Gravitational Lensing

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ABSTRACT

In this thesis we study one of the most interesting aspects of the Einstein's theory of gravity which can be examined practically by astronomers. In this natural phenomena, when a beam of light passes nearby a heavy object, due to the gravity of the object the beam of light bends toward the center of the object which causes a deflection from its original path. This is much like a lens located in the path of the beam of light which deflects it and that is why it is called *Lensing Effect*. The implications of the theoretical results, even from the very beginning of the general relativity, have been realized by the astronomers where they could not explain many unusual observatory datas which could not be analyzed without considering this effect.

Nowadays, applying the theory of lensing, helps us to localize the observed objects in the space surrounding our Earth. Therefore, our preliminary aim is to go through the theory of lensing by considering the general relativity concepts introduced by Einstein and developed by other great physicists. Next we apply the theory for some known spacetimes like the Schwarzschild black hole, the Grumiller spacetime and regular Bardeen black hole. To complete our assessment, we shall study lensing by a wormhole too.

Keywords: Lensing Effect, Einstein General Relativity, Schwarzschild Black Hole, Grumiller Spacetime, Bardeen Black Hole

ÖZ

Bu tezde Einstein çekim yasasının astronomlar tarafından test edilebilecek bir özelliği inceleniyor. Bir ışık hüzmesi kütleli bir gök cismi yanından geçerken yer çekimine kapılıp bükülmektedir. Merceğin ışığı saptırmasından ötürü bu olaya Mercek etkisi denmektedir. Bu etki göz önüne alınmadan pek çok uzay gözlemi yanıltıcı sonuçlar vermektedir. Son zamanlarda Mercek etkisi kullanılarak Dünya etrafındaki gök cisimlerinin yerleri doğru olarak tesbit edilmektedir. Bu tezde Mercek etkisinin ayrıntıları incelenmektedir. Schwarzschild, Grumiller ve tekilsiz Bardeen karadelik gibi iyi bilinen uzaylara bu etki uygulanmaktadır. Son olarak bir uzay solucan deliğinde de Mercek etkisi incelenmektedir.

Anahtar Kelimeler: Mercek Etkisi, Einstein Genel Görecelik Kuramı, Schwarzschild, Grumiller ve Bardeen Solucan Uzayları

To My Family

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Chapter 1

INTRODUCTION

Gravitational lensing is the phenomena which causes the light or any other electromagnetic radiation deflect in a gravitational field. The object which causes the deflection is called a *lens*. Gravitational lensing is one of the consequences of Einstein's General Relativity. Newton and Laplace were among the first physicist who suspected the existance of this deflection. The first publication on the subject was in 1804 by a german geodesist and astronomer Johann Soldner who calculated the magnitude of the bending angle due to the sun by considering that light rays are made of particles and applying newtonian gravity. The calculation was the same as Einstein's in 1911 who use an equivalence principle, Newtonian mechanics and the quantum properties of light. In 1915 after completion of General Theory of Relativity, he found that the deflection angle is actually twice the previous result due to the cervature of the metric. In 1919 Eddington measured the deflection angle during two observations of total solar eclipse and confirmed the value participated by Einstein. In 1920 Eddington suggested that gravitational lensing can create multiple images. In 1937 Zwicky noted that galaxies as lenses can create images far enough from background sources that makes them obsevable. They would also magnify far galaxies and make them detectable. At last real examples of lensing were discovered by finding quasars as sources. In 1979 Walsh, Carswell and Weymann discovered the first examples of gravitational lensing.

Nowadays there are many examples of mutiple-imaged quasars known to the scientist. In 1986 The idea of *microlensing* was introduced by Paczyński. *Enstein rings* were discovered by Hewitt in the radio waveband in 1987.

The application of gravitational lensing can be put into three categories. firstly there is the magnification which enables us to observe distant objects. Secondly lensing is a perfect way of detecting dark matter and studying the mass in the universe. Finally, constraints can be put through lensing on physical constants like Hubble constant or cosmological constant.

Chapter 2

Bending Of Light In Schwarzschild And Schwarzschild-de Sitter Geometry

2.1 Introduction

In this chapter we will concentrate on bending of light produced by spherically symmetric mass using a method proposed by W. Rindler and M. Ishak [8] for Schwarzschild-de Sitter (SdS) geometry. The SdS line element is given by

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right), \tag{2.1}$$

where

$$f(r) \equiv 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}. \tag{2.2}$$

in which Λ is the *cosmological constant*. Relativistic units $c = G = 1$ are used here. Schwarzschild metric can be obtained by putting $\Lambda = 0$. A constant time equatorial plane ($\theta = \frac{\pi}{2}$) of Schwarzschild spacetime has the line element

$$dl^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\varphi^2. \tag{2.3}$$

Now, for we imagine that there exists an Euclidean dimension z, which has no physical meaning. If we replace our plane, equation (2.3), with a surface with $z = 2\sqrt{2M(r-2M)}$, then we have a surface on which distances are the same as distances on Shwarzschild line element.

$$dz^2 + dr^2 + r^2 d\varphi^2 = dl^2 = \left(\frac{1}{1-\frac{2M}{r}}\right) dr^2 + r^2 d\varphi^2 \tag{2.4}$$

