

A Study of Integer Partitions and their Derivations

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ABSTRACT

This M.Sc. thesis studies the partitions of integers, mainly restricted integers and how to derive them methodically. Analysis is made of different theories of calculating integers which are generating functions, Euler's identity, McMahon's recurrence, Sylvester's approach, Frobenius partitions and generalized partitions. Chapter 1 shows how to obtain partition identities using Ferrer's diagram, Durfee square and Jacobi's triple product identity. The basic generation of partition of integers is considered first. This is followed by the expression of partitions using Ferrer's diagram in chapter 2. In chapter 3, the number of partitions in a set of integers is calculated using the method of function generation. Using the preceding chapters, partition identities are obtained and further explained them in chapter 5 using Durfee squares and its relation to Ferrer's diagram. Euler's identity is proven combinatorically by means of bijection in chapter 6 and Euler's pentagonal number is used to represent a special case of Jacobi's triple product identity in chapter 7. When the pattern of a pentagonal number is notable, McMahon's approach is used to generate functions to calculate partitions in restricted integers as discussed in chapter 8. The first Sylvester wave is defined which is an explicit formula for the polynomial part of a restricted partition function. The last three chapters looks at special cases in generalized partitions and use Euler's result for identically distributed partitions.

Keywords: Generating functions, Restricted integers, Euler's identity, Euler's pentagonal number, Ferrer's diagram, Durfee square, McMahon's approach, Bijection, Sylvester wave, Generalized partitions, Euler's result.

ÖZ

Bu tez, tamsayıların, özellikle kısıtlı tamsayıların bölümlerini ve bunların metodik olarak nasıl türetilceğini inceler. Analiz, üreten fonksiyonlar, Euler'in kanunu, McMahon'un tekrarı, Sylvester'ın yaklaşımı, Frobenius bölümleri ve genelleştirilmiş bölümler gibi tamsayıları hesaplamak için farklı teorilerden yapılmıştır. Bölüm 1, Ferrer diyagramı, Durfee karesi ve Jacobi'nin üçlü ürün kimliği kullanılarak bölüm kimliklerinin nasıl elde edileceğini gösterir. İlk önce tamsayıların bölünmesinin temel nesli düşünülür. Bunu, bölüm 2'deki Ferrer diyagramını kullanarak bölümlerin ifadesi takip eder. Bölüm 3'te, bir tamsayı kümesindeki bölümlerin sayısı, fonksiyon oluşturma yöntemi kullanılarak hesaplanır. Önceki bölümleri kullanarak, bölüm kimlikleri elde edilir ve Bölüm 5'te Durfee kareleri ve bunun Ferrer diyagramıyla ilişkisi kullanılarak daha ayrıntılı olarak açıklanır. Euler'in kanunu, 6. bölümde alıntılama yoluyla kombinatorik olarak kanıtlanmıştır ve Euler'in beşgen sayısı, 7. bölümde Jacobi'nin üçlü ürün kimliğinin özel bir durumunu temsil etmek için kullanılmıştır. Beşgen bir sayının modeli dikkate değer olduğunda, McMahon'un yaklaşımı, 8. bölümde tartışılan kısıtlı tamsayılardaki bölümleri hesaplamak için fonksiyonlar oluşturmak için kullanılır. Kısıtlı bir bölme fonksiyonunun polinom kısmı için açık bir formül olan ilk Sylvester dalgası tanımlanmıştır. Son üç bölüm, genelleştirilmiş bölmelerdeki özel durumlara bakar ve özdeş olarak dağıtılmış bölümler için Euler'in sonucunu kullanır.

Anahtar Kelimeler: Üreten fonksiyonlar, Kısıtlı tamsayılar, Euler'in özdeşliği, Euler'in beşgen sayısı, Ferrer'in diyagramı, Durfee karesi, McMahon'un yaklaşımı, Bijection, Sylvester dalgası, Genelleştirilmiş bölümler, Euler'in sonucu.

DEDICATION

To My Late Mother Beletu Kebede

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Firstly, I am grateful to my supervisor Prof. Dr. Benedek Nagy for his unhindered assistance. I am thankful for his conscientious work in checking my paper is proper and satisfactory.

In addition, I would also want to extend my gratitude to my father for his unwavering support and a source of strong encouragement since the beginning of my thesis. I will not be able to reach my goals without your patience that cannot be underestimated. Thank you for teaching me the concept of for every n time I fall, to get back up $n+1$ times.

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Chapter 1

INTRODUCTION

In 1674, Gottfried Leibniz, a mathematician introduced the study of integer partitions. His main objective was to get a clear understanding of how many times a positive integer can be subdivided into partitions and still find the sum of the number (Cameron, 1994). For instance, let us consider n as the partition, it becomes a positive integer, having its parts as positive integers $p_1, p_2, p_3, \dots, P_n$ which when added together totals to be the m value. The p_1 or p_2 is what is referred to as partition parts. The number of partitions is denoted by $p(m)$ for m which is the integer. This paper examines what is integer partition identities, their terms, theorems, and applications.

Example 1: $p(5) = 7$, this means all the 7 partitions of integer $m = 5$ are: $5 = (5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)$.

- $p(m, r)$ represents how many division of m with r segments, for example $p(5, 1) = 1, p(5, 2) = 2, p(5, 3) = 2, p(5, 4) = 3, p(5, 5) = 3, p(5, 6) = 4$ and $p(5, 7) = 5$.
- $p(m)$ represents the whole quantity of m .
- $q(m, r)$ represents how many divisions of m with r distinct parts.
- $p(m)$ represents the total quantity of division of m with distinct parts.

The number of integer partition $p(m)$ was difficult to find when m was a large value. The challenge was finally solved by the development of function generations which made it simple to calculate partitions total in an integer (Wilson & Van Lint, 2001).

The generating function for the number of partitions of m would be:

$$p(z) = (1 + z + z^2 + z^3 + \dots)(1 + z^2 + z^4 + z^6 + \dots)(1 + z^3 + z^6 + z^9 + \dots) \dots$$

$$p(z) = \left(\frac{1}{1-z}\right) \left(\frac{1}{1-z^2}\right) \left(\frac{1}{1-z^3}\right) \dots \left(\frac{1}{1-z^m}\right) \dots$$

$$p(z) = \prod_{u \geq 1} \frac{1}{1-z^u} \quad (1.1)$$

where $(1 + z + z^2 + z^3 + \dots)$ gives number of 1s in the partition,

$(1 + z^2 + z^4 + z^6 + \dots)$ gives number of 2s in the partition, and so on.

