Asymptotic Behavior of Solutions to Nonlinear Neutral Differential Equations

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ABSTRACT

In Chapter 2 of this thesis, in the first part, we deal with asymptotic behavior of non-oscillatory solutions to higher order nonlinear neutral differential equations of the form

$$\left(x\left(t\right)+p\left(t\right)x\left(t-\tau\right)\right)^{(n)}+f\left(t,x\left(t\right),x\left(\rho\left(t\right)\right),x'\left(t\right),x'\left(\sigma\left(t\right)\right)\right)=0,$$

for $n\geq 2$. We formulate sufficient conditions for all non-oscillatory solutions to behave like polynomial functions at infinity. For the higher order differential equation

$$\left(x\left(t\right)+p\left(t\right)x\left(t-\tau\right)\right)^{(n)}+f\left(t,x\left(t\right),x\left(\rho\left(t\right)\right)\right)=0,$$

we provide necessary and sufficient conditions that guarantee existence of non-oscillatory solutions with polynomial-like behavior at infinity.

In Chapter 3, we look into oscillation problem of second order nonlinear neutral differential equations

$$\big(r\left(t\right)\psi\left(x\left(t\right)\right)\left(x\left(t\right)+p\left(t\right)x\left(\tau\left(t\right)\right)\right)'\big)'+q\left(t\right)f\left(x\left(t\right),x\left(\sigma\left(t\right)\right)\right)=0$$

and

$$\big(r\left(t\right)\left(x\left(t\right)+p\left(t\right)x\left(\tau\left(t\right)\right)\right)'\big)'+q\left(t\right)f\left(x\left(t\right),x\left(\sigma\left(t\right)\right)\right)=0.$$

Keywords: asymptotic behavior, oscillation, positive solutions, neutral equations

ÖZ

Bu tezin ilk kısmında şekli, $n \geq 2$ için,

$$\left(x\left(t\right)+p\left(t\right)x\left(t-\tau\right)\right)^{(n)}+f\left(t,x\left(t\right),x\left(\rho\left(t\right)\right),x'\left(t\right),x'\left(\sigma\left(t\right)\right)\right)=0$$

olan lineer olmayan yüksek dereceli nötr diferansiyel denklemlerin salınımlı olmayan çözümlerinin asimptotik davranışları incelendi. Buna ek olarak şekli

$$\left(x\left(t\right)+p\left(t\right)x\left(t-\tau\right)\right)^{(n)}+f\left(t,x\left(t\right),x\left(\rho\left(t\right)\right)\right)=0$$

olan diferansiyel denklemin çözümlerinin sonsuzda polinom gibi davranmalarını garanti edecek gerek ve yeter koşullar elde edilmiştir.

İkinci kısımda ise şekilleri

$$\left(r\left(t\right)\psi\left(x\left(t\right)\right)\left(x\left(t\right)+p\left(t\right)x\left(\tau\left(t\right)\right)\right)'\right)'+q\left(t\right)f\left(x\left(t\right),x\left(\sigma\left(t\right)\right)\right)=0$$

ve

$$\left(r\left(t\right)\left(x\left(t\right)+p\left(t\right)x\left(\tau\left(t\right)\right)\right)'\right)'+q\left(t\right)f\left(x\left(t\right),x\left(\sigma\left(t\right)\right)\right)=0$$

olan diferansiyel denklemlerin salınım problemine bakılmıştır.

Anathar Kelimeler: asimtotik davranış, salınım, pozitif çözümler, nötr denklemler

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Chapter 1

INTRODUCTION

In many applications, one assumes the system under consideration is governed by a principle of causality; that is, the future state of the system is independent of the past states and is determined solely by the present. If it is also assumed that the system is governed by an equation involving the state and rate of change of the state, then, generally, one is considering either ordinary or partial differential equations. However, under closer examination, it becomes apparent that the principle of causality is often only a first approximation to the true situation and that a more realistic model would include some of the past states of the system. Also, in some problems it is meaningless not to have dependence on the past. This has been known for some time but the theory for such systems has only been developed recently.

Delay differential equations arise in many areas of mathematical modeling, for example, population dynamics (taking into account the gestation times), infectious diseases (accounting for the incubation periods), physiological and pharmaceutical kinetics (modeling, for instance, the body's reaction to CO_2 in circulating blood) and chemical kinetics such as mixing reactants, the navigational control of ships and aircraft with, respectively, large and short lags and more general control problems. There are many of books that address applications of delay differential equations, see, for example, Driver [19], Gopalsamy [28], Halanay [36], Kolmanovskii and Myshkis [47], Kolmanovskii and Nosov [48] and Kuang [51].

In what follows, we mention only a few possible applications. It is well known that there are many problems appearing in biological models which are related with delay differential equations, see, for instance, [33] and [68]. In 1948, Hutchinson [43] suggested to use the following delay logistic equation for describing the dynamics of a single species

$$x'(t) = ax(t)\left(1 - \frac{x(t-\tau)}{K}\right),$$

where the delay τ includes various factors influencing the increase of species such as hatching period, pregnancy period and the time of renewal of food. Based on biological considerations, ecologists predict that there are solutions with small positive initial values which will steadily approach the environmental capacity $x(t) = K$ when $a > 0$ and $\tau \ll 1$. On the other hand, for a larger τ, the solution may exceed the capacity and start oscillating around $x(t) = K$. It is known that if $a\tau > e^{-1}$, then every solution is oscillatory. This result provides many tools for ecologists to determine limits for the delay τ which causes oscillatory phenomenon. In respect to industry, the oscillation of the contacts of electromagnetic switches is described by the following second order delay differential equation

$$x''(t) + ax'(t) + bx(t) + cx(t-\tau) = 0.$$

In 1951, Goodwin [27] constructed a business cycle model with nonlinear acceleration principle of investment and showed that model gives rise to cyclic oscillations when its stationary state is locally unstable. Goodwin's basic model is summarized as the following nonlinear differential equation

$$\varepsilon x'(t) - \varphi(x'(t)) + (1-\alpha)x(t) = 0, \tag{1.0.1}$$

where time dependent variable x is national income, α the national propensity to consume such that $\alpha \in (0,1)$, ε a positive adjustment coefficient of x and $\varphi(x'(t))$ denotes the induced investment that is dependent on the rate of change in national income. Goodwin's model adopts the nonlinear acceleration principle, according to which investment is proportional to the change in national income in a neighborhood of the equilibrium income but becomes inflexible for the extremely larger and smaller values of income. In order to come close to reality, Goodwin introduced the production lag τ between decisions to invest and corresponding outlays. As a result, model in (1.0.1) resulted in the following nonlinear neutral delay differential equation

$$\varepsilon x'(t) - \varphi(x'(t-\tau)) + (1-\alpha)x(t) = 0.$$

The oscillation theory of functional differential equations differs from that of ordinary differential equations and, in fact, the former reveals the oscillation or non-oscillation of solutions caused by the appearance of deviating arguments in the differential equation. Fite's paper [24] was among the first papers on the oscillation of functional differential equations. It deals with the n-th order differential equation with a deviating argument

$$x^{(n)}(t) + p(t)x(\sigma(t)) = 0, \qquad -\infty < t < +\infty, \tag{1.0.2}$$

for $n \geq 1$, $p \in C(-\infty,+\infty)$, $\sigma(t) = k - t$, $k \in \mathbb{R}$. Fite [24] proved that under the assumption $p(t) > h > 0$ for sufficiently large $|t|$, if

1. n is odd, then every solution of Eq. (1.0.2) oscillates infinitely;
2. n is even, then every solution of Eq. (1.0.2) oscillates either odd number of times or infinitely.

The first book written in English on oscillation theory of functional differential equations was by Ladde et. al [55] where achievements in this field up to the year 1984 were systematically summarized.

Neutral differential equations play an important role in theory of functional differential equations. In recent years, the theory of neutral differential equations has become an independent area of research and literature on this subject comprises over 1000 titles. Many results concerning the theory of neutral functional differential equations were given in the monographs by Hale and Lunel [34, 35]. These equations find numerous applications in natural sciences and technology but, as a rule, they are characterized by specific properties which make their study difficult both in aspects of ideas and techniques.

Investigation of the oscillation and non-oscillation of neutral differential equations has already been initiated in sixties and became a popular subject in eighties, see, for instance, Norkin's book [67], papers by Zahariev and Bainov [7, 94] and references there. Among the problems that attracted the attention of many mathematicians around the world, we mention obtaining of the necessary and sufficient conditions of oscillation of all solutions to neutral differential equations, the classification of non-oscillatory solutions, existence of positive solutions, comparison theorems and linearized criteria. In 1991, two books, one written by Bainov and Mishev [6], the other by Györi and Ladas [32], were published collecting many results of the oscillation theory of neutral differential equations between the years 1980 and 1990.

Qualitative analysis of several classes of neutral differential equations is the main subject of this thesis which is organized as follows. Chapter 2 presents a wide range of results from literature as well as our recently obtained results. For the second order nonlinear neutral differential equation

$$\left(x\left(t\right)+p\left(t\right)x\left(t-\tau\right)\right)''+f\left(t,x\left(t\right),x\left(\rho\left(t\right)\right),x'\left(t\right),x'\left(\sigma\left(t\right)\right)\right)=0, \tag{1.0.3}$$

we provide sufficient conditions for the existence of asymptotically linear solutions which behave like non-trivial linear functions, or, equivalently, solutions of the form

$$x\left(t\right)=At+o\left(t\right)\qquad\text{as}\qquad t\rightarrow+\infty.$$

For a higher order equation of the form

$$(x(t)+p(t)x(t-\tau))^{(n)}+f(t,x(t),x(\rho(t)),x'(t),x'(\sigma(t)))=0, \tag{1.0.4}$$

we present sufficient conditions that ensure polynomial-like asymptotic behavior of non-oscillatory solutions $x(t)$. As a particular case of Eq. (1.0.4), we also consider a neutral differential equation

$$(x(t)+p(t)x(t-\tau))^{(n)}+f(t,x(t),x(\rho(t)))=0$$

and obtain a new necessary and sufficient condition for the existence of polynomial-like non-trivial solutions. Results reported in Chapter 2 complements research on asymptotic behavior of non-oscillatory solutions of functional differential equations reported by Dahiya and Singh [16], Dahiya and Zafer [17], Graef et al. [29], Graef and Spikes [30], Grammatikopoulos et al. [31], Kong et al. [49], Kulcsár [52], Ladas [54], M. Naito [65], Y. Naito [66], Tanaka [83] and many other authors.

Chapter 3 focuses on oscillatory behavior of solutions of nonlinear neutral differential equations of the forms

$$\left(r(t)\psi(x(t))(x(t)+p(t)x(\tau(t)))'\right)'+q(t)f(x(t),x(\sigma(t)))=0 \tag{1.0.5}$$

and

$$\left(r(t)(x(t)+p(t)x(\tau(t)))'\right)'+q(t)f(x(t),x(\sigma(t)))=0. \tag{1.0.6}$$

In 1986, Yan [92] proved several important oscillation results for the linear differential equation with linear damping term

$$(r(t)x'(t))'+p(t)x'(t)+q(t)x(t)=0 \tag{1.0.7}$$

by extending celebrated Kamenev's oscillation criterion [44]. Yan's [92] results proved to be among the most efficient tools for studying oscillatory behavior of solutions not only for Eq. (1.0.7) but even for linear differential equations

$$x''(t) + q(t)\, x(t) = 0$$

and

$$(r(t)\, x'(t))' + q(t)\, x(t) = 0.$$

Yan's paper [92] boosted extensive investigation in the field and stimulated further development of a so-called integral averaging technique opening a hallway to important contributions to the Theory of Oscillation.

For more than three decades, conditions like the one used by Yan [92],

$$\limsup_{t\to+\infty} t^{-\alpha} \int_{t_0}^{t} (t-s)^{\alpha}\, h(s)\, q(s)\, ds < +\infty, \tag{1.0.8}$$

were necessary to prove oscillatory behavior of solutions of various classes of differential equations. Very recently, Rogovchenko and Tuncay [76] enhanced results due to Yan [92] by removing condition (1.0.8) thanks to a refined integral averaging technique developed in [74] and [75]. Following an idea similar to developed by Rogovchenko and Tuncay, we formulate new oscillatory results for Eqs. (1.0.5) and (1.0.6).

We conclude the introduction by mentioning that results reported in this thesis are published in the papers [37, 38, 39, 40] and presented at the following international conferences:

- The 7th AIMS (American Institute of Mathematical Sciences) Conference on Dynamical Systems and Differential Equations (May 18-21, 2008, Arlington, Texas, USA);
- The 6th International Conference On Differential Equations and Dynamical Systems (May 22-26, 2008, Baltimore, Maryland, USA);

- The 4th International Conference on Mathematical Analysis, Differential Equations and Their Applications (September 12-15, 2008, Famagusta, North Cyprus);
- Conference on Differential and Difference Equations and Applications 2010 (June 21-25, 2010, Rajecké Teplice, Slovak Republic).

Chapter 2

ASYMPTOTIC BEHAVIOR

Behavior of solutions of differential equations at infinity attracted many researchers. In many cases, the main idea is to obtain conditions that ensure behavior of solutions at infinity similar to that of much simpler differential equations. As a consequence, this topic resulted in numerous papers. For the differential equation

$$x''(t) + q(t)\,x(t) = 0, \tag{2.0.1}$$

Fubini [25] has posed the following question: what could be said about asymptotic behavior of solutions of Eq. (2.0.1) if we suppose that

$$\lim_{t\to+\infty} q(t) < +\infty?$$

Eq. (2.0.1) is asymptotic to

$$x''(t) = 0 \tag{2.0.2}$$

when $q(t)$ vanishes at infinity. Does this mean that all solutions of Eq. (2.0.1) behave like linear functions at infinity? The answer is negative. Consider the following classical example by Sansone [78]. The linear differential equation

$$x''(t) + \left(\frac{1}{4t} + \frac{3}{16t^2}\right)x(t) = 0$$

has a two-paramater family of solutions

$$x(t) = A\sqrt{t}\sin\left(\sqrt{t} + B\right),$$

where $A \neq 0$ and $A, B \in \mathbb{R}$, which is not asymptotic to the solution

$$x(t) = at + b \tag{2.0.3}$$

of Eq. (2.0.2), although

$$\lim_{t \to +\infty} \left(\frac{1}{4t} + \frac{3}{16t^2} \right) = 0.$$

Clearly, the problem of finding asymptotically linear solutions is related to finding sufficient conditions for the existence of non-oscillatory solutions of differential equations. The situation is very simple for the linear equations with constant coefficients and in the case of varying coefficients there is a massive array of results which help to classify the equation as oscillatory or non-oscillatory. The simplest oscillation and non-oscillation criteria can be built up by using the classical Sturm theory developed for second order self-adjoint linear differential equations. However, the things become more complicated if we have to work with nonlinear differential equations.

2.1 Brief History

There are many reasons why one might be interested in studying seemingly simple type of asymptotic behavior like the one described by (2.0.3). We note that existence of asymptotically linear solutions is related, for example, to

1. existence of non-oscillatory solutions,
2. existence of bounded solutions,
3. existence of square integrable solutions and limit point/limit circle classification,
4. existence of monotonic solutions,
5. existence of eventually positive (negative) solutions.

The asymptotic behavior of solutions of nonlinear equation

$$x''(t) + f(t, x(t)) = 0, \tag{2.1.1}$$

has been studied by Cohen [12], Constantin [13], Tong [84], Waltman [88] and Wong [89]. Some results for the linear case are also known, see, for instance, Trench [85] and Waltman [88]. Cohen [12] proved the following result for Eq. (2.1.1).

Theorem 2.1.1 ([12, p.608, Theorem 1]). *Suppose that*

(i) $f(t, u)$ *is continuous on* $D = \{(t, u) : t \geq 1, u \in \mathbb{R}\}$;

(ii) the derivative $f_u(t, u)$ *exists and is positive on* D;

(iii) $|f(t, u)| < f_u(t, u)|u|$ *on* D.

In addition, suppose that

$$\int_1^{+\infty} s f_u(s, 0)\, ds < +\infty.$$

Then every solution $x(t)$ *of Eq. (2.1.1) is asymptotic to* $at + b$ *as* $t \to +\infty$.

In the proof of Theorem 2.1.1, Cohen [12] used Bellman's method [9, p. 114-115] based on Gronwall's inequality. Using Bihari inequality, Tong [84] proved the following generalization of the results due to Cohen [12].

Theorem 2.1.2 ([84, p. 235, Theorem B]). *Let* $f(t, u)$ *be continuous on*

$$D = \{(t, u) : t \geq 0, u \in \mathbb{R}\}.$$

If there are two nonnegative continuous functions $v(t), \varphi(t)$ *for* $t \geq 0$ *and a continuous function* $g(x)$ *for* $x > 0$, *such that*

(i) $\int_1^{+\infty} v(s)\varphi(s)\, ds < +\infty$;

(ii) for $x > 0$, $g(x)$ *is positive and nondecreasing;*

(iii) $|f(t, u)| < v(t)\varphi(t)\, g\left(\frac{|u|}{t}\right)$, *for* $t \geq 1$, $u \in \mathbb{R}$,

then Eq. (2.1.1) has solutions which are asymptotic to $at + b$, *where* $a, b \in \mathbb{R}$ *and* $a \neq 0$.

Remark 2.1.3. *Notice that, in Theorem 2.1.2, if we let* $v(t) = f_u(t,0)$, $\varphi(t) = t$ *and* $g(x) = x$, *we obtain Theorem 2.1.1.*

On the other hand, Constantin [13] proved, among other, the following criterion for the asymptotic behavior of solutions to Eq. (2.1.1).

Theorem 2.1.4 ([13, p 633, Corollary 2])**.** *Let* $f(t,u)$ *be continuous on*

$$D = \{(t,u) : t \geq 1, u \in \mathbb{R}\}.$$

Suppose there exists functions $\varphi, w \in C(\mathbb{R}_+, \mathbb{R}_+)$, w *nondecreasing on* $\mathbb{R}_+$, $w(x) > 0$ *for* $x > 0$, *such that*

$$|f(t,u)| \leq \varphi(t)\, w\left(\frac{|u|}{t}\right), \qquad t \geq 1, \qquad u \in \mathbb{R},$$

and

$$\int_1^{+\infty} \varphi(s)\, ds < +\infty, \qquad \int_1^{+\infty} \frac{ds}{w(s)} = +\infty.$$

Then if $x(t)$ *is a solution of Eq. (2.1.1) we have that* $x(t) = at + b + o(t)$ *as* $t \to +\infty$ *where* $a, b \in \mathbb{R}$.