This plane is called *Flamm's paraboloid*. It can be derived by considering Euclidean line element in cylindrical coordinates (r, φ, z) and writing

$$ds^2 = dz^2 + dr^2 + r^2 d\varphi^2. \tag{2.5}$$

If z is a function of r, then

$$ds^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\varphi^2. \tag{2.6}$$

Comparing this equation with (2.3) we can conclude that

$$1 + \left(\frac{dz}{dr}\right)^2 = \left(1 - \frac{2M}{r}\right)^{-1}. \tag{2.7}$$

or

$$dz = \frac{dr}{\frac{r}{2M} - 1} \tag{2.8}$$

Integrating from both side will give us

$$z = 4m\sqrt{\frac{r}{2m} - 1} + constant. \tag{2.9}$$

which is the Flamm's paraboloid equation. SdS spacetime geometry is different. The equatorial coordinate plane has a 2-demensional metric

$$dl^2 = \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\varphi^2. \tag{2.10}$$

If r is small, the second term in $f(r)$ is dominant and we get the Flamm's geometry egain. When we are far from the center, the third term of $f(r)$ is dominant, so we get the geometry of a sphere with radius $r = \sqrt{\frac{3}{\Lambda}}$. Thus our surface for SdS is a Flamm's paraboloid and a sphere as shown in the Fig.2.1. In Fig.2.1, $\sum^1$ is a flat plane with the photon orbit $\mathcal{L}^1$ on it. The photon orbit $\mathcal{L}^2$ and $\mathcal{L}^3$ are vertical projection of $\mathcal{L}^1$ on surfaces $\sum^2$ (the SdS sphere) and $\sum^3$ (the Flamm's paraboloid), respectively. Since Shwarzschild spacetime, $\sum^3$ becomes flat at infinity, the asymptotes of $\mathcal{L}^3$ are exactly the same as $\mathcal{L}^1$ and as a result we can find the total deflection in $\sum^1$ instead of Schwarzschild spacetime. However this approach does not work for SdS since $\mathcal{L}^2$ is not flat at infinity. So our approach is going to differ.

2.2 Light Bending in Scharzschild Metric

First we write the Lagrangian for null geodesics as

$$L = -\frac{1}{2} f(r) \dot{t}^2 + \frac{1}{2} f^{-1}(r) \dot{r}^2 + \frac{1}{2} r^2 \dot{\varphi}^2, \tag{2.11}$$

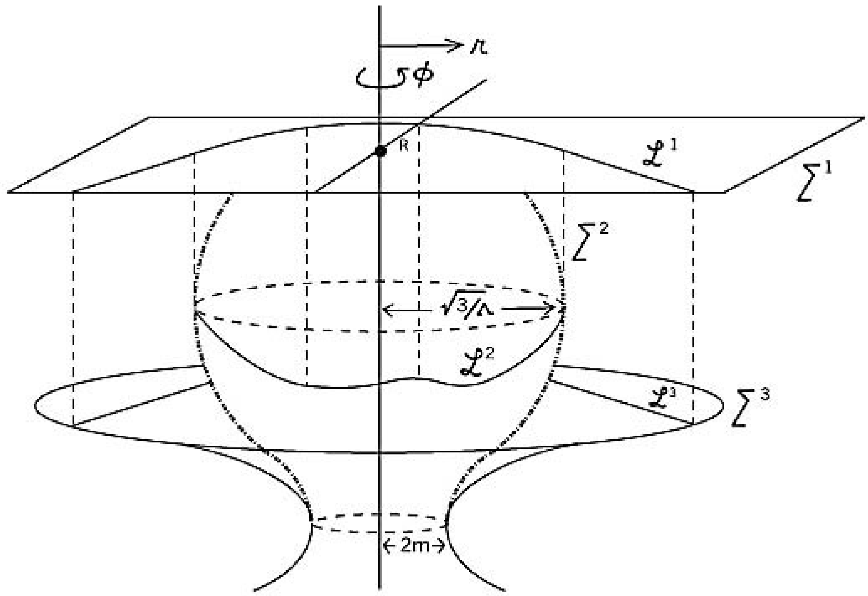

Figure 2.1:Schwarzschild and Schwarzschild-de Sitter Geometries.

where overdot demonstrates differentiation wih respect to an affine parameter λ along geodesics.

$$-\frac{1}{2}f(r)\dot{t}^2+\frac{1}{2}f^{-1}(r)\dot{r}^2+\frac{1}{2}r^2\dot{\varphi}^2=0. \tag{2.12}$$

Then we write Euler-Lagrange equations

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial L}{\partial\dot{t}}\right)=\frac{\partial L}{\partial t},$$

and

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial L}{\partial\dot{\varphi}}\right)=\frac{\partial L}{\partial\varphi}.$$

Since $\partial L/\partial t = 0$

$$\frac{\partial L}{\partial \dot{t}} = f(r)\,\dot{t} = \varepsilon, \tag{2.13}$$

in which ε is a conserved quantity i.e. the energy. This equation helps us to find

$$\dot{t} = \frac{\varepsilon}{f(r)}. \tag{2.14}$$

From $\partial L/\partial \varphi = 0$ one obtains

$$\frac{\partial L}{\partial \dot{\varphi}} = r^2\dot{\varphi} = \ell, \tag{2.15}$$

or consequently

$$\dot{\varphi} = \frac{\ell}{r^2}. \tag{2.16}$$

Here energy ε and momentum in φ direction ℓ are conserved quantities. Now we subtitute (2.14) and (2.16) in (2.12) and get an equation for r as

$$\dot{r}^2 = \varepsilon^2 - \frac{\ell^2}{r^2} f(r). \tag{2.17}$$

We are looking for $dr/d\varphi$ so we should devide (2.17) by (2.16) and write

$$\frac{\dot{r}^2}{\dot{\varphi}^2} = \left(\frac{dr}{d\varphi}\right)^2 = \frac{\varepsilon^2 r^4}{\ell^2} - r^2\left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right). \tag{2.18}$$

