# Stable Magnetic Universes Revisited

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A regular class of static, cylindrically symmetric pure magnetic field metrics is rederived in a different metric ansatz in all dimensions. Radial, time dependent perturbations show that for dimensions $d > 3$ such spacetimes are stable at both near $r \approx 0$ and large radius $r \to \infty$. In a different gauge these stability analysis and similar results were known beforehand. For $d = 3$, however, simultaneous stability requirement at both, near and far radial distances can not be reconciled for time - dependent perturbations. Restricted, numerical geodesics for neutral particles reveal a confinement around the center in the polar plane. Charged, time-like geodesics for $d = 4$ on the other hand are shown numerically to run toward infinity.

## I. INTRODUCTION

In $4-$dimensional spacetimes the electric - magnetic duality symmetry of the Maxwell equations is an important property which can not be satisfied in other dimensions unless different form fields other than $2-$forms are introduced. For this reason a dyonic solution admits a meaningful interpretation only in $d = 4$. The Reissner - Nordestrom (RN) solution constitutes in this regard the best example which has both electric and magnetic solutions in a symmetric manner. In other dimensions ( $d \neq 4$ ) similar duality properties can, in principle, be defined as well but physical interpretation corresponding to electric and magnetic fields turn out to be rather abstract. For such reasons, in order to avoid complications due to the absence of a tangible duality, pure electric or pure magnetic solutions seemed to attract considerable attention. This amounts to only half of the Maxwell equations, the other half being trivially satisfied. From this token we wish to resort here to the pure magnetic solutions which yield a completely solvable class without much effort. From the physical side, occurrence of pure and very strong magnetic fields associated with astronomical objects such as magnetars motivate us to search for such solutions in general relativity.

It was Melvin, who first studied such cylindrically symmetric parallel magnetic lines of force remaining in equilibrium under their mutual gravitational attraction in $d = 4$[1]. Later on, generalized version of the Melvin's magnetic universe was also considered[2]. The Melvin universe is invariant under rotation and translation along the $z_i$ axis orthogonal to the polar plane $(r, \varphi)$. Thorne popularized the Melvin universe further by showing its absolute stability against small radial perturbations[3]. Due to this stability property it can be presumed that astrophysical objects emitting strong beams of magnetic fields may everlast in an accelerating universe. Additionally, in $d = 3$ [4] and 5 dimensional [5] cases also pure magnetic field solutions were found and their energy content investigated [6]. Pure magnetic solutions in higher dimensions are also known to exist in string, Lovelock, Yang-Mills, Born-Infeld and other theories [7].

In this paper, we present in a particular cylindrically symmetric metric ansatz, a class of non-singular, source-free, static, pure magnetic solutions to Einstein-Maxwell (EM) equations in all dimensions. In a different metric ansatz these solutions were known previously [4, 8]. Our principal aim is to investigate the stability of such magnetic universes against time dependent small radial perturbations and explore the possible role of dimensionality of spacetime in such matters. It has been known for a long time that for $d > 3$ these kind of magnetic solutions are all stable [8] . We verify these results once more in a different metric (i.e. non canonical) ansatz with supplement of the $d = 3$ case.. We show that for $d > 3$ the metrics are stable against small perturbations at both near axis $z_i$ and at far distance away from $z_i$. We observe also that when $d = 3$ these two regions behave differently. Namely, the metric can be made stable at $r \approx 0$ or, at $r \to \infty$, but not simultaneously, which we phrase as 'weakly' stable. The solutions justify once more the impossibility of cylindrical magnetic field lines implosion and therefore formation of such black holes. We investigate the time-like ( $d \geq 4$,with fixed polar angle) and null ( $d = 3, 4$ ) geodesics for neutral particles numerically. Only for $d = 4$ and 5 we were able to obtain exact integrals, albeit in non-invertible forms, of the geodesics equation. In each case a confinement of geodesics is observed to take place near the central region. Due to its physical importance we consider also the time-like geodesics of a charged particle. It turns out that such geodesics can not be confined and in their proper time they diverge to infinity.

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Organization of the paper is as follows. In Section II we present our metric, field equations and solve them in $d-$dimensions. Perturbation analysis of our system follows in Section III. Geodesics motion is studied in Section IV. Our results are summarized in Conclusion which appears in Section V.

## II.  METRIC AND SOLUTIONS OF FIELD EQUATIONS IN $d-$DIMENSIONS

Our $d-$dimensional static, cylindrically symmetric line element ansatz is given by

$$ds^2 = f(r) \left( dt^2 - dr^2 - \sum_{i=1}^{d-3} dz_i^2 \right) - \frac{r^2 b_0^2}{f(r)^k} d\varphi^2, \qquad\qquad d \geq 3 \qquad (1)$$

in which $f(r)$ is a function of $r$ to be found and $b_0$ and $k$ are constant parameters. Also the pure magnetic $2-$form field is chosen to be

$$\mathbf{F} = F_{r\varphi} dr \wedge d\varphi, \qquad (2)$$

where $F_{r\varphi}$ is the only non-zero component of the electromagnetic field. The energy momentum tensor is defined by

$$4\pi T_i^j = -F_{ik} F^{jk} + \frac{1}{4} \delta_i^j F_{mn} F^{mn} \qquad (3)$$

which admits the non-zero components

$$T_i^j = \mathrm{diag} \left[ T_0^0 = T_2^2 = T_3^3 = \cdots = -T_1^1 = -T_{d-1}^{d-1} \right] = \frac{1}{8\pi} F_{r\varphi} F^{r\varphi}. \qquad (4)$$