Another particular case of Eq. (2.1.1) is the autonomous differential equation

$$x''(t) + f(x(t), x'(t)) = 0,$$

which has been studied by Rogovchenko and Villari [77] using the phase plane analysis.

In the study of asymptotic behavior of solutions to differential equation

$$x''(t) + f(t, x(t), x'(t)) = 0, \tag{2.1.2}$$

it is usually supposed that the nonlinearity f in Eq. (2.1.2) satisfies

$$|f(t, x(t), x'(t))| \leq F(t, |x(t)|, |x'(t)|),$$

where the real-valued function $F(t,u,v)$ is continuous, monotone in the last two arguments and vanishes at infinity with the condition of decay expressed in terms of convergent improper integrals, see, for instance Constantin [13], Mustafa and Rogovchenko [63], S. Rogovchenko and Yu. Rogovchenko [72], Rogovchenko [73] and Tong [84]. In particular, S. Rogovchenko and Yu. Rogovchenko [72] studied Eq. (2.1.2) assuming that

$$|f(t,u,v)| \leq h_1(t)\, g_1\left(\frac{|u|}{t}\right) + h_2(t)\, g_2(|v|) + h_3(t) \tag{2.1.3}$$

or

$$|f(t,u,v)| \leq h_4(t)\, g_3\left(\frac{|u|}{t}\right) g_4(|v|) + h_5(t),$$

where the functions h_i are nonnegative, continuous and integrable over $[1,+\infty)$, for all $i = 1,\ldots 5$, while g_j are nonnegative, continuous and monotone nondecreasing for all $j = 1,\ldots 4$. It has been proved, among, other results, that all continuable solutions of Eq. (2.1.2) behave like linear functions at infinity provided that $G_1(+\infty) = +\infty$ and $G_2(+\infty) = +\infty$, where

$$G_1(x) = \int_1^x \frac{ds}{g_1(s) + g_2(s)} \qquad \text{and} \qquad G_2(x) = \int_1^x \frac{ds}{g_3(s)\, g_4(s)} ds.$$

The results obtained in [72] extend those by Constantin [13], Meng [59], Rogovchenko [73] and Tong [84]. Using a different approach based on the fixed point theory, Mustafa and Rogovchenko [63] have established that assumptions used in [72] are sufficient for global existence of solutions.

Dannan [18] and S. Rogovchenko and Yu. Rogovchenko [72] studied Eq. (2.1.2) where the nonlinearity f satisfies (2.1.3) but condition $G_1(+\infty) = +\infty$ fails to hold. In this case, differential equation usually has local non-extendable solutions and the set of departure points for global solutions of Eq. (2.1.2) that behave like linear functions at infinity is in many cases a bounded subset of the phase plane. However, Mustafa and Rogovchenko [62] have proved for a class of nonlinear equations that this set can be also unbounded and proper, that is, neither void, nor coinciding with $\mathbb{R}^2$. In 2004, Mustafa and Rogovchenko [61] established existence of asymptotically linear solutions of Eq. (2.1.2) locally near $+\infty$ assuming that f satisfies inequality similar to (2.1.3) without requiring that $G_1(+\infty) = +\infty$.

Theorem 2.1.5 ([61, p. 313, Theorem 2.1])**.** *Suppose that the real-valued function $f(t, u, v)$ is continuous in $D = \{(t, u, v) : t \geq 1, u, v \in \mathbb{R}\}$ and satisfies*

$$|f(t, u, v)| \leq h_1\left(t, \frac{|u|}{t}\right) + h_2(t, |v|),$$

where the functions $h_1(t, s)$ and $h_2(t, s)$ are continuous, nonnegative and monotone non-decreasing in s. Assume that there exists a constant $c > 0$ such that

$$\int_1^{+\infty} (h_1(t, c) + h_2(t, c))\, dt < +\infty.$$

Then, for every pair of real numbers x_0, x_1, where $\max(|x_0|, |x_1|) < c/4$, there exists a $t_0 \geq 1$ such that every solution $x(t)$ of Eq. (2.1.2) satisfying initial conditions $x(t_0) = x_0$, $x'(t_0) = x_1$ is defined on $[t_0, +\infty)$ and has asymptotic development $x(t) = a_x t + o(t)$ at infinity, where a_x is a real constant that depends on $x(t)$. Furthermore, if $x_1 \neq 0$, then $a_x \neq 0$.

Interesting results regarding asymptotic properties of solutions of different classes of functional differential equations have been obtained by Dahiya and Singh [16], Dahiya and Zafer [17], Džurina [20], Graef and Spikes [30], Grammatikopoulos et al. [31], Kong et al. [49], Kulcsár [52], Ladas [54], M. Naito [65], Y. Naito [66] and Tanaka [83].

In particular, Kulcsár [52] obtained sufficient conditions for the convergence to zero of non-oscillatory solutions of the second order linear neutral differential equations

$$(x(t) - p(t)\, x(t - \tau))'' + q(t)\, x(t) = 0.$$

Graef and Spikes [30] derived two sets of sufficient conditions which guarantee that any bounded non-oscillatory solutions of a forced nonlinear neutral differential equation

$$(x(t) + p(t)x(\rho(t)))'' + q(t)f(x(t-\sigma)) = r(t) \tag{2.1.4}$$

tends to zero as $t \to +\infty$, while Grammatikopoulos et al. [31] established similar conditions for non-oscillatory solutions of Eq. (2.1.4) in the case $r(t) \equiv 0$. Further studies in this direction have been undertaken by Graef et al. [29] who derived sufficient conditions for solutions of neutral differential equation (2.1.4) with $r(t) \equiv 0$ to have one of the following properties:

1. the non-oscillatory solutions are bounded or tend to zero;
2. the bounded solutions are either oscillatory or tend to zero;
3. the unbounded solutions are either oscillatory or tend to infinity.

Recently, Džurina [20] extended results of Rogovchenko [73] on asymptotic integration of Eq. (2.1.2) to second order nonlinear neutral differential equation

$$(x(t) + p(t)x(t-\tau))'' + f(t, x(t)) = 0$$

establishing conditions under which all non-oscillatory solutions behave like linear functions $at + b$ as $t \to +\infty$ for some $a, b \in \mathbb{R}$ and stated without proof a similar theorem for equations of the form

$$(x(t) + p(t)x(t-\tau))'' + f(t, x(t), x'(t)) = 0.$$

For higher order equations, Kong et al. [49] gave a classification of non-oscillatory solutions of odd order linear neutral differential equation

$$(x(t) - x(t-\tau))'' + p(t)x(t-\sigma) = 0$$

and established conditions for the existence of each type of non-oscillatory solution. Finally, we note that M. Naito [65] proved that an n-th order nonlinear neutral differential equation

$$(x(t) + \lambda x(t-\tau))^{(n)} + \sigma f(t, x(\rho(t))) = 0$$

has a solution satisfying

$$\lim_{t \to +\infty} \frac{x(t)}{t^k} = c > 0$$

if and only if

$$\int_{t_0}^{+\infty} t^{n-k-1} f\left(t, c(\rho(t))^k\right) dt < +\infty$$

for some $c > 0$, whereas Y. Naito [66] derived a necessary and sufficient condition for a neutral differential equation

$$(x(t) - p(t)x(\tau(t)))^{(n)} + f(t, x(\rho(t))) = 0$$

to have a positive solution satisfying

$$\lim_{t \to +\infty} \frac{x(t) - p(t)x(\tau(t))}{t^k} = c > 0.$$

2.2 Second Order Nonlinear Neutral Differential Equations

In this section, we consider the neutral differential equations of the form

$$(x(t) + p(t)x(t-\tau))'' + f(t, x(t), x(\rho(t)), x'(t), x'(\sigma(t))) = 0, \tag{2.2.1}$$

where $t \geq t_0 > 0$, $t_0 \in \mathbb{R}$, $\tau > 0$, $p \in C([t_0, +\infty), \mathbb{R})$, $\rho, \sigma \in C([t_0, +\infty), [t_0, +\infty))$ and $f \in C([t_0, +\infty) \times \mathbb{R}^4, \mathbb{R})$. Firstly, we prove that solutions of Eq. (2.2.1) can be indefinitely continued to the right. Secondly, using the celebrated Bihari integral inequality, we obtain conditions for all non-oscillatory solutions to behave like nontrivial linear functions at infinity. The following are preliminary results together with the Bihari integral inequality which has an important role in the proofs of the main results.

Definition 2.2.1. *A function $\omega : [0, +\infty) \to [0, +\infty)$ is said to belong to the class H if*

(i) $\omega(t)$ is nondecreasing and continuous for $t \geq 0$ and positive for $t > 0$;

(ii) there is a continuous function ϕ defined on $[0, +\infty)$ such that

$$\omega(\alpha t) \leq \phi(\alpha)\,\omega(t)$$

for $\alpha > 0$, $t \geq 0$.

Some important properties of functions from the class H are collected in the following result due to Dannan [18, Lemma 1].

Lemma 2.2.1. *Let $f(u)$ and $g(u)$ belong to the class H with the corresponding multiplier functions $\varphi(\alpha)$ and $\psi(\alpha)$. Then*

(i) $f(u) + g(u)$, $f(u)\,g(u)$ and $f(g(u))$ belong to the class H;

(ii) $h(u) = \int_0^u f(s)\,ds$ belongs to the class H.

Following is the celebrated Bihari inequality.

Lemma 2.2.2. *Let $K \geq 0$, $f(t)$ and $g(t)$ be continuous on the interval $I = [0, +\infty)$, and let $\omega(t)$ belong to the class H. Then the inequality*

$$f(t) \leq K + \int_{t_0}^{t} g(s)\,\omega(f(s))\,ds \qquad (2.2.2)$$

implies

$$f(t) \leq G^{-1}\left(G(K) + \int_{t_0}^{t} g(s)\,ds\right),$$

where $t \geq t_0 \geq 0$, $G(t)$ is defined by

$$G(t) \stackrel{\text{def}}{=} \int_{t_*}^{t} \frac{ds}{\omega(s)}$$

and $G^{-1}(t)$ denotes the inverse of $G(t)$.

Proof. Let us denote the right hand side of the inequality (2.2.2) by $h(t)$. Then, one can easily see that

$$h'(t) = g(t)\,\omega(f(t)).$$

Dividing both sides of the latter equality by $\omega(h(t))$ and taking into account that $f(t) \le h(t)$, we get

$$\frac{h'(t)}{\omega(h(t))} = g(t)\frac{\omega(f(t))}{\omega(h(t))} \le g(t)\frac{\omega(h(t))}{\omega(h(t))} = g(t).$$

Next, we integrate both sides from t_0 to t to obtain

$$\int_{t_0}^{t} \frac{h'(s)}{\omega(h(s))}ds \le \int_{t_0}^{t} g(s)\,ds,$$

or, equivalently,

$$G(h(t)) - G(h(t_0)) \le \int_{t_0}^{t} g(s)\,ds. \tag{2.2.3}$$

Let $h(t_0) = K$. Then, inequality (2.2.3) assumes the form

$$G(h(t)) \le G(K) + \int_{t_0}^{t} g(s)\,ds.$$

Applying the inverse of G to both sides of the latter inequality, we obtain

$$h(t) \le G^{-1}\left(G(K) + \int_{t_0}^{t} g(s)\,ds\right).$$

Hence, the conclusion of the lemma follows immediately. □

Although independent of Eq. (2.2.1), the next result helps us to study non-oscillatory nature of solutions of this equation, cf. Džurina [20, Lemma 1], Györi and Ladas [32, p. 17-18, Lemma 1.5.1].

Lemma 2.2.3. *Let $x(t) > 0$ (or $x(t) < 0$) eventually, $\tau > 0$, and $p(t)$ be a continuous function, $0 \le p(t) \le p < 1$, such that*

$$\lim_{t\to+\infty} p(t) = p_0.$$

Define

$$w\left(t\right)=x\left(t\right)+p\left(t\right)\frac{t-\tau}{t}x\left(t-\tau\right). \tag{2.2.4}$$

If there exists a finite limit $\lim_{t\to+\infty}w\left(t\right)=c$, *then*

$$\lim_{t\to+\infty}x\left(t\right)=\frac{c}{1+p_0}. \tag{2.2.5}$$

Proof. Suppose that $x\left(t\right)>0$. It is clear from (2.2.4) that $c\geq0$ and (2.2.5) yields

$$\liminf_{t\to+\infty}x\left(t\right)\leq\frac{c}{1+p_0}\leq\limsup_{t\to+\infty}x\left(t\right).$$

Assume that there exist $\alpha_1,\alpha_2\geq0$ and sequences μ_n,ν_n diverging to $+\infty$ such that

$$\begin{aligned}\limsup_{t\to+\infty}x\left(t\right)&=\lim_{n\to+\infty}x\left(\mu_n\right)=\frac{c+\alpha_1}{1+p_0},\\ \liminf_{t\to+\infty}x\left(t\right)&=\lim_{n\to+\infty}x\left(\nu_n\right)=\frac{c-\alpha_2}{1+p_0}.\end{aligned}$$

We have to prove that $\alpha_1=\alpha_2=0$. Consider the following two cases.

Case 1. Assume that $\alpha_1>0$ and $\alpha_1\geq\alpha_2\geq0$. It follows from (2.2.4) that, for any $\varepsilon>0$,

$$w\left(t\right)\geq x\left(t\right)+p\left(t\right)\frac{t-\tau}{t}\frac{c-\alpha_2-\varepsilon}{1+p_0}. \tag{2.2.6}$$

Letting in (2.2.6) $t=\mu_n$ and passing to the limit as $n\to+\infty$, we obtain

$$c\geq\frac{c+\alpha_1}{1+p_0}+p_0\frac{c-\alpha_2-\varepsilon}{1+p_0},$$

or, equivalently,

$$\alpha_1\leq p_0\left(\alpha_2+\varepsilon\right). \tag{2.2.7}$$

Choose now $\varepsilon=\left(2p_0\right)^{-1}\left(1-p_0\right)\alpha_2$. Since $p_0<1$, (2.2.7) yields

$$\alpha_1\leq\frac{1}{2}\alpha_2\left(p_0+1\right)<\alpha_2,$$

which contradicts our initial assumption that $\alpha_1\geq\alpha_2$.

Case 2. Assume now that $\alpha_2 > 0$ and $\alpha_2 \geq \alpha_1 \geq 0$. Similarly to Case 1, (2.2.4) implies that, for any $\varepsilon > 0$,

$$w(t) \leq x(t) + p(t) \frac{t-\tau}{t} \frac{c + \alpha_1 + \varepsilon}{1 + p_0}. \tag{2.2.8}$$

Let in (2.2.8) $t = \nu_n$ and pass to the limit as $n \to +\infty$ to obtain

$$c \leq \frac{c - \alpha_2}{1 + p_0} + p_0 \frac{c + \alpha_1 + \varepsilon}{1 + p_0},$$

which is equivalent to

$$\alpha_2 \leq p_0 (\alpha_1 + \varepsilon). \tag{2.2.9}$$

Choose $\varepsilon = (2p_0)^{-1} (1 - p_0) \alpha_1$. Using (2.2.9) and the fact that $p_0 < 1$, we conclude that

$$\alpha_2 \leq \frac{1}{2} \alpha_1 (p_0 + 1) < \alpha_1,$$

which contradicts our assumption that $\alpha_2 \geq \alpha_1$. The proof is complete. □

Remark 2.2.1. *In the case $p(t) = p$, Lemma 2.2.3 reduces to Džurina's result [20, Lemma 1].*

The following lemma, due to Mustafa and Rogovchenko [63], is used to prove solutions of Eq. (2.2.1) can be continued to the right indefinitely.

Lemma 2.2.4 ([63, p. 346, Lemma 7])**.** *Suppose that the function $g(s)$ is a continuous, positive and nondecreasing on $(0, +\infty)$. Assume further that*

$$\int_{t_0}^{+\infty} \frac{1}{g(s)} ds = +\infty.$$

Then, for every $k > 0$ one has

$$\int_{t_0}^{+\infty} \frac{1}{k + g(s)} ds = +\infty.$$

In the sequel, we suppose that the following conditions hold:

(A1) $f(t, u_1, u_2, v_1, v_2)$ is continuous in

$$D = \{(t, u_1, u_2, v_1, v_2) : t \geq t_0 \geq 1,\ u_1, u_2, v_1, v_2 \in \mathbb{R}\};$$

(A2) there exist continuous functions $h_1, \ldots, h_5, g_1, \ldots, g_4 : [t_0, +\infty) \to [t_0, +\infty)$ such that either

$$|f(t, u_1, u_2, v_1, v_2)| \leq h_1(t)\, g_1\left(\frac{|u_1|}{t}\right) + h_2(t)\, g_2\left(\frac{|u_2|}{\rho(t)}\right) + h_3(t), \tag{2.2.10}$$

or

$$|f(t, u_1, u_2, v_1, v_2)| \leq h_4(t)\, g_3\left(\frac{|u_1|}{t}\right) g_4\left(\frac{|u_2|}{\rho(t)}\right) + h_5(t), \tag{2.2.11}$$

where, for $s > 0$, the functions $g_i(s)$, $i = 1, \ldots, 4$, are nondecreasing and

$$\int_{t_0}^{+\infty} h_i(s)\, ds = H_i < +\infty, \qquad i = 1, \ldots, 5;$$

(A3) $\rho, \sigma \in C([t_0, +\infty), [t_0, +\infty))$, $\rho(t) \leq t$, $\sigma(t) \leq t$, $\lim_{t\to+\infty} \rho(t) = +\infty$, and $\lim_{t\to+\infty} \sigma(t) = +\infty$.