Now, we introduce $u \equiv 1/r$ which yields $du/u^2 = -dr$ and subtitute it into (2.18).

$$\left(\frac{du}{d\varphi}\right)^2 = 2Mu^3 - u^2 + \frac{\varepsilon^2}{\ell^2} + \frac{\Lambda}{3}. \tag{2.19}$$

Next, we differentiate (2.19) with respect to φ which reads

$$\frac{d^2u}{d\varphi^2} + u = 3Mu^2. \tag{2.20}$$

Equation (2.20) does not have an exact solution, therefore we shall use an approximation method. We first solve (2.20) without considering the right hand side, i.e.,

$$\frac{d^2u}{d\varphi^2} + u = 0. \tag{2.21}$$

A solution to this equation is

$$u = \frac{1}{r} = \frac{sin\varphi}{R}, \tag{2.22}$$

where R=constant and is called *impact parameter*. This is the first approximation to (2.20). We subtitute this approximation to the right hand side of (2.20)

$$\frac{d^2u}{d\varphi^2} + u = 3M\frac{sin^2\varphi}{R^2}, \tag{2.23}$$

or consequently

$$\frac{d^2u}{d\varphi^2} + u = \frac{3M}{2R^2}\left(1 - cos2\varphi\right). \tag{2.24}$$

Equation (2.24) is the approximate null geodesic equation for the line element (2.1). Next, we solve this equation, using the standard method to find

$$u = \frac{1}{r} = \frac{sin\varphi}{R} + \frac{M}{2R^2}\left(3 + cos2\varphi\right). \tag{2.25}$$

Let's consider r_0 the *distance of closest approach* and can be found by putting $\varphi = \frac{\pi}{2}$ in (2.25), i.e.,

$$\frac{1}{r_0} = \frac{1}{R} + \frac{M}{R^2}. \tag{2.26}$$

As we have disscused earlier, to find deflection angle in Schwarzschild spacetime we can let $r \to \infty$. For small φ we use the approximation $sin\varphi_\infty \approx \varphi_\infty$ to rewrite (2.25) as

$$\frac{1}{R}\varphi_\infty + \frac{M}{2R^2}\left(4 - 2\varphi_\infty^2\right) = 0.$$

This equation gives us the diflection angle

$$\varphi_\infty = -\frac{2M}{R}. \tag{2.27}$$

As a result we find the total bending of light between asymptotes $\frac{4M}{R}$.

2.3 Light Bending in Schwarzschild-de Sitter Metric

To find the SdS deflection, let's consider the plane $t = const.$ and $\theta = \frac{\pi}{2}$ as shown in Fig.2.2.

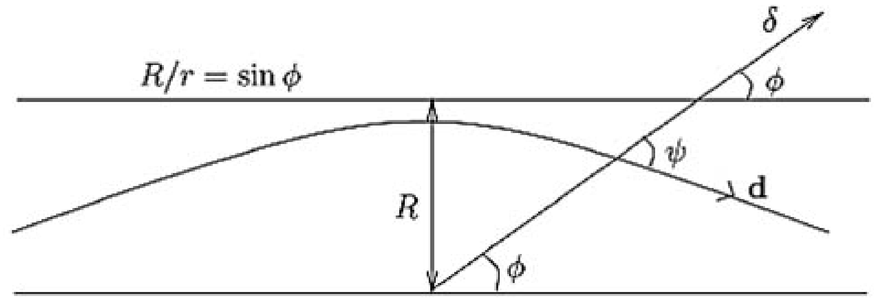

Figure 2.2:The Graph Showing the Orbit of Photon and the Bending Angle.

The angle ψ is between the two direction $\mathbf{d} = d^i = (dr, d\varphi)$ and $\delta = \delta^j = (\delta r, 0)$. To find ψ we get help from the definition of dot product,

$$cos\psi = \frac{g_{ij}d^i\delta^j}{\sqrt{g_{ij}d^id^j}\sqrt{g_{ij}\delta^i\delta^j}} \tag{2.28}$$

where g_{ij} is the metric tensor and from (2.1) we get

$$g_{11} = f(r)^{-1} = \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1}, \qquad g_{22} = r^2.$$

Differentitation of (2.25) with respect to φ gives

$$A(r,\varphi) = \frac{dr}{d\varphi} = \frac{Mr^2}{R^2}sin2\varphi - \frac{r^2}{R}cos\varphi, \tag{2.29}$$

which together with (2.28), we obtain

$$cos\psi = \frac{dr}{\sqrt{g_{11}dr^2 + g_{22}d\varphi^2}} = \frac{dr}{\sqrt{dr^2 + g^{11}g_{22}d\varphi^2}}\frac{dr/d\varphi}{\sqrt{\left(\frac{dr}{d\varphi}\right)^2 + g^{11}r^2}}, \tag{2.30}$$

or consequently

$$cos\psi = \frac{|A|}{\sqrt{A^2 + g^{11}r^2}}. \tag{2.31}$$

Knowing $1 + tan^2\psi = \frac{1}{cos^2\psi}$ we can find ψ in terms of $tan\psi$, i.e.

