\alpha-\beta,\varphi}_{k,s}f\right)(x).
$$

Theorem 5.3: Let f be a continuous function on $[0,\infty)$ and let $\phi \in C^1[0,\infty)$ be an increasing function with $\phi^{'}(x) \neq 0,\ \forall x \in [0,\infty)$. $s,\beta \in \mathbb{R}^+$, and $n,k \in \mathbb{N}$ with $0 \leq \beta < 1$. Then $\forall 0 < a < x$,

$$
{}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\beta,\phi}_{k,s}\left({}^{\mathfrak{R}}_{a^+}\mathfrak{D}^{\beta,\phi}_{k,s}f\right)(x) = \frac{1}{k}f(x) - \frac{s^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\left({}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{k-\beta,\phi}_{k,s}f\right)\left(a^+\right)\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\beta}{k}-1}.
$$

Proof. Under given conditions, we have

$$
\begin{aligned}
&{}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{\beta,\phi}_{k,s}\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{D}^{\beta,\phi}_{k,s}f\right)(x)\\
&=\frac{s^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^x\left(\phi^s(x)-\phi^s(t)\right)^{\frac{\beta}{k}-1}\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{D}^{\beta,\phi}_{k,s}f\right)(t)\,d\phi^s(t)\\
&=\frac{s^{-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^x\left(\phi^s(x)-\phi^s(t)\right)^{\frac{\beta}{k}-1}\left(\phi^{1-s}(t)\frac{d}{d\phi(t)}\right)^{1}\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{k-\beta,\phi}_{k,s}f\right)(t)\,d\phi^s(t)\\
&=\frac{s^{1-\frac{\beta}{k}}}{k^{\frac{\beta}{k}}\Gamma\left(\frac{\beta}{k}\right)}\int_a^x\left(\phi^s(x)-\phi^s(t)\right)^{\frac{\beta}{k}-1}\left(\frac{d}{d\phi^s(t)}\right)^{1}\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{k-\beta,\phi}_{k,s}f\right)(t)\,d\phi^s(t)\\
&=\frac{d}{d\phi^s(x)}\left(\frac{s^{1-\frac{\beta}{k}}}{k^{\frac{\beta}{k}}\Gamma\left(\frac{\beta}{k}+1\right)}\int_a^x\left(\phi^s(x)-\phi^s(t)\right)^{\frac{\beta}{k}}\left(\frac{d}{d\phi^s(t)}\right)^{1}\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{k-\beta,\phi}_{k,s}f\right)(t)\,d\phi^s(t)\right),
\end{aligned}
$$

applying integration by parts, we acquire

$$
\begin{aligned}
&{}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{\beta,\phi}_{k,s}\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{D}^{\beta,\phi}_{k,s}f\right)(x)\\
&=\frac{d}{d\phi^s(x)}\left(-\frac{s^{1-\frac{\beta}{k}}}{k^{\frac{\beta}{k}}\Gamma\left(\frac{\beta}{k}+1\right)}\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{k-\beta,\phi}_{k,s}f\right)\left(a^{+}\right)\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\beta}{k}}\right)\\
&+\frac{d}{d\phi^s(x)}\left(\frac{s^{1-\frac{\beta}{k}}}{k^{\frac{\beta}{k}}\Gamma\left(\frac{\beta}{k}\right)}\int_a^x\left(\phi^s(x)-\phi^s(t)\right)^{\frac{\beta}{k}-1}\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{k-\beta,\phi}_{k,s}f\right)(t)\,d\phi^s(t)\right)\\
&=-\frac{s^{1-\frac{\beta}{k}}}{k^{\frac{\beta}{k}}\Gamma\left(\frac{\beta}{k}\right)}\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{k-\beta,\phi}_{k,s}f\right)\left(a^{+}\right)\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\beta}{k}-1}\\
&+\frac{d}{d\phi^s(x)}\left(\frac{s^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\int_a^x\left(\phi^s(x)-\phi^s(t)\right)^{\frac{\beta}{k}-1}\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{k-\beta,\phi}_{k,s}f\right)(t)\,d\phi^s(t)\right),
\end{aligned}
$$