For $t \geq t_0$, we introduce the functions G_1 and G_2 by

$$G_1(t) \stackrel{\text{def}}{=} \int_{t_0}^{t} \frac{ds}{g_1(s) + g_2(s)}, \qquad G_2(t) \stackrel{\text{def}}{=} \int_{t_0}^{t} \frac{ds}{g_3(s)\, g_4(s)}.$$

Let

$$z(t_0) = c_1 \qquad \text{and} \qquad z'(t_0) = c_2.$$

In what follows, we shall use the notation

$$c_* \stackrel{\text{def}}{=} |c_1| + |c_2|.$$

Further, define $z(t)$ by

$$z(t) = x(t) + p(t)\, x(t - \tau). \tag{2.2.12}$$

The next result provides useful estimates for solutions of Eq. (2.2.1).

Lemma 2.2.5. *(i) Assume that* $f(t, u_1, u_2, v_1, v_2)$ *satisfies* (2.2.10). *Then, for all* $t \geq t_0$, *one has*

$$\max\left[\frac{|z(t)|}{t}, \frac{|z(\rho(t))|}{\rho(t)}\right] \leq \Phi_1(t), \tag{2.2.13}$$

where

$$\Phi_1(t) \overset{\text{def}}{=} c_* + \int_{t_0}^{t} h_1(s)\, g_1\left(\frac{|z(s)|}{s}\right) ds$$
$$+ \int_{t_0}^{t} h_2(s)\, g_2\left(\frac{|z(\rho(s))|}{\rho(s)}\right) ds + \int_{t_0}^{t} h_3(s)\, ds. \tag{2.2.14}$$

(ii) Assume that $f(t, u_1, u_2, v_1, v_2)$ *satisfies* (2.2.11). *Then, for all* $t \geq t_0$, *one has*

$$\max\left[\frac{|z(t)|}{t}, \frac{|z(\rho(t))|}{\rho(t)}\right] \leq \Phi_2(t), \tag{2.2.15}$$

where

$$\Phi_2(t) \overset{\text{def}}{=} c_* + \int_{t_0}^{t} h_4(s)\, g_3\left(\frac{|z(s)|}{s}\right) g_4\left(\frac{|z(\rho(s))|}{\rho(s)}\right) ds + \int_{t_0}^{t} h_5(s)\, ds. \tag{2.2.16}$$

Proof. Part (i). Let $x(t)$ be a non-oscillatory solution of Eq. (2.2.1). Clearly,

$$|z(t)| \geq |x(t)|, \tag{2.2.17}$$

and it follows from Eq. (2.2.1) that

$$z''(t) = -f(t, x(t), x(\rho(t)), x'(t), x'(\sigma(t))), \tag{2.2.18}$$

where $z(t)$ is defined as in (2.2.12). Integrating (2.2.18) twice from t_0 to t, we obtain

$$z'(t) = c_2 - \int_{t_0}^{t} f(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s)))\, ds, \tag{2.2.19}$$
$$z(t) = c_2(t - t_0) + c_1 - \int_{t_0}^{t} (t - s)\, f(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s)))\, ds. \tag{2.2.20}$$

It follows from (2.2.19) and (2.2.20) that, for $t \geq t_0$,

$$|z'(t)| \leq |c_2| + \int_{t_0}^{t} |f(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s)))|\, ds,$$
$$|z(t)| \leq t\left(c_* + \int_{t_0}^{t} |f(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s)))|\, ds\right).$$

Using (2.2.10), (2.2.17) and monotonicity of the functions g_1 and g_2, we have

$$\begin{aligned}|f\left(t,x\left(t\right),x\left(\rho\left(t\right)\right),x'\left(t\right),x'\left(\sigma\left(t\right)\right)\right)| &\le h_3\left(t\right)+h_1\left(t\right)g_1\left(\frac{|x\left(t\right)|}{t}\right)\\ &\quad + h_2\left(t\right)g_2\left(\frac{|x\left(\rho\left(t\right)\right)|}{\rho\left(t\right)}\right)\le h_1\left(t\right)g_1\left(\frac{|z\left(t\right)|}{t}\right)\\ &\quad + h_2\left(t\right)g_2\left(\frac{|z\left(\rho\left(t\right)\right)|}{\rho\left(t\right)}\right)+h_3\left(t\right).\end{aligned}$$

Hence, for all $t \ge t_0$,

$$\begin{aligned}|z'\left(t\right)| &\le |c_2| + \int_{t_0}^{t} h_1\left(s\right)g_1\left(\frac{|z\left(s\right)|}{s}\right)ds\\ &\quad + \int_{t_0}^{t} h_2\left(s\right)g_2\left(\frac{|z\left(\rho\left(s\right)\right)|}{\rho\left(s\right)}\right)ds + \int_{t_0}^{t} h_3\left(s\right)ds,\end{aligned} \tag{2.2.21}$$

and

$$\begin{aligned}\frac{|z\left(t\right)|}{t} &\le c_* + \int_{t_0}^{t} h_1\left(s\right)g_1\left(\frac{|z\left(s\right)|}{s}\right)ds\\ &\quad + \int_{t_0}^{t} h_2\left(s\right)g_2\left(\frac{|z\left(\rho\left(s\right)\right)|}{\rho\left(s\right)}\right)ds + \int_{t_0}^{t} h_3\left(s\right)ds,\end{aligned} \tag{2.2.22}$$

from which (2.2.13) follows.

Part (ii). Assume now that f satisfies (2.2.11). Following the same lines as above, we conclude that, for $t \ge t_0$,

$$|z'\left(t\right)| \le |c_2| + \int_{t_0}^{t} h_4\left(s\right)g_3\left(\frac{|z\left(s\right)|}{s}\right)g_4\left(\frac{|z\left(\rho\left(s\right)\right)|}{\rho\left(s\right)}\right)ds + \int_{t_0}^{t} h_5\left(s\right)ds, \tag{2.2.23}$$

and

$$\frac{|z\left(t\right)|}{t} \le c_* + \int_{t_0}^{t} h_4\left(s\right)g_3\left(\frac{|z\left(s\right)|}{s}\right)g_4\left(\frac{|z\left(\rho\left(s\right)\right)|}{\rho\left(s\right)}\right)ds + \int_{t_0}^{t} h_5\left(s\right)ds, \tag{2.2.24}$$

which immediately yields (2.2.15). □

The following lemma establishes existence of solutions of Eq. (2.2.1) for all $t \ge t_0 \ge 1$ and resembles the result proved by Mustafa and Rogovchenko [61, p. 318-319, Lemma 3.6] for the differential equation

$$x''\left(t\right)+f(t,x\left(t\right),x'\left(t\right))=0, \qquad t\ge t_0\ge 1$$

in the case f satisfies the condition

$$|f(t,u,v)| \le h_1(t)\, g_1\left(\frac{|u|}{t}\right) + h_2(t)\, g_2(|v|) + h_3(t).$$

Lemma 2.2.6. *Suppose that there exists a solution $x(t)$ of Eq. (2.2.1) defined on $[1,T)$, $1 < T < +\infty$, which cannot be continued to the right of T.*

(i) If $f(t,u_1,u_2,v_1,v_2)$ satisfies (2.2.10), then $G_1(+\infty) < +\infty$.

(ii) If $f(t,u_1,u_2,v_1,v_2)$ satisfies (2.2.11), then $G_2(+\infty) < +\infty$.

Proof. Part (i). Let $x(t)$ be a solution of Eq. (2.2.1) which is defined on $[1,T)$, $1 < T < +\infty$, and cannot be continued to the right of T, and let $z(t)$ be defined by (2.2.12). Using estimates (2.2.21) and (2.2.22), we conclude that, for $t \in [1,T)$,

$$\max\left[\frac{|z(t)|}{T}, \frac{|z(\rho(t))|}{\rho(T)}\right] \le \max\left[\frac{|z(t)|}{t}, \frac{|z(\rho(t))|}{\rho(t)}\right] \le \gamma(t), \tag{2.2.25}$$

where $\gamma(t)$ is the maximal solution of the initial value problem

$$\begin{cases} \xi' = (h_1(t) + h_2(t) + h_3(t))\,(g_1(\xi) + g_2(\xi) + 1), \\ \xi(1) = \xi_0 \overset{\text{def}}{=} c_*. \end{cases} \tag{2.2.26}$$

Since solution $x(t)$ of Eq. (2.2.1) cannot be continued to the right,

$$\lim_{t\to T-} |x(t)| = +\infty,$$

which, in virtue of (2.2.17) and (2.2.25), implies $\gamma(t) \to +\infty$ as $t \to T-$. Integration of (2.2.26) yields, for $t \in [1,T)$,

$$\int_{\xi_0}^{\gamma(t)} \frac{ds}{g_1(s) + g_2(s) + 1} = \int_1^t (h_1(s) + h_2(s) + h_3(s))\, ds. \tag{2.2.27}$$

Passing in (2.2.27) to the limit as $t \to T-$, we deduce that

$$\int_{\xi_0}^{+\infty} \frac{ds}{g_1(s) + g_2(s) + 1} = \int_1^T (h_1(s) + h_2(s) + h_3(s))\, ds < +\infty. \tag{2.2.28}$$

If $G_1(+\infty) = +\infty$, then, according to Lemma 2.2.4, one has

$$\int_{\xi_0}^{+\infty} \frac{ds}{g_1(s) + g_2(s) + 1} = +\infty,$$

which contradicts (2.2.28). Thus, Part (i) is proved.

Part (ii). Let $x(t)$ and $z(t)$ be as in Part (i). Using estimates (2.2.23) and (2.2.24), we conclude that, for $t \in [1, T)$, inequality (2.2.25) holds, where this time $\gamma(t)$ is the maximal solution of the initial value problem

$$\begin{cases} \xi' = (h_4(t) + h_5(t))\,(g_3(\xi)g_4(\xi) + 1)\,, \\ \xi(1) = \xi_0, \end{cases} \tag{2.2.29}$$

and ξ_0 is as above. Integrating ordinary differential equation in (2.2.29) and taking into account that $\gamma(t) \to +\infty$ as $t \to T-$, we obtain, for $t \in [1, T)$,

$$\int_{\xi_0}^{\gamma(t)} \frac{ds}{g_3(s)g_4(s) + 1} = \int_1^t (h_4(s) + h_5(s))\,ds. \tag{2.2.30}$$

Passing in (2.2.30) to the limit as $t \to T-$, we conclude that

$$\int_{\xi_0}^{+\infty} \frac{ds}{g_3(s)g_4(s) + 1} = \int_1^T (h_4(s) + h_5(s))\,ds < +\infty. \tag{2.2.31}$$

Another application of Lemma 2.2.4 yields

$$\int_{\xi_0}^{+\infty} \frac{ds}{g_3(s)g_4(s) + 1} = +\infty$$

provided that $G_2(+\infty) = +\infty$, which, in virtue of (2.2.31), leads to contradiction. This completes the proof of lemma. □

As an immediate consequence of Lemma 2.2.6, we obtain the following important continuation result.

Corollary 2.2.1. *Assume that the nonlinearity f satisfies (2.2.10) (respectively, (2.2.11)) and $G_1(+\infty) = +\infty$ (respectively, $G_2(+\infty) = +\infty$). Then all solutions of Eq. (2.2.1) can be indefinitely continued to the right.*

Next, we present a theorem related with the existence of asymptotically linear solutions.

Theorem 2.2.2. *Suppose that (2.2.10) holds and $G_1(+\infty) = +\infty$. Then any non-oscillatory solution of Eq. (2.2.1) has the asymptotic representation*

$$x(t) = At + o(t), \tag{2.2.32}$$

and there exist solutions for which $A \neq 0$.

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (2.2.1) and $z(t)$ be defined by (2.2.12). Then, by virtue of Lemma 2.2.5, (2.2.13) holds. Since $g_1(s)$ and $g_2(s)$ are nondecreasing for $s > 0$, one has

$$g_1\left(\frac{|z(t)|}{t}\right) \leq g_1(\Phi_1(t)) \qquad \text{and} \qquad g_2\left(\frac{|z(\rho(t))|}{\rho(t)}\right) \leq g_2(\Phi_1(t)). \tag{2.2.33}$$

Taking into account (2.2.33) and the definition of $\Phi_1(t)$, we conclude that

$$\Phi_1(t) \leq M + \int_{t_0}^{t} h_1(s)\, g_1(\Phi_1(s))\, ds + \int_{t_0}^{t} h_2(s)\, g_2(\Phi_1(s))\, ds,$$

where $M \stackrel{\text{def}}{=} c_* + H_3$. Observing further that

$$\begin{aligned} h_1(s)\, g_2(\Phi_1(s)) + h_2(s)\, g_2(\Phi_1(s)) \leq{} & (h_1(s) + h_2(s)) \\ & \times (g_1(\Phi_1(s)) + g_2(\Phi_1(s))), \end{aligned}$$

we obtain

$$\Phi_1(t) \leq M + \int_{t_0}^{t} (h_1(s) + h_2(s))\,(g_1(\Phi_1(s)) + g_2(\Phi_1(s)))\, ds. \tag{2.2.34}$$

Application of Lemma 2.2.2 to (2.2.34) yields

$$\Phi_1(t) \leq G_1^{-1}\left(G_1(M) + \int_{t_0}^{t} (h_1(s) + h_2(s))\, ds\right),$$

where G_1^{-1} is the inverse of G_1 defined for $x \in (G_1(+\infty), +\infty)$. Let

$$K_1 \stackrel{\text{def}}{=} G_1(M) + H_1 + H_2 < +\infty.$$

Since G_1^{-1} is increasing, we conclude that

$$\Phi_1(t) \leq G_1^{-1}(K_1) \stackrel{\text{def}}{=} K_2 < +\infty.$$

Thus,

$$\frac{|z(t)|}{t} \leq K_2 \qquad \text{and} \qquad \frac{|z(\rho(t))|}{\rho(t)} \leq K_2,$$

where, in virtue of (A3), the second inequality follows from the fact

$$|z(\rho(t))| \leq \rho(t)\,\Phi_1(t) \leq \rho(t)\,\Phi_1(\rho(t)).$$

On the other hand, for $t \geq t_0$,

$$\int_{t_0}^{t} |f(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s)))|\, ds \leq g_1(K_2)\, H_1 \\ + g_2(K_2)\, H_2 + H_3 \stackrel{\text{def}}{=} K_3 < +\infty.$$

Therefore,

$$\lim_{t\to+\infty} \int_{t_0}^{t} |f(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s)))|\, ds$$

exists, and it follows from (2.2.19) that there exists a number $\mu \in \mathbb{R}$ such that

$$\lim_{t\to+\infty} z'(t) = \mu.$$

Choosing t_0 appropriately, one can always ensure that $\mu \neq 0$. Furthermore, application of l'Hospital's rule implies that

$$\lim_{t\to+\infty} \frac{z(t)}{t} = \lim_{t\to+\infty} z'(t) = \mu.$$

Set $w(t) = z(t)/t$ and $u(t) = x(t)/t$. Then (2.2.12) yields

$$w(t) = u(t) + p(t)\,\frac{t-\tau}{t}\,u(t-\tau).$$

Taking into account that

$$\lim_{t\to+\infty} w(t) = \lim_{t\to+\infty} \frac{z(t)}{t} = \mu \neq 0$$

and using Lemma 2.2.3, we conclude that

$$\lim_{t\to+\infty} u(t) = \lim_{t\to+\infty} \frac{x(t)}{t} = \frac{\mu}{1+p_0} \stackrel{\text{def}}{=} A.$$

The proof is complete now. □

Theorem 2.2.3. *Suppose that (2.2.11) holds and $G_2(+\infty) = +\infty$. Then the conclusion of Theorem 2.2.2 holds.*

Proof. Let $x(t)$ and $z(t)$ be as be as in Theorem 2.2.2. By Lemma 2.2.5,

$$\frac{|z(t)|}{t} \leq \Phi_2(t) \qquad \text{and} \qquad \frac{|z(\rho(t))|}{\rho(t)} \leq \Phi_2(t). \tag{2.2.35}$$

Using (2.2.15), (2.2.35), and monotonicity of the functions g_3 and g_4, we obtain

$$\Phi_2(t) \leq N + \int_{t_0}^{t} h_4(s)\, g_3(\Phi_2(s))\, g_4(\Phi_2(s))\, ds, \tag{2.2.36}$$

where $N \overset{\text{def}}{=} c_* + H_5$. Application of the Bihari inequality to (2.2.36) yields

$$\Phi_2(t) \leq G_2^{-1}\left(G_2(N) + \int_{t_0}^{t} h_4(s)\, ds\right),$$

where G_2^{-1} is the inverse of G_2 defined for $x \in (G_2(+\infty), +\infty)$. Let

$$K_4 \overset{\text{def}}{=} G_2(N) + H_4 < +\infty.$$

Then,

$$\Phi_2(t) \leq G_2^{-1}(K_4) \overset{\text{def}}{=} K_5 < +\infty,$$

and the proof is completed in the same manner as in Theorem 2.2.2. $\square$

2.3 Higher Order Nonlinear Neutral Differential Equations

In this section, we discuss asymptotic behavior of solutions for higher order nonlinear neutral differential equation

$$(x(t) + p(t)\, x(t-\tau))^{(n)} + f(t, x(t), x(\rho(t)), x'(t), x'(\sigma(t))) = 0. \tag{2.3.1}$$

In addition, as a particular case of Eq. (2.3.1), we also consider the differential equation

$$(x(t) + p(t)\, x(t-\tau))^{(n)} + f(t, x(t), x(\rho(t))) = 0. \tag{2.3.2}$$

2.3.1 Asymptotic Behavior of Solutions of Eq. (2.3.1)

Let $\mathbb{R}^+ = [0, +\infty)$. In what follows, we suppose that

(A1) $f \in C(\mathbb{R}^+ \times \mathbb{R}^4, \mathbb{R})$, and there exist functions $\phi_k, \omega_l \in C(\mathbb{R}^+, \mathbb{R}^+)$, $k = 1, \dots, 5$, $l = 1, \dots 4$, such that either

$$|f(t, u_1, u_2, v_1, v_2)| \le \phi_1(t) + \phi_2(t)\,\omega_1\left(\frac{|u_1|}{t^{n-1}}\right) + \phi_3(t)\,\omega_2\left(\frac{|u_2|}{[\rho(t)]^{n-1}}\right), \qquad (2.3.3)$$

or

$$|f(t, u_1, u_2, v_1, v_2)| \le \phi_4(t) + \phi_5(t)\,\omega_3\left(\frac{|u_1|}{t^{n-1}}\right)\omega_4\left(\frac{|u_2|}{[\rho(t)]^{n-1}}\right), \qquad (2.3.4)$$

where, for $s > 0$, the functions $\omega_l(s)$ are positive, nondecreasing and

$$\int_{t_0}^{+\infty} \phi_k(s)\,ds = A_k < +\infty, \qquad k = 1, \dots, 5; \qquad (2.3.5)$$

(A2) $\rho, \sigma \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\rho(t) \le t$, $\sigma(t) \le t$, $\lim_{t\to+\infty} \rho(t) = +\infty$, and $\lim_{t\to+\infty} \sigma(t) = +\infty$;

(A3) $p \in C(\mathbb{R}^+, \mathbb{R}^+)$, $0 \le p(t) \le p_* < 1$, and $\lim_{t\to+\infty} p(t) = p_0$.