However, various theories arose while trying to calculate the total number of partitions in all forms belonged to the integers.

Chapter 2

FERRER'S DIAGRAM

(Yee, 2003) defined Ferrer's diagram as a model of integer partition that gives a great overview of visualizing the parts and giving identities. Ferrer came up with the model by constructing a stack on the left side row cells. The total cells in every row aligns with the part sizes. On the other hand, rows were aligned according to their size with the first-row corresponding to the largest part and so forth. For instance,

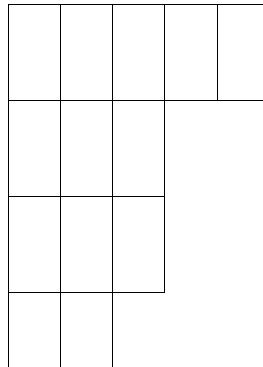


Figure 1: The Ferrer's diagram for $(5, 3, 3, 2)$

When the conjugate is inverted for columns to be rows and rows to be columns, the diagram appears as Figure 2 and is represented as the λ^* to denote the conjugate of λ , and $\lambda^* = (5, 3, 3, 2)^* = (4, 4, 3, 1, 1)$ is obtained.

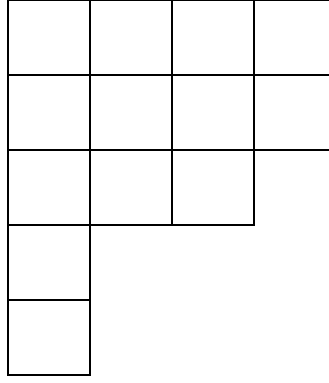


Figure 2: Conjugate of figure 1 partition

Firstly, we should notice that our original partition is the conjugate of the conjugate, meaning $(\lambda^*)^* = \lambda$. So $(4, 4, 3, 1, 1)^* = (5, 3, 3, 2) = \lambda$. Secondly, λ total parts is the same to the λ^* biggest part. In **example 1**, $m=5$ has $r=4$ parts which is λ^* biggest part. Hence $p(m, r)$ is likely the quantity of divisions of m with r parts, with the biggest part $\lambda_1 = r$. The form λ^* is constructed straightforwardly excluding the Ferrer's charts. Consider $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ and its form be $\lambda^* = (\lambda_1^*, \lambda_2^* \dots \lambda_h^*)$, we find out $\lambda_1^* = k$ and $\lambda_1 = h$.

We know that the size of λ_u^* is the size of column u in λ , likewise in this column, there is a single cell for each row of size at least u . Hence λ_u^* is equivalent to the quantity of parts $\geq u$ in λ , or same to the biggest v so that $\lambda_v \geq u$.

Example 2: By computing $(4, 4, 3, 3, 3, 1)^*$ to get $(6, 5, 5, 2)$, λ_6 of λ becomes the last 1, λ_5 is the last part ≥ 2 and the last 3, λ_2 is the last 4 in λ .

Example 3: Using Figure 1 for $(5, 3, 3, 2)$, λ_4 is the last 2, λ_3 is the last 3, λ_1 is the last part ≥ 4 and the last 5.

We refer a partition as a self-conjugate when $\lambda^* = \lambda$, implying that the Ferrer's graph is symmetric, as shown in Figure 3, where $\lambda = (5,4,2,2,1)$:

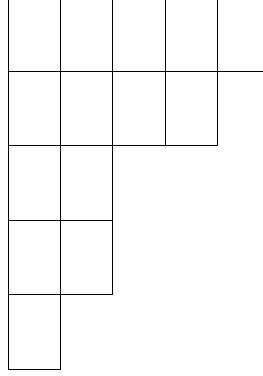


Figure 3: Ferrer's graph of $\lambda = (5,4,2,2,1)$

Using the Ferrer's diagrams, the following partition theory are proven;

First theory: $k(m)$ is another quantity of divisions of m into distinct, odd parts.

Proof: $\phi(\lambda)$ is referred as the partition with hooks parts in the diagram of λ , by the self-conjugate λ when a bijection ϕ from $\{\text{self-conjugate } \lambda \mid m\}$ to $\{\lambda \mid m \text{ with distinct odd parts}\}$ is given.

This gives us $\phi((5,4,2,2,1)) = (9,5)$. Each hook contains an odd size since it is symmetric in the middle, and each hook is bigger than the following one. Hence $\phi(\lambda)$ has unique odd parts. Likewise, we can create corresponding symmetric hooks by having the distinct odd a given distinct odd numbers, a related symmetric can be made to attach and rest them as one into a graph. We can have a bijection if ϕ^{-1} is defined.

Second theory: The m total partitions with l , the parts are same to integer m total partition with l as its largest part. Considering Ferrer's diagram, function $p(m, l)$ defines the numbers of m partitions with its greatest part equaling to l .

Let's consider looking for a simple way of defining the function which leads to $\sum_{m=0}^{\infty} p(m)z^m$ by generating the function from Euler's theory for the sequence $\{p(m)\}_{m=0}^{\infty}$.

Let's consider, obtaining $\sum_{m=0}^{\infty} p(m)z^m$ as our result by expanding to

$$(1 + z + z^2 + z^3 \dots) (1 + z^2 + z^4 + z^6 \dots) (1 + z^3 + z^6 \dots) (1 + z^4 + z^8 \dots) \dots \quad (2.1)$$

From the product of equation (2.1), we get a geometric series from every term and its products can be as:

$$\frac{1}{1-z} \cdot \frac{1}{1-z^2} \cdot \frac{1}{1-z^3} \dots$$

The above observation can lead to Euler's Theorem that will be discussed later.

Another example for similar variations can be derived;

Example 4: By taking z^3 as a coefficient and choose z for the primary parenthesis, z^2 following, together with 1 from the other parentheses, the summation of 1 is gotten as the coefficient of z^3 . Likewise, if z^3 is chosen from the third parenthesis, and 1 from all others, the result is the coefficient of z^3 from the contribution of 1 .

The example is related to integer partition if the total times part u appear in the partition is denoted by monomial chosen from the u -th which is $1+x^u+x^{2u}+x^{3u} \dots$, in (2.1) above. The value of u will be seen c_u times in the partition if the monomial chosen from the u -th which is $1+x_u+x_{2u}+x_{3u} \dots$, in (2.1) above. The value of u will be seen c_u times in the partition if the monomial $z_{c_u u}$ from the u -th parenthesis is considered.