We note that our choice of indices $\{0, 1, 2, \ldots, (d-1)\}$ denote $\{t, r, z_1, \cdots, z_{d-3}, \varphi\}$ and the energy conditions satisfied by this energy-momentum tensor are discussed in the Appendix A. From the Einstein equations, $T_i^j = G_i^j$ Eq. (4) implies that

$$G_0^0 = G_2^2 = G_3^3 = \cdots = -G_1^1 = -G_{d-1}^{d-1}. \qquad (5)$$

From these relations we can write

$$G_1^1 = G_{d-1}^{d-1} \qquad (6)$$
$$G_1^1 = -G_0^0 \qquad (7)$$
$$G_0^0 = -G_{d-1}^{d-1}. \qquad (8)$$

where

$$G_0^0 = \frac{1}{4f^3 r} \left[ 2ff''r (k - (d-3)) + rf'^2 \left( -k^2 + (d-6)k - (d-8)\frac{(d-3)}{2} \right) + 2ff'(2k - (d-4)) \right] \qquad (9)$$

$$G_1^1 = \frac{1}{4f^3 r} \left[ rf'^2 \left( (d-2)k - (d-2)\frac{(d-3)}{2} \right) - 2ff'(d-2) \right] \qquad (10)$$

$$G_{d-1}^{d-1} = \frac{1}{4f^3 r} \left[ -2ff''r (d-2) - rf'^2 \left( (d-2)\frac{(d-7)}{2} \right) \right]. \qquad (11)$$

From equation (6) we obtain the differential equation

$$2rf''f + (k-2)rf'^2 - 2ff' = 0. \qquad (12)$$

whose solution for $d \geq 4$ is

$$f(r) = (kr^2 + C_1)^{\frac{2}{k}}. \qquad (13)$$

By putting this result into Eq. (7) for finding $k$ we find out that for $d \geq 4$, $k$ is $d-3$. On the other hand, for $d=3$ the solution turns out to be

$$f(r) = C_2 e^{cr^2} \tag{14}$$

for the integration constants $C_2$ and $c$. For convenience we make the choices $C_1 = d-3$ and $C_2 = 1$, so that the solution can be expressed by

$$f(r) = \begin{cases} (d-3)^{\frac{2}{d-3}}(r^2+1)^{\frac{2}{d-3}} & d \geq 4 \\ e^{cr^2} & d = 3 \end{cases}. \tag{15}$$

Accordingly, our line element takes the form

$$ds^2 = \begin{cases} (d-3)^{\frac{2}{d-3}}(r^2+1)^{\frac{2}{d-3}}\left(dt^2 - dr^2 - \sum_{i=1}^{d-3}dz_i^2\right) - \frac{r^2 b_0^2}{(d-3)^2(r^2+1)^2}d\varphi^2, & d \geq 4 \\ e^{cr^2}\left(dt^2 - dr^2\right) - r^2 b_0^2 d\varphi^2, & d = 3 \end{cases}. \tag{16}$$

We note that these solutions are not new, for they coincide with those of [8] (for $d \geq 4$) and [4] for $(d=3)$, respectively. It can easily be seen that for $r \to 0$ it reduces to the following form

$$ds^2 \approx \begin{cases} (d-3)^{\frac{2}{d-3}}(dt^2 - dr^2 - \sum_{i=1}^{d-3}dz_i^2) - \frac{r^2 b_0^2}{(d-3)^2}d\varphi^2, & d \geq 4 \\ (dt^2 - dr^2) - r^2 b_0^2 d\varphi^2, & d = 3 \end{cases} \tag{17}$$

This represents a conical geometry signalling the existence of a cosmic string near $r=0$. By choosing $\frac{b_0^2}{(d-3)^2} = 1$ for $d \geq 4$ and $b_0 = 1$ for $d=3$ , we have Minkowskian metrics as one approaches the axes $z_i$. The solution (16) is a singularity free magnetic universe in $d-$ dimensions in analogy with the Melvin space time.