For $t \ge t_0$, let

$$\Psi_1(t) \stackrel{\text{def}}{=} \phi_1(t) + \phi_2(t) + \phi_3(t), \quad \Omega_1(t) \stackrel{\text{def}}{=} \omega_1(t) + \omega_2(t), \quad \tilde{G}_1(t) \stackrel{\text{def}}{=} \int_{t_0}^{t} \frac{ds}{\Omega_1(s)},$$
$$\Psi_2(t) \stackrel{\text{def}}{=} \phi_4(t) + \phi_5(t), \quad \Omega_2(t) \stackrel{\text{def}}{=} \omega_3(t)\,\omega_4(t), \quad \tilde{G}_2(t) \stackrel{\text{def}}{=} \int_{t_0}^{t} \frac{ds}{\Omega_2(s)}.$$

The following result is a generalization of Lemma 2.2.3. Its proof follows a similar pattern and is therefore omitted, cf. Džurina [20, Lemma 1], Györi and Ladas [32, p. 17-18, Lemma 1.5.1].

Lemma 2.3.1. *Let $u(t) > 0$ (or $u(t) < 0$) eventually, $p(t)$ satisfy (A3), and $w(t)$ be defined by*

$$w(t) = u(t) + p(t)\frac{(t-\tau)^{n-1}}{t^{n-1}}u(t-\tau). \qquad (2.3.6)$$

If there exists a finite limit $\lim_{t\to+\infty} w(t) = c$, then

$$\lim_{t\to+\infty} u(t) = \frac{c}{1+p_0}.$$

The following result has an independent interest and is used to assure that any non-oscillatory solution of Eq. (2.3.1) can be indefinitely continued to the right.

Theorem 2.3.1. *Suppose that there exists a non-oscillatory solution $x(t)$ of Eq. (2.3.1) defined on $[t_0, T)$, $t_0 < T < +\infty$, which cannot be continued to the right beyond T.*

(i) If $f(t, u_1, u_2, v_1, v_2)$ satisfies (2.3.3), then $\tilde{G}_1(+\infty) < +\infty$.

(ii) If $f(t, u_1, u_2, v_1, v_2)$ satisfies (2.3.4), then $\tilde{G}_2(+\infty) < +\infty$.

Proof. **(i)** Let $x(t)$ be a non-oscillatory solution of Eq. (2.3.1) defined on $[t_0, T)$, $t_0 < T < +\infty$, which cannot be continued to the right beyond T. Then,

$$\lim_{t \to T-} |x(t)| = +\infty. \tag{2.3.7}$$

Define $z(t)$ as in (2.2.12). Clearly,

$$|z(t)| \geq |x(t)|,$$

and it follows from Eq. (2.3.1) that

$$z^{(n)}(t) = -f(t, x(t), x(\rho(t)), x'(t), x'(\sigma(t))).$$

Integrating the latter equation n times from t_0 and t, one obtains, for $t \geq t_0$,

$$\begin{aligned} z(t) =_{i=1}^{n-1} \frac{z^{(i)}(t_0)}{i!}(t - t_0)^i \\ - \int_{t_0}^{t} \frac{(t-s)^{n-1}}{(n-1)!} f(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s)))\, ds. \end{aligned}$$

Then, for $t \geq t_0$,

$$|z(t)| \leq t^{n-1}\left(M + \int_{t_0}^{t} |f(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s)))|\, ds\right),$$

where

$$M = \sum_{i=1}^{n-1} \frac{\left|z^{(i)}(t_0)\right|}{i!}. \tag{2.3.8}$$

Using (2.3.3), (2.2.17), and monotonicity of the functions $\omega_i(s)$, we obtain

$$\begin{aligned} |f(t,x(t),x(\rho(t)),x'(t),x'(\sigma(t)))| \leq \phi_1(t) & \\ + \phi_2(t)\,\omega_1\left(\frac{|z(t)|}{t^{n-1}}\right) & \\ + \phi_3(t)\,\omega_2\left(\frac{|z(\rho(t))|}{[\rho(t)]^{n-1}}\right), & \end{aligned} \tag{2.3.9}$$

which yields

$$\begin{aligned} \frac{|z(t)|}{t^{n-1}} \leq \tilde{\Phi}_1(t) \stackrel{\text{def}}{=} M + \int_{t_0}^{t} \phi_1(s)\,ds & \\ + \int_{t_0}^{t} \phi_2(s)\,\omega_1\left(\frac{|z(s)|}{s^{n-1}}\right) ds & \\ + \int_{t_0}^{t} \phi_3(s)\,\omega_2\left(\frac{|z(\rho(s))|}{[\rho(s)]^{n-1}}\right) ds. & \end{aligned}$$

Clearly, $\tilde{\Phi}_1(t)$ is increasing because, for all $t \geq t_0$,

$$\tilde{\Phi}_1'(t) = \phi_1(t) + \phi_2(t)\,\omega_1\left(\frac{|z(t)|}{t^{n-1}}\right) + \phi_3(t)\,\omega_2\left(\frac{|z(\rho(t))|}{[\rho(t)]^{n-1}}\right) > 0.$$

By the assumption (A2), one has

$$|z(\rho(t))| \leq [\rho(t)]^{n-1}\,\tilde{\Phi}_1(\rho(t)) \leq [\rho(t)]^{n-1}\,\tilde{\Phi}_1(t),$$

or

$$\frac{|z(\rho(t))|}{[\rho(t)]^{n-1}} \leq \tilde{\Phi}_1(t),$$

and thus, for all $t \geq t_0$,

$$\max\left[\frac{|z(t)|}{t^{n-1}}, \frac{|z(\rho(t))|}{[\rho(t)]^{n-1}}\right] \leq \tilde{\Phi}_1(t). \tag{2.3.10}$$

It follows from (2.3.10) that, for $t \in [t_0, T)$,

$$\max\left[\frac{|z(t)|}{T^{n-1}}, \frac{|z(\rho(t))|}{[\rho(T)]^{n-1}}\right] \leq \max\left[\frac{|z(t)|}{t^{n-1}}, \frac{|z(\rho(t))|}{[\rho(t)]^{n-1}}\right] \leq \vartheta(t), \tag{2.3.11}$$

where $\vartheta(t)$ is the maximal solution of the initial value problem

$$\begin{cases} \zeta' = \Psi_1(t)\,(\Omega_1(\zeta) + 1), \\ \zeta(t_0) = \zeta_0 = M. \end{cases} \tag{2.3.12}$$

By virtue of (2.2.17) and (2.3.11), (2.3.7) implies

$$\lim_{t\to T-} \vartheta(t) = +\infty. \tag{2.3.13}$$

Integration of (2.3.12) yields, for $t \in [t_0, T)$,

$$\int_{\zeta_0}^{\vartheta(t)} \frac{ds}{\Omega_1(s)+1} = \int_{t_0}^{t} \Psi_1(s)\, ds.$$

Passing to the limit as $t \to T-$, one has

$$\int_{\zeta_0}^{+\infty} \frac{ds}{\Omega_1(s)+1} = \int_{t_0}^{T} \Psi_1(s)\, ds < +\infty. \tag{2.3.14}$$

Since the integrals

$$\int_{\zeta_0}^{+\infty} \frac{ds}{\Omega_1(s)+1} \qquad \text{and} \qquad \int_{\zeta_0}^{+\infty} \frac{ds}{\Omega_1(s)}$$

converge or diverge simultaneously, it follows from (2.3.14) that $\tilde{G}_1(+\infty) < +\infty$.

(ii) Assume now that f satisfies (2.3.4). By an argument similar to the one used in part (i), we conclude that, for $t \geq t_0$,

$$\frac{|z(t)|}{t^{n-1}} \leq \tilde{\Phi}_2(t) \stackrel{\text{def}}{=} M + \int_{t_0}^{t} \left\{ \phi_4(s) + \phi_5(s)\, \omega_3\left(\frac{|z(s)|}{s^{n-1}}\right) \omega_4\left(\frac{|z(\rho(s))|}{[\rho(s)]^{n-1}}\right)\right\} ds,$$

where M is given by (2.3.8). With the same reasoning as above, we arrive at the estimate

$$\max\left[\frac{|z(t)|}{t^{n-1}}, \frac{|z(\rho(t))|}{[\rho(t)]^{n-1}}\right] \leq \tilde{\Phi}_2(t). \tag{2.3.15}$$

Furthermore, we conclude that, for $t \in [t_0, T)$, inequality (2.3.11) holds, where $\vartheta(t)$ is the maximal solution of the initial value problem

$$\begin{cases} \zeta' = \Psi_2(t)\left(\Omega_2(\zeta)+1\right), \\ \zeta(t_0) = \zeta_0 = M. \end{cases}$$

The rest of the proof follows the same lines as in part (i). $\square$

An immediate consequence of Theorem 2.3.1 and [63, Lemma 7] is the following extension result.

Corollary 2.3.1. *Assume that nonlinearity f satisfies (2.3.3) (respectively (2.3.4)) and $\tilde{G}_1(+\infty) = +\infty$ (respectively $\tilde{G}_2(+\infty) = +\infty$). Then all non-oscillatory solutions of Eq. (2.3.1) can be indefinitely continued to the right.*

Theorem 2.3.2. *Suppose that (2.3.3) holds and*

$$\tilde{G}_1(+\infty) = +\infty. \tag{2.3.16}$$

Then any non-oscillatory solution $x(t)$ of Eq. (2.3.1) satisfies

$$\lim_{t\to+\infty} \frac{x(t)}{t^{n-1}} = a, \tag{2.3.17}$$

and there exist non-oscillatory solutions for which $a \neq 0$.

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (2.3.1) and $z(t)$ be defined by (2.2.12). Then, (2.3.10) holds, and

$$\tilde{\Phi}_1(t) \le M + A_1 + \int_{t_0}^{t} [\phi_2(s)\,\omega_1(\Phi_1(s)) + \phi_3(s)\,\omega_2(\Phi_1(s))]\,ds.$$

Observing that

$$\phi_2(t)\,\omega_1\left(\tilde{\Phi}_1(t)\right) + \phi_3(t)\,\omega_2\left(\tilde{\Phi}_1(t)\right) \le [\phi_2(t) + \phi_3(t)]\,\Omega_1\left(\tilde{\Phi}_1(t)\right),$$

one has

$$\tilde{\Phi}_1(t) \le M + A_1 + \int_{t_0}^{t} [\phi_2(s) + \phi_3(s)]\,\Omega_1\left(\tilde{\Phi}_1(s)\right) ds.$$

An application of Lemma 2.2.2 yields

$$\tilde{\Phi}_1(t) \le \tilde{G}_1^{-1}\left(\tilde{G}_1(M + A_1) + \int_{t_0}^{t} [\phi_2(s) + \phi_3(s)]\,ds\right),$$

where $\tilde{G}_1^{-1}$ is the inverse of $\tilde{G}_1$ defined for $x \in \left(\tilde{G}_1(0+), +\infty\right)$. Let

$$\tilde{K}_1 \stackrel{\text{def}}{=} \tilde{G}_1(M + A_1) + A_2 + A_3 < +\infty.$$

Since $\tilde{G}_1^{-1}$ is increasing, we conclude that

$$\tilde{\Phi}_1(t) \le \tilde{G}_1^{-1}\left(\tilde{K}_1\right) \stackrel{\text{def}}{=} \tilde{K}_2 < +\infty.$$

Thus, it follows from (2.3.10) and the latter inequality that

$$\max\left[\frac{|z(t)|}{t^{n-1}}, \frac{|z(\rho(t))|}{[\rho(t)]^{n-1}}\right] \le \tilde{K}_2.$$

On the other hand, by virtue of (2.3.9), for $t \ge t_0$,

$$\begin{aligned}\int_{t_0}^{t} |f(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s)))|\, ds \\ \le A_1 + A_2\omega_1\left(\tilde{K}_2\right) \\ + A_3\omega_2\left(\tilde{K}_2\right) \overset{\text{def}}{=} \tilde{K}_3 < +\infty.\end{aligned}$$

Therefore, the limit

$$\lim_{t\to+\infty}\int_{t_0}^{t} |f(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s)))|\, ds$$

is finite, and there exists a number $q \in \mathbb{R}$ such that

$$\begin{aligned}q = \lim_{t\to+\infty} z^{(n-1)}(t) = z^{(n-1)}(t_0) \\ - \int_{t_0}^{+\infty} f(s, x(s), x(\rho(s)), x'(s), x'(\sigma(s)))\, ds.\end{aligned}$$

Choosing t_0 appropriately, one can always ensure that $q \neq 0$, see, for instance, Dahiya and Singh [16], Džurina [20], or Ladas [54]. Repeated application of the l'Hôpital's rule yields

$$\lim_{t\to+\infty}\frac{z(t)}{t^{n-1}} = \frac{q}{(n-1)!}. \tag{2.3.18}$$

Let $z(t) = t^{n-1}w(t)$ and $x(t) = t^{n-1}u(t)$. It is easy to see that, by virtue of (2.2.12), $w(t)$ satisfies (2.3.6), and it follows from (2.3.18) that

$$\lim_{t\to+\infty} w(t) = \frac{q}{(n-1)!}.$$

Using Lemma 2.3.1, we conclude that

$$\lim_{t\to+\infty} u(t) = \lim_{t\to+\infty}\frac{x(t)}{t^{n-1}} = \frac{q}{(1+p_0)(n-1)!} \overset{\text{def}}{=} a \neq 0,$$

which completes the proof. □

Theorem 2.3.3. *Suppose that (2.3.4) is satisfied and*

$$\tilde{G}_2(+\infty) = +\infty. \tag{2.3.19}$$

Then the conclusion of Theorem 2.3.2 holds.

Proof. Let $x(t)$ and $z(t)$ be as in Theorem 2.3.1. By virtue of (2.3.15),

$$\tilde{\Phi}_2(t) \le M + A_4 + \int_{t_0}^{t} \phi_5(s)\,\Omega_2\left(\tilde{\Phi}_2(s)\right) ds.$$

An application of Lemma 2.2.2 yields

$$\tilde{\Phi}_2(t) \le \tilde{G}_2^{-1}\left(\tilde{G}_2(M + A_4) + \int_{t_0}^{t} \phi_5(s)\,ds\right),$$

where $\tilde{G}_2^{-1}$ is the inverse of $\tilde{G}_2$ defined for $x \in \left(\tilde{G}_2(0+), +\infty\right)$. Let

$$\tilde{K}_4 \overset{\text{def}}{=} \tilde{G}_2(M + A_4) + A_5 < +\infty.$$

Then, it is not hard to prove that

$$\tilde{\Phi}_2(t) \le \tilde{G}_2^{-1}\left(\tilde{K}_4\right) < +\infty,$$

and the rest of the proof resembles that of Theorem 2.3.2. □

2.3.2 Asymptotic Behavior of Solutions of Eq. (2.3.2)

In this section, we study asymptotic behavior of solutions of Eq. (2.3.2). In what follows, we suppose that

(B1) $f \in C(\mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R})$, and there exist functions $\phi_k, \eta_l \in C(\mathbb{R}^+, \mathbb{R}^+)$, $k = 1, \ldots, 5$, $l = 1, \ldots, 4$, such that, for $s > 0$, $\eta_j(s)$ are nondecreasing, and either

$$|f(t, u_1, u_2)| \ge \phi_1(t) + \phi_2(t)\,\eta_1\left(\frac{|u_1|}{t^{n-1}}\right) + \phi_3(t)\,\eta_2\left(\frac{|u_2|}{[\rho(t)]^{n-1}}\right), \tag{2.3.20}$$

or

$$|f(t, u_1, u_2)| \ge \phi_4(t) + \phi_5(t)\,\eta_3\left(\frac{|u_1|}{t^{n-1}}\right)\eta_4\left(\frac{|u_2|}{[\rho(t)]^{n-1}}\right); \tag{2.3.21}$$

(B2) $\rho \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\rho(t) \leq t$, and $\lim_{t\to+\infty} \rho(t) = +\infty$;

(B3) $p \in C(\mathbb{R}^+, \mathbb{R}^+)$, $0 \leq p(t) \leq p_* < 1$, and $\lim_{t\to+\infty} p(t) = p_0$;

(B4) if u_1 and u_2 have the same sign, then $f(t, u_1, u_2)$ has that sign for all t sufficiently large.

The following lemma, due to Kiguradze [45], is essential for the proof of the main result of this section.