We generate a single contribution to the coefficient of z^n and in general by choosing every monomial. If only the contributions occur in the form $z^{1c_1} \cdot z^{2c_2} \cdot z^{3c_3} \cdots = z^{c_1+2c_2+3c_3} \cdots$

Hence, the coefficient of z^n is the number of ways of representing $n = c_1 + 2c_2 + 3c_3 + \cdots$ where each $c_u \geq 0$.

2.1 The Frobenius Partitions

Given λ , Frobenius saw the opportunity to give an immediate conjugate of a partition using Ferrer's diagram of λ .

Example 5: the Frobenius notation for $7+7+5+4+2+2$ is given as follows:

First, we draw the Ferrer's graph for the partition

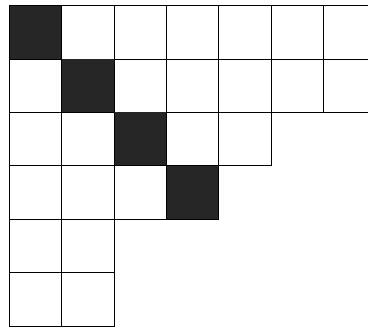


Figure 4: Ferrer's graph of $\lambda = (7, 7, 5, 4, 2, 2)$

Then we delete the main diagonal shown in Figure 4. The columns and rows to the right and left of the diagonal, respectively, are computed as strictly decreasing sequence of integers.

If there are no ‘cells’ to the right and below of the diagonal’s last cell, we produce 0.

Thus, for our example, we obtain the Frobenius notation as $\begin{pmatrix} 6 & 5 & 2 & 0 \\ 5 & 4 & 1 & 0 \end{pmatrix}$ and the

conjugate partition will be $\begin{pmatrix} 5 & 4 & 1 & 0 \\ 6 & 5 & 2 & 0 \end{pmatrix}$.

Example 6: Given the partition $5+5+4+4+2+2+1$, we draw the Ferrer’s graph and give the Frobenius notation as follows:

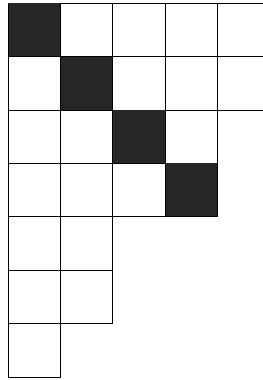


Figure 5: Ferrer’s graph of $\lambda = (5, 5, 4, 4, 2, 2, 1)$

Frobenius notation: $\begin{pmatrix} 4 & 3 & 1 & 0 \\ 6 & 4 & 1 & 0 \end{pmatrix}$

Example 7: Using $m=27$ in example 5, another partition for 27 is $7+6+6+3+3+2$ and we give the Frobenius notation as follows:

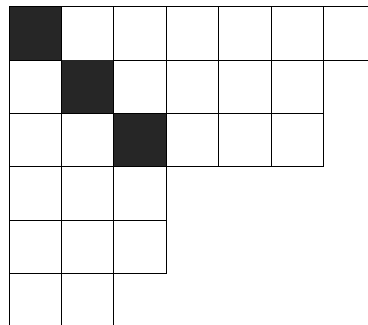


Figure 6: Ferrer’s graph of $\lambda = (7, 6, 6, 3, 3, 2)$

Frobenius notation: $\begin{pmatrix} 6 & 4 & 3 \\ 5 & 4 & 2 \end{pmatrix}$

Chapter 3

GENERATING FUNCTIONS FOR PARTITIONS

By generating function, the elements of one identity function are the same as the numbers of elements in another different identity (Yee,2003). However, this approach is different from the bijective proof where a bijection between the identity of two different sets of objects is used.

We can consider generating function $P(z)$ which is equal to $\sum_{m=0}^{\infty} p(m)z^m$ that represent the total partitions of m numbers in relation to quantity z^m for a segment of m .

By considering a form segment λ of unrestricted m , each positive integer u can be chosen individually how many times it can be included as a part of λ . We contribute each part of u to the overall size of m by using u as a part.

The generating function for the choice of using any quantity of reoccurrence of part u , the generating function is represented as $1 + z^u + z^{2u} + \dots = 1/(1-z^u)$. Multiplication of all u , the result obtained is;

$$p(z) = \sum_n p(m)z^m = \prod_{u=1}^{\infty} \frac{1}{1-z^u} = \frac{1}{(1-z)(1-z^2)(1-z^3) \dots} \quad (3.1)$$

To get any certain $p(m)$ as a coefficient, all the terms having z to a power m or reduced are multiplied.

An example to prove the above, consider multiplying the product,

$$\frac{1}{1-z} \cdot \frac{1}{1-z^2} \cdot \frac{1}{1-z^3} \frac{1}{1-z^4} \frac{1}{1-z^5} \dots$$

taking z^7 as our final consideration, every segments of m for $m = 0$ to 7 is computed by comparing the coefficient of the values of $p(0)$ through $p(7)$. We obtain endless variations through the strategy used to represent $p(z)$.

For instance;

I. Work only with the terms for $u = 1, 2, \dots, n$ to obtain

$$p_{\leq n}(z) = \sum p_{\leq n}(m)z^m = \prod_{u=1}^n \frac{1}{1-z^u} = \frac{1}{(1-z)(1-z^2)\dots(1-z^n)} \quad (3.2)$$

and to count partitions whose parts are represented by $\leq n$. The division between divisions of m with most n parts and a partition of m with parts $\leq n$ is given by taking the conjugate partition. Hence $P_{\leq n}(z)$ estimates also divisions no more than n parts.

II. We use conjugates and quantity of segments having the biggest part equivalent to n to get the segments with specifically n parts. A similar case to **equation 3.2**, except that the factor $1/(1-z^n)$ is replaced with $z^n + z^{2n} + \dots = z^n / (1-z^n)$ to prove that a single part should be same as n . Thus, giving the generating function as

$$p_n(z) = \sum_m p(m, n)z^m = \frac{z^n}{(1-z)(1-z^2)\dots(1-z^n)} \quad (3.3)$$

III. We determine the number $o(m)$ belonging to m as a partition working only odd parts by using the terms of odd values u , then obtain the function generated as

$$O(z) = \sum_n O(m)z^m = \prod_{u \text{ odd}} \frac{1}{1-z^u} = \frac{1}{(1-z)(1-z^3)(1-z^5)} \dots \quad (3.4)$$

IV. We can obtain the generating function by counting distinct parts partitions, by only selecting for every u by determining either to use the part u singly or not use it completely, meaning, our partition will be set instead of a multiset.