arranging the last equality

$$
\begin{aligned}
&{}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{\beta,\phi}_{k,s}\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{D}^{\beta,\phi}_{k,s}f\right)(x)\\
&=\frac{d}{d\phi^s(x)}\left(s\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{\beta,\phi}_{k,s}f\right)\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{k-\beta,\phi}_{k,s}f\right)(x)\right)\\
&-\frac{s^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{k-\beta,\phi}_{k,s}f\right)\left(a^{+}\right)\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\beta}{k}-1}\\
&=\frac{d}{d\phi^s(x)}\left(s\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{k,\phi}_{k,s}f\right)(x)\right)-\frac{s^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{k-\beta,\phi}_{k,s}f\right)(a)\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\beta}{k}-1}\\
&=\frac{1}{k}f(x)-\frac{s^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)}\left({}^{\mathfrak{R}}_{a^{+}}\mathfrak{I}^{k-\beta,\phi}_{k,s}f\right)\left(a^{+}\right)\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\beta}{k}-1}.
\end{aligned}
$$

□

In the following theorem, semi-group property of ϕ-GRL k-FD is demonstrated.

Theorem 5.4: Let $\phi \in C^1[0,\infty)$ be an increasing function with $\phi'(x) \neq 0,\ \forall x \in [0,\infty)$. For continuous f on $[0,\infty)$, $s,\alpha,\beta \in \mathbb{R}^+$, and $k \in \mathbb{N}$ with $0 \leq \alpha < 1$, $0 \leq \beta < 1$ such that $\alpha+\beta < k$. Assume that ${}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{k-\beta,\phi} f(a^+) = 0$. Then $\forall 0 < a < x$,

$$ {}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\beta,\phi} f\right)(x) = \frac{1}{k}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha+\beta,\phi} f\right)(x). $$

Proof. From the inverse and semi-group properties of ϕ-GRL k-FD and ϕ-GRL k-FI, respectively, we obtain

$$ \begin{aligned} {}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\beta,\phi} f\right)(x) &= \left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^1 {}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{k-\alpha,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\beta,\phi} f\right)(x) \\ &= \left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^1 {}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{k-\alpha-\beta,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\beta,\phi}\right)\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\beta,\phi} f\right)(x) \\ &= \frac{1}{k}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^1 {}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{k-\alpha-\beta,\phi} f(x) \\ &= \frac{1}{k}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^1 \left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{k-(\alpha+\beta),\phi} f\right)(x) \\ &= \frac{1}{k}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha+\beta,\phi} f\right)(x) \end{aligned} $$

which is the wanted result. □

Here is the commutativity and linearity of ϕ-GRL k-FD.

Corollary 5.3: Let $\phi \in C^1[0,\infty)$ be an increasing function with $\phi'(x) \neq 0,\ \forall x \in [0,\infty)$. For continuous f on $[0,\infty)$, $s,\alpha,\beta \in \mathbb{R}^+$, and $k \in \mathbb{N}$ with $0 \leq \alpha < 1$, $0 \leq \beta < 1$ such that $\alpha+\beta < k$. If $\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{k-p,\phi} f\right)(a^+) = 0$ for $p = \alpha,\beta$, then $\forall 0 < a < x$,

$$ {}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\beta,\phi} f\right)(x) = {}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\beta,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi} f\right)(x). $$

Corollary 5.4: Let $\phi \in C^1[0,\infty)$ be an increasing function with $\phi'(x) \neq 0,\ \forall x \in [0,\infty)$. For continuous g,h on $[0,\infty)$, $s,\alpha \in \mathbb{R}^+$, $\mu \in \mathbb{R}^+$ and $n,k \in \mathbb{N}$ with $n = [\alpha]+1$, then $\forall 0 < a < x$,

$$ {}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}[g(x)+\mu h(x)] = {}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi} g(x) + \mu\, {}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi} h(x). $$

Lemma 5.2: Let $\phi \in C^1[0,\infty)$ be an increasing function with $\phi'(x) \neq 0,\ \forall x \in [0,\infty)$, and let $s,\alpha,\gamma \in \mathbb{R}^+$, $n,k \in \mathbb{N}$ with $n = [\alpha]+1$. Then $\forall 0 < a < x$,

$$ {}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\gamma}{k}} = \frac{s^{\frac{\alpha}{k}}\Gamma_k(k+\gamma)}{\Gamma_k(nk+k+\gamma-\alpha)}\left(\frac{d}{d\phi^s(x)}\right)^n\left(\phi^s(x)-\phi^s(a)\right)^{n+\frac{\gamma}{k}-\frac{\alpha}{k}}. $$