Lemma 2.3.2 ([45]). *Let $z(t)$ be an n times differentiable function on $\mathbb{R}_+$ of constant sign, $z(t) \not\equiv 0$ on $[t_0, +\infty)$ which satisfies $z^{(n)}(t)\, z(t) \leq 0$. Then there is an integer l, $0 \leq l \leq n-1$, such that $n + l$ is even and*

$$\begin{array}{ll} z(t)\, z^{(i)}(t) > 0, & 0 \leq i \leq l, \\ (-1)^{n+i+1} z(t)\, z^{(i)}(t) > 0, & l+1 \leq i \leq n. \end{array}$$

Theorem 2.3.4. *Assume that (2.3.20) holds. If Eq. (2.3.2) has a solution $x(t)$ satisfying (2.3.17), then (2.3.5) holds for $k = 1, 2, 3$.*

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (2.3.2). Without loss of generality, we may assume that $x(t) > 0$, for $t \geq t_1 \geq t_0$. It follows from (2.2.12) that there exists a t_2 such that, for $t \geq t_2$, one has $z(t) > x(t) > 0$, whereas

$$z^{(n)}(t) = -f(t, x(t), x(\rho(t))) \tag{2.3.22}$$

yields that $z^{(n)}(t) < 0$, for $t \geq t_2$. Consequently, by Lemma 2.3.2, all derivatives $z'(t)$, $z''(t), \ldots, z^{(n-1)}(t)$ are of constant sign for sufficiently large t. We claim that $z^{(n-1)}(t)$ is eventually nonnegative. Indeed, assuming that there exists a $T \geq t_2$ such that $z^{(n-1)}(T) < 0$ and using the fact that $z^{(n-1)}(t)$ is decreasing, we conclude that, for $t \geq T$,

$$z^{(n-1)}(t) < z^{(n-1)}(T) < 0. \tag{2.3.23}$$

It follows from (2.3.23) that $\lim_{t\to+\infty} z^{(n-2)}(t) = -\infty$ and $\lim_{t\to+\infty} z(t) = -\infty$. Therefore, by (2.2.12), $x(t)$ is eventually negative, which contradicts our assumption of eventual positivity of $x(t)$. Thus, we have established that there exists a $t_3 \geq t_2$ such that, for all $t \geq t_3$,

$$z^{(n-1)}(t) \geq 0. \tag{2.3.24}$$

Integration of Eq. (2.3.22) yields

$$z^{(n-1)}(t) = z^{(n-1)}(t_3) - \int_{t_3}^{t} f(s, x(s), x(\rho(s)))\, ds,$$

which, by (2.3.24), immediately implies that

$$\int_{t_3}^{+\infty} f(s, x(s), x(\rho(s)))\, ds \leq z^{(n-1)}(t_3) < +\infty.$$

On the other hand, by (2.3.17) and (B2), there exists a $t_4 \geq t_3$ such that

$$\frac{x(t)}{t^{n-1}} > \frac{a}{2} \qquad \text{and} \qquad \frac{x(\rho(t))}{[\rho(t)]^{n-1}} > \frac{a}{2}, \tag{2.3.25}$$

for all $t \geq t_4$. Taking into account (2.3.20), (2.3.25), and using monotonicity of the functions η_1 and η_2, we observe that

$$\begin{aligned} +\infty &> \int_{t_4}^{+\infty} f(s, x(s), x(\rho(s)))\, ds \\ &\geq \int_{t_4}^{+\infty} \left[\phi_1(s) + \phi_2(s)\,\eta_1\left(\frac{x(s)}{s^{n-1}}\right) + \phi_3(s)\,\eta_2\left(\frac{x(\rho(s))}{[\rho(s)]^{n-1}}\right)\right] ds \\ &\geq \int_{t_4}^{+\infty} \left[\phi_1(s) + \phi_2(s)\,\eta_1\left(\frac{a}{2}\right) + \phi_3(s)\,\eta_2\left(\frac{a}{2}\right)\right] ds, \end{aligned}$$

which yields the desired conclusion. □

Theorem 2.3.5. *Assume that (2.3.21) holds. If Eq. (2.3.2) has a solution $x(t)$ satisfying (2.3.17), then property (2.3.5) holds for $k = 4, 5$.*

Proof. The proof is similar to that of Theorem 2.3.4 and is therefore omitted. □

Combining Theorems 2.3.2 and 2.3.4 (respectively, Theorems 2.3.3 and 2.3.5), we obtain necessary and sufficient conditions for existence of solutions of Eq. (2.3.2) that satisfy (2.3.17).

Theorem 2.3.6. *Let conditions (2.3.3), (2.3.16), and (2.3.20) (respectively, (2.3.4), (2.3.19), and (2.3.21)) be satisfied. Then, a necessary and sufficient condition for Eq. (2.3.2) to have solutions $x(t)$ with the asymptotic property (2.3.17) is that (2.3.5) holds for $k = 1, 2, 3$ (respectively, for $k = 4, 5$).*

Remark 2.3.7. *We conclude this section by noting that Džurina formulated without proof a result [20, Theorem 2] stating that all non-oscillatory solutions of*

$$(x(t) + p(t)\, x(t-\tau))'' + f(t, x(t), x'(t)) = 0 \tag{2.3.26}$$

are asymptotic to $at + b$ as $t \to +\infty$ for some $a, b \in \mathbb{R}$, under the assumption that

$$|f(t, u, v)| \le h(t)\, g\left(\frac{|u|}{t}\right) |v|,$$

where $h(t)$ is integrable on $[t_0, +\infty)$ and $\int_{t_0}^{x} 1/g(s)ds \to +\infty$ as $x \to +\infty$. However, in order to prove this assertion, in addition to the estimate (2.2.17), one has to use the inequality $|x'(t)| \le |z'(t)|$ which, in general, is not satisfied for solutions of Eq. (2.3.26). This fact explains our main assumptions (2.3.3) and (2.3.4) on the nonlinearity f.

2.4 Examples

In the following examples, classification of the solutions has been done according to Kong et. al [49, Definition 2.1].

Definition 2.4.1. *For $t \in [T, +\infty)$, a non-oscillatory solution $x(t)$ of equation*

$$(x(t) - x(t-\tau))^{(n)} + p(t)\, x(t-\sigma) = 0 \tag{2.4.1}$$

is said to be of type $\mathcal{A}_k$, $k \in \{0, \dots, n\}$ if $x(t) = at^k + b(t)$, where $a \ne 0$ and $b(t)$ is a bounded function on $[T, +\infty)$.

For $t \in [T, +\infty)$, a non-oscillatory solution $x(t)$ of Eq. (2.4.1) is said to be of type $\mathcal{B}_{k,l}$, $k \in \{1, \dots, n\}$, $l \in \{1, \dots, k\}$ if $x(t) = at^k + b(t)$, where $a \ne 0$ and $b(t) = o(t^l)$ as $t \to +\infty$.

For $t \in [T, +\infty)$, a non-oscillatory solution $x(t)$ of Eq. (2.4.1) is said to be of type $\mathcal{C}_h$ for an odd number $h \in \{1, 3, \ldots, n,\}$ if

$$\lim_{t\to+\infty} \frac{x(t)}{t^{h-1}} = +\infty \qquad \textit{and} \qquad \lim_{t\to+\infty} \frac{x(t)}{t^{h}} = 0.$$

Example 2.4.1. *For $t \geq 2$, consider the nonlinear neutral differential equation*

$$(x(t) + p(t)\, x(t-1))'' + a(t) \tanh\left(x'(\sigma(t))\right) + b(t) = 0, \tag{2.4.2}$$

where

$$\alpha(t) = \left[(2t+1)^3 (t-1)^2\right]^{-1}, \qquad a(t) = \frac{12t^3 \alpha(t)}{\tanh(1 + 2/t)},$$

$$b(t) = \alpha(t)\, t^{-2} \left[(4\ln(t-1) - 10)\, t^4 + (5 - 8\ln(t-1))\, t^3 \right.$$
$$\left. + (4\ln(t-1) - 3)\, t^2 + 4t + 1\right],$$

$$p(t) = \frac{t}{2t+1} \qquad \textit{and} \qquad \sigma(t) = \frac{t}{2}.$$

By Theorem 2.2.2, for any non-oscillatory solution of Eq. (2.4.2), (2.2.32) holds. In fact, $x(t) = t + \ln t$ is such a solution. Observe also that this solution belongs to the class $\mathcal{B}_{1,1}$.

Example 2.4.2. *For $t \geq 2$, consider the nonlinear neutral differential equation*

$$(x(t) + p(t)\, x(t-1))'' + a(t) \left[\frac{x^2(t)}{x^2(t)+1}\right]^{3/4} \left[\frac{(x'(t))^2}{(x'(t))^2 + 1}\right]^{1/4} = b(t) \tag{2.4.3}$$

where

$$a(t) = \frac{28t^3 \left(t^4 - t^2 + 1\right)^{3/4} \left(2t^4 + 2t^2 + 1\right)^{1/4}}{\left(t^2 - 1\right)^{3/2} \left(t^2 + 1\right)^{1/2} \left(2t^2 - t - 1\right)^3},$$

$$b(t) = \frac{2\left(18t^5 - 6t^4 - 8t^3 - 3t^2 + 3t + 1\right)}{t^3 \left(2t^2 - t - 1\right)^3} \qquad \textit{and} \qquad p(t) = \frac{1}{2t+1}.$$

By Theorem 2.2.3, for any non-oscillatory solution $x(t)$ of Eq. (2.4.3), (2.2.32) holds. In fact, $x(t) = t - t^{-1}$ is such a solution. In addition, according to Definition 2.4.1, this solution is in the class $\mathcal{A}_1$ since $b(t) = -1/t$ is a bounded function on $[2, +\infty)$.

Remark 2.4.1. *We note that neither results reported by Džurina in [20], nor those in the references [29], [30], [31], [42], [52], [65], [66] apply to Eqs. (2.4.2) and (2.4.3).*

Example 2.4.3. *For* $t \geq 4$, *consider the second order nonlinear neutral differential equation*

$$\left(x(t) + \frac{1}{2t+1} x(t-1)\right)'' + a(t)\, x(t) + b(t) \frac{x'(t/2)}{\sqrt{(x'(t/2))^2 + 1}} = 0, \tag{2.4.4}$$

where $\alpha(t) = [(2t+1)(t-1)]^{-3}$, $a(t) = -4t^4 (t^2+1)^{-1} \alpha(t)$, *and*

$$b(t) = \frac{\sqrt{2}\left(-36t^5 + 60t^4 - 40t^3 - 6t^2 + 6t + 2\right)\left(t^4 - 4t^2 + 8\right)^{1/2}}{t^3 (t^2 - 4)} \alpha(t).$$

By Theorem 2.3.2, any non-oscillatory solution of Eq. (2.4.4) satisfies (2.3.17). In fact, $x(t) = t + t^{-1}$ *is such a solution. According to classification of Kong et al. [49], this solution belongs to the class* $\mathcal{A}_1$ *because* $x(t) = t + b(t)$, *where* $b(t)$ *is a bounded function on* $[4, +\infty)$.

Example 2.4.4. *For* $t \geq 6$, *consider the third order nonlinear neutral differential equation*

$$\begin{aligned} &\left(x(t) + \frac{t}{2t+1} x(t-1)\right)''' + a(t)\, x(t) + b(t)\, x(t-2) \\ &\qquad + c(t) \frac{(x'(t))^2 + 1}{(x'(t))^2 + 2} + d(t) = 0, \end{aligned} \tag{2.4.5}$$

where $\beta(t) = [(t-1)(2t+1)]^{-4}$, $a(t) = 126t^5 (t^3 + t^2 + 1)^{-1} \beta(t)$,

$$\begin{aligned} b(t) &= -120t^3 (t-2)\left(t^3 - 5t^2 + 8t - 3\right)^{-1} \beta(t), \\ c(t) &= -12 \frac{4t^6 + 4t^5 + 3t^4 - 4t^3 - 2t^2 + 1}{4t^6 + 4t^5 + 2t^4 - 4t^3 - 2t^2 + 1} t^2 \beta(t), \\ d(t) &= \left(264t^5 + 6t^4 - 120t^3 - 12t^2 + 24t + 6\right) t^{-4} \beta(t) \end{aligned}$$

By Theorem 2.3.2, any non-oscillatory solution of Eq. (2.4.5) satisfies (2.3.17). In fact, $x(t) = t^2 + t + t^{-1}$ *is such a solution. In the classification suggested by Kong et al. [49], this solution belongs to the class* $\mathcal{B}_{2,2}$ *because* $x(t) = t^2 + b(t)$, *and* $b(t) = \mathrm{o}(t^2)$ *as* $t \to \infty$.

Example 2.4.5. *For $t \geq 5$, consider the third order nonlinear neutral differential equation*

$$\left(x\left(t\right)+\frac{t}{3t+2}x\left(t-1\right)\right)^{\prime\prime\prime}+a\left(t\right)\sqrt{x\left(t\right)x\left(\frac{t-1}{2}\right)}+b\left(t\right)=0, \tag{2.4.6}$$

where

$$\begin{aligned}a\left(t\right)=&\,12\sqrt{\frac{t\left(t-1\right)}{\left(t^2-t+1\right)\left(t^2-4t+7\right)}}\times\\&\times\frac{92t^7-354t^6+380t^5+109t^4-136t^3-72t^2+32t+16}{t^4\left(t-1\right)^4\left(t+1\right)\left(3t+2\right)^4},\\b\left(t\right)=&\,\frac{348t^4}{\left(t-1\right)^4\left(3t+2\right)^4}.\end{aligned}$$

By Theorem 2.3.3, any non-oscillatory solution of Eq. (2.4.6) satisfies (2.3.17), and $x\left(t\right)=t^2+t^{-1}$ is such a solution, which, according to Definition 2.4.1, belongs to the class $\mathcal{A}_2$ since $x(t)=t^2+b(t)$, where $b(t)$ is a bounded function on $[5,+\infty)$.

Remark 2.4.2. *We would like to stress that theorems reported by Dahiya and Singh [16], Dahiya and Zafer [17], Džurina [20], Graef and Spikes [30], Grammatikopoulos et al. [31], Kong et al. [49], Kulcsár [52], Lacková [53], Ladas [54], M. Naito [65], Y. Naito [66], Tanaka [83] do not apply to neutral differential equations considered in Examples 2.4.3, 2.4.4, and 2.4.5.*

Chapter 3

OSCILLATION

Oscillation theory is a rapidly developing branch of the qualitative theory of differential equations. Its foundations were laid down by the well-known results regarding zeros of solutions of self-adjoint second order differential equations obtained by Sturm [81]. Since then oscillatory properties of solutions to different classes of linear and nonlinear ordinary, functional, partial, discrete and impulsive differential equations have attracted attention of many researchers. An elevated interest to this topic has been reflected, for instance, in monographs on oscillation by Agarwal et al. [1, 2], Bainov and Mishev [6], Erbe et al. [23], Győri and Ladas [32], Kreith [50], Ladde et al. [55], Swanson [82]; chapters in monographs on asymptotic behavior of solutions of differential equations by Bellman [9], Coppel [15], Kiguradze and Chanturiya [46], Norkin [67] and survey papers by Wong [89, 90].

In this chapter, we focus on second order nonlinear neutral differential equations

$$\left(r\left(t\right)\left(x\left(t\right)+p\left(t\right)x\left(t-\tau\right)\right)'\right)'+q\left(t\right)f\left(x\left(t\right),x\left(\sigma\left(t\right)\right)\right)=0, \tag{3.0.1}$$

and

$$\left(r\left(t\right)\psi\left(x\left(t\right)\right)\left(x\left(t\right)+p\left(t\right)x\left(\delta\left(t\right)\right)\right)'\right)'+q\left(t\right)f\left(x\left(t\right),x\left(\sigma\left(t\right)\right)\right)=0, \tag{3.0.2}$$

where $t\geq t_0>0$, $\tau\geq 0$ is a constant, $\delta\geq 0$, $r,\sigma\in C^1\left(\left[t_0,+\infty\right),\left(0,+\infty\right)\right)$, $p,q\in C\left(\left[t_0,+\infty\right),\mathbb{R}\right)$, $\psi\in C^1\left(\mathbb{R},\mathbb{R}\right)$ and $f\in C\left(\mathbb{R}^2,\mathbb{R}\right)$. Our aim is to provide new efficient oscillation results for Eqs. (3.0.1) and (3.0.2).

3.1 Brief History

In what follows, we briefly review several important oscillation results obtained for second order neutral differential equations. Grammatikopoulos et al. [31] established that condition

$$\int_{t_0}^{+\infty} q(s)(1 - p(s-\sigma))\, ds = +\infty,$$

ensures oscillation of a linear neutral differential equation

$$(x(t) + p(t)\, x(t-\tau))'' + q(t)\, x(t-\sigma) = 0.$$

Sufficient conditions for the oscillation of solutions of a slightly more general neutral differential equation

$$(x(t) + p(t)\, x(t-\tau))'' + q(t)\, x(\sigma(t)) = 0,$$

including the case when $p = 1$, were obtained by Džurina and Mihalíková [22]. By using the integral averaging method, Ruan [70] derived a number of general oscillation criteria for a nonlinear neutral differential equation

$$\left(r(t)\,(x(t) + p(t)\, x(t-\tau))'\right)' + q(t)\, f(x(t-\sigma)) = 0, \tag{3.1.1}$$

whereas Li [56] provided classification of nonoscillatory solutions of the equation (3.1.1) and established necessary and/or sufficient conditions for the existence of eventually positive solutions. Ruan's results for equation (3.1.1) have been further improved by Li and Liu [57] who exploited a generalized Riccati transformation. Interesting applications of the integral averaging technique to oscillation of several classes of nonlinear neutral differential equations can be found in the papers by Džurina and Lacková [21], Gai et al. [26], and Xu et al. [91]. In particular, the latter paper addresses the oscillation of a nonlinear neutral differential equation

$$\left(r(t)\left(x(t)+p(t)x(t-\tau)\right)'\right)'+f\left(t,x(t),x(t-\sigma),x'(t)\right)=0,$$

where

$$f\left(t,x(t),x(t-\sigma),x'(t)\right)\geq q(t)\,f_1(x(t))\,f_2(x(t-\sigma))\,g(x'(t)),$$

$f_1(x)\geq k_1>0$, $f_2(x)/x\geq k_2>0$, $g(x)\geq k_3>0$.

Recently, Shi and Wang [86] proved several oscillation criteria for equation (3.0.1), one of which we present below.

Theorem 3.1.1. *Let the following conditions hold:*

(A_1) for all $t\geq t_0$, $0\leq p(t)\leq 1$, $r(t)>0$, $q(t)\geq 0$, and $q(t)$ is not identically zero for large t,

(A_2) $\int^{+\infty} r^{-1}(s)\,ds=+\infty$,

(A_3) for all $t\geq t_0$, $\sigma(t)\leq t$, $\sigma'(t)>0$, and $\lim_{t\to+\infty}\sigma(t)=+\infty$,

(A_4) $\left|\dfrac{f(x,y)}{y}\right|\geq K>0$, for $y\neq 0$, and $f(x,y)$ has the sign of x and y if they have the same sign.