Therefore, we replace the factor $1/(1-z^u)$ in $P(z)$ with $(1+z^u)$,

$$Q(z) = \sum_m q(m)z^m = \prod_{u=1}^{\infty} (1+z^u) = (1+z)(1+z^2)(1+z^3)\dots \quad (3.5)$$

V. A generating function can be obtained by combining the formula of examples **III** and **IV** by counting partitions with distinct, odd parts.

$$K(z) = \sum_m k(m)z^m = \prod_{u=odd} (1+z^u) = (1+z)(1+z^3)(1+z^5)\dots \quad (3.6)$$

This is also an example of obtaining a function generated by recording self-conjugate segments:

$$K(z) = \sum_m k(m)z^m \quad (3.7)$$

VI. Considering two-variable generating function for all partitions, $p(m, n)$ can be obtained as a generating function by counting segment $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \vdash m$ with quality of $s^n z^m$ having m as the size and n as the quantity of parts. Then sz^u becomes the factor that gives the quantity summation for one part instead of just z^u and accordingly, the generating function is obtained

$$p(z, s) = \sum_{m,n} p(m, n) s^n z^m = \prod_{u=1}^{\infty} \frac{1}{1-sz^u} = \frac{1}{(1-sz)(1-sz^2)(1-sz^3)} \dots \quad (3.8)$$

We go back to our actual generating function $P(z)$ by setting $s = 1$.

Chapter 4

PARTITION IDENTITIES

According to (Andrews & Eriksson, 2004), partition identities can be determined in several ways.

For instance, $p(m, n)$ was counted into two essentially different ways in the previous section. By applying the 2-variable generating function directly to get,

$$p(z, s) = \sum_{m,n} p(m, n) s^n z^m = \prod_{u=1}^{\infty} \frac{1}{1-sz^u} \quad (4.1)$$

and by applying conjugation indirectly as our second method to get

$$p_n(z) = \sum_m p(m, n) z^n = \frac{z^n}{(1-z) \cdots (1-z^n)}. \quad (4.2)$$

By summing these two equations we obtain a 2-variable generating function with a factor s^n for all n :

$$p(z, s) = \sum_{m,n} p(m, n) s^n z^m = \sum_n p_n(z) s^n = \sum_n \frac{s^n z^n}{(1-z) \cdots (1-z^n)}$$

getting our partition identity as

$$\prod_{u=1}^{\infty} \frac{1}{1-sz^u} = \sum_{n=0}^{\infty} \frac{s^n z^n}{(1-z) \cdots (1-z^n)} \quad (4.3)$$

The result is a purely algebraic identity, though we derived it combinatorically.

Mainly in combinatorics, we focus on finding combinatorial elaborations for algebraic identities however, the above identities are primarily derived with completely different methods which can be explained in future combinatorically.

A subtler combinatorial analysis of the Ferrer's diagram enables us to obtain further identities. For instance, an analog for partitions with distinct parts like the above work.

If n distinct parts belong to λ , its diagram should obtain a “staircase” segment $(n-1, n-2, \dots, 1)$. Besides, it should be noted that, the segments of a common division with n parts are the same as the rows of the difference, and the former partition might be arbitrary. This is illustrated in the figure below using $\lambda = (6, 5, 3, 1)$.

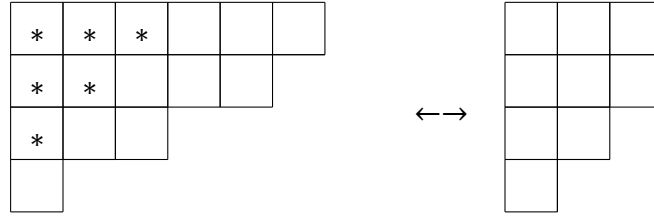


Figure 7: Ferrer's diagram for $\lambda = (6, 5, 3, 1)$

On the left of Figure 7 is the model of λ together with some staircase diagram denoted as *'s found inside it. On the right is the corresponding difference partition. A partition with n distinct parts can be chosen by considering an ordinary partition with n parts and later use a staircase to boost it. This leads to the addition of $\binom{n}{2}$ on n whole estimate or multiplication of the function generated by $z^{\binom{n}{2}}$.

This gives the function generated below:

$$Q_n(z) = \sum_m q(m, n) z^n = \frac{z^{n+\binom{n}{2}}}{(1-z) \cdots (1-z)^n} \quad (4.4)$$

For partitions with distinct parts, we work out as above and result to partition identity for:

$$Q(z, s) = \sum_{m,n} q(m, n) s^n z^m = \prod_{u=1}^{\infty} (1 + s z^u) = \sum_{n=0}^{\infty} \frac{s^n z^n + \binom{n}{2}}{(1-z) \cdots (1-z^n)}. \quad (4.5)$$

Reversibly, we can also apply the connection between partition combinatorics and algebraic identities and obtain complex combinatorial facts.

We can use the generating functions found in the previous section that were applied on segments with distinct parts,

$$Q(z) = (1 + z) (1 + z^2) (1 + z^3) \cdots \quad (4.6)$$

and those applied on partitions with odd parts

$$O(z) = \frac{1}{(1-z)(1-z^3)(1-z^5) \cdots} \quad (4.7)$$

We write $O(z)$ with the results of $(1-z^u)$ with every u in the lower part (denominator), canceled with the result for even u in the upper part (numerator) in order to simplify it,

$$\begin{aligned} O(z) &= \frac{(1-z^2)(1-z^4)(1-z^6) \cdots}{(1-z)(1-z^2)(1-z^3) \cdots} \\ &= \frac{\prod_u (1 - z^{2u})}{\prod_u (1 - z^u)} \end{aligned}$$

but $(1 - z^{2u}) / (1 - z^u) = 1 + z^u$, giving

$$O(z) = \prod_u (1 + z^u) = Q(z) \quad (4.8)$$

Thus, using simple algebraic manipulation together with the generating function we have proved the combinatorial identity that is equation (4.8).

The total $o(m)$ segments of m into odd parts is equivalent to the $q(m)$ partitions of m into specific parts for every m .