Proof. Because of its definition, we have

$$ \begin{aligned} &{}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\gamma}{k}} \\ &= \frac{s^{\frac{\alpha}{k}}}{k\Gamma_k(nk-\alpha)}\left(\frac{d}{d\phi^s(x)}\right)^n\int_a^x\left(\phi^s(x)-\phi^s(t)\right)^{\frac{nk-\alpha}{k}-1}\left(\phi^s(t)-\phi^s(a)\right)^{\frac{\gamma}{k}}d\phi^s(t) \end{aligned} $$

By substituting $z = \frac{\phi^s(t)-\phi^s(a)}{\phi^s(x)-\phi^s(a)}$, we obtain $z=0$, $z=1$, $[\phi^s(x)-\phi^s(a)]z = d\phi^s(t)$, and

$$ \begin{aligned} &{}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\gamma}{k}} \\ &= \frac{s^{\frac{\alpha-nk}{k}}}{k\Gamma_k(nk-\alpha)}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^n\int_0^1\left(\phi^s(x)-\phi^s(a)\right)^{n+\frac{\gamma}{k}-\frac{\alpha}{k}}(1-z)^{\frac{nk-\alpha}{k}-1}z^{\frac{\gamma}{k}}dz \\ &= \frac{s^{\frac{\alpha-nk}{k}}}{k\Gamma_k(nk-\alpha)}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^n\left(\phi^s(x)-\phi^s(a)\right)^{n+\frac{\gamma}{k}-\frac{\alpha}{k}}kB_k(nk-\alpha,\gamma+k). \end{aligned} $$

which grants the desired result. $\square$

5.2 The ϕ-Generalized Caputo k-Fractional Derivative

In this section, we introduce the ϕ-generalized Caputo k-fractional derivative(ϕ-GC k-FD) of order $\alpha > 0$. We will discuss relations of ϕ-GC k-FD with ϕ-GRL k-FD and ϕ-GRL k-FI as well as some simple properties.

Here is the definition of ϕ-GC k-FD.

Definition 5.3: Let $f,\phi \in C^n[0,\infty)$ be two functions such that ϕ is increasing and $\phi'(x), x \in [0,\infty)$ and let $s,\alpha \in \mathbb{R}^+$, $n,k \in \mathbb{N}$ such that $n := [\alpha]+1$ and $k(n-1) < \alpha < nk$. Then $\forall 0 < a < x$, the ϕ-generalized Caputo k-fractional derivative(ϕ-GC k-FD) of order $\alpha > 0$ is

$$ \begin{aligned} &\left({}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\phi}f\right)(x) \\ &= \frac{s^{\frac{\alpha-nk}{k}}}{k\Gamma_k(nk-\alpha)}\int_a^x\left(\phi^s(x)-\phi^s(t)\right)^{\frac{nk-\alpha}{k}-1}\left[\left(\phi^{1-s}(t)\frac{d}{d\phi(t)}\right)^n f(t)\right]d\phi^s(t). \end{aligned} $$

It can be expressed as follows

$$\left({}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\phi}f\right)(x) = {}_{a^+}^{\mathfrak{R}}\,\mathfrak{I}_{k,s}^{nk-\alpha,\phi}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^n f(x).$$

Again, we want to emphasize that we can obtain different kinds of fractional derivatives apart from the above-mentioned ones depending on selecting the choices of k,s,ϕ. For instance, ϕ-GC k-FD reduces to the a generalized Caputo type k-fractional derivative [117] when $\phi(x) = x$. On choosing $k \to 1$, $s = 1$, it coincides with the ϕ-Caputo fractional derivative [127] [139] [116] [115]. ϕ-GC k-FD with $\phi(x) = x$, $k \to 1$, $s = 1$ corresponds to the well-known Caputo fractional derivative. With the appropriate selections of ϕ,s,k, one can derive a k-Caputo fractional derivative [119], the k-Caputo Hadamard fractional derivative, the Caputo modification of the Hadamard fractional derivative [122], the Caputo type Weyl fractional derivative in addition to the Caputo–Hadamard fractional derivative [122] [125], theCaputo–Erdélyi–Kober fractional derivative [129]. One can find more details in the references [127] [139] [116] [115].