Suppose further that there exist functions $H\in C^1(D,\mathbb{R})$, $h\in C(D_0,\mathbb{R})$, and $k,\rho\in C^1([t_0,+\infty),(0,+\infty))$ satisfying

(i) $H(t,t)=0, \qquad t\geq t_0; \qquad H(t,s)>0, \qquad t>s\geq t_0;$

(ii) $\dfrac{\partial H}{\partial s}(t,s)\leq 0, \qquad (t,s)\in D_0;$

(iii) $\dfrac{\partial(H(t,s)k(s))}{\partial s}+H(t,s)k(s)\dfrac{\rho'(s)}{\rho(s)}=-h(t,s)\sqrt{H(t,s)k(s)}.$

Assume also that

$$0<\inf_{s\geq t_0}\left[\liminf_{t\to+\infty}\frac{H(t,s)}{H(t,t_0)}\right]\leq+\infty \tag{3.1.2}$$

and

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{r(\sigma(s))\rho(s)}{\sigma'(s)}h^2(t,s)\,ds<+\infty. \tag{3.1.3}$$

If there exists a function $B \in C([t_0, +\infty), \mathbb{R})$ *such that*

$$\limsup_{t\to+\infty} \int_{t_0}^{t} \frac{\sigma'(s) B_+^2(s)}{k(s)\rho(s) r(\sigma(s))} ds = +\infty$$

and

$$\limsup_{t\to+\infty} \frac{1}{H(t,T)} \int_T^t \Big[KH(t,s) k(s) \rho(s) q(s) (1 - p(\sigma(s))) \cdot \\ - \frac{r(\sigma(s))\rho(s)}{4\sigma'(s)} h^2(t,s) \Big] ds \geq B(T),$$

for any $T \geq t_0$, *where* $B_+(t) = \max(B(t), 0)$, *then equation (3.0.1) is oscillatory.*

Džurina and Lacková [21] proved a number of oscillation criteria for differential equation

$$\big(r(t)\psi(x(t))(x(t) + p(t)x(\tau(t)))'\big)' + q(t) f(x(\sigma(t))) = 0. \tag{3.1.4}$$

The following theorem is one of the results due to Džurina and Lacková [21].

Theorem 3.1.2 ([21, Theorem 1.1]). *Let the following conditions be satisfied:*

(A_1) for all $t \geq t_0$, $0 \leq p(t) \leq p < 1$;

(A_2) $q(t) \geq 0$ *and* $q(t)$ *is not identically zero for large* t;

(A_3) $R(+\infty) = +\infty$, *where* $R(t) = \int_{t_0}^{t} \frac{1}{r(s)} ds$;

(A_4) for all $t \geq t_0$, $\tau(t) \leq t$, $\sigma(t) \leq t$, $\lim_{t\to+\infty} \sigma(t) = \lim_{t\to+\infty} \tau(t) = +\infty$, *and* $\sigma'(t) \geq 0$;

(A_5) $0 < m \leq \psi(u) \leq M$;

(A_6) f *is nondecreasing,* $f \in C^1(\mathbb{R}\backslash\{0\}, \mathbb{R})$ *and, for* $y \neq 0$, $yf(y) > 0$;

(A_7) f' *is nondecreasing in* $(-\infty, -t^*)$ *and nonincreasing in* $(t^*, +\infty)$, *for some* $t^* \geq 0$.

Suppose also that

$$\int^{+\infty} q(s) f(\pm NR(\sigma(s))) ds = +\infty, \tag{3.1.5}$$

for all $N > 0$, *and*

$$\int^{+\infty}\left[R(\sigma(s))\,q(s)-\frac{M}{4(1-p)\,R(\sigma(s))}\right.$$
$$\left.\times\frac{\sigma'(s)}{r(\sigma(s))\,f'(\pm KR(\sigma(s)))}\right]ds=+\infty,$$

for some $K > 0$. *Then Eq. (3.1.4) is oscillatory.*

To the best of our knowledge, apart from the paper by Džurina and Lacková [21], oscillation results for equations with a nonlinear neutral term involving the function $\psi(x)$ were discussed only by Wang and Yu [87] for a class of neutral differential equations with continuously distributed deviating arguments of the form

$$\left(r(t)\,\psi(x(t))\left(x(t)+\sum_{i=1}^{n}p_i(t)\,x(\tau_i(t))\right)'\right)'$$
$$+\int_a^b q(t,s)f(y(g(t,s)))\,d\sigma(s)=0.$$

3.2 Second Order Nonlinear Neutral Differential Equations

In this section, our purpose is to strengthen oscillation results obtained for equation (3.0.1) by Shi and Wang [86] using a generalized Riccati transformation and developing ideas exploited by Rogovchenko and Tuncay [74].

Definition 3.2.1. *A solution* $x(t)$ *is called continuable if* $x(t)$ *exists for all* $t \geq t_0$. *A non-constant continuable solution* $x(t)$ *called proper if*

$$\sup_{t\geq t_0}|x(t)|>0.$$

A proper solution $x(t)$ *is called oscillatory if there exits a sequence of real numbers* $\{t_n\}_{n=1}^{+\infty}$ *diverging to* $+\infty$ *such that* $x(t_n)=0$ *for all* $n \in \mathbb{N}$. *Otherwise it is called non-oscillatory. An equation is said to be oscillatory if all its proper solutions are oscillatory.*

In what follows, we use the following notations:

$$D_0=\{(t,s):t_0\leq s<t<+\infty\}$$

and

$$D = \{(t,s) : t_0 \le s \le t < +\infty\}.$$

Definition 3.2.2. *We say that a continuous function $H : D \to [0,+\infty)$ belongs to the class $\Im$ if:*

(i) $H(t,t) = 0$ and $H(t,s) > 0$ for $(t,s) \in D_0$;

(ii) H has a continuous partial derivative with respect to the second variable satisfying, for some locally integrable function h,

$$\frac{\partial H}{\partial s}(t,s) = -h(t,s)\sqrt{H(t,s)}. \tag{3.2.1}$$

By a solution of equation (3.0.1) we mean a continuous function $x(t)$, defined on $[t_x, +\infty)$, such that $r(t)(x(t) + p(t)x(t-\delta))'$ is continuously differentiable and $x(t)$ satisfies (3.0.1) for $t \ge t_x$. In the sequel, we assume that solutions of equation (3.0.1) exist and can be continued indefinitely to the right.

Theorem 3.2.1. *Let conditions (A_1)-(A_3) of Theorem 3.1.1 hold with (A_4) replaced by*

(A_4^*) $\dfrac{f(x,y)}{y} \ge \kappa > 0$, for $y \ne 0$, and $yf(x,y) > 0$, for $xy > 0$.

Suppose that there exits a function $\rho \in C^1([t_0,+\infty),\mathbb{R})$ such that, for some $\beta \ge 1$ and for some $H \in \Im$,

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\psi(s) - \frac{\beta v(s) r(\sigma(s))}{4\sigma'(s)} h^2(t,s)\right] ds = +\infty, \tag{3.2.2}$$

where

$$\psi(t) = v(t)\left[\kappa q(t)[1 - p(\sigma(t))] + \sigma'(t)\frac{r^2(t)\rho^2(t)}{r(\sigma(t))} - (r(t)\rho(t))'\right] \tag{3.2.3}$$

and

$$v(t) = \exp\left(-2\int^{t} \sigma'(s)\frac{r(s)\rho(s)}{r(\sigma(s))} ds\right). \tag{3.2.4}$$

Then equation (3.0.1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of equation (3.0.1). Then, there exists a $T_0 \geq t_0$ such that $x(t) \neq 0$ for all $t \geq T_0$. Without loss of generality, we may assume that $x(t) > 0$ and $x(\sigma(t)) > 0$ for all $t \geq T_0 \geq t_0$. Define

$$z(t) = x(t) + p(t)\, x(t - \delta), \qquad t \geq T_0.$$

Obviously, for all $t \geq T_0$, $z(t) \geq x(t) > 0$, and $r(t)\, z'(t)$ is nonincreasing because

$$(r(t)\, z'(t))' = -q(t)\, f(x(t), x(\sigma(t))) \leq 0.$$

We claim that $z'(t) \geq 0$, for all $t \geq T_0$. Otherwise, there should exist a $T_1 \geq T_0 \geq t_0$ such that $z'(T_1) < 0$, which implies that $r(T_1)\, z'(T_1) < 0$. Since $r(t)\, z'(t)$ is nonincreasing and $q(t)$ does not eventually vanish, there exists a $T_2 \geq T_1$ such that $r(T_2)\, z'(T_2) < 0$ and $r(t)\, z'(t) \leq r(T_2)\, z'(T_2) < 0$, for all $t \geq T_2$. Thus,

$$z'(t) \leq r(T_2)\, z'(T_2) \frac{1}{r(t)}.$$

Integration of the latter inequality from T_2 to t yields

$$z(t) \leq z(T_2) + r(T_2)\, z'(T_2) \int_{T_2}^{t} \frac{1}{r(s)} ds. \tag{3.2.5}$$

Passing in (3.2.5) to the limit as $t \to +\infty$ and using (A_2), we conclude that

$$\lim_{t \to +\infty} z(t) = -\infty,$$

which contradicts the fact that $z(t) > 0$.

Note that condition (A_4^*) implies that

$$(r(t)\, z'(t))' + \kappa q(t)\, x(\sigma(t)) \leq 0. \tag{3.2.6}$$

On the other hand,

$$x(t) = z(t) - p(t)\, x(t - \delta) \geq z(t) - p(t)\, z(t - \delta) \geq (1 - p(t))\, z(t).$$

Since $\lim_{t\to+\infty}\sigma(t)=+\infty$, there exists a $T_3 \geq T_2 > 0$ such that, for all $t \geq T_3$,

$$x(\sigma(t)) \geq (1-p(\sigma(t)))\, z(\sigma(t)). \tag{3.2.7}$$

It follows from (3.2.6) and (3.2.7) that, for all $t \geq T_3$,

$$(r(t)\, z'(t))' \leq -\kappa q(t)(1-p(\sigma(t)))\, z(\sigma(t)). \tag{3.2.8}$$

Introduce the generalized Riccati transformation by

$$u(t) = v(t)\, r(t)\left[\frac{z'(t)}{z(\sigma(t))} + \rho(t)\right], \tag{3.2.9}$$

where ρ is a C^1 function and v is defined by (3.2.4). Differentiating (3.2.9) and using (3.0.1), (A_3) and (A_4^*), after some algebra we conclude that, for all $t \geq T_3$,

$$\begin{aligned}
u'(t) &= \frac{v'(t)}{v(t)}u(t) + v(t)\frac{(r(t)\, z'(t))'}{z(\sigma(t))} \\
&\quad - v(t)\, r(t)\frac{\sigma'(t)\, z'(\sigma(t))}{z'(t)}\left(\frac{z'(t)}{z(\sigma(t))}\right)^2 + v(t)(r(t)\,\rho(t))' \\
&\leq \frac{v'(t)}{v(t)}u(t) - \kappa v(t)\, q(t)(1-p(\sigma(t))) \\
&\quad - v(t)\,\sigma'(t)\frac{r^2(t)}{r(\sigma(t))}\left(\frac{u(t)}{v(t)\, r(t)} - \rho(t)\right)^2 + v(t)(r(t)\,\rho(t))'.
\end{aligned}$$

The latter inequality yields, for all $t \geq T_3$,

$$u'(t) \leq -\psi(t) - \sigma'(t)\frac{u^2(t)}{v(t)\, r(\sigma(t))}, \tag{3.2.10}$$

where ψ is defined by (3.2.3). Multiplying (3.2.10) by $H(t,s)$ and integrating between T_3 and t, we have, for all $\beta \geq 1$ and for all $t \geq T_3$,

$$\begin{aligned}
&\int_{T_3}^{t}\left[H(t,s)\,\psi(s) - \frac{\beta v(s)\, r(\sigma(s))}{4\sigma'(s)}h^2(t,s)\right]ds \\
&\leq H(t,T_3)\, u(T_3) - \int_{T_3}^{t}\frac{(\beta-1)\,\sigma'(s)\, H(t,s)}{\beta v(s)\, r(\sigma(s))}u^2(s)\, ds \\
&- \int_{T_3}^{t}\left[\sqrt{\frac{\sigma'(s)\, H(t,s)}{\beta v(s)\, r(\sigma(s))}}u(s) + \sqrt{\frac{\beta v(s)\, r(\sigma(s))}{4\sigma'(s)}}h(t,s)\right]^2 ds.
\end{aligned} \tag{3.2.11}$$

Using monotonicity of H, we conclude that, for all $t \geq T_3$,

$$\int_{T_3}^{t} \left[H(t,s)\,\psi(s) - \frac{\beta v(s)\, r(\sigma(s))}{4\sigma'(s)} h^2(t,s) \right] ds \leq H(t,T_3)\,|u(T_3)| \leq H(t,t_0)\,|u(T_3)|,$$

and, correspondingly,

$$\int_{t_0}^{t} \left[H(t,s)\,\psi(s) - \frac{\beta v(s)\, r(\sigma(s))}{4\sigma'(s)} h^2(t,s) \right] ds \leq H(t,t_0) \left[|u(T_3)| + \int_{t_0}^{T_3} |\psi(s)|\, ds \right]. \tag{3.2.12}$$

By virtue of (3.2.12),

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\,\psi(s) - \frac{\beta v(s)\, r(\sigma(s))}{4\sigma'(s)} h^2(t,s) \right] ds \leq |u(T_3)| + \int_{t_0}^{T_3} |\psi(s)|\, ds < +\infty,$$

which contradicts (3.2.2). Therefore, all solutions of equation (3.0.1) are oscillatory. □

Remark 3.2.2. *Efficient oscillation tests can be derived from Theorem 3.2.1 with the appropriate choice of the functions H and h. For instance, the standard choice for many handy oscillation results is a Kamenev-type function H defined by*

$$H(t,s) = (t-s)^{n-1}, \qquad (t,s) \in D, \tag{3.2.13}$$

where $n > 2$ is an integer. It is easily seen that the function H belongs to the class $\Im$, and one chooses the function

$$h(t,s) = (n-1)(t-s)^{(n-3)/2}, \qquad (t,s) \in D,$$

to meet the condition (ii) of Definition 3.2.2.

A consequence of Theorem 3.2.1 is the following oscillation criterion.

Corollary 3.2.1. *Suppose that there exists a function $\rho \in C^1([t_0,+\infty),\mathbb{R})$ such that, for some integer $n>2$ and for some $\beta \geq 1$,*

$$\limsup_{t\to+\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-3}\left[(t-s)^2\psi(s)-\beta(n-1)^2\frac{v(s)r(\sigma(s))}{4\sigma'(s)}\right]ds=+\infty,$$

where ψ and v are as in Theorem 3.2.1. Then equation (3.0.1) is oscillatory.

The following theorem brings improvements to the result due to Shi and Wang [86] by removing a condition similar to (3.1.3).

Theorem 3.2.3. *Suppose that (3.1.2) holds. Assume also that there exist functions $H \in \Im$, $\rho \in C^1([t_0,+\infty),\mathbb{R})$ and $\phi \in C([t_0,+\infty),\mathbb{R})$ such that, for all $T \geq t_0$ and for some $\beta > 1$,*

$$\limsup_{t\to+\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)\psi(s)-\frac{\beta v(s)r(\sigma(s))}{4\sigma'(s)}h^2(t,s)\right]ds\geq\phi(T),$$

where ψ and v are as in Theorem 3.2.1. Suppose further that

$$\limsup_{t\to+\infty}\int_{t_0}^{t}\frac{\sigma'(s)\phi_+^2(s)}{v(s)r(\sigma(s))}ds=+\infty, \tag{3.2.14}$$

where $\phi_+(t)=\max(\phi(t),0)$. Then equation (3.0.1) is oscillatory.