$$tan\psi = \frac{\sqrt{g^{11}}r}{|A|} = \frac{f(r)^{\frac{1}{2}}r}{|A|} \tag{2.32}$$

We note that the one-sided bending angle is given by $\varepsilon = \psi - \varphi$, however we can easily calculate the deflection by putting $\varphi = 0$ and therefore $\varepsilon = \psi_0$. Upon these (2.25) becomes

$$r = \frac{R^2}{2M}, \tag{2.33}$$

while (2.29) becomes

$$|A| = \frac{R^3}{4M^2}. \tag{2.34}$$

We know that for small angle ψ_0

$$\psi_0 \approx tan\psi_0, \tag{2.35}$$

and consequently Eq.(2.32) implies,

$$\psi_0 \approx \frac{2M\sqrt{1 - \frac{4M^2}{R^2} - \frac{\Lambda R^4}{12M^2}}}{R} \approx \frac{2M}{R}\left(1 - \frac{2M^2}{R^2} - \frac{\Lambda R^4}{24M^2}\right). \tag{2.36}$$

Finally the total bending angle becomes:

$$2\psi_0 \approx \frac{4M}{R}\left(1 - \frac{2M^2}{R^2} - \frac{\Lambda R^4}{24M^2}\right). \tag{2.37}$$

The classical Einstein bending angle is the first term in (2.37). Note that a positive Λ diminishes the bending angle. However we should notice that the contribution of Λ in this expression is very weak.

Chapter 3

Bending Of Light In Grumiller Model

3.1 Grumiller Model of Gravity at Large Distances

In this chapter we find the deflection angle of the light rays using a model of gravity at large distances which has been proposed by D. Grumiller [4]. This effective model was suggested due to the disagreement between the observed trajectoty of the test particles and the computed trajectory in the gravitational field of a central object. The line element proposed by Grumiller reads

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right), \tag{3.1}$$

where

$$f(r) = 1 - \frac{2M}{r} - \Lambda r^2 + 2ar. \tag{3.2}$$

Here a is the *Rindler acceleration* that can be an explaination for the apparent anomalous and constant acceleration observed on Pioneer 10 and 11 with the magnitude $\approx 8.5 \times 10^{-}8\ cm/s^2$ having a direction towards the sun [5, 6].

3.2 Light Bending at Large Distances in Grumiller Metric

The approach in this section was originated from [8] and then used by [10] recently. The line element on equatorial plane with constant time is given by

$$dl^2 = \frac{dr^2}{f(r)} + r^2 d\varphi^2. \tag{3.3}$$

Let's start with the Lagrangian for null geodesics which is given by

$$L = -\frac{1}{2} f(r)\dot{t}^2 + \frac{1}{2f(r)}\dot{r}^2 + \frac{1}{2} r^2 \dot{\varphi}^2. \tag{3.4}$$

For null geodesics we have the following constant

$$-\frac{1}{2} f(r)\dot{t}^2 + \frac{1}{2f(r)}\dot{r}^2 + \frac{1}{2} r^2 \dot{\varphi}^2 = 0, \tag{3.5}$$

where overdot demonstrates differentiation with respect to affine parameter λ along the geodesics. Here we go through the same steps as the previous chapter. We write the Euler-Lagrange equations as

$$\frac{\partial}{\partial \lambda}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t},$$

and

$$\frac{\partial}{\partial \lambda}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\partial L}{\partial \varphi}.$$

Then we solve these equations to find

$$\dot{t} = \frac{\varepsilon}{f(r)}, \tag{3.6}$$

and

$$\dot{\varphi} = \frac{\ell}{r^2}. \tag{3.7}$$

We substitute (3.6) and (3.7) into (3.5) to get

$$\dot{r}^2 = \varepsilon^2 - f(r)\frac{\ell^2}{r^2}. \tag{3.8}$$

To obtain $dr/d\varphi$ we divide (3.8) by (3.7) which gives

$$\frac{\dot{r}^2}{\dot{\varphi}^2} = \left(\frac{dr}{d\varphi}\right)^2 = \frac{\varepsilon^2 r^4}{\ell^2} - r^2\left(1 - \frac{2M}{r} - \Lambda r^2 + 2ar\right). \tag{3.9}$$

Now, we introduce $u = \frac{1}{r}$, put it in (3.9) and diffrentiate with respect to φ which yeilds

$$\frac{d^2u}{d\varphi^2} + u = 3Mu^2 - a. \tag{3.10}$$

To solve this equation we need to use the same approximation as the one we used to solve (2.17) i.e. we find solution without the right hand side which is

$$u = \frac{1}{r} = \frac{sin\varphi}{R}. \tag{3.11}$$

Herein R is called the *impact parameter*, which is the closest distance between spherically symmetric matter distribution (lens) and the light ray without deflection. We replace the first approximation in the right hand side to get

$$\frac{d^2u}{d\varphi^2}+u=3M\frac{sin^2\varphi}{R^2}-a,$$

or equivalently

$$\frac{d^2u}{d\varphi^2}+u=\frac{3M}{2R^2}\left(1-cos2\varphi\right)-a. \tag{3.12}$$

Next, we solve the resulting equation with the usual methods which admits

$$u=\frac{1}{r}=\frac{sin\varphi}{R}+\frac{M}{2R^2}\left(3+cos2\varphi\right)-a. \tag{3.13}$$

