Approach to the shifted $1/N$ expansion for the Klein-Gordon equation

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A different approach to the shifted $1/N$ expansion technique is developed to deal with the Klein-Gordon particle trapped in a spherically symmetric potential. Properly modifying the definition of the perturbative expansion of the energy eigenvalue, and without making any approximation in the determination of the parameters involved, we obtain sufficiently good results compared with the exact ones for the Coulomb problem. The calculations are carried out to the second-order correction of the energy series.

I. INTRODUCTION

The $1/N$ expansion technique has proved itself in solving the Schrödinger equation for a large number of physically interesting potentials yielding sufficiently accurate results.[1-6] It has also been applied to solid-state physics[7,8] and quantum-field theory.[9-11]

This technique has recently been modified by Imbo and co-workers[12,13] and called the shifted $1/N$ expansion. A suitable shifting parameter was introduced, which has the meaning of an additional degree of freedom, that considerably improves the analytical structure of the perturbation series for the eigenvalues and surpasses most approximation methods in its domain of applicability and accuracy as well.[12-18]

To the best of our knowledge, only a few groups[19-23] have so far applied the $1/N$ expansion technique (unshifted and shifted) to study the relativistic bound-state energies of spin-0 and spin-$\frac{1}{2}$ particles. It has been noted, however, that the rate of convergence of the unshifted expansion is very slow for the relativistic part of the energy eigenvalue as compared to that for the nonrelativistic part. Panja and Dutt[24] have extended this technique and introduced a shifting parameter to deal with relativistic particles (with and without spin). For the Coulomb case, exact analytical expressions and highly convergent expansion were restored for the relativistic correction of order $1/c^2$.

In this paper, an alternative approach to the shifted $1/N$ expansion technique is introduced to work out the energy eigenvalues of a Klein-Gordon (KG) particle. We have defined the energy eigenvalue series as $E=E_0+E_1/\bar{k}+E_2/\bar{k}^2+\cdots$ and determined the shift parameter requiring that $E_1=0$.

This approach provides remarkably very accurate results for the energy eigenvalues for the Coulomb potential. The calculations of the energy eigenvalues were carried out to the second-order correction.

In Sec. II we develop the formalism of this technique for the Klein-Gordon particle. Exact numerical results for the KG-Coulomb potential[25,26] are presented in Sec. III, together with our results. Section IV contains conclusions.

II. THE METHOD

The radial part of the KG equation (in units $\hbar=c=1$) for a scalar particle of mass m moving in a spherically symmetric potential $V(r)$ is given by[18,19]

$$\left[-\frac{d^2}{dr^2}+\frac{(k-1)(k-3)}{4r^2}-\{[E-V(r)]^2-m^2\}\right]U_{n_r}(r)=0\ , \tag{1}$$

where $k=N+2l$ and $U_{n_r}(r)$ is the reduced radial wave function. In terms of the shifting parameter a, i.e., $\bar{k}=k-a$, Eq. (1) becomes

$$\left[-\frac{d^2}{dr^2}+\frac{(\bar{k}+a-1)(\bar{k}+a-3)}{4r^2}+[2EV(r)-V(r)^2]\right]U_{n_r}(r)=(E^2-m^2)U_{n_r}(r)\ . \tag{2}$$

It is convenient to shift the origin by defining

$$x=\bar{k}^{1/2}(r-r_0)/r_0\ , \tag{3}$$

and to use the following expansions:

$$V(r)=(\bar{k}^2/Q)[V(r_0)+V'(r_0)r_0x/\bar{k}^{1/2}+V''(r_0)r_0^2x^2/2\bar{k}+\cdots], \tag{4a}$$

$$E=E_0+E_1/\bar{k}+E_2/\bar{k}^2+\cdots, \tag{4b}$$

where Q is a scale, whose magnitude is to be determined later. Equations (4), when substituted in Eq. (2), yield