Lemma 5.3: Let $\phi \in C^n[0,\infty)$ be a function with $\phi' \neq 0$, $x \in [0,\infty)$ and let $\alpha,\beta,s \in \mathbb{R}^+$, $n,k \in \mathbb{N}$. Then $0 < a < x$,

$${}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\beta}{k}-1} = s^{\frac{\alpha}{k}}\frac{\Gamma_k(\beta-nk)\Gamma\left(\frac{\beta}{k}\right)}{\Gamma_k(\beta-\alpha)\Gamma\left(\frac{\beta}{k}-n\right)}\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\beta-\alpha}{k}-1}.$$

Proof. It is easy to calculate the following equality

$$s^{-n}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^n\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\beta}{k}-1} = \frac{\Gamma\left(\frac{\beta}{k}\right)}{\Gamma\left(\frac{\beta}{k}-n\right)}\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\beta}{k}-n-1}.$$

By using the given definition of ϕ-GC k-FD

$$
\begin{aligned}
&{}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\phi}_{k,s}\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\beta}{k}-1}\\
&=\frac{s^{\frac{\alpha}{k}}}{k\Gamma_k(nk-\alpha)}\int_a^x\left(\phi^s(x)-\phi^s(t)\right)^{\frac{nk-\alpha}{k}-1}\left(\frac{d}{d\phi^s(x)}\right)^n\left(\phi^s(t)-\phi^s(a)\right)^{\frac{\beta}{k}-1}d\phi^s(t)\\
&=\frac{s^{\frac{\alpha}{k}}\Gamma\left(\frac{\beta}{k}\right)}{k\Gamma_k(nk-\alpha)\Gamma\left(\frac{\beta}{k}-n\right)}\int_a^x\left(\phi^s(x)-\phi^s(t)\right)^{\frac{nk-\alpha}{k}-1}\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\beta}{k}-n-1}d\phi^s(t).
\end{aligned}
$$

The desired thing is obtained from substituting $y=\frac{\phi^s(t)-\phi^s(a)}{\phi^s(x)-\phi^s(a)}$, and using k-Beta function and its properties as follows,

$$
\begin{aligned}
&=\frac{s^{\frac{\alpha}{k}}\Gamma\left(\frac{\beta}{k}\right)}{k\Gamma_k(nk-\alpha)\Gamma\left(\frac{\beta}{k}-n\right)}\left(\phi^s(x)-\phi^s(a)\right)^{\frac{nk-\alpha}{k}-1+\frac{\beta}{k}-n-1+1}kB_k(nk-\alpha,\beta-nk)\\
&=s^{\frac{\alpha}{k}}\frac{\Gamma_k(\beta-nk)\Gamma\left(\frac{\beta}{k}\right)}{\Gamma_k(\beta-\alpha)\Gamma\left(\frac{\beta}{k}-n\right)}\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\beta-\alpha}{k}-1}.
\end{aligned}
$$

□

Theorem 5.5: Let $f,\phi\in C^n[0,\infty)$ be two functions such that ϕ is increasing and $\phi'(x)$, $x\in[0,\infty)$ and let $s,\alpha\in\mathbb{R}^+$, $n,k\in\mathbb{N}$ such that $n:=[\alpha]+1$ and $k(n-1)<\alpha<nk$. Then $\forall 0<a<x$,

$$
{}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s}\left({}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\phi}_{k,s}f\right)(x)=\frac{1}{k^n}\left(f(x)-\sum_{m=0}^{n-1}\frac{1}{m!}\left(\phi^s(x)-\phi^s(a)\right)^m f^{(m)}_{\phi}(a)\right)
$$

where $f^{(m)}_{\phi}(x)=s^{-m}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^m f(x)$.

Proof. We have

$$
\begin{aligned}
{}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s}\left({}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{\alpha,\phi}_{k,s}f\right)(x)&={}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s}\,{}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{nk-\alpha,\phi}_{k,s}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^n f(x)\\
&={}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{nk,\phi}_{k,s}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^n f(x)\\
&=\frac{s^{-n}}{k\Gamma_k(nk)}\int_a^x\left(\phi^s(x)-\phi^s(t)\right)^{n-1}\left[\left(\phi^{1-s}(t)\frac{d}{d\phi(t)}\right)^n f(t)\right]d\phi^s(t)\\
&=:\left({}^{\mathfrak{C}}_{a^+}\mathfrak{D}^{0,\phi}_{k,s}f\right)(x)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{k\Gamma_k(nk)} \int_a^x (\phi^s(x) - \phi^s(t))^{n-1} \left[\left(\frac{d}{d\phi^s(t)} \right)^n f(t) \right] d\phi^s(t) \\
&= \frac{1}{k^n\Gamma(n)} \int_a^x (\phi^s(x) - \phi^s(t))^{n-1} \left[\left(\frac{d}{d\phi^s(t)} \right)^n f(t) \right] d\phi^s(t). \qquad (5.1)
\end{aligned}
$$