Chapter 5

THE DURFEE SQUARE

Through the analysis of the Ferrer's diagrams another method of obtaining partition identities using the Durfee square is discovered (Gessel, 1984). The only biggest model of partition that can occupy the above left side in the Ferrer's graph is the Durfee square of λ . For instance, in Figure 8, the Durfee square is denoted with $*$'s:

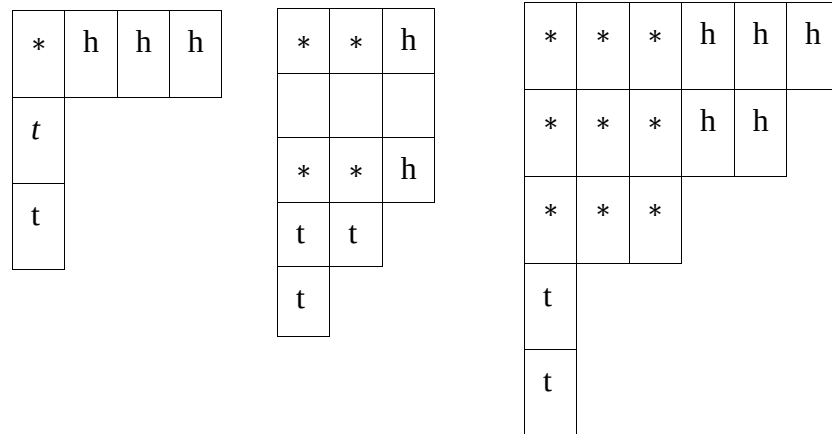


Figure 8: Durfee Square in Ferrer's diagram

When the square is d by d , d becomes λ representing the Durfee quantity. The other parts of the illustration λ have double parts, known as the hand (denoted by h 's) and another the toes (denoted by t 's). Both the hand and the toes represent diagrams themselves. The partition with at most d segments can have both the hand and the toes.

In case Durfee number d with a part of unrestricted n is shown, the D square is started later the hand and toes can be chosen independently. The D square adds up d^2 to the whole n size. The choice of the hand and of a toe generating function is given as;

$$P_{\leq d}(z) = \frac{1}{(1-z)(1-z^2)\dots(1-z^d)} \quad (5.1)$$

By multiplication, we obtain the Durfee number d together with the generating function as;

$$z^{d^2} P_{\leq d}(z)^2 = \frac{z^{d^2}}{(1-z)^2(1-z^2)^2 \dots (1-z^d)^2} \quad (5.2)$$

We get the identity below by totaling all the values of d which by addition principle gives $P(z)$.

$$\begin{aligned} \prod_{i=1}^{\infty} \frac{1}{1-x^i} &= \sum_{d=0}^{\infty} \frac{x^{d^2}}{(1-x)^2(1-x^2)^2 \dots (1-x^d)^2} \\ &= 1 + \frac{x}{(1-x)^2} + \frac{x^4}{(1-x)^2(1-x^2)^2} + \frac{x^9}{(1-x)^2(1-x^2)^2(1-x^3)^2} + \dots \end{aligned} \quad (5.3)$$

Chapter 6

THE EULER'S IDENTITY

Euler discovered the partition equivalences below,

$$\prod_u (1 - z^u) = 1 + \sum_{n=1}^{\infty} (-1)^n (z^{(3n^2-n)/2} + z^{(3n^2+n)/2}) \quad (6.1)$$

Starting from the right, it elaborates a power series having many terms as zero, the rest having a coefficient ± 1 ; this occurs from the exponents z^m having m as the form $(3n^2 \pm n)/2$ (Andrews, 1986). The equation can be written clearer by writing terms to read as:

$$(1 - z) (1 - z^2) (1 - z^3) \cdots = 1 - z - z^2 + z^5 + z^7 - z^{12} - z^{15} + z^{22} + z^{26} - \cdots$$

Euler's identity can be interpreted combinatorically and proven to employ a bijection

$$Q(z, s) = \prod_{u=1}^{\infty} (1 + sz^u) \quad (6.2)$$

which represent partition into specific parts from the generating function, considered having weight of $s^n z^m$ for the partition λ belonging to m with n parts. λ is represented as a partition of m into distinct parts by using a notation $\lambda \models m$.

Using $s = -1$ in $Q(z, s)$, if a partition into distinct parts has an even number of parts then its weight is set to z^m and $-z^m$, if it possesses an odd number of parts. We know, $Q(z, -1)$ is the left hand side of Euler's identity.

Therefore, the coefficient of z^m in $Q(z, -1)$ is

$$|\{\lambda \models m, \text{ even number of parts}\}| - |\{\lambda \models m, \text{ odd number of parts}\}|. \quad (6.3)$$

We must show the equivalence by proving the contrast in **equation (6.3)** is likewise estimated by the right-hand side of the generating function in **equation (6.2)**.

Identity **(6.1)** holds when we can prove that $(\lambda \models m, \text{even number of parts})$ equals the same number as $(\lambda \models m, \text{odd number of parts})$ unless m is a pentagonal number. A number is a pentagonal number if it can be represented as a pattern of pentagonal dots. The first few are 0,1,2,5,7,12,15. In the case m is a pentagonal number, when n is even, we will have an additional segment λ of even parts and when n is odd, we will have similar additional segment but with odd parts. We obtain our “wedge-shaped” segments $(2n-1, 2n-2, \dots, n)$ and $(2n, 2n-1, \dots, n+1)$, for every n for the additional partitions (George E. Andrews and Kimmo Eriksson, 2004). For $n=3$, we have the following:

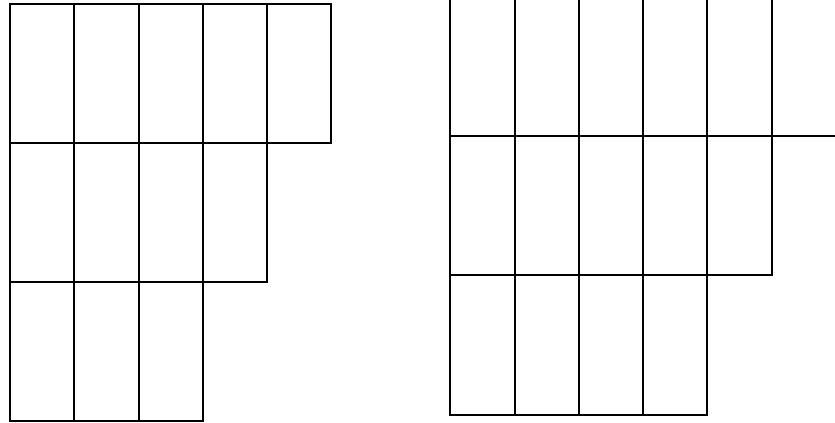


Figure 9: Ferrer's diagram for $n=3$

From the above, the initial wedge (left) is a segment of $(3n^2 - n)/2$ and the following (right) is a segment of $(3n^2 + n)/2$.