From Maxwell's Eq. it fallows that

$$F^{r\varphi} = \begin{cases} \frac{B_0}{(d-3)^{\frac{2}{d-3}}rb_0(1+r^2)^{\frac{2}{d-3}}}, & d \geq 4 \\ \frac{B_0}{rb_0}e^{-cr^2}, & d = 3 \end{cases} \tag{18}$$
$$(B_0 = \text{ an integration constant})$$

which implies that the magnetic field behaviors as a function of $r$ are

$$F_{r\varphi} \sim \begin{cases} \frac{r}{(1+r^2)^2} & d \geq 4 \\ r & d = 3 \end{cases}.$$

The marked distinction between $d=3$ and $d \geq 4$ cases can already be seen from these behaviors. Accordingly the energy density reads

$$T_0^0 = \begin{cases} \frac{B_0^2}{8\pi(d-3)^{\frac{2d-4}{d-3}}(1+r^2)^{\frac{2d-4}{d-3}}} & d \geq 4 \\ \frac{B_0^2}{8\pi}e^{-cr^2} & d = 3 \end{cases} \tag{19}$$

while the Ricci scalar for the metric (16) is

$$R = \begin{cases} \frac{-4(d-4)(d-3)^{\frac{1-d}{d-3}}}{(1+r^2)^{\frac{2(d-2)}{d-3}}} & d \geq 4 \\ 2ce^{-cr^2} & d = 3 \end{cases}. \tag{20}$$

Similarly, the Kretchmann scalar has the behavior

$$K \sim \begin{cases} \frac{1}{(d-3)^{\frac{4(d-2)}{d-3}}(1+r^2)^{\frac{4(d-2)}{d-3}}} & d \geq 4 \\ 12c^2 e^{-2cr^2} & d = 3 \end{cases}. \tag{21}$$

It is observed that regularity at $r \to \infty$ dictates us to make the choice $c > 0$ for the integration constant.

## III.   PERTURBATION ANALYSIS

In this section we perturb the metric and magnetic potential. Since the case $d = 3$ forms a special case we consider it separately. Similar analysis was carried out by Gibbons and Wiltshire [8] where they used the canonical metric ansatz. We shall show below that their results can also be obtained in a different metric ansatz.

### 1.   The case for $d = 3$

Our line element is

$$ds^2 = f(r,t)\left(dt^2 - dr^2\right) - \frac{r^2 b_0^2}{g(r,t)} d\varphi^2 \tag{22}$$

where

$$
\begin{aligned}
f(r,t) &= f_0(r) + \epsilon u(r,t) \\
g(r,t) &= g_0(r) + \epsilon w(r,t)
\end{aligned} \tag{23}
$$

and the magnetic potential is expressed by

$$A_\varphi(r,t) = A_\varphi(r) + \epsilon a(r,t). \tag{24}$$

The unperturbed functions are

$$
\begin{aligned}
f_0(r) &= e^{cr^2} \tag{25} \\
g_0(r) &= 1 \tag{26} \\
A_\varphi(r) &= \frac{B_0 b_0 r^2}{2} \tag{27}
\end{aligned}
$$

where $u(r,t), w(r,t)$ and $a(r,t)$ are the perturbed functions. Since $\epsilon$ is a small parameter we assume that $\epsilon^2 \approx 0$ in our analysis. We use Einstein's equations to find the perturbed functions. The differential equations satisfied by the perturbed functions are

$$\left(cr^2 \frac{\partial w(r,t)}{\partial r} - r\frac{\partial^2 w(r,t)}{\partial t^2}\right)e^{cr^2} + b_0\left[r\frac{\partial^2 u(r,t)}{\partial r^2} - r\frac{\partial^2 u(r,t)}{\partial t^2} - 4r^2 c\frac{\partial u(r,t)}{\partial r} + 4r^3 c^2 u(r,t) - \frac{\partial u(r,t)}{\partial r}\right] = 0 \tag{28}$$

$$r\frac{\partial^2 w(r,t)}{\partial r^2} + 2\frac{\partial w(r,t)}{\partial r} - r\frac{\partial^2 w(r,t)}{\partial t^2} = 0 \tag{29}$$

$$-\frac{\partial^2 a(r,t)}{\partial t^2} + \frac{\partial^2 a(r,t)}{\partial r^2} - \frac{1}{r}\frac{\partial a(r,t)}{\partial r} + \frac{B_0 b_0 r}{2}\frac{\partial w(r,t)}{\partial r} = 0. \tag{30}$$

This system of differential equations admits the solutions

$$
\begin{aligned}
u(r,t) = {}& e^{-\alpha t + cr^2}\left(\frac{4rB_0}{b_0}\left[B_1 I_1(\alpha r) + B_2 K_1(\alpha r)\right] + \right. \tag{31} \\
& cre^{(-\alpha r - \alpha)}\left\{-(\sinh \alpha r + \cosh \alpha r)(E_1 + E_2)e^{2\alpha r} + e^{2\alpha}(\sinh \alpha r - \cosh \alpha r)(-E_1 + E_2)\right\} + \\
& \left. \left\{(crE_2 - \alpha E_1)\cosh \alpha r + (crE_1 - \alpha E_2)\sinh \alpha r\right\}\right)
\end{aligned}
$$

$$w(r,t) = e^{-\alpha t}(E_1 \frac{\sinh \alpha r}{r} + E_2 \frac{\cosh \alpha r}{r}) \tag{32}$$