Applying n-times integration by parts, we get

$$
\begin{aligned}
&= (\phi^s(x) - \phi^s(t))^{n-1} \left(\frac{d}{d\phi^s(t)} \right)^{n-1} f(t) \Big|_a^x \\
&+ (n-1) \int_a^x (\phi^s(x) - \phi^s(t))^{n-2} \left[\left(\frac{d}{d\phi^s(t)} \right)^{n-1} f(t) \right] d\phi^s(t) \\
&= -(\phi^s(x) - \phi^s(a))^{n-1} \left(\frac{d}{d\phi^s(t)} \right)^{n-1} f(t) \Big|_a \\
&+ (n-1) \int_a^x (\phi^s(x) - \phi^s(t))^{n-2} \left[\left(\frac{d}{d\phi^s(t)} \right)^{n-1} f(t) \right] d\phi^s(t) \\
&= -(\phi^s(x) - \phi^s(a))^{n-1} \left(\frac{d}{d\phi^s(t)} \right)^{n-1} f(t) \Big|_a \\
&- (n-1)(\phi^s(x) - \phi^s(a))^{n-2} \left(\frac{d}{d\phi^s(t)} \right)^{n-2} f(t) \Big|_a \\
&+ (n-1)(n-2) \int_a^x (\phi^s(x) - \phi^s(t))^{n-3} \left[\left(\frac{d}{d\phi^s(t)} \right)^{n-2} f(t) \right] d\phi^s(t) \\
&\vdots \\
&= -(\phi^s(x) - \phi^s(a))^{n-1} \left(\frac{d}{d\phi^s(t)} \right)^{n-1} f(t) \Big|_a \\
&- (n-1)(\phi^s(x) - \phi^s(a))^{n-2} \left(\frac{d}{d\phi^s(t)} \right)^{n-2} f(t) \Big|_a \\
&- (n-1)(n-2)(\phi^s(x) - \phi^s(a))^{n-3} \left(\frac{d}{d\phi^s(t)} \right)^{n-3} f(t) \Big|_a \\
&\vdots \\
&+ (n-1)! \int_a^x \frac{d}{d\phi^s(t)} f(t)\, d\phi^s(t). \qquad (5.2)
\end{aligned}
$$

Combining (5.1) with (5.2), we reach to the craved result. □

Corollary 5.5: Let $f, \phi \in C^n[0,\infty)$ be two functions such that ϕ is increasing and $\phi'(x), x \in [0,\infty)$ and let $s, \alpha \in \mathbb{R}^+$, $n, k \in \mathbb{N}$ such that $n := [\alpha] + 1$ and $k(n-1) < \alpha < nk$. Then $\forall 0 < a < x$,

$$ {}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\beta,\phi}\left({}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\phi}f\right)(x)=\left({}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha-\beta,\phi}f\right)(x). $$

Corollary 5.6: Let $g,h,\phi\in C^n[0,\infty)$ be two functions such that ϕ is increasing and $\phi^{'}(x)$, $x\in[0,\infty)$ and let $c_1,c_2\in\mathbb{R}$, $s,\alpha\in\mathbb{R}^+$, $n,k\in\mathbb{N}$ such that $n:=[\alpha]+1$ and $k(n-1)<\alpha<nk$. Then $\forall 0<a<x$,

$$ {}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\phi}[c_1g(x)+c_2h(x)]=c_1{}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\phi}g(x)+c_2{}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\phi}h(x). $$

Corollary 5.7: Let $f,\phi\in C^n[0,\infty)$ be two functions such that ϕ is increasing and $\phi^{'}(x)$, $x\in[0,\infty)$ and let $s,\alpha\in\mathbb{R}^+$, $n,k\in\mathbb{N}$ such that $n:=[\alpha]+1$ and $k(n-1)<\alpha<nk$. Then $\forall 0<a<x$,

$$ \left({}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\phi}f\right)(x)={}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left(f(x)-\sum_{m=0}^{n-1}\frac{1}{m!}(\phi^s(x)-\phi^s(a))^m f_{\phi}^{(m)}(a)\right). $$