It is then proven that the properties below are specifically for wedges partition. First, every one of the parts is successive. Secondly, the final part can be equivalent to the quantity of parts n , or $n+1$.

The hypothesis is proven by making a bijection between non-wedge having distinct even parts of λ and non-wedge odd distinct odd parts of λ .

The involution S is used for all non-wedge parts having distinct parts and a bijection is defined. S involution is defined as an activity with an end goal making $S \circ S$ the identity, for instance, in the event that it is done two times, it is returned to where it started. The involution is arranged to convert the quantity of parts by 1 or -1, therefore $S(\lambda)$ should consistently consist precisely one less or more part than λ .

Double numbers dictated by the outline of $\lambda \models n$ is utilized to characterize S , and $a(\lambda)$ define as the most little piece of λ or the length of the last row in the illustration. We have $b(\lambda)$ defined as the length of the slanting line to the left bottom from the last cell of first row of the illustration. The letters 'a' and 'b' are some marked cells on the diagram and not numbers.

An instance, when λ of Figure 10 is considered, $|a| = 3$, and $|b| = 3$ on the grounds that the first 3 section (8,7,6) are back-to-back. Outline will appear as it is shown on Figure 10.

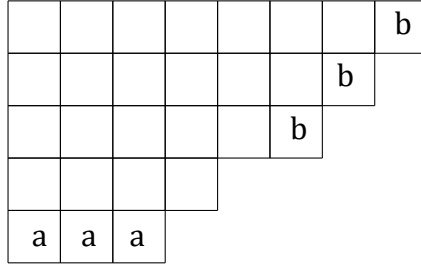


Figure 10: Ferrer's diagram for $\lambda = (8,7,6,4,3)$

The activity S can transfer all 'a' cell to the 'b' cell position or the other way, depending on which one makes sense. We end up with different cases:

First model ($|a| \leq |b|$). In such situation, as shown in Figure 11, the last column of λ or all marked 'a' cells are eliminated and every one of the main 'b' line is expanded by one cell.

Example 8;

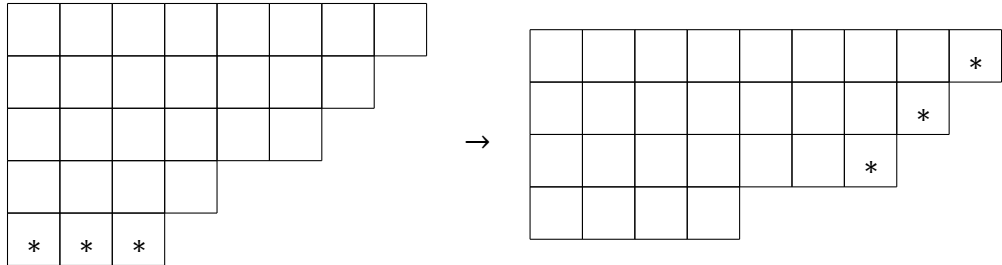


Figure 11: Partition identity of $\lambda = (8,7,6,4,3)$

The method works only when the row being removed, that is the last one, is not amongst the one being extended, which is the first one and can make sense if $|a| \leq |b|$, hence $|a| = |b|$, which makes λ a wedge of the first type, that is removed. With the application of S in the first case, b for the introduced segment $S(\lambda)$ is a for old λ .

Also, a for $S(\lambda)$ becomes the length of the next to the last row of λ , hence $b(\lambda)$ becomes larger.

Second model ($|a| > |b|$). This means, from every first a rows, 1 cell is removed, and an addition length $|a|$ of new last row.

Example 9;

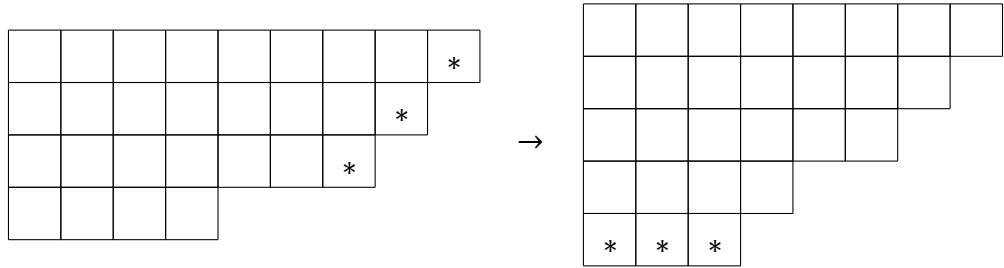


Figure 12: Partition identity of $\lambda = (9,8,7,4)$

The method works only when the row we are putting below is bigger than the new row of size $|a|$ that is being added.

Length l represent the old last row, by peeling off cells from the principle ‘ a ’ line, it is lessened to $b-l$, if there were only ‘ a ’ row, to start with. An instance where $|b| > |a|$, the impossibility occurs when $|b| = |a| + 1$ and the last row is included by the ‘ a ’ rows. Meaning the part is a wedge of the subsequent form, that is removed.

However, the second model, a for $S(\lambda)$ is b for λ . From every first ‘ a ’ row, we eliminated 1 and they are still sequential, so ‘ a ’ for $S(\lambda)$ is at least ‘ a ’(λ). We can now prove S belongs to an involution.

In the first model, we notice that $'b'(S(\lambda)) = 'a'(\lambda)$ and $'a'(S(\lambda)) > 'a'(\lambda)$. Hence $S(\lambda)$ is categorized in the second model if λ is in the first model. In the same way, in the second model, $'a'(S(\lambda)) = 'b'(\lambda)$ and $'b'(S(\lambda)) \geq 'b'(\lambda)$.

Therefore, $S(\lambda)$ is categorized in the first model if λ is in the second model.

A complete proof of Euler's identity is concluded when the operations in both the first and second model undo each other, and $S(S(\lambda)) = \lambda$, in either case, is concluded (George E. Andrews and Kimmo Eriksson, 2004).