$$a(r,t) = re^{-\alpha t}\left(B_1 I_1(\alpha r) + B_2 K_1(\alpha r) - \right. \tag{33}$$
$$\left. \frac{B_0 b_0}{4} e^{-\alpha r - \alpha}\left\{(\sinh\alpha r + \cosh\alpha r)(E_1 + E_2)e^{2\alpha r} + e^{2\alpha}(\sinh\alpha r - \cosh\alpha r)(-E_1 + E_2)\right\}\right),$$

where $B_1, B_2, E_1, E_2$ and $\alpha$ are all integration constants while $I_1(\alpha r)$ and $K_1(\alpha r)$ are the modified Bessel functions of order one. At all times $t$, to a first order in $\epsilon$ , the locally flat nature of the metric near $r = 0$ will not be altered [1]. This implies when $r \to 0$

$$u(r,t) = e^{-\alpha t + cr^2}\left(\frac{4rB_0}{b_0}B_2 K_1(\alpha r) - crE_1 e^{-\alpha r - \alpha}\left\{(\sinh\alpha r + \cosh\alpha r)e^{2r\alpha} + e^{2\alpha}(\sinh\alpha r - \cosh\alpha r)\right\}\right. \tag{34}$$
$$\left. + E_1\left\{cr\sinh\alpha r - \alpha\cosh\alpha r\right\}\right),$$

$$w(r,t) = e^{-\alpha t}E_1\frac{\sinh\alpha r}{r}, \tag{35}$$

$$a(r,t) = re^{-\alpha t}\left(B_2 K_1(\alpha r) - \frac{b_0 B_0}{4}E_1 e^{-\alpha r - \alpha}\left\{(\sinh\alpha + \cosh\alpha)e^{2\alpha r} - e^{2\alpha}(\sinh\alpha - \cosh\alpha)\right\}\right). \tag{36}$$

It is seen that we can choose $B_1 = 0$ and $E_2 = 0$ to have finite limit when $r \to 0$. However, for $r \to \infty$ it can be checked that $w(r,t)$ diverges and since $g_0(r) = 1$ this implies that the ratio of $\frac{w(r,t)}{g_0(r)}$ grows indefinitely. No other choice of constants suffice to eliminate this divergence. As a result the perturbation converges for $r \to 0$ but diverges for $r \to \infty$ in $d = 3$ case. Let us note that for the time - independent perturbation both, near and far - region perturbation terms become convergent.

### 2. The case for $d = 5$ and $d > 5$

Our perturbed line element now for $d = 5$ is of the form

$$ds^2 = f(r,t)\left(dt^2 - dr^2 - dx^2 - dy^2\right) - \frac{r^2 b_0^2}{g(r,t)}d\varphi^2 \tag{37}$$

From Eq. (23) and (24) we introduce in analogy, the perturbed functions in which the unperturbed functions are given by

$$f_0(r) = 2(r^2 + 1) \tag{38}$$
$$g_0(r) = 4(r^2 + 1)^2 \tag{39}$$
$$A_\varphi(r) = -\frac{B_0 b_0}{8}\frac{1}{(r^2 + 1)} \tag{40}$$

From the Einstein's equations we obtain to the first order in $\epsilon$ the following differential equations for $u(r,t)$ and $w(r,t)$

$$r\frac{\partial^2 u(r,t)}{\partial r^2} - \frac{\partial u(r,t)}{\partial r} - ru(r,t) = 0 \tag{41}$$

$$2(r^2 + 1)\frac{\partial^2 u(r,t)}{\partial t^2} - \frac{1}{2}\frac{\partial^2 w(r,t)}{\partial t^2} = 0 \tag{42}$$

Integration of these equations yield

$$u(r,t) = e^{-\alpha t}\left[r(pI_1(\alpha r) + qK_1(\alpha r))\right] \tag{43}$$
$$w(r,t) = 4e^{-\alpha t}\left[r(pI_1(\alpha r) + qK_1(\alpha r))\right](r^2 + 1) \tag{44}$$

for integration constants $p, q$ and $\alpha$, and $I_1(\alpha r)$ and $K_1(\alpha r)$ are the modified Bessel functions of order one. At all times $t$, to a first order in $\epsilon$, the locally flat nature of the metric near $r = 0$ remains intact. Also, for all times to a first order in $\epsilon$, the static metric for $r \to \infty$ will not be altered. In this case we see that perturbed functions are finite at $r = 0$, and when $r \to \infty$ they go to zero.

A similar analysis has been carried out for $d = 6, 7, ...$ so that we found the general solution for the metric functions. In each case we have obtained the following relations between $w(r, t)$ and $u(r, t)$

$$w(r, t) = (d - 3) \frac{g_0(r)}{f_0(r)} u(r, t) \tag{45}$$

$$f_0(r) = (d - 3)^{\frac{2}{d-3}} (r^2 + 1)^{\frac{2}{d-3}} \tag{46}$$

$$g_0(r) = (d - 3)^2 (r^2 + 1)^2. \tag{47}$$

Solutions for $w(r, t)$ and $u(r, t)$ are given in all dimensions as follow

$$w(r, t) = (d - 3)^{\frac{3d-11}{d-3}} (r^2 + 1)^{\frac{2d-8}{d-3}} u(r, t), \tag{48}$$