Theorem 5.6: $\phi\in C^1[a,b], a>0$ is increasing with $\phi^{'}(x)\neq 0$, $x\in[a,b]$ and let $s,\alpha\in\mathbb{R}^+$, $n,k\in\mathbb{N}$ such that $n:=[\alpha]+1$ and $k(n-1)<\alpha<nk$. If $f\in C^1[a,b]$, then $\forall 0<a<x$,

$$ {}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)(x)=\frac{1}{k^n}f(x). $$

Proof. By using corollary 5.7, we have

$$ \begin{aligned} &{}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)(x)\\ &={}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left(\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)(x)-\sum_{m=0}^{n-1}\frac{1}{m!}(\phi^s(x)-\phi^s(a))^m\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)_{\phi}^{(m)}(a)\right) \end{aligned} $$

Considering that

$$ \begin{aligned} \left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)_{\phi}^{(m)}(x)&=s^{-m}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^m\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)(x)\\ &=s^{-m}\left(\phi^{1-s}(x)\frac{d}{d\phi(x)}\right)^m\frac{s^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_a^x(\phi^s(x)-\phi^s(t))^{\frac{\alpha}{k}-1}f(t)\,d\phi^s(t). \end{aligned} $$

One can easily infer the following inequality from the above equation

$$\left|\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)_{\phi}^{(m)}(x)\right| \leq \frac{s^{-\frac{\alpha}{k}}}{k^{\frac{\alpha}{k}}\Gamma_k\left(\frac{\alpha}{k}-m+1\right)}\left(\phi^s(x)-\phi^s(a)\right)^{\frac{\alpha}{k}-(m+1)}\|f\|_C,$$

and so $\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)_{\phi}^{(m)}(a)=0$ for all $m=0,1,...,n-1$. Therefore,

$${}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)(x) = {}_{a^+}^{\mathfrak{R}}\mathfrak{D}_{k,s}^{\alpha,\phi}\left({}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}f\right)(x) = \frac{1}{k^n}f(x).$$

This completes the proof. $\square$

Define the norm on $C^n([a,b],\mathbb{R})$ $\|.\|_{C_{\phi}^{[n]}}: C^n([a,b],\mathbb{R}) \to \mathbb{R}$ by

$$\|f\|_{C_{\phi}^{[n]}} := \sum_{m=0}^{n}\left\|f_{\phi}^{[m]}\right\|_C,$$

where $n \in \mathbb{N}$.

Theorem 5.7: The ϕ generalized Caputo k-fractional derivatives of order $\alpha > 0$ are bounded operators, i.e let $f,\phi \in C^n[a,b], a > 0$ be two functions such that ϕ is increasing and $\phi'(x)$, $x \in [a,b]$ and let $s,\alpha \in \mathbb{R}^+$, $n,k \in \mathbb{N}$ such that $n := [\alpha]+1$ and $k(n-1) < \alpha < nk$.

$$\left\|\left({}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\phi}f\right)(x)\right\|_C \leq M\|f\|_{C_{\phi}^{[n]}}$$

where

$$M = \frac{s^{\frac{\alpha-nk}{k}}}{\Gamma_k(nk-\alpha+1)}\left[\frac{\phi^{1-s}(b)}{\min_{x\in[a,b]}\left|\phi'(x)\right|}\right]^n\left(\phi^s(x)-\phi^s(a)\right)^{\frac{nk-\alpha}{k}}.$$

Proof. Since $\left\|f_{\phi}^{[m]}\right\|_C \leq \|f\|_{C_{\phi}^{[n]}}$ for all $0 < a < x$, we get