To get the *distance of closest approach* we put $\varphi=\frac{\pi}{2}$, which gives

$$\frac{1}{r_0}=\frac{1}{R}+\frac{M}{R^2}-a \tag{3.14}$$

Now, we define ψ to be the angle between the two directions $\mathbf{d}=d^i=(dr,d\varphi)$ and $\delta=\delta^j=(\delta r,0)$ as we can see in Fig.2.2. From the definition of dot product we get

$$cos\psi=\frac{g_{ij}d^i\delta^j}{\sqrt{g_{ij}d^id^j}\sqrt{g_{ij}\delta^i\delta^j}}, \tag{3.15}$$

where g_{ij} is the metric tensor. Differentiaion from (3.13) gives

$$A\left(r,\varphi\right)=\frac{dr}{d\varphi}=\frac{Mr^2}{R^2}sin2\varphi-\frac{r^2}{R}cos\varphi, \tag{3.16}$$

Which together with (3.15) one finds

$$cos\psi = \frac{dr}{\sqrt{g_{11}dr^2 + g_{33}d\varphi^2}} = \frac{dr}{\sqrt{dr^2 + g^{11}g_{33}d\varphi^2}} = \frac{\frac{dr}{d\varphi}}{\sqrt{\frac{dr^2}{d\varphi^2} + g^{11}r^2}},$$

or consequently

$$cos\psi = \frac{|A|}{\sqrt{A^2 + g^{11}r^2}}. \tag{3.17}$$

Knowing that

$$1 + tan^2\psi = \frac{1}{cos^2\psi},$$

we find

$$tan\psi = \frac{\sqrt{g^{11}}r}{|A|}. \tag{3.18}$$

The deflection angle will be obtained through $\varepsilon = \psi - \varphi$. Therefore the deflection angle corresponds to $\varphi = 0$ or $\varepsilon = \psi_0$. As a result (3.13) and (3.16) become

$$r = \frac{R^2}{2M - aR^2}, \tag{3.19}$$

and

$$|A| = \frac{r^2}{R} = \frac{R^3}{(2M - aR^2)^2}. \tag{3.20}$$

Substituting these two equations into (3.18) and considering that ψ_0 is a small angle, one gets

$$\psi_0 \approx tan\psi_0 = \frac{\sqrt{g^{11}}r}{|A|} \tag{3.21}$$

$$\psi_0 \approx \frac{\sqrt{4R^2M^2 - R^6\Lambda - 16M^4}}{R^2} + 12\frac{M^3}{\sqrt{4R^2M^2 - R^6\Lambda - 16M^4}}a + O\left(a^2\right), \tag{3.22}$$

where a is a small number and thus we can neglect higher order of it. Finally we find

$$\psi_0 \approx \frac{2M}{R} - \frac{\Lambda R^3}{4M} + \frac{6M^2}{R}a, \tag{3.23}$$

and therefore the total deflection angle is twice of this value, and reads

$$2\psi_0 \approx \frac{4M}{R} - \frac{\Lambda R^3}{2M} + \frac{12M^2}{R}a. \tag{3.24}$$

As one can see, beisde the *Einstein angle* $\frac{4M}{R}$, our bending angle contains two other terms contributed from cosmological constant and Rindler acceleration. A positive Rindler acceleration increases the bending angle while cosmological constant decreases it. Moreover, in the second term a is inversely proportional to the distance of closest aproach whereas in [9] a is proportinal to r_0 which means that as r_0 increases light bending increases too. This result contradics with the observations.

Chapter 4

Bending Of Light In Ellis Wormhole Geometry

4.1 Introduction

We know gravitational lensing as the most important test of general relativity. But it is also known as one of the most significant tools in cosmology. Moreover, it has been proposed as a method of detecting a wormhole. One particular type of wormhole was introduced by H. G. Ellis in 1973 [3]. This type of wormhole is special due to its asymptotic mass which is zero at infinity. The line element of Ellis wormhole is given as

$$ds^2 = -dt^2 + dr^2 + \left(r^2 + a^2\right)\left(d\theta^2 + sin^2\theta d\varphi^2\right). \tag{4.1}$$

Unlike black holes, the coordinate r is from $-\infty$ to $+\infty$ but here we only cosider the positive half of our interval for simplicity, $r = 0$ is called *throat* of the wormhole.

4.2 Light Bending in Ellis Wormhole

The Lagrangian reads

$$L = -\frac{\dot{t}^2}{2} + \frac{\dot{r}^2}{2} + \frac{\left(r^2 + a^2\right)}{2}\left(\dot{\theta}^2 + sin^2\theta\dot{\varphi}^2\right), \tag{4.2}$$

where the dot shows derivation with respect to λ, the affine parameter. Looking at equation (4.1) we easily realize that Ellis wormhole is spherically symmetric, which means that we can simplify our task even more by considering the equatorial plane $\theta = \frac{\pi}{2}$. Hence the Lagrangian changes into

$$L = -\frac{\dot{t}^2}{2} + \frac{\dot{r}^2}{2} + \frac{\left(r^2 + a^2\right)}{2}\dot{\varphi}^2. \tag{4.3}$$

We write the Euler-Lagrange equations for parameters t and φ, respectively, to find the constants of motion;

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t} \quad \Rightarrow \quad \dot{t} = \varepsilon = const, \tag{4.4}$$