$$u(r, t) = e^{-\alpha t} (d - 3)^{\frac{d-1}{d-3}} (pI_1(\alpha r) + qK_1(\alpha r)) \frac{r}{(r^2 + 1)^{\frac{d-5}{d-3}}}, \tag{49}$$

in which $p$ and $q$ are constants. These are both finite at $r \to 0$ and the ratio of them when $r \to \infty$ go to zero. Now, from Maxwell's equations we attempt to find the perturbative solution for the magnetic potential. We take the potential in the form

$$A_\varphi(r, t) = A_\varphi(r) + \epsilon a(r, t)$$

where

$$A_\varphi(r) = -\frac{B_0 b_0}{2(d - 3)^2} \frac{1}{(r^2 + 1)}. \tag{50}$$

The differential equation satisfied by $a(r, t)$ becomes

$$-\frac{\partial^2 a(r, t)}{\partial t^2} + \frac{\partial^2 a(r, t)}{\partial r^2} + \frac{(3r^2 - 1)}{r(r^2 + 1)} \frac{\partial a(r, t)}{\partial r} + \frac{b_0 B_0 r}{(d - 3)^{\frac{d-1}{d-3}} (r^2 + 1)^2} \frac{d}{dr} \left[ \frac{u(r, t)}{(r^2 + 1)^{\frac{2}{d-3}}} \right] = 0 \tag{51}$$

in which

$$u(r, t) = e^{-\alpha t} (d - 3)^{\frac{d-1}{d-3}} (pI_1(\alpha r) + qK_1(\alpha r)) \frac{r}{(r^2 + 1)^{\frac{d-5}{d-3}}}. \tag{52}$$

Upon substitution for $u(r, t)$ we see that for all dimensions the equation satisfied by $a(r, t)$ takes the form

$$-\frac{\partial^2 a(r, t)}{\partial t^2} + \frac{\partial^2 a(r, t)}{\partial r^2} + \frac{(3r^2 - 1)}{r(r^2 + 1)} \frac{\partial a(r, t)}{\partial r} + \frac{b_0 B_0 r e^{-\alpha t}}{(r^2 + 1)^2} \frac{d}{dr} \left[ \frac{r}{(r^2 + 1)} (pI_1(\alpha r) + qK_1(\alpha r)) \right] = 0. \tag{53}$$

An exact solution for $a(r, t)$ for all $r$ is not at our disposal, therefore, we shall search for solutions near $r = 0$ and for $r \to \infty$. The solution for the homogenous part is

$$a_H(r, t) = \frac{r e^{-\alpha t}}{(r^2 + 1)} (C_1 I_1(\alpha r) + C_2 K_1(\alpha r)). \tag{54}$$

Since a particular solution is not available we proceed to study the answers for a limited case, when $r$ is small (to order $r$)

$$-\frac{\partial^2 a\left(r,t\right)}{\partial t^2} + \frac{\partial^2 a\left(r,t\right)}{\partial r^2} - \frac{1}{r}\frac{\partial a\left(r,t\right)}{\partial r} = 0, \tag{55}$$
$$a\left(r,t\right) = e^{-\alpha t} r\left(C_1 I_1\left(\alpha r\right) + C_2 K_1\left(\alpha r\right)\right).$$

If we go to higher orders of $r$, (order $r^3$ for instance) we obtain

$$-\frac{\partial^2 a\left(r,t\right)}{\partial t^2} + \frac{\partial^2 a\left(r,t\right)}{\partial r^2} + \left[4r - \frac{1}{r}\right]\frac{\partial a\left(r,t\right)}{\partial r} + b_0 B_0 r^2\left\{-p\alpha\left(-\ln\left(\frac{1}{2}\alpha\right) - \ln r - \gamma\right) + p\alpha - \frac{2q}{\alpha}\right\} = 0 \tag{56}$$

whose solution can be expressed as

$$a\left(r,t\right) = e^{-r^2-\alpha t}\left\{C_3 Whitta\ker M(-\frac{1}{8}\alpha^2, \frac{1}{2}, 2r^2) + C_4 Whitta\ker M(-\frac{1}{8}\alpha^2, \frac{1}{2}, 2r^2)\right\} - \frac{B_0 b_0 e^{-\alpha t}}{\alpha\left(-8 + \alpha^2\right)^2}\left(\Xi + \Pi\right) \tag{57}$$

in which

$$\Xi = -\alpha^2 r^2 q\left(-8 + \alpha^2\right)\ln\alpha - \alpha^2 r^2 q\left(-8 + \alpha^2\right)\ln r + \left[q\left(\ln 2 - \gamma\right) - p\right]\alpha^4 r^2,$$
$$\Pi = \left(\left[\left(8\gamma - 2 - 8\ln 2\right)r^2 - 2\right]q + 8r^2 p\right)\alpha^2 + 16q - 16r^2 q.$$