$$\begin{aligned}
&\left|\left({}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\phi}f\right)(x)\right| \\
&\leq \frac{s^{\frac{\alpha-nk}{k}}}{k\Gamma_k(nk-\alpha)}\int_a^x\left(\phi^s(x)-\phi^s(t)\right)^{\frac{nk-\alpha}{k}-1}d\phi^s(t)\left[\frac{\phi^{1-s}(b)}{\min_{x\in[a,b]}\left|\phi'(x)\right|}\right]^n\|f\|_{C_{\phi}^{[n]}} \\
&= \frac{s^{\frac{\alpha-nk}{k}}}{k\Gamma_k(nk-\alpha)}\left[\frac{\phi^{1-s}(b)}{\min_{x\in[a,b]}\left|\phi'(x)\right|}\right]^n\frac{\left(\phi^s(x)-\phi^s(a)\right)^{\frac{nk-\alpha}{k}}}{\frac{nk-\alpha}{k}}\|f\|_{C_{\phi}^{[n]}} \\
&= \frac{s^{\frac{\alpha-nk}{k}}}{(nk-\alpha)\Gamma_k(nk-\alpha)}\left[\frac{\phi^{1-s}(b)}{\min_{x\in[a,b]}\left|\phi'(x)\right|}\right]^n\left(\phi^s(x)-\phi^s(a)\right)^{\frac{nk-\alpha}{k}}\|f\|_{C_{\phi}^{[n]}},
\end{aligned}$$

$$= \frac{s^{\frac{\alpha-nk}{k}}}{\Gamma_k(nk-\alpha+1)} \left[\frac{\phi^{1-s}(b)}{\min_{x\in[a,b]}|\phi'(x)|}\right]^n (\phi^s(x)-\phi^s(a))^{\frac{nk-\alpha}{k}} \|f\|_{C_\phi^{[n]}}.$$

which is obtained the desired result. □

The solution of non-homogenous linear differential equation with the ϕ-generalized Caputo k-fractional derivative under special choices of parameters is given in following section.

5.3 Applications

In this section, we look for a solution of the Cauchy-type problem for non-homogeneous linear ϕ- generalized Caputo k- fractional differential equation. A solution of the Cauchy-type problem for ϕ-GRL k-FDEs in the same form can be examined in the similar manner.

Theorem 5.8: For two functions $y,\phi \in C[0,\infty)$ such that ϕ is increasing and $\phi'(x)$, $x\in[0,\infty)$ and $s\in\mathbb{R}^+$, $0<\alpha<1$, $k\in\mathbb{N}$ such that $0<\alpha<k$ and $\lambda, c\in\mathbb{R}$. The following fractional initial value problem

$$ {}_{a^+}^{\mathfrak{C}}\mathfrak{D}_{k,s}^{\alpha,\phi} y(x) - \lambda y(x) = f(x), \tag{5.3}$$

$$y(a) = c, \tag{5.4}$$

is of the solution

$$y(x) = cE_{\frac{\alpha}{k}}\left(\varphi_{k,s}^{\alpha,\phi}(x,a)\right) + \frac{s^{-\frac{\alpha}{k}}}{k^{\frac{\alpha}{k}-1}}\int_a^x (\phi^s(x)-\phi^s(t))^{\frac{\alpha}{k}-1} E_{\frac{\alpha}{k},\frac{\alpha}{k}}\left(\varphi_{k,s}^{\alpha,\phi}(x,t)\right) f(t)\,d\phi^s(t), \tag{5.5}$$

where $\varphi_{k,s}^{\alpha,\phi}(x,y) := k^{1-\frac{\alpha}{k}}\lambda\left(\frac{\phi^s(x)-\phi^s(y)}{s}\right)^{\frac{\alpha}{k}}$.

Proof. By applying ${}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi}$ to both sides of 5.3 and using Theorem 5.2 and Corollary 5.7, we get

$$y(x) = y(a) + k\lambda\, {}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi} y(x) + k\, {}_{a^+}^{\mathfrak{R}}\mathfrak{I}_{k,s}^{\alpha,\phi} f(x).$$

To solve this integral equation, we use the method of successive approximation. According to this method, we set:

$$y_0(x) = y(a),$$

$$y_m(x) = y_0(x) + k\lambda\, {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s} y_{m-1}(x) + k\, {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s} f(x)$$

where $m \geq 1$. For $m = 1$, we have

$$y_1(x) = y_0(x) + k\lambda\, {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s} y_0(x) + k\, {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s} f(x).$$

By rewriting and regulating

$$y_1(x) = y(a) + k\lambda y(a) \frac{s^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \left(\phi^s(x) - \phi^s(a)\right)^{\frac{\alpha}{k}} + k\, {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s} f(x).$$