Chapter 7

THE JACOBI TRIPLE PRODUCT IDENTITY

Here we use Euler's pentagonal number method which is

$$\prod_{j=1}^{\infty} (1 - z^j) = \sum_{m=-\infty}^{\infty} (-1)^m z^{m(3m+1)/2} = 1 - z - z^2 + z^5 + z^7 - z^{12} - z^{15} + \dots \quad (7.1)$$

which represents a special case of the Jacobi triple product identity (Yee, 2015) represented as follows;

$$\prod_{m=1}^{\infty} (1 - z^{2m}) (1 + x^{2m-1} z^2) (1 + x^{2m-1} z^{-2}) = \sum_{m=-\infty}^{\infty} x^{m^2} z^{2m} \quad \dots \quad (7.2)$$

To prove this identity is lengthy, and in this section we get a new expression by replacing z with z^n and v^2 with $-x^\ell$ as represented below:

$$\prod_{m=1}^{\infty} (1 - z^{2nm-n-\ell}) (1 + z^{2nm-n+\ell} v^2) (1 - z^{2nm}) = \sum_{m=-\infty}^{\infty} (-1)^m z^{nm^2+lm} \quad \dots \quad (7.3)$$

We obtain the pentagonal number theorem delivered by Euler by taking $n = 3/2$ and $\ell = 1/2$ and work with the obtained results for the expression and the above expression (7.3) is considered as a formal product.

Chapter 8

MACMAHON'S RECURRENCE

This is another theorem of Euler's identity that is useful in the calculation of partitions in restricted integers. McMahon also noted that an efficient recurrence of the partition number $p(m)$ can be obtained using this theorem (Rapudi, 2019).

Let's consider $1/P(z)$ as the one side of Euler's identity(left) the following functions identity is obtained.

$$(1 - z - z^2 + z^5 + z^7 - z^{12} - z^{15} + z^{22} + z^{26} - \dots) P(z) = 1$$

Having z^m as a coefficient of the identity, it is observed as below

$$p(m) - p(m-1) - p(m-2) + p(m-5) + p(m-7) - p(m-12) - p(m-15) + \dots = 0.$$

In another case,

$$p(m) = p(m-1) + p(m-2) - p(m-5) - p(m-7) + p(m-12) + p(m-15) - \dots$$

The total for every m simply proceeds so long as the forms remain $p(h)$ for $h \geq 0$ and initial mark is $p(0) = 1$. This proves that the sum should not be infinite.

Considering the quantity of terms being repeated for $p(m)$. They are $p(h)$ with $h = m - (3n^2 \pm n)/2$, making the final one relates with biggest n for which $(3n^2 - n)/2 \leq m$. This might be around $\sqrt{2m/3}$, in that case, a repeat that just includes around 3 terms. Therefore, we need to use 36 previous values on about $2\sqrt{2m/3}$ to compute $p(500)$.

Chapter 9

THE SYLVESTER'S APPROACH

The theory is applied when calculating the partitions in restricted integers forming an explicit formula for the polynomial part of the restricted partition function also known as the first Sylvester wave (Agnarsson, 2002). Let $W(n, d)$ be the number of partitions of n with parts in d , where $d = (d_1, d_2, \dots, d_s)$ of positive integers.

$$f(t, d) = \prod_{v=1}^s \frac{1}{1-t^{d_v}} = \sum_{n=0}^{\infty} W(n, d) t^n \quad (9.1)$$

Where $W(n, d)$ is taken as a sum of “waves”

$$W(n, d) = \sum_{v \geq 1} W_v(n, d) \quad (9.2)$$

and this sum is taken over distinct divisors of v of the components d and for each v , $W_v(n, d)$ is the coefficient of t^{-1} in the series expansion of equation (9.1) in ascending powers of t of

$$F(n, d^s, m; t) = e^{n w_m} \prod_{r=1}^m \frac{1}{1-e^{d_r u_m}}, \quad (9.3)$$

$$w_m = 2\pi i \frac{p_m}{q} + t,$$

$$w_m = 2\pi i \frac{p_m}{q} - t$$

where $P_1, P_2, \dots, P_{max\ m}$ are all numbers less than q and prime to it.

Chapter 10

GENERALIZED PARTITIONS

(Exponential structures)

The general structure deals with special cases such as,

10.1 Integer partitions

In this case, we follow the sequence of a segment (part) which is the number n that is being divided. Overall, from the blend of double objects, we acquire our outcome through recording the double items next to each other. Considering the instance of whole number divisions, where the combination $6 = 4 + 2$ and $7 = 5 + 1 + 1$ segments, we basically get $13 = 5 + 4 + 2 + 1 + 1$. Hence the segments belong to the primes $1 = 1, 2 = 2, 3 = 3, 4 = 4$, and continues.

10.2 Rooted unlabeled forests

Considering rooted forests as another example, we obtain the order of a rooted forest by counting how many hubs or vertices are in the woodland. The result of the synthesis of two woodland is the woodland that is obtained when the two given woodlands are written next to each other, and the unlabeled rooted trees are given as the primes.

10.3 Plane partitions

It is a segment whose partition of the whole number m is into two-dimensional array of integers $P_{u,v}$ for $u, v \geq 0$ to make every $P_{u,v}$ a non-negative integer (Bodini, Fusy & Pivoteau, 2010).

That is to say, $P_{u,v} \geq P_{u+1,v}$ and $P_{u,v} \geq P_{u,v+1}$ where $\forall u,v \in \mathbb{N}^2$ and

$$|p| := \sum_{u,v} p_{u,v} = m \quad (10.1)$$

Example 10: We consider partitions of $5 = (5), (4,1), (2,2,1), (1,1,1,1,1)$. Each section (row) is a plane partition of $m=5$ as shown below in Figure 13. Using equation 10.1, we describe the top row plane partition as

$$p = (p_{u,v}) = p_{1,1} + p_{1,2} + p_{1,3} + p_{1,4} + p_{1,5} = 1+1+1+1+1 = 5 = m, \text{ and for the second row,}$$

$$p = (p_{u,v}) = p_{2,1} + p_{2,2} + p_{2,3} = 2+2+1 = 5 = m.$$

1	1	1	1	1
2	2	1		
4	1			
5				

Figure 13: Plane partitions of $m = 5$

Moving high upward the section, the segment is none expanding, likewise, going across a line (row), the segments are additionally none expanding.

Example 11: We consider partitions of $m=8 = (8), (1,7), (2,6), (3,5), (4,4)$ and looking at partition $(3,5)$, we can further expand it to get more partitions of 8 as $(3,4,1), (3,3,2), (3,2,1,1,1), (3,1,1,1,1,1)$, where each represent a row carrying a plane partition of 8. For plane partition $(3,3,2)$, we have $p_{3,v} = p_{3,1} + p_{3,2} + p_{3,3} = 3+3+2=8=m$.

Chapter 11

AN EXTENSION OF EULER'S RESULT

This is an example of identically distributed partition statistics where the quantity of segment of m with precisely v reoccurred part sizes is equivalent to the quantity of partitions of m with precisely v even part estimates. The ‘odd-particular hypothesis’ originally by Euler’s theorem is considered a unique situation where $v = 0$.