Here $p, q, C_3, C_4, \alpha$ and $\gamma$ are all constants and $Whittaker M\left(-\frac{1}{8}\alpha^2, \frac{1}{2}, 2r^2\right)$ stands for the Whittaker function[9]. For $r \to 0$ the function $a\left(r,t\right)$ is finite. Now, to see the case when $r$ goes to infinity we solve the differential equation

$$-\frac{\partial^2 a\left(r,t\right)}{\partial t^2} + \frac{\partial^2 a\left(r,t\right)}{\partial r^2} = 0, \qquad \text{(to order } 1/r\text{)} \tag{58}$$

whose solution reads

$$a\left(r,t\right) = e^{-\alpha t}\left(C_5 e^{\alpha r} + C_6 e^{-\alpha r}\right). \tag{59}$$

Here $C_5$ and $C_6$ are new integration constants. It can easily be checked that the ratio of this solution goes to zero for $r \to \infty$ if we choose $C_5 = 0$.

## IV.  GEODESIC MOTION

In this section we shall investigate the time-like (for $d \geq 4$) and null (for $d = 3, 4$) geodesics by employing our line element given in Eq. (15). For $d \geq 4$ we divide the line element by $d\tau$ ($\tau$ is proper time) and for $d = 3$ by $d\lambda$ ($\lambda$ is an affine parameter) so that the Lagrangian can be expressed in the form

$$L = \begin{cases} (d-3)^{\frac{2}{d-3}}(r^2+1)^{\frac{2}{d-3}}\left[\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - \sum\limits_{i=1}^{d-3}\left(\frac{dz_i}{d\tau}\right)^2\right] - \frac{r^2 b_0^2}{(d-3)^2(r^2+1)^2}\left(\frac{d\varphi}{d\tau}\right)^2, & d \geq 4 \\ e^{cr^2}\left[\left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dr}{d\lambda}\right)^2\right] - r^2 b_0^2\left(\frac{d\varphi}{d\lambda}\right)^2, & d = 3 \end{cases} \tag{60}$$

For the equations of motion with constant azimuthal angle ($\varphi = $ constant for $d \geq 4$) and null geodesics for $d = 3$ we obtain

$$\begin{cases} \frac{dt}{d\tau} = \frac{d_0}{(d-3)^{\frac{2}{d-3}}(r^2+1)^{\frac{2}{d-3}}} \\ \frac{dz_i}{d\tau} = \frac{d_i}{(d-3)^{\frac{2}{d-3}}(r^2+1)^{\frac{2}{d-3}}} & d \geq 4 \\ \frac{dt}{d\lambda} = H_0 e^{-cr^2} \\ \frac{d\varphi}{d\lambda} = \frac{H_1}{r^2 b_0^2} & d = 3 \end{cases} \tag{61}$$

in which $d_0$, $d_i$, $H_0$ and $H_1$ are all constants of integration. From the metric condition we find $\frac{dr}{d\tau}$ (for $d \geq 4$, $\varphi =$ constant) and $\frac{dr}{d\lambda}$ (for $d = 3$) as follow

$$
\begin{cases}
\frac{dr}{d\tau} = \pm \frac{1}{(d-3)^{\frac{2}{d-3}}(r^2+1)^{\frac{2}{d-3}}} \sqrt{a_0^2 - (d-3)^{\frac{2}{d-3}}(r^2+1)^{\frac{2}{d-3}}} & \\
a_0^2 = d_0^2 - \sum_{i=1}^{d-3} d_i^2 & d \geq 4 \\
\frac{dr}{d\lambda} = \pm H_0 e^{-cr^2} \sqrt{1 - \frac{H_2^2}{r^2} e^{cr^2}} & d = 3 \\
H_2 = \frac{H_1}{H_0 b_0}
\end{cases}
\tag{62}
$$

In effect, we obtain for $d = 4$ the relation between $r$ and $\tau$ but for $d = 3$ we want to find the relation between $r$ and $\varphi$.

$$
\begin{cases}
\pm (\tau - \tau_0) = (d-3)^{\frac{1}{d-3}} \int \frac{(r^2+1)^{\frac{2}{d-3}} dr}{\sqrt{k_0^2 - (r^2+1)^{\frac{2}{d-3}}}} & \\
\left( k_0 = \frac{a_0}{(d-3)^{\frac{1}{d-3}}} \right) & d \geq 4
\end{cases}
\tag{63}
$$

$$
\pm (\varphi - \varphi_0) = \frac{1}{H_0} \int \frac{e^{cr^2} dr}{r\sqrt{r^2 - H_2^2 e^{cr^2}}} \qquad\qquad d = 3 \tag{64}
$$

in which, $\tau_0$ and $\varphi_0$ are initial constants and we impose the restrictions so that

$$
\begin{cases}
(r^2 + 1)^{\frac{2}{d-3}} < k_0^2 & d \geq 4 \\
\frac{H_2^2}{r^2} e^{cr^2} < 1 & d = 3
\end{cases}.
\tag{65}
$$