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\partial L}{\partial \varphi} \quad \Rightarrow \quad \left(r^2 + a^2\right)\dot{\varphi} = \ell = const. \tag{4.5}$$

Now, we subtitute (4.4) and (4.5) into (4.3) to get

$$\dot{r}^2 = \varepsilon^2 - \frac{\ell^2}{r^2 + a^2}. \tag{4.6}$$

Equation (4.5) gives us

$$\dot{\varphi}^2 = \frac{\ell^2}{\left(r^2 + a^2\right)^2}. \tag{4.7}$$

In order to obtain an expression for $\frac{dr}{d\varphi}$ we use (4.7) and (4.6) to write

$$\frac{\dot{r}^2}{\dot{\varphi}} = \left(\frac{dr^2}{d\varphi}\right)^2 = \left(r^2 + a^2\right)^2\left(\frac{\varepsilon^2}{\ell^2} - \frac{1}{r^2 + a^2}\right). \tag{4.8}$$

Let's introduce b as

$$b \equiv \frac{\ell}{\varepsilon}, \tag{4.9}$$

and rewrite (4.8) using (4.9) to find

$$\frac{1}{\left(r^2+a^2\right)^2}\left(\frac{dr}{d\varphi}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2+a^2}. \tag{4.10}$$

It is clear that in Ellis wormhole if $a \to 0$ there is no deflection so in (4.10) the minimum value for r is b. Therefore b can be considered as the *impact parameter*, which is the distance between the center of the lens and the undeflected trajectory of the light ray. Also r_0 which is the *Distance of closest approach* is found by considering (4.6) to be equal to zero. Thus we write

$$\frac{1}{b^2} - \frac{1}{r_0^2 - a^2} = 0,$$

which gives

$$r_0 = \sqrt{b^2 - a^2}. \tag{4.11}$$

Next, we find an expression for φ in terms of r using equation (4.10)

$$\left(\frac{d\varphi}{dr}\right)^2 = \frac{b^2}{\left(r^2+a^2\right)^2 - b^2\left(r^2+a^2\right)},$$

and thus we obtain

$$d\varphi = \frac{b\,dr}{\sqrt{\left(r^2+a^2\right)^2 - b^2\left(r^2+a^2\right)}}. \tag{4.12}$$

An Integration from both sides gives φ. To Find the deflection angle we write [11]

$$\psi(b) = 2\int_{r_0}^{\infty} \frac{b\,dr}{\sqrt{\left(r^2+a^2\right)^2 - b^2\left(r^2+a^2\right)}} - \pi, \tag{4.13}$$

where ψ is the deflection angle. Now, we choose the transformation $R^2 = r^2 + a^2$ So that $dr = R\;dR/r$. That gives us

$$r = \sqrt{R^2 - a^2}.$$

This transformation changes (4.13) to

$$\psi(b) = 2\int_{r_0}^{\infty} \frac{b\,dR}{\sqrt{\left(R^2-a^2\right)\left(R^2-b^2\right)}} - \pi. \tag{4.14}$$

With two more transformations this integral can be transformed into a *complete elliptic integral of first kind*. First we introduce α as

$$\alpha \equiv \frac{b}{R}, \tag{4.15}$$

and therefore

$$dR = -\frac{R^2}{b}d\alpha. \tag{4.16}$$

Next, we introduce β as

$$\beta \equiv \frac{a}{b}. \tag{4.17}$$

Subtituting these two, changes (4.14) to

$$\psi(b) = \int_1^0 \frac{-R^2\, d\alpha}{\sqrt{R^4\left(1-\frac{a^2}{R^2}\right)\left(1-\frac{b^2}{R^2}\right)}} - \pi, \tag{4.18}$$

Upon using

$$\frac{a^2}{R^2} = \alpha^2\beta^2, \tag{4.19}$$

we reach to

$$\psi(b) = 2\int_0^1 \frac{d\alpha}{\sqrt{(1-\alpha^2\beta^2)(1-\alpha^2)}} - \pi. \tag{4.20}$$

The integral term in (4.20) is called a *complete elliptic integral of first kind* and can be expressed in terms of a power series

$$K(b) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \beta^{2n}. \tag{4.21}$$

The term inside summation is the definition of *Legendre polynomials*. Hence we find

$$K(b) = \frac{\pi}{2}\sum_{n=0}^{\infty}[P_{2n}(0)]\beta^{2n}, \tag{4.22}$$

Therefore we expand the deflection angle as

$$\psi(b) = \pi \left[\left(\frac{1}{2}\right)^2 \beta^2 + \left(\frac{1\times 3}{2\times 4}\right)^2 \beta^4 + ... + \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \beta^{2n} + ... \right] \tag{4.23}$$

Chapter 5

Bending Of Light In A Regular Bardeen Black Hole

5.1 Strong Field Limit

Gravitational lensing can be classified into three categories: strong lensing, weak lensing and microlensing. In privous chapters we have discusted the weak lensing in different geometries, where the distortion of the source is not much. But here we discuss the strong lensing where distortion of the source is noticeable. This includes *Einstein rings* and *multiple images*.