Proof. Without loss of generality, assume again that equation (3.0.1) possesses a solution $x(t)$ such that $x(t)>0$ and $x(\sigma(t))>0$ on $[T_0,+\infty)$, for some $T_0 \geq t_0$. Proceeding as in the proof of Theorem 3.2.1, we arrive at the inequality (3.2.11), which yields, for all $t \geq T_3$ and for any $\beta > 1$,

$$\begin{aligned}\phi(T_3) &\leq \limsup_{t\to+\infty}\frac{1}{H(t,T_3)}\int_{T_3}^{t}\left[H(t,s)\psi(s)-\frac{\beta v(s)r(\sigma(s))}{4\sigma'(s)}h^2(t,s)\right]ds\\ &\leq u(T_3)-\liminf_{t\to+\infty}\frac{1}{H(t,T_3)}\int_{T_3}^{t}\frac{(\beta-1)\sigma'(s)H(t,s)}{\beta v(s)r(\sigma(s))}u^2(s)\,ds.\end{aligned}$$

The latter inequality implies that, for all $t \geq T_3$,

$$\phi(T_3)+\liminf_{t\to+\infty}\frac{1}{H(t,T_3)}\int_{T_3}^{t}\frac{(\beta-1)\sigma'(s)H(t,s)}{\beta v(s)r(\sigma(s))}u^2(s)\,ds\leq u(T_3).$$

Consequently,

$$\phi\left(T_3\right) \leq u\left(T_3\right) \tag{3.2.15}$$

and

$$\begin{aligned}\liminf_{t\to+\infty} \frac{1}{H\left(t,T_3\right)} \int_{T_3}^{t} \frac{\sigma'\left(s\right) H\left(t,s\right)}{v\left(s\right) r\left(\sigma\left(s\right)\right)} u^2\left(s\right) ds \\ \leq \frac{\beta}{\beta-1}\left(u\left(T_3\right)-\phi\left(T_3\right)\right) < +\infty.\end{aligned} \tag{3.2.16}$$

Assume now that

$$\int_{T_3}^{+\infty} \frac{\sigma'\left(s\right) u^2\left(s\right)}{v\left(s\right) r\left(\sigma\left(s\right)\right)} ds = +\infty. \tag{3.2.17}$$

Condition (3.1.2) implies existence of a $\vartheta > 0$ such that

$$\liminf_{t\to+\infty} \frac{H\left(t,s\right)}{H\left(t,t_0\right)} > \vartheta > 0. \tag{3.2.18}$$

It follows from (3.2.17) that, for any positive constant η, there exists a $T_4 > T_3$ such that, for all $t \geq T_4$,

$$\int_{T_3}^{t} \frac{\sigma'\left(s\right) u^2\left(s\right)}{v\left(s\right) r\left(\sigma\left(s\right)\right)} ds \geq \frac{\eta}{\vartheta}. \tag{3.2.19}$$

Using integration by parts and (3.2.19), we have, for all $t \geq T_4$,

$$\begin{aligned}&\frac{1}{H\left(t,T_3\right)} \int_{T_3}^{t} H\left(t,s\right) \frac{\sigma'\left(s\right) u^2\left(s\right)}{v\left(s\right) r\left(\sigma\left(s\right)\right)} ds \\ &= \frac{1}{H\left(t,T_3\right)} \int_{T_3}^{t} H\left(t,s\right) d\left[\int_{T_3}^{s} \frac{\sigma'\left(\xi\right) u^2\left(\xi\right)}{v\left(\xi\right) r\left(\sigma\left(\xi\right)\right)} d\xi\right] \\ &= \frac{1}{H\left(t,T_3\right)} \int_{T_3}^{t} \left[\int_{T_3}^{s} \frac{\sigma'\left(\xi\right) u^2\left(\xi\right)}{v\left(\xi\right) r\left(\sigma\left(\xi\right)\right)} d\xi\right] \left[-\frac{\partial H}{\partial s}\left(t,s\right)\right] ds \\ &\geq \frac{1}{H\left(t,T_3\right)} \int_{T_4}^{t} \left[\int_{T_3}^{s} \frac{\sigma'\left(\xi\right) u^2\left(\xi\right)}{v\left(\xi\right) r\left(\sigma\left(\xi\right)\right)} d\xi\right] \left[-\frac{\partial H}{\partial s}\left(t,s\right)\right] ds \\ &\geq \frac{\eta}{\vartheta} \frac{1}{H\left(t,T_3\right)} \int_{T_4}^{t} \left[-\frac{\partial H}{\partial s}\left(t,s\right)\right] ds \\ &= \frac{\eta}{\vartheta} \frac{H\left(t,T_4\right)}{H\left(t,T_3\right)} \geq \frac{\eta}{\vartheta} \frac{H\left(t,T_4\right)}{H\left(t,t_0\right)}.\end{aligned}$$

By virtue of (3.2.18), there exists a $T_5 \geq T_4$ such that

$$\frac{H\left(t,T_4\right)}{H\left(t,t_0\right)} \geq \vartheta,$$

for all $t \geq T_5$, which implies that

$$\frac{1}{H(t,T_3)} \int_{T_3}^{t} H(t,s) \frac{\sigma'(s) u^2(s)}{v(s) r(\sigma(s))} ds \geq \eta.$$

Since η is an arbitrary positive constant,

$$\liminf_{t\to+\infty} \frac{1}{H(t,T_3)} \int_{T_3}^{t} H(t,s) \frac{\sigma'(s) u^2(s)}{v(s) r(\sigma(s))} ds = +\infty,$$

and the latter contradicts (3.2.16). Consequently,

$$\int_{T_3}^{+\infty} \frac{\sigma'(s) u^2(s)}{v(s) r(\sigma(s))} ds < +\infty,$$

and, by virtue of (3.2.15),

$$\int_{T_3}^{+\infty} \frac{\sigma'(s) \phi_+^2(s)}{v(s) r(\sigma(s))} ds \leq \int_{T_3}^{+\infty} \frac{\sigma'(s) u^2(s)}{v(s) r(\sigma(s))} ds < +\infty,$$

which contradicts (3.2.14). Therefore, equation (3.0.1) is oscillatory. □

Choosing H as in Corollary 3.2.1, we observe that condition (3.1.2) holds because

$$\lim_{t\to+\infty} \frac{H(t,s)}{H(t,t_0)} = \lim_{t\to+\infty} \frac{(t-s)^{n-1}}{(t-t_0)^{n-1}} = 1.$$

Thus, we derive from Theorem 3.2.3 a useful oscillation test for equation (3.0.1).

Corollary 3.2.2. *Assume that there exist functions $\rho \in C^1([t_0,+\infty),\mathbb{R})$ and $\phi \in C([t_0,+\infty),\mathbb{R})$ such that, for all $T \geq t_0$, some integer $n > 2$ and for some $\beta > 1$,*

$$\limsup_{t\to+\infty} \frac{1}{t^{n-1}} \int_{T}^{t} (t-s)^{n-3} \left[(t-s)^2 \psi(s) - \beta (n-1)^2 \frac{v(s) r(\sigma(s))}{4\sigma'(s)} \right] ds \geq \phi(T).$$

Suppose also that (3.2.14) holds, where ψ, v and ϕ_+ are as in Theorem 3.2.3. Then equation (3.0.1) is oscillatory.

The result that follows is an immediate consequence of properties of the limits.

Theorem 3.2.4. *Let (3.1.2) be satisfied. Assume also that there exist functions $H \in \Im$, $\rho \in C^1([t_0,+\infty),\mathbb{R})$ and $\phi \in C([t_0,+\infty),\mathbb{R})$ such that, for all $T \geq t_0$ and for some $\beta > 1$,*

$$\liminf_{t\to+\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\psi(s) - \frac{\beta v(s) r(\sigma(s))}{4\sigma'(s)} h^2(t,s) \right] ds \geq \phi(T),$$

where ψ, v, and ϕ_+ are as in Theorem 3.2.3. Suppose further that (3.2.14) holds. Then equation (3.0.1) is oscillatory.

Proof. The conclusion of the theorem follows immediately from the properties of the limits

$$\begin{aligned} \phi(T) &\leq \liminf_{t\to+\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\psi(s) - \frac{\beta v(s) r(\sigma(s))}{4\sigma'(s)} h^2(t,s) \right] ds \\ &\leq \limsup_{t\to+\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\psi(s) - \frac{\beta v(s) r(\sigma(s))}{4\sigma'(s)} h^2(t,s) \right] ds \end{aligned}$$

and Theorem 3.2.3. □

The following result, analogous to Corollary 3.2.2, is derived by choosing a Kamenev-type function $H(t,s) = (t-s)^{n-1}$.

Corollary 3.2.3. *Assume that there exist functions $\rho \in C^1([t_0,+\infty),\mathbb{R})$ and $\phi \in C([t_0,+\infty),\mathbb{R})$ such that, for all $T \geq t_0$, for some integer $n > 2$, and for some $\beta > 1$,*

$$\liminf_{t\to+\infty} \frac{1}{t^{n-1}} \int_T^t (t-s)^{n-3} \left[(t-s)^2 \psi(s) - \beta(n-1)^2 \frac{v(s) r(\sigma(s))}{4\sigma'(s)} \right] ds \geq \phi(T).$$

Suppose also that (3.2.14) holds, where ψ, v, and ϕ_+ are as in Theorem 3.2.3. Then equation (3.0.1) is oscillatory.

Consider now Eq. (3.0.2). Similarly, by a solution of Eq. (3.0.2) we mean a continuous function $x(t)$, defined on $[t_*, T)$, such that

$$r(t)\psi(x(t))(x(t) + p(t)x(\delta(t)))'$$

is continuously differentiable and $x(t)$ satisfies (3.0.2) for $t \in [t_*, T)$. Our aim is to extend and improve oscillation results obtained by Džurina and Lacková [21] by developing ideas suggested by Rogovchenko and Tuncay [76]. Oscillation criteria we will provide for Eq. (3.0.2) prove to be more efficient compared to known results even for less general classes of equations, including, for instance, Eq. (3.1.4) and alike.

We start our discussion with an auxiliary lemma.

Lemma 3.2.1. *Assume that conditions (A_2)-(A_5) of Theorem 3.1.2 hold with (A_1) and (A_6) replaced respectively by*

(A_1^*) for all $t \geq t_0$, $0 \leq p(t) \leq 1$;

(A_6^*) for $y \neq 0$, $\frac{f(x,y)}{y} \geq \kappa > 0$ and, for $xy > 0$, $yf(x,y) > 0$.

Let $x(t)$ be an eventually positive solution of Eq. (3.0.2). For $t \geq t_0$, define

$$z(t) = x(t) + p(t)\,x(\delta(t)). \tag{3.2.20}$$

Then $z'(t) \geq 0$.

Proof. Without loss of generality, we may assume that $x(t) > 0$ and $x(\sigma(t)) > 0$, for all $t \geq T_0 \geq t_0$. Similar arguments apply when $x(t) < 0$ and $x(\sigma(t)) < 0$. Then, for all $t \geq T_0$, $z(t) \geq x(t) > 0$ and $r(t)\,\psi(x(t))\,z'(t)$ is nonincreasing since

$$(r(t)\,\psi(x(t))\,z'(t))' = -q(t)\,f(x(t), x(\sigma(t))) \leq 0.$$

Suppose, contrary to our assertion, that there exists a $T_1 \geq T_0$ such that $z'(T_1) < 0$. Consequently, $r(T_1)\,\psi(x(T_1))\,z'(T_1) < 0$. Since $r(t)\,\psi(x(t))\,z'(t)$ is nonincreasing and $q(t)$ does not eventually vanish, there exists a $T_2 \geq T_1$ such that

$$r(T_2)\,\psi(x(T_2))\,z'(T_2) < 0$$

and

$$r(t)\,\psi(x(t))\,z'(t) \leq r(T_2)\,\psi(x(T_2))\,z'(T_2) < 0,$$

for all $t \geq T_2$. Thus,

$$z'(t) \leq \frac{r(T_2)\psi(x(T_2))z'(T_2)}{\psi(x(t))}\frac{1}{r(t)}$$
$$\leq \frac{r(T_2)\psi(x(T_2))z'(T_2)}{m}\frac{1}{r(t)}.$$

Integration of the latter inequality from T_2 to t yields

$$z(t) \leq z(T_2) + \frac{r(T_2)\psi(x(T_2))z'(T_2)}{m}\int_{T_2}^{t}\frac{1}{r(s)}ds. \tag{3.2.21}$$

Passing in (3.2.21) to the limit as $t \to +\infty$ and using (A_2), we conclude that

$$\lim_{t\to+\infty} z(t) = -\infty,$$

which contradicts the fact that the function $z(t)$ is eventually positive. □

Remark 3.2.5. *Under the conditions of Lemma 3.2.1, for an eventually negative solution $x(t)$ of Eq. (3.0.2), one has $z'(t) \leq 0$.*

Theorem 3.2.6. *Let the conditions (A_1^*), (A_2)-(A_5) and (A_6^*) hold. Suppose that there exists a function $\rho \in C^1([t_0, +\infty), \mathbb{R})$ such that, for some $\beta \geq 1$ and for some $H \in \Im$,*

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\vartheta(s) - \frac{\beta M v(s)}{4}\right.$$
$$\left.\times\frac{r(\sigma(s))}{\sigma'(s)}h^2(t,s)\right]ds = +\infty, \tag{3.2.22}$$

where

$$\vartheta(t) = v(t)\left[\kappa q(t)(1 - p(\sigma(t))) + \right.$$
$$\left.\frac{1}{M}\frac{\sigma'(t)\rho^2(t)}{r(\sigma(t))} - \rho'(t)\right] \tag{3.2.23}$$

and

$$v(t) = \exp\left(-\frac{2}{M}\int^{t}\frac{\sigma'(s)\rho(s)}{r(\sigma(s))}ds\right). \tag{3.2.24}$$

Then Eq. (3.0.2) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (3.0.2). Then, there exists a $T_0 \geq t_0$ such that $x(t) \neq 0$ for all $t \geq T_0$. Without loss of generality, suppose that $x(t) > 0$ and $x(\sigma(t)) > 0$, for all $t \geq T_0$. It follows from (A_6^*) that

$$(r(t)\psi(x(t))z'(t))' + \kappa q(t)x(\sigma(t)) \leq 0. \tag{3.2.25}$$

By Lemma 1, $z(t)$ is nondecreasing. Therefore,

$$\begin{aligned} x(t) &= z(t) - p(t)x(\delta(t)) \\ &\geq z(t) - p(t)z(\delta(t)) \geq (1-p(t))z(t). \end{aligned} \tag{3.2.26}$$

Since $\sigma(t) \to +\infty$ as $t \to +\infty$, there exists a $T_1 \geq T_0$ such that, for all $t \geq T_1$,

$$x(\sigma(t)) \geq (1-p(\sigma(t)))z(\sigma(t)).$$

By virtue of (3.2.25) and (3.2.26), it follows that, for all $t \geq T_1$,

$$(r(t)\psi(x(t))z'(t))' \leq -\kappa q(t)(1-p(\sigma(t)))z(\sigma(t)).$$

Let

$$u(t) = v(t)\left[\frac{r(t)\psi(x(t))z'(t)}{z(\sigma(t))} + \rho(t)\right], \tag{3.2.27}$$

where ρ is a C^1 function and v is defined by (3.2.24). Differentiating (3.2.27), using (3.0.2) and conditions (A_3)-(A_5), we conclude that, for all $t \geq T_1$,

$$\begin{aligned} u'(t) &= \frac{v'(t)}{v(t)}u(t) + v(t)\left[\frac{(r(t)\psi(x(t))z'(t))'}{z(\sigma(t))}\right. \\ &\qquad \left. - \frac{\sigma'(t)z'(\sigma(t))}{r(t)\psi(x(t))z'(t)}\left[\frac{u(t)}{v(t)} - \rho(t)\right]^2 + \rho'(t)\right] \\ &\leq v(t)\left[-\kappa q(t)(1-p(\sigma(t))) - \frac{\sigma'(t)u^2(t)}{Mr(\sigma(t))v^2(t)}\right. \\ &\qquad \left. + \frac{2\sigma'(t)u(t)\rho(t)}{Mr(\sigma(t))v(t)} - \frac{\sigma'(t)\rho^2(t)}{Mr(\sigma(t))} + \rho'(t)\right] \\ &\qquad + \frac{v'(t)}{v(t)}u(t). \end{aligned}$$

The latter inequality yields, for all $t \geq T_1$,

$$u'(t) \leq -\vartheta(t) - \sigma'(t)\frac{u^2(t)}{Mr(\sigma(t))\,v(t)}, \tag{3.2.28}$$

where ϑ is defined by (3.2.23). Multiplying (3.2.28) by $H(t,s)$ and integrating from T_1 to t, we have, for all $\beta \geq 1$ and for all $t \geq T_1$,

$$\begin{aligned}\int_{T_1}^{t}&\left[H(t,s)\vartheta(s) - \frac{\beta Mr(\sigma(s))\,v(s)}{4\sigma'(s)}h^2(t,s)\right]ds \\ &\leq H(t,T_1)\,u(T_1) \\ &\qquad - \int_{T_1}^{t}\frac{(\beta-1)\,\sigma'(s)\,H(t,s)}{\beta Mr(\sigma(s))\,v(s)}u^2(s)\,ds \\ &- \int_{T_1}^{t}\left[\sqrt{\frac{\sigma'(s)\,H(t,s)}{\beta Mr(\sigma(s))\,v(s)}}u(s)\right. \\ &\qquad \left. + \sqrt{\frac{\beta Mr(\sigma(s))\,v(s)}{4\sigma'(s)}}h(t,s)\right]^2 ds.\end{aligned} \tag{3.2.29}$$

Using monotonicity of H, we conclude that, for all $t \geq T_1$,

$$\begin{aligned}\int_{T_1}^{t}&\left[H(t,s)\vartheta(s) - \frac{\beta Mr(\sigma(s))\,v(s)}{4\sigma'(s)}h^2(t,s)\right]ds \\ &\leq H(t,T_1)\,|u(T_1)| \leq H(t,t_0)\,|u(T_1)|\end{aligned}$$

and

$$\begin{aligned}\int_{t_0}^{t}&\left[H(t,s)\vartheta(s) - \frac{\beta Mr(\sigma(s))\,v(s)}{4\sigma'(s)}h^2(t,s)\right]ds \\ &\leq H(t,t_0)\left[|u(T_1)| + \int_{t_0}^{T_1}\vartheta(s)\,ds\right].\end{aligned}$$

The latter inequality implies that, for all $t \geq t_0$,

$$\begin{aligned}\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}&\left[H(t,s)\vartheta(s) - \frac{\beta Mv(s)}{4}\right. \\ &\left.\times\frac{r(\sigma(s))}{\sigma'(s)}h^2(t,s)\right]ds \\ &\leq |u(T_1)| + \int_{t_0}^{T_1}\vartheta(s)\,ds < +\infty,\end{aligned}$$

which contradicts (3.2.22). Thus, all solutions of Eq. (3.0.2) are oscillatory. □

Due to Theorem 3.2.6, useful oscillation criteria can be deduced by introducing a Kamenev-type function. For instance, the following result is obtained by choosing the function (3.2.13).

Corollary 3.2.4. *Suppose that there exist functions $\rho \in C^1([t_0,+\infty),\mathbb{R})$ and $\varphi \in C([t_0,+\infty),\mathbb{R})$ such that, for all $T \geq t_0$, for some integer $n > 3$ and for some $\beta \geq 1$,*

$$\limsup_{t\to+\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-3}\left[(t-s)^2\vartheta(s)-\frac{\beta(n-1)^2}{4}\times\frac{Mv(s)\,r(\sigma(s))}{\sigma'(s)}\right]ds=+\infty, \quad (3.2.30)$$

where ϑ and v are as in Theorem 3.2.6. Then Eq. (3.0.2) is oscillatory.

Theorem 3.2.7. *Suppose that (3.1.2) holds. Assume also that there exist functions $H \in \Im$, $\rho \in C^1([t_0,+\infty),\mathbb{R})$ and $\varphi \in C([t_0,+\infty),\mathbb{R})$ such that, for all $T \geq t_0$ and for some $\beta > 1$,*

$$\limsup_{t\to+\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)\vartheta(s)-\frac{\beta M v(s)}{4}\times\frac{r(\sigma(s))}{\sigma'(s)}h^2(t,s)\right]ds\geq\varphi(T), \quad (3.2.31)$$

where ϑ, φ_+ and v are as in Theorem 3.2.6. Suppose finally that

$$\limsup_{t\to+\infty}\int_{t_0}^{t}\frac{\sigma'(s)\,\varphi_+^2(s)}{v(s)\,r(\sigma(s))}ds=+\infty. \quad (3.2.32)$$

Then Eq. (3.0.2) is oscillatory.