$$-\frac{d^2U_{n_r}(x)}{dx^2}+\left[\frac{\bar{k}}{4}-\frac{(2-a)}{2}+\frac{(1-a)(3-a)}{4\bar{k}}\right]\left[1-\frac{2x}{\bar{k}^{1/2}}+\frac{3x^2}{\bar{k}}-\cdots\right]U_{n_r}(x)$$
$$+\frac{2r_0^2\bar{k}}{Q}\left[E_0+\frac{E_1}{\bar{k}}+\frac{E_2}{\bar{k}^2}+\cdots\right]\left[V(r_0)+V'(r_0)\frac{r_0x}{\bar{k}^{1/2}}+V''(r_0)\frac{r_0^2x^2}{2\bar{k}}+\cdots\right]U_{n_r}(x)$$
$$-\frac{r_0^2\bar{k}}{Q}\left[V(r_0)+V'(r_0)\frac{r_0x}{\bar{k}^{1/2}}+V''(r_0)\frac{r_0^2x^2}{2\bar{k}}+\cdots\right]^2U_{n_r}(x)=\varepsilon_{n_r}U_{n_r}(x), \tag{5}$$

where

$$\varepsilon_{n_r}=(r_0^2/Q)[\bar{k}(E_0^2-m^2)+2E_0E_1+(E_1^2+2E_0E_2)/\bar{k}+\cdots]. \tag{6}$$

Equation (5) is a Schrödinger-like equation for the one-dimensional anharmonic oscillator problem which has been discussed in detail by Imbo, Pagnamenta, and Sukhatme.[12] Therefore, following their formalism, we obtain

$$\varepsilon_{n_r}=\bar{k}[\tfrac{1}{4}+2r_0^2E_0V(r_0)/Q-r_0^2V(r_0)^2/Q]+[(1+2n_r)w/2-(2-a)/2]$$
$$+(1/\bar{k})\{(1-a)(3-a)/4+(1+2n_r)\tilde{\varepsilon}_2+3(1+2n_r+2n_r^2)\tilde{\varepsilon}_4-(1/w)[\tilde{\varepsilon}_1^2+6(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_3+(11+30n_r+30n_r^2)\tilde{\varepsilon}_3^2]\}, \tag{7}$$

where n_r is the radial quantum number and

$$\tilde{\varepsilon}_j=\varepsilon_j/w^{j/2}, \quad j=1,2,3,4 \tag{8}$$

and

$$\varepsilon_1=2-a, \quad \varepsilon_2=-3(2-a)/2,$$
$$\varepsilon_3=-1+(r_0^5/3Q)[E_0V'''(r_0)-V(r_0)V'''(r_0)-3V'(r_0)V''(r_0)], \tag{9}$$
$$\varepsilon_4=\tfrac{5}{4}+(r_0^6/12Q)[E_0V''''(r_0)-V(r_0)V''''(r_0)-4V'(r_0)V'''(r_0)-3V''(r_0)V''(r_0)].$$

Comparing the terms of Eq. (7) with those of Eq. (6) and equating terms of same order in $\bar{k}$ implies

$$(r_0^2/Q)(E_0^2-m^2)=[\tfrac{1}{4}+2r_0^2E_0V(r_0)/Q-r_0^2V(r_0)^2/Q], \tag{10}$$

$$(r_0^2/Q)(2E_0E_1)=[(1+2n_r)w/2-(2-a)/2], \tag{11}$$

$$(r_0^2/Q)(E_1^2+2E_0E_2)=\{(1-a)(3-a)/4+(1+2n_r)\tilde{\varepsilon}_2+3(1+2n_r+2n_r^2)\tilde{\varepsilon}_4$$
$$-(1/w)[\tilde{\varepsilon}_1^2+6(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_3+(11+30n_r+30n_r^2)\tilde{\varepsilon}_3^2]\}. \tag{12}$$