In this chapter first we present a method of finding deflection angle in any given spherically symmetric spacetime for standard null geodesic equation in strong field limit [12]. Next, by implementing this method we reach to the bending angle in a regular black hole geometry. A Spherically symmetric spacetime reads

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + C(r)\left(d\theta^2 + sin^2\theta d\varphi^2\right), \tag{5.1}$$

with the condition

$$\frac{C'(r)}{C(r)} = \frac{A'(r)}{A(r)}. \tag{5.2}$$

The largest positive solution to equation (5.2) is called *radius of photon sphere* r_{ps}, where Photon trajectory becomes an orbit. The *impact parameter* and the *distance of* *closest approach* are shown by b and r_0, respectively. The impact parameter can be expressed as

$$b = \sqrt{\frac{C(r_0)}{A(r_0)}}. \tag{5.3}$$

The Lagrangian for metric (5.1) is

$$L = -\frac{1}{2}A(r)\dot{t}^2 + \frac{1}{2}B(r)\dot{r}^2 + \frac{1}{2}C(r)\dot{\varphi}^2, \tag{5.4}$$

therefore for the null geodesics on equatorial plane we write

$$-\frac{1}{2}A(r)\dot{t}^2 + \frac{1}{2}B(r)\dot{r}^2 + \frac{1}{2}C(r)\dot{\varphi}^2 = 0. \tag{5.5}$$

The Euler-Lagrange equations give us $\dot{t} = \frac{\varepsilon}{A(r)}$ and $\dot{\varphi} = \frac{\ell}{C(r)}$, where ε and ℓ are constants. Subtituting these two relations in (5.5), we obtain

$$\frac{\dot{r}^2}{2} + \frac{\ell^2 A(r)}{2C(r)} = E, \tag{5.6}$$

where $E = \varepsilon^2/2$ is constant. Now, we define $V_{eff} = \frac{\ell^2 A(r)}{2C(r)}$ and thus equation (5.6) turns into

$$\frac{\dot{r}^2}{2} + V_{eff} = E. \tag{5.7}$$

When $r = r_0$ then $\dot{r} = 0$ so we can find E as

$$E = V_{eff} = \frac{\ell^2 A(r_0)}{2C(r_0)}. \tag{5.8}$$

Hence we rewrite (5.7) to get

$$\dot{r}^2 = \ell^2 \left[\frac{A(r_0)}{C(r_0)} - \frac{A(r)}{C(r)} \right]. \tag{5.9}$$

Using equation (5.9) we reach to

$$\left(\frac{d\varphi}{dr} \right)^2 = \frac{\frac{\ell^2}{C(r)^2}}{\ell^2 \left[\frac{A(r_0)}{C(r_0)} - \frac{A(r)}{C(r)} \right]},$$

or equivalently

$$\left(\frac{d\varphi}{dr} \right)^2 = \frac{1}{A(r)C(r) \left[\frac{A(r_0)C(r)}{A(r)C(r_0)} - 1 \right]} \tag{5.10}$$

The deflection angle can be found by [11]

$$\alpha(r_0) = \psi(r_0) - \pi, \tag{5.11}$$

where

$$\psi(r_0) = \int_{r_0}^{\infty} \frac{2\sqrt{B(r)}dr}{\sqrt{C(r) \left[\frac{A(r_0)C(r)}{A(r)C(r_0)} - 1 \right]}}. \tag{5.12}$$

We should note that as b decreases, the bending angle increases until it is larger than 2π, which means that the light rays encircle the black hole completely. If we decrease b even more, it causes the photons to rotate around the black hole many times and then come out of the loop. At $r = r_{ps}$ and consequently $b = b_{ps}$ the photon will be trapped inside an orbit. We introduce a new parameter

$$y = \frac{A(r) - A(r_0)}{1 - A(r_0)}, \tag{5.13}$$

which implies

$$dr = \frac{1 - A(r_0)}{A'(r)} dy, \tag{5.14}$$

and

$$r = A^{-1}\left[(1 - A(r_0))\, y + A(r_0)\right]. \tag{5.15}$$

Therefore equation (5.12) becomes

$$\psi(r_0) = \int_0^1 \frac{2\sqrt{A(r)B(r)C(r_0)}\,(1 - A(r_0))\, dy}{A'(r)C(r)\sqrt{A(r_0) - \left[(1 - A(r_0))\, y + A(r_0)\right]\frac{C(r)}{C(r_0)}}},$$

or simply

$$\psi(r_0) = \int_0^1 I_1(y, r_0) I_2(y, r_0) dy, \tag{5.16}$$

where

$$I_1(y, r_0) = 2\frac{\sqrt{A(r)B(r)C(r_0)}}{C(r)A'(r)}\,(1 - A(r_0)), \tag{5.17}$$

and

$$I_2(y,r_0) = \frac{1}{\sqrt{A(r_0) - \left[(1-A(r_0))\,y + A(r_0)\right] \frac{C(r)}{C(r_0)}}}. \tag{5.18}$$

As we can clearly see if $y \to 0$ then $r \to r_0$ and thus $I_2(y,r_0)$ diverges, while $I_1(y,r_0)$ is regular. We apply Taylor expansion on $I_2(y,r_0)$ at $y=0$ to get

$$I_2(y,r_0) \approx I_0(y,r_0) = \frac{1}{\sqrt{\alpha y + \beta y^2}}, \tag{5.19}$$

in which

$$\alpha = \frac{A(r_0)C'(r_0) - A'(r_0)C(r_0)}{A'(r_0)C(r_0)} \left(1 - A(r_0)\right), \tag{5.20}$$

and

$$\begin{aligned} \beta = & \frac{(1-A(r_0))^2}{2A'^2(r_0)C^2(r_0)} \left[\left(C(r_0)C''(r_0) - 2C'^2(r_0)\right) A(r_0)A'(r_0)\right] \\ & + \frac{(1-A(r_0))^2}{2A'^2(r_0)C^2(r_0)} \left[2A'^2(r_0)C(r_0)C'(r_0) - A(r_0)A''(r_0)C(r_0)C'(r_0)\right]. \end{aligned} \tag{5.21}$$