For $d = 4$ and 5, we have exact integrals given by

$$
\tau - \tau_0 = \begin{cases}
-\frac{1}{3} r \sqrt{k_0^2 - (r^2+1)^2} + \frac{\sqrt{1 + \frac{r^2}{(k_0+1)}}(k_0-1)\sqrt{1 - \frac{r^2}{(k_0-1)}}}{3\sqrt{k_0^2 - (r^2+1)^2}\sqrt{k_0 - 1}} \Omega & d = 4 \\
\sqrt{2} \left[ -\frac{1}{2} r \sqrt{k_0^2 - (r^2+1)} + \frac{1}{2} (k_0^2 + 1) \arctan\left( \frac{r}{\sqrt{k_0^2 - (r^2+1)}} \right) \right] & d = 5
\end{cases}
\tag{66}
$$

in which

$$
\Omega = (k_0^2 - 2k_0)\, \text{EllipticF}\left( \frac{r}{\sqrt{k_0 - 1}}, \sqrt{-1 + \frac{2}{(k_0+1)}} \right) + (2k_0 + 2)\, \text{EllipticF}\left( \frac{r}{\sqrt{k_0 - 1}}, \sqrt{-1 + \frac{2}{(k_0+1)}} \right).
$$

Fig.s 1a and 1b depict the behaviors of (63) (for $d = 4$) and (66) (for $d = 5$), respectively. Now, we wish to consider the $d = 4$ null geodesics as well. The line element is

$$
ds^2 = (r^2 + 1)^2 \left( dt^2 - dr^2 - dz^2 \right) - \frac{r^2 b_0^2}{(r^2 + 1)^2} d\varphi^2 \tag{67}
$$

Eq.s of motion imply for the affine parameter $\lambda$

$$
\frac{dt}{d\lambda} = \frac{\alpha_0}{(r^2 + 1)^2}, \qquad \frac{dz}{d\lambda} = \frac{\beta_0}{(r^2 + 1)^2}, \qquad \frac{d\varphi}{d\lambda} = \gamma_0 \frac{(r^2 + 1)^2}{r^2} \tag{68}
$$

in which $\alpha_0$, $\beta_0$ and $\gamma_0$ are integration constants. We note that (67) and (68) correspond to Eq.s (1) and (6) of [3], respectively. From the null-metric condition $ds^2 = 0$ and upon shifting the independent variable to $\varphi$ we obtain

$$
\pm (\varphi - \varphi_0) = \int \frac{(r^2 + 1)^4 dr}{r\sqrt{\lambda_0^2 r^2 - b_0^2 (r^2 + 1)^4}}, \tag{69}
$$

where we have introduced the constant $\lambda_0$ as $\lambda_0^2 = \frac{\alpha_0^2 - \beta_0^2}{\gamma_0^2}$. Fig.s 2a and 2b are the plots corresponding to (64) and (69), respectively.

Finally, we study the 4-dimensional time-like geodesics of a charged particle with charge $e$ whose Lagrangian is given by

$$L = (r^2 + 1)^2 \left[ \left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dr}{d\tau}\right)^2 - \left(\frac{dz}{d\tau}\right)^2 \right] - \frac{r^2 b_0^2}{(r^2+1)^2} \left(\frac{d\varphi}{d\tau}\right)^2 - \frac{eB_0 b_0}{2} \frac{1}{(r^2+1)} \left(\frac{d\varphi}{d\tau}\right). \tag{70}$$

The Euler-Lagrange equations of motion yield

$$\frac{dt}{d\tau} = \frac{l_0}{(r^2+1)^2}, \ \frac{dz}{d\tau} = \frac{\mu_0}{(r^2+1)^2}, \ \frac{d\varphi}{d\tau} = \frac{1}{2} \frac{\left(-\sigma_0 r^2 - \sigma_0 + b_0 a_1\right)(r^2+1)}{b_0^2 r^2}, \ a_1 = -\frac{eB_0}{2}, \tag{71}$$

for the integration constants $l_0$, $\mu_0$ and $\sigma_0$. Since we shall be interested only in the $r(\tau)$ behavior of the motion we derive the second order equation as follows

$$-2(r^2+1)^2 \frac{d^2 r}{d\tau^2} + \left[ -8r(r^2+1) + \frac{4r}{(r^2+1)^3}\left(r^8 + 4r^6 + 6r^4 + 4r^2 + 1\right) \right] \left(\frac{dr}{d\tau}\right)^2 \tag{72}$$

$$+ \frac{1}{2r^3 b_0^2 (r^2+1)^3} \left[ \sigma_0^2 r^{10} + 3\sigma_0^2 r^8 + r^6 \left(2\sigma_0^2 + b_0^2 a_1^2\right) - r^4 \left(2\sigma_0^2 + b_0^2 \left[-3a_1^2 + 2\mu_0^2 - 2l_0^2\right]\right) \right.$$

$$\left. -r^2 \left(3\sigma_0^2 - 3a_1^2 b_0^2\right) - \sigma_0^2 + a_1^2 b_0^2 \right] = 0.$$

For a set of chosen constants and initial values we plot the behavior of $r(\tau)$ as depicted in Fig. 3. Our overall analysis shows that irrespective of the initial conditions $r(\tau) \to \infty$, with the increasing proper time.