Similarly we find for $y_2(x)$ that

$$y_2(x) = y_0(x) + k\lambda\, {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s} \left[y(a) + k\lambda y(a) \frac{s^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \left(\phi^s(x) - \phi^s(a)\right)^{\frac{\alpha}{k}} + k\, {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s} f(x) \right]$$

$$+ k\, {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{\alpha,\phi}_{k,s} f(x).$$

With the help of Lemma 5.1, one can easily reach to

$$y_2(x) = y(a) \sum_{j=1}^{3} \frac{k^{j-1}\lambda^{j-1} s^{-(j-1)\frac{\alpha}{k}}}{\Gamma_k((j-1)\alpha+k)} \left(\phi^s(x) - \phi^s(a)\right)^{\frac{(j-1)\alpha}{k}} + \sum_{j=1}^{2} k^j \lambda^{j-1}\, {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{j\alpha,\phi}_{k,s} f(x).$$

By keeping on this process, we derive the following equation for $y_m(x)$, $m \geq 1$

$$y_m(x) = y(a) \sum_{j=1}^{m+1} \frac{k^{j-1}\lambda^{j-1} s^{-(j-1)\frac{\alpha}{k}}}{\Gamma_k((j-1)\alpha+k)} \left(\phi^s(x) - \phi^s(a)\right)^{\frac{(j-1)\alpha}{k}} + \sum_{j=1}^{m} k^j \lambda^{j-1}\, {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{j\alpha,\phi}_{k,s} f(x).$$

Taking the limit while m tends to ∞, we get the following explicit pattern of $y(x)$ to the solution of 5.3 and 5.4:

$$y(x) = y(a) \sum_{j=1}^{\infty} \frac{k^{j-1}\lambda^{j-1} s^{-(j-1)\frac{\alpha}{k}}}{\Gamma_k((j-1)\alpha+k)} \left(\phi^s(x) - \phi^s(a)\right)^{\frac{(j-1)\alpha}{k}} + \sum_{j=1}^{\infty} k^j \lambda^{j-1}\, {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{j\alpha,\phi}_{k,s} f(x).$$

By replacing the index of summation j by $j-1$, we have

$$y(x) = y(a) \sum_{j=0}^{\infty} \frac{k^{j}\lambda^{j} s^{-j\frac{\alpha}{k}}}{\Gamma_k(j\alpha+k)} \left(\phi^s(x) - \phi^s(a)\right)^{\frac{j\alpha}{k}} + \sum_{j=0}^{\infty} k^{j+1} \lambda^{j}\, {}^{\mathfrak{R}}_{a^+}\mathfrak{I}^{(j+1)\alpha,\phi}_{k,s} f(x).$$

which provides us the required result by keeping in mind the given definition of ϕ-GRL k-FI and k-Gamma and k-Beta functions and their features . □

Corollary 5.8: For $0 < \alpha < 1$, the following special case of the fractional initial value problem (5.3)-(5.4):

$$
{}^{\mathfrak{C}}_{0^+}\mathfrak{D}^{\alpha,x^2}_{2,1}y(x) - 2^{\frac{\alpha}{2}-1}y(x) = 1,
$$

$$
y(0) = 1,
$$

is of the solution obtained from the equation 5.6

$$
y(x) = E_{\frac{\alpha}{2}}\left(\left(x^2\right)^{\frac{\alpha}{2}}\right) + 2^{\frac{\alpha}{2}-1}\int_0^x \left(x^2 - t^2\right)^{\frac{\alpha}{2}-1} E_{\frac{\alpha}{2},\frac{\alpha}{2}}\left(\left(x^2 - t^2\right)^{\frac{\alpha}{2}}\right) dt^2, \tag{5.6}
$$

which is equal to

$$
y(x) = E_{\frac{\alpha}{2}}\left(x^{\alpha}\right) + 2^{1-\frac{\alpha}{2}}x^{\alpha}E_{\frac{\alpha}{2}\frac{\alpha}{2}+1}\left(x^{\alpha}\right).
$$

5.4 Suggested Problems

It is surely beyond doubt that there are lots of new things over the basement of theory consisting of new fractional derivatives and integrals. One can research for chain rule, exponential functions, Gronwall's inequality, integration by parts, Taylor power series expansions, Laplace transforms, the Rolle's, Cauchy, Lagrange's and Darboux's theorem, and all the rest in the context of both ϕ-generalized Riemann-Liouville and Caputo k-fractional derivatives.

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