Proof. Without loss of generality, we may assume that Eq. (3.0.2) has a solution $x(t)$ such that $x(t) > 0$ and $x(\sigma(t)) > 0$ on $[T_0,+\infty)$, for some $T_0 \geq t_0$. Then (3.2.29) holds and, for all $t \geq T_1$ and for all $\beta > 1$,

$$u(T_1)\geq\varphi(T_1)+\liminf_{t\to+\infty}\frac{1}{H(t,T_1)}\int_{T_1}^{t}H(t,s)\frac{(\beta-1)\,\sigma'(s)}{\beta M r(\sigma(s))\,v(s)}u^2(s)\,ds.$$

Consequently,

$$u(T_1) \geq \varphi(T_1)$$

and

$$\liminf_{t\to+\infty} \frac{1}{H(t,T_1)} \int_{T_1}^{t} H(t,s) \frac{\sigma'(s)}{r(\sigma(s))\,v(s)} u^2(s)\,ds \\ \leq \frac{\beta M}{\beta-1}\left(u(T_1)-\varphi(T_1)\right) < +\infty. \quad (3.2.33)$$

Assume that

$$\int_{T_1}^{+\infty} \frac{\sigma'(s)\,u^2(s)}{r(\sigma(s))\,v(s)} ds = +\infty. \quad (3.2.34)$$

By condition (3.1.2), there exists a constant η such that

$$\inf_{s\geq t_0}\left[\liminf_{t\to+\infty} \frac{H(t,s)}{H(t,t_0)}\right] > \eta > 0. \quad (3.2.35)$$

If follows from the assumption (3.2.34) that, for any positive constant ν, there exists a $T_2 > T_1$ such that, for all $t \geq T_2$,

$$\int_{T_1}^{t} \frac{\sigma'(s)\,u^2(s)}{r(\sigma(s))\,v(s)} ds \geq \frac{\nu}{\eta} > 0. \quad (3.2.36)$$

Using integration by parts and (3.2.36), we have, for all $t \geq T_2$,

$$\frac{1}{H(t,T_1)} \int_{T_1}^{t} H(t,s) \frac{\sigma'(s)\,u^2(s)}{r(\sigma(s))\,v(s)} ds \\ = \frac{1}{H(t,T_1)} \int_{T_1}^{t} H(t,s)\, d\left[\int_{T_1}^{s} \frac{\sigma'(\xi)\,u^2(\xi)}{r(\sigma(\xi))\,v(\xi)} d\xi\right] \\ \geq \frac{1}{H(t,T_1)} \int_{T_2}^{t} \left[\int_{T_1}^{t} \frac{\sigma'(\xi)\,u^2(\xi)}{r(\sigma(\xi))\,v(\xi)} d\xi\right]\left[-\frac{\partial H}{\partial s}(t,s)\right] ds$$

$$\geq \frac{\nu}{\eta} \frac{1}{H(t,T_1)} \int_{T_2}^{t} \left[-\frac{\partial H}{\partial s}(t,s)\right] ds \\ = \frac{\nu}{\eta} \frac{H(t,T_2)}{H(t,T_1)} \geq \frac{\nu}{\eta} \frac{H(t,T_2)}{H(t,t_0)}.$$

Inequality (3.2.35) implies that there exists a $T_3 \geq T_2$ such that

$$\frac{H(t,T_2)}{H(t,t_0)} \geq \eta,$$

for all $t \geq T_3$, which yields

$$\frac{1}{H(t,T_1)}\int_{T_1}^{t} H(t,s)\frac{\sigma'(s)\,u^2(s)}{r(\sigma(s))\,v(s)}ds \geq \nu.$$

Since ν is an arbitrary positive constant,

$$\liminf_{t\to+\infty}\frac{1}{H(t,T_1)}\int_{T_1}^{t} H(t,s)\frac{\sigma'(s)\,u^2(s)}{r(\sigma(s))\,v(s)}ds = +\infty.$$

But the latter contradicts (3.2.33). Therefore,

$$\int_{T_1}^{+\infty}\frac{\sigma'(s)\,u^2(s)}{r(\sigma(s))\,v(s)}ds < +\infty.$$

Then,

$$\int_{T_1}^{+\infty}\frac{\sigma'(s)\,\varphi_+^2(s)}{r(\sigma(s))\,v(s)}ds \leq \int_{T_1}^{+\infty}\frac{\sigma'(s)\,u^2(s)}{r(\sigma(s))\,v(s)}ds < +\infty.$$

Hence, a contradiction with the assumption (3.2.32). □

Choosing H as in (3.2.13), one obtains from Theorem 3.2.7 the following useful proposition.

Corollary 3.2.5. *Suppose that there exist functions $\rho \in C^1([t_0,+\infty),\mathbb{R})$ and $\varphi \in C([t_0,+\infty),\mathbb{R})$ such that, for all $T \geq t_0$, for some integer $n > 3$ and for some $\beta > 1$,*

$$\limsup_{t\to+\infty}\frac{1}{t^{n-1}}\int_{T}^{t}(t-s)^{n-3}\left[(t-s)^2\vartheta(s) - \frac{\beta M(n-1)^2}{4}\times\frac{v(s)\,r(\sigma(s))}{\sigma'(s)}\right]ds \geq \varphi(T). \quad (3.2.37)$$

Assume also that (3.2.32) holds, where ϑ, v and φ_+ are as in Theorem 3.2.6. Then Eq. (3.0.2) is oscillatory.

Theorem 3.2.8. *Assume that conditions of Theorem 3.2.7 hold except that (3.2.31) is replaced with*

$$\liminf_{t\to+\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)\vartheta(s) - \frac{\beta M v(s)}{4}\times\frac{r(\sigma(s))}{\sigma'(s)}h^2(t,s)\right]ds \geq \varphi(T).$$

Then, assertion of Theorem 3.2.7 remains intact.

Proof. The conclusion follows immediately from the properties of the limits and Theorem 3.2.7. □

The following result is derived by choosing again a Kamenev-type function (3.2.13).

Corollary 3.2.6. *Assume that conditions of Corollary 3.2.5 hold with the condition (3.2.37) being replaced by*

$$\liminf_{t\to+\infty}\frac{1}{t^{n-1}}\int_T^t (t-s)^{n-3}\left[(t-s)^2\vartheta(s)-\frac{\beta M(n-1)^2}{4}\right.$$
$$\left.\times\frac{v(s)\,r(\sigma(s))}{\sigma'(s)}\right]ds\geq\varphi(T).$$

Then, conclusion of Corollary 3.2.5 holds.

Remark 3.2.9. *Observe that a variety of new oscillation criteria can be obtained from the general Theorems 3.2.1, 3.2.3, 3.2.4, 3.2.6, 3.2.7 and 3.2.8 with different choices of H and h. For instance, one may take the pairs*

$$H(t,s)=\ln^{n-1}\left(\frac{t}{s}\right)$$

and

$$h(t,s)=\frac{n-1}{s}\left[\ln\left(\frac{t}{s}\right)\right]^{(n-3)/2}$$

or

$$H(t,s)=\left(e^{t-s}-e^{s-t}\right)^{n-1}$$

and

$$h(t,s)=(n-1)\left(e^{t-s}+e^{s-t}\right)\left(e^{t-s}-e^{s-t}\right)^{(n-3)/2}.$$

Remark 3.2.10. *In the stated results, we enjoy the advantages of the technique developed in [76] and would like to stress that it is very important that the parameter β in Theorems 3.2.3, 3.2.4 and 3.2.7 is strictly larger than one. This allows us to eliminate in the stated results condition similar to (3.1.3) which has been assumed in most papers on the subject.*

Furthermore, modifications of the proofs through the refinement of the standard integral averaging method allowed us to shorten significantly the proofs of Theorems 3.2.3, 3.2.4 and 3.2.7, cf. [86]. If one selects $\beta = 1$ in Theorems 3.2.3, 3.2.4 and 3.2.7, all advantages of a new technique are lost, and assumptions similar to (3.1.3) should be introduced. We also note that a different approach, which allows one to eliminate assumption (3.1.3) or alike using an elementary quadratic inequality, is suggested in the papers [76] and [91].

3.3 Examples

Example 3.3.1. *For $t \geq 1$, consider the second order neutral differential equation*

$$\begin{aligned}\left[\frac{1}{t^2}\left(x(t) + \frac{t}{2t+1}x(t-1)\right)'\right]'&\\ +\frac{1}{t+2}\left(2 + x^4(t)\right)x\left(\frac{t}{2}\right) &= 0. \end{aligned} \tag{3.3.1}$$

Let

$$\rho(t) = -\frac{8}{t} \qquad \textit{and} \qquad v(t) = t^2.$$

Then,

$$\psi(t) = \frac{t^4 - 16t - 16}{t^2(t+1)}.$$

An application of Corollary 3.2.1 with $n = 3$ establishes the oscillation of equation (3.3.1) since, for any $\beta \geq 1$,

$$\limsup_{t\to+\infty} \frac{1}{t^2}\int_1^t \left[(t-s)^2 \frac{s^4 - 16s - 16}{s^2(s+1)} - 8\beta\right] ds = +\infty.$$

Example 3.3.2. *For $t \geq 1$, consider the nonlinear neutral differential equation*

$$\left(r(t)\left(x(t) + p(t)x(t-1)\right)'\right)' + q(t)\left(x^2(t) + 2\right)x\left(\frac{t}{2}\right) = 0, \tag{3.3.2}$$

where

$$\begin{aligned} r(t) = \frac{1}{4}\left(\frac{1}{4t^2} + \frac{1}{3}\right)(2 + \cos 2t), \qquad \sigma(t) = \frac{t}{2}, \qquad q(t) = t^2 + 3,\\ p(t) = 1 + \frac{(4t^2(4t(t-1) - 1) - 1)\cos 2t}{32t^4(4t^2+3)} - \frac{2 + t\sin 2t}{96t^4}. \end{aligned}$$

We apply Corollary 3.2.2 with $n = 3$ and

$$\rho(t) = -\frac{8(t^2+3)(2+\cos t)}{t(4t^2+3)(2+\cos 2t)}.$$

Then, $v(t) = t^2$ and

$$\psi(t) = \left(2t - t^2 + 1\right)\cos t + \frac{2}{3}.$$

Let $\beta = 2$. Then,

$$\begin{aligned}
&\limsup_{t\to+\infty}\frac{1}{t^2}\int_T^t \left[(t-s)^2\left(\left(2s - s^2 + 1\right)\cos s + \frac{2}{3}\right) - \left(\frac{s^2}{3}+1\right)(2+\cos s)\right] ds \\
&\quad \stackrel{\text{def}}{=} \phi(T) = \frac{5}{3} - \frac{2T}{3} + T^2\sin T + 2T\cos T - 3\sin T - 2\cos T - 2T\sin T,
\end{aligned}$$

and it follows from

$$\limsup_{t\to+\infty}\int_1^t 2\frac{\phi_+^2(s)}{(s^2/3+1)(2+\cos s)}ds \geq 2\limsup_{t\to+\infty}\int_1^t \frac{\phi_+^2(s)}{s^2+3}ds = +\infty$$

that equation (3.3.2) is oscillatory. Note that in this example

$$\limsup_{t\to+\infty}\frac{1}{t^2}\int_1^t \left(\frac{s^2}{3}+1\right)(2+\cos s)\,ds = +\infty,$$

which means that an analogue of the condition (3.1.3) in Theorem 3.1.1, not requested for our oscillation criterion, fails to hold.

Remark 3.3.1. *Note that it is much simpler to determine an appropriate function $h(t,s)$ coupled with one's selection of $H(t,s)$ using (3.2.1) rather than the condition (iii) in Theorem 3.1.1 since the latter condition involves, in addition to functions H and h, functions k and ρ. Examples 3.3.1 and 3.3.2 clearly demonstrate that our approach leads to more flexible and easily verifiable criteria for oscillation.*

Example 3.3.3. *For $t \geq 2$, consider the second order nonlinear neutral differential equation,*

$$\begin{aligned}
&\left(r(t)\left(x(t) + p(t)x(\delta(t))\right)'\right)' \\
&\qquad\qquad + \frac{1}{t^2}x(\sigma(t))\left(x^2(\sigma(t)) + 1\right) = 0, \quad (3.3.3)
\end{aligned}$$

where $r(t) = p(t) = (t+1)^{-1}$ *and* $\sigma(t) = t-2$. *We apply Corollary 3.2.4 choosing*

$$\rho(t) = -\frac{2}{t^2 - t} \qquad \textit{and} \qquad v(t) = t^4.$$

Then,

$$\vartheta(t) = \frac{t^3(t-3)}{(t-1)^2}$$

and Eq. (3.3.3) is oscillatory since, for $n = 3$ *and any* $\beta \geq 1$,

$$\limsup_{t\to+\infty} \frac{1}{t^2} \int_1^t \left[(t-s)^2 \frac{s^3(s-3)}{(s-1)^2} - \beta \frac{s^4}{s-1} \right] = +\infty.$$

Observe that in this example two conditions of Theorem 3.1.2 fail to hold. Namely, contrary to assumption (A_7), f' *is decreasing on* $(-\infty, 0)$, *increasing on* $(0, +\infty)$. *Consequently, Theorem 3.1.2 cannot be applied to establish oscillatory nature of Eq. (3.3.3).*

Note that our example is a particular case of a more general equation

$$\left(r(t) \left(p_0(t)x(t) + \sum_{i=1}^{k} p_i(t)\, x(t - t_i) \right)' \right)' + q(t)\, f(x(t - t_0)) = 0, \quad (3.3.4)$$

studied in the paper of Budinčević [11]. However, none of the three oscillation theorems reported in the cited paper apply to Eq. (3.3.3). In fact, since the integral

$$\int_{t_0}^{\infty} q(s)ds = \int_{t_0}^{\infty} s^{-2}ds$$

converges, [11, Theorems 1 and 2] cannot be used. On the other hand, the integral

$$\int_{t_0}^{\infty} \frac{1}{r(s)}ds = \int_{t_0}^{\infty} (s+1)\, ds$$

diverges, and one of the assumptions in [11, Theorem 3] fails to hold.

Example 3.3.4. *For* $t \geq 1$, *consider a neutral differential equation*

$$\left(r(t)\, \psi(x(t))\, (x(t) + p(t)\, x(\delta(t)))' \right)' + q(t)\, x(\sigma(t)) \left(1 + \ln\left(1 + x^2(\sigma(t))\right) \right) = 0, \quad (3.3.5)$$

where

$$r(t) = \frac{(4t^2+3)(2+\sin 2t)}{192t^2}, \qquad \sigma(t) = \frac{t}{2},$$
$$\psi(t) = \frac{3t^2+2}{2t^2+1}, \qquad q(t) = t^2+1$$

and

$$p(t) = \frac{192t^6+48t^4-4t^2-3}{48t^4(4t^2+1)} + \frac{(4t^2+3)\cos 2t}{96t^3(4t^2+1)} + \frac{(16t^4-16t^3-4t^2-1)\sin 2t}{32t^4(4t^2+1)}.$$

Let

$$\rho(t) = -\frac{(t^2+3)(2+\sin t)}{12t^3} \qquad \textit{and} \qquad v(t) = t^2.$$

Then,

$$\vartheta(t) = \left(t - \frac{t^2}{2} + \frac{1}{2}\right)\sin t + \frac{1}{3}.$$

An application of Corollary 3.2.5 with $n = 3$ *and* $\beta = 2$ *yields*

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_T^t \left[(t-s)^2 \left(\left(s - \frac{s^2}{2} + \frac{1}{2}\right)\sin s + \frac{1}{3}\right) - \left(\frac{s^2}{6} + \frac{1}{2}\right)(2+\sin s)\right] ds$$
$$\stackrel{\text{def}}{=} \varphi(T) = \frac{5}{6} - \frac{T}{3} + T\sin T - \sin T + \frac{3}{2}\cos T + T\cos T - \frac{T^2}{2}\cos T,$$

and Eq. (3.3.5) is oscillatory because

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_1^t 3\frac{\varphi_+^2(s)}{(s^2+3)(2+\sin s)} ds \geq 3\limsup_{t\to\infty} \frac{1}{t^2} \int_1^t \frac{\varphi_+^2(s)}{(s^2+3)} ds = +\infty.$$

Chapter 4

CONCLUSIONS

The main results in this thesis are collected in Chapters 2 and 3. In Chapter 2, using the Bihari integral inequality, we formulate sufficient conditions for all non-oscillatory solutions of Eqs. (2.2.1) to behave like nontrivial linear functions at infinity. We prove that all non-oscillatory solutions of Eq. (2.3.1) satisfy (2.3.17). In addition, for a particular case of Eq. (2.3.1), Eq. (2.3.2), we provide necessary and sufficient conditions that guarantee existence of non-oscillatory solutions with the same property. In Chapter 3, we address oscillation problem of solutions of second order nonlinear neutral differential equations of the forms (3.0.1) and (3.0.2). By using a generalized Riccati transformation and techniques developed by Rogovchenko and Tuncay [74, 75, 76], we state new efficient oscillation criteria for Eqs. (3.0.1) and (3.0.2). Chapters 2 and 3 conclude with a number of carefully selected examples which illustrate main results obtained in this thesis. Routine calculations are performed by computer algebra system *Wolfram Research Mathematica*.

Some of the features that characterize this thesis are as follows:

- results provided in Chapter 2 apply to wider classes of neutral differential equations; they improve and extend many results known in the literature;
- oscillation criteria formulated in Chapter 3 improve most related results reported in literature by removing conditions similar to (3.1.3) which have been traditionally

used for almost three decades;

- proofs of Theorems 3.2.1, 3.2.3, 3.2.6, 3.2.7 and alike are significantly shorter due to technique improved by Rogovchenko and Tuncay [74, 75, 76];
- proofs of Theorems 3.2.4 and 3.2.8 are reduced to a few lines from several pages;
- our general results are flexible in applications;
- a variety of simple and efficient oscillation criteria are obtained by choosing, for instance, appropriate Kamenev-type functions.

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