From Eq. (10) we obtain

$$E_0=V(r_0)+m(1+Q/4m^2r_0^2)^{1/2}, \tag{13}$$

where r_0 is chosen to minimize E_0. That is,

$$\frac{dE_0}{dr_0}=0, \quad \frac{d^2E_0}{dr_0^2}>0, \tag{14}$$

therefore, r_0 satisfies the equation

$$r_0^3V'(r_0)(1+Q/4m^2r_0^2)^{1/2}=Q/4m. \tag{15}$$

To solve for the shifting parameter a, the next contribution to the energy eigenvalue is chosen to vanish,[12] i.e., $E_1=0$, which implies that

$$a=2-(1+2n_r)w, \tag{16}$$

where w is given by

$$w=[3+r_0V''(r_0)/V'(r_0)-4r_0^4V'(r_0)V'(r_0)/Q]^{1/2} \tag{17}$$

and Q satisfies Eq. (15), which can be written as

$$Q=[r_0^2V'(r_0)]^2(2+2\gamma), \tag{18}$$

where

$$\gamma=\{1+[2m/r_0V'(r_0)]^2\}^{1/2}. \tag{19}$$

Equations (16) and (18) along with Eqs. (17) and (19), with $Q=\bar{k}^2$, read

$$1+2l+(1+2n_r)w=r_0^2V'(r_0)(2+2\gamma)^{1/2}, \tag{20}$$

which is an explicit equation in r_0. Once r_0 is determined, Eq. (13) gives E_0 and Eq. (12) gives E_2. Finally, Eq. (4) gives

TABLE I. The energy levels for spin-0 particle in the Coulomb potential in units of $10^{-8}mc^2$.

States	E_0	E_0+E_2/k^2	Exact (Ref. 25)
1*s*	99 997 335.93	99 997 335.89	99 997 355.86
2*s*	99 999 334.00	99 999 333.99	99 999 333.98
2*p*	99 999 334.00	99 999 334.00	99 999 334.00
3*s*	99 999 704.00	99 999 704.00	99 999 703.99
3*p*	99 999 704.00	99 999 704.00	99 999 704.00
3*d*	99 999 704.00	99 999 704.00	99 999 704.00

$$E=E_0+(1/2E_0r_0^2)\{(1-a)(3-a)/4+(1+2n_r)\tilde{\varepsilon}_2+3(1+2n_r+2n_r^2)\tilde{\varepsilon}_4 -(1/w)[\tilde{\varepsilon}_1^2+6(1+2n_r)\tilde{\varepsilon}_1\tilde{\varepsilon}_3+(11+30n_r+30n_r^2)\tilde{\varepsilon}_3^2]\} . \tag{21}$$

III. APPLICATION TO THE COULOMB POTENTIAL

For the Coulomb potential,[25]

$$V(r)=-\beta/r, \quad \beta=e^2 , \tag{22}$$

Eqs. (17), (19), and (20) yield

$$w=[(\gamma-1)/(\gamma+1)]^{1/2} , \tag{23}$$

$$\gamma=[1+(2mr_0/\beta)^2]^{1/2} , \tag{24}$$

and

$$\beta(2\gamma+2)^{1/2}=1+2l+(1+2n_r)[(\gamma-1)/(\gamma+1)]^{1/2} . \tag{25}$$

We have numerically solved Eq. (25) for r_0 in terms of mc^2 and found the energy eigenvalues.

In Table I, the numerical results for the energy eigenvalues calculated by the leading term E_0 of the energy series and by $E_0+E_2/\bar{k}^2$ are compared with the exact ones given by Ref. 25.

The agreement of our results with the exact ones is better than that of Panja and Dutt,[24] especially for small l states. Moreover, the convergence of the results listed in Table I seems to be fast in a sense that the second-order contribution to the energy series, $E_2/\bar{k}^2$, is very small (of the order of 10^{-8}–10^{-11}) compared with the contribution of the leading term, E_0. It should be pointed out, however, that the accuracy as well as the convergence of our results increases as the principal quantum number n of the state increases.

IV. CONCLUSION

In this paper we have developed a formalism of the shifted $1/N$ expansion technique for the Klein-Gordon equation with radially symmetric potentials. For the Coulomb potential the method looks quite attractive as it yields highly accurate results. We have also seen that the convergence increases as the principal quantum number increases.

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