From a sieve point of view, it is noted as a simple consequence which can be stated more clearly, as follows. By considering the measurement of a partition is nonnegative integer-esteemed capacity illustrated on the integer’s partitions. The quantity of parts is an illustration of how many even parts, how many repeated parts are multiples of 6, and the list continues (George E. Andrews and Kimmo Eriksson, 2004).

By defining the equation below, in relation to every partition measurement Z a likelihood arrangement is formed.

$$\text{prob}_m(Z = v) \stackrel{\text{def}}{=} |\{\pi \in p(m) : z(\pi) = v\}|$$

The arrangement of segment of m is represented by $p(m)$. We get the Remmel’s theorem which states that let C, D are dual sieve-equivalent arrangement of non-empty multisets, and $C = \{C_i\}_{i \in \omega}$, $D = \{D_u\}_{u \in \omega}$ such that the following holds.

$$|\bigcup_{u \in S} C_u| = |\bigcup_{u \in S} D_u| \quad (\forall S \subseteq \omega)$$

Then the number of partitions of m that contain no C_u is equal to the number of partitions of m that contain no D_u .

For example, if set $\mathbf{S} = \{3,5,7\}$, we see that

$$|\bigcup_{u \in \mathbf{S}} C_u| = |\bigcup_{u \in \mathbf{S}} D_u| = 3$$

since $6+10+14 = 3+3+5+5+7+7$.

Let P, Q be two comparable appropriated partition measurements, the quality of the measurement $P(\pi)$ (resp. $Q(\pi)$) on a partition π is the quantity of u with the end goal π have the multiset C_u (resp. D_u).

From the sieve-equivalence (the principle of inclusion-exclusion) machinery, the following are constructed examples for the pairs of similar distributed partition measurements:

- a. The total of even part measures $=P$, the total reoccurring part sizes $=Q$.
- b. The total successive even part measures $=P$, the total successive reoccurred part sizes $=Q$.
- c. All total part estimate having the complete squares $=P$; all total part measures u whose variety is $\geq u = Q$
- d. All total segment measures with number $\equiv 2,3,4 \pmod 6 = P$, all total segment estimates that are either an odd multiple of 3 or, likelihood reoccurred and not a multiple of 3 $=Q$.
- e. By fixing a whole number $d > 1$. We allow the total part estimates belonging to the product of $d = P$; the total part measures whose multiplicity is $\geq d = Q$.
- f. Sets of positive integers are represented by G_1, G_2 . Address $2G_1 = \{v: (v/2) \in G_1\}$. Assume that $2G_1 \subseteq G_1$ and $G_2 = G_1 - 2G_1$.

At that point we represent the measurement P the quantity of part estimates that are not in G_2 ; Q = the quantity of part measure u such an extent that possibly $u \in G_1$ and is rehashed.

We can obtain this in the other direction by considering the example below;

Example 12: In this case, we use the theory of **Rogers-Ramanujan Identity**, defined as the quantity for m partitions in forms of parts consistence to 1 or 4 mod 5 is equivalent to the partitions quantity that its segments cannot reoccur or be continuous. In our case, the hypothesis of Remmel lack significance, and it can be shown by recording down the properties lists as:

$gaps = 0 \text{ or } 1$	$parts \equiv 1 \text{ or } 4 \pmod{5}$
1 1	2
2 1	3
2 2	5
3 2	7
3 3	8
4 3	10

Figure 14: Properties of Rogers-Ramanujan Identity

It tends to be noticed the Remmel's theorem will not apply by ordering the properties. The sieve strategy cannot apply here and this is shown when, the segments of 4 with precisely single space of estimate 0 or 1 are (22,1111) and the division of 4 with precisely single-part estimate congruent to 0, 2, or 3 mod 5 are (31,22,211).

Hence, the above arrangement of terms cannot be sieve-equally because these quantities of divisions appear not the same.

Chapter 12

ESTIMATES OF THE RATIONAL QUANTITIES

This additional section shows the application of whole number division estimating functions in a unique way checking the rationales (Verdooleage & Woods, 2008).

Below is a constructed list being of the positive rational numbers and is represented as:

1/1, 1/2, 2/1, 1/3, 3/2, 2/3, 3/1, 1/4, 4/3, 3/5, 5/2, 2/5, 5/3, 3/4, 4/1, 1/5, 5/4, 4/7, 7/3, 3/8, 8/5, 5/7, 7/2, 2/7, 7/5, ...

From the list and **Figure 14**, we observe the following;

1. numerators of each fraction are from the previous denominators. Meaning the m^{th} rational quantity in the arrangement is represented as $c(m)/c(m+1)$ where ($m = 0, 1, 2, \dots$), having c to represent a particular function of the nonnegative whole numbers having terms as $\{c(m)\} m \geq 0 = \{1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7 \dots\}$.
2. We can write integer m as the total of power of 2 using it at most twice by applying the function values $c(m)$, an example, by representing $6 = 5 + 1 = 2 + 2 + 1 + 1$. Therefore, 6 can be represented by two such ways, so $c(6) = 2$. Considering $c(m)$ as the quantity of hyper binary representations of the integer m .
3. When every rational occurs, it occurs in reduced form when the back-to-back estimations of the function c are generally prime.
4. On the list each positive rational only occur once and not any other time.

Chapter 13

CONCLUSION

This thesis was about the definition of different terms in partition identity such as generating function, bijective, and Durfee diagram. We used Ferrer's diagram to represent models of different integer partitions and obtain partition identities with distinct parts. A discussion and generalization of basic theories that are used in partition identities such as the Euler's identity, Jacobi triple product identity and the Sylvester's approach was made. The theories and their application were also discussed.

Euler's identity theorem was used to show there are the same number of partitions of m into an even number of distinct parts as there are partitions into an odd number of distinct parts unless m is a pentagonal number. We also introduced the concept of 'wedge-shaped' partitions to produce such identities.

In McMahon's approach, Euler's identity was used to note a recurrence of partition number $p(m)$. In Sylvester's approach, we used the first Sylvester wave to obtain partitions of restricted integers.

In exponential structures, we looked at generalized structures such as rooted unlabeled forests and plane partitions.

Remmel's theorem was discussed for partitions of m into odd parts by excluding elements of the multisets holding even parts, and vice versa for odd parts of a partition.

In Rogers-Ramanujan Identity, it was shown that sieve strategy or equivalence does not appear and Remmel's theory cannot be applied. Besides the latter, an additional concept was also discussed which is counting the rational numbers using whole number divisions.

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