## V. CONCLUSION

We rederive the family of cylindrically symmetric magnetic universes in a particular metric ansatz which is conformally flat on each constant azimuthal angle. These are non-black hole solutions where unlike their spherical counterparts the gravity of magnetic fields is not strong enough to make black holes. The energy conditions (in the Appendix) of the magnetic field are satisfied only in particular dimensions. Being inspired by the stability properties of the original 4-dimensional Melvin's magnetic universe and those of Gibbons and Wiltshire we prove also the stability of present universes in a different gauge and in all dimensions including $d = 3$. Small radial perturbations of metric functions and the magnetic field (which automatically yields an electric field in accordance with the Maxwell equations) result in convergent expansions. Stability of the 3-dimensional case which was not considered in previous studies turns out to be weaker (i.e. convergence of perturbations are not satisfied simultaneously at $r = 0$ and at $r \to \infty$).

Geodesics show numerically that in running proper time uncharged particles are confined while null geodesics spiral around the center. Exact, particular geodesics in terms of the elementary functions are available in $d = 5$, whereas in $d = 4$ we have elliptic functions.

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**Apendix A**

**Energy conditions**

When a matter field couples to any system, energy conditions must be satisfied for physically acceptable solutions. We follow the steps as given in[10] .

*Weak Energy Condition (WEC):*

The WEC states that

$$\rho \geq 0$$
$$\rho + p_i \geq 0 \tag{A1}$$

In which $\rho$ is the energy density and $p_i$ are the principle pressure given by

$$\rho = T_0^0$$
$$p_i = -T_i^i, i = 1, 2, \cdots, (d-1) \tag{A2}$$

The WEC conditions are trivially satisfied.

*Strong Energy Condition (SEC):*

This condition states that

$$\rho + \sum_{i=1}^{d-1} p_i \geq 0$$
$$\rho + p_i \geq 0 \tag{A3}$$

For d=3, it means that

$$\rho + \sum_{i=1}^{d-1} p_i \geq 0 \Rightarrow 3T_0^0 \geq 0$$
$$\rho + p_i \geq 0 \Rightarrow 2T_0^0 \geq 0 \tag{A4}$$

which are satisfied. For $4 \leq d \leq 6$ it is also satisfied because $\rho + \sum_{i=1}^{d-1} p_i = -(d-6)T_0^0$. It can easily be seen also that for $7 \leq d$

$$\rho + \sum_{i=1}^{d-1} p_i \geq 0 \Rightarrow -T_0^0 \geq 0$$
$$\rho + p_i \geq 0 \Rightarrow T_0^0 \geq 0 \tag{A5}$$

i.e. the SEC is violated.

*Dominant Energy Condition (DEC):*
In accordance with DEC, the effective pressure should not be negative. This amounts to

$$p_{eff} = \frac{1}{d-1} \sum_{i=1}^{d-1} T_i^i = \frac{(d-5)}{d-1} T_0^0 \geq 0 \tag{A6}$$

For having $p_{eff} \geq o$ it is clear that $d \geq 5$.

*Causality Condition:*
In addition to the energy conditions one can impose the causality condition

$$0 \leq \frac{p_{eff}}{\rho} < 1 \tag{A7}$$

This implies that

$$0 \leq \frac{(d-5)}{d-1} < 1 \tag{A8}$$

which is satisfied for $d \geq 5$.

**Figure captions**
Fig. 1 : Radial distance behavior as a function of proper time for time-like, neutral particle geodesics. From Eq. (63) in the text, the behavior of $r(\tau)$ as the proper time $\tau$ runs from zero to infinity. The starting points are chosen such that $r = 0$ at $\tau = 0$, in both $d = 4$ (Fig. 1a) and $d = 5$ (Fig. 1b)cases. The fact that $r$ is confined is clearly seen from these plots. This particular property is already implied from Eq. (65). Let us note that in both cases for simplicity we choose $(+)$ sign and $\tau_0 = 0$. Further, we choose the constant $k_0 = 2$.

Fig. 2 : Radial behavior as a function of the azimuthal angle $\varphi$ for null geodesics in $d = 3$ (Fig 2a) and $d = 4$ dimensional (Fig. 2b) magnetic universes. The horizontal axis $x$ $(= r \cos \varphi)$ and vertical axis $y$ $(= r \sin \varphi)$ are plotted numerically in each case from the expressions given in Eq. (64) and Eq. (69) . For simplicity we choose the constants $H_2 = 0.01$, $b_0 = 0.01$ , $H_0 = 1$, $c = 1$ and $\lambda_0 = 1$. Let us note that we have chosen the $(+)$ sign in both (64) and (69) , which give outward orbits around the center. Obviously, the choice $(-)$ should yield inward orbits.

Fig. 3 : This is a numerical plot of the intricate second order differential equation (72) in $d = 4$. For technical reason we choose the constants $a_1$, $b_0$, $l_0$, $\mu_0$ and $\sigma_0$ all equal to one. Two different initial conditions are displayed (A and B) which reveal the pattern of increasing radial distance in proper time.

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