Herein when α is zero, r_0 should be equal to r_{ps} and therefore the divergence is of order y^{-1}, but when α is not vanishing the order of divergence is $y^{-1/2}$ which can be integrated. So we can separate (5.16) into two parts like

$$\psi(r_0) = \psi_D(r_0) + \psi_R(r_0), \tag{5.22}$$

where the divergent part is given by

$$\psi_D(r_0) = \int_0^1 I_1(0, r_{ps}) I_0(y, r_0), \tag{5.23}$$

and the regular part reads as

$$\psi_R(r_0) = \int_0^1 f(y, r_0) dy, \tag{5.24}$$

where

$$f(y, r_0) = I_1(y, r_0) I_2(y, r_0) - I_1(0, r_{ps}) I_0(y, r_0). \tag{5.25}$$

We note that $\psi_D(r_0)$ has an exact solution

$$\psi_D(r_0) = I_1(0, r_{ps}) \int_0^1 \frac{dy}{\sqrt{\alpha y + \beta y^2}} = \frac{2I_1(0, r_{ps})}{\sqrt{\beta}} \left[ln\left(2\left(\sqrt{\beta(\alpha+\beta y)} + \beta\sqrt{y}\right)\right) - ln\left(2\sqrt{\alpha\beta}\right)\right],$$

or simply

$$\psi_D(r_0) = \frac{2I_1(0, r_{ps})}{\sqrt{\beta}} \, ln\left(\frac{\sqrt{\beta} + \sqrt{\alpha+\beta}}{\sqrt{\alpha}}\right). \tag{5.26}$$

Let's write Taylor expansion of α about r_{ps} and obtain

$$\alpha' = \frac{1 - A(r_0)}{A'^2(r_0) C^2(r_0)} \times$$
$$\left[A(r_0)A'(r_0)C(r_0)C''(r_0) - A(r_0)A'(r_0)C'^2(r_0) - A(r_0)A''(r_0)C(r_0)C'(r_0) + A'^2(r_0)C(r_0)C'(r_0)\right],$$

or

$$\alpha = \frac{1-A(r_{ps})}{A'(r_{ps})C(r_{ps})}\left(A(r_{ps})C'(r_{ps})-A'(r_{ps})C(r_{ps})\right)+\alpha'_{ps}\left(r_0-r_{ps}\right)+O\left(r_0-r_{ps}\right)^2. \tag{5.26}$$

Using equation (5.2) one finds

$$\alpha = \frac{(1-A(r_{ps}))A'(r_{ps})C(r_{ps})}{A^2(r_{ps})C'^2(r_{ps})}\left(A(r_{ps})C''(r_{ps})-A''(r_{ps})C(r_{ps})\right)\left(r_0-r_{ps}\right)+O\left(r_0-r_{ps}\right)^2. \tag{5.26}$$

If $r_0 = r_{ps}$, in equation (5.21) we get

$$\beta_{ps} = \frac{(1-A(r_{ps}))^2 C(r_{ps})}{2A^2(r_{ps})C'^2(r_{ps})}\left(A(r_{ps})C''(r_{ps})-A''(r_{ps})C(r_{ps})\right). \tag{5.26}$$

Subtituting (5.29) into (5.28) yields

$$\alpha = \frac{2\beta_{ps}A'(r_{ps})}{1-A(r_{ps})}\left(r_0-r_{ps}\right)+O\left(r_0-r_{ps}\right)^2. \tag{5.26}$$

Upon this (5.26) becomes

$$\psi_D(r_0) = \frac{I_1(0,r_{ps})}{\sqrt{\beta}}\, ln\left(\frac{\alpha+2\beta+2\sqrt{\beta\left(\alpha+\beta\right)}}{\alpha}\right),$$

or equivalently

$$\psi_D(r_0) = \frac{I_1(0,r_{ps})}{\sqrt{\beta}}\left[ln\left(\frac{2\left(1-A(r_{ps})\right)}{A'(r_{ps})r_{ps}}\right)-ln\left(\frac{r_0}{r_{ps}}-1\right)+O\left(r_0-r_{ps}\right)\right].$$

After some rearrangment we obtain

$$\psi_D(r_0) = -m\, ln\left(\frac{r_0}{r_{ps}} - 1\right) + n_D + O\left(r_0 - r_{ps}\right), \tag{5.24}$$

in which

$$m = \frac{I_1(0, r_{ps})}{\sqrt{\beta}}, \tag{5.24}$$

and

$$n_D = \frac{I_1(0, r_{ps})}{\sqrt{\beta}}\, ln\left(\frac{2\left(1 - A(r_{ps})\right)}{A'(r_{ps}) r_{ps}}\right). \tag{5.24}$$

Now, we expand (5.24) around r_{ps} to get

$$\psi_R(r_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(r_0 - r_{ps}\right)^2 \int_0^1 \frac{\partial^n f}{\partial r_0^n}\Big|_{r_0 = r_{ps}} dy.$$

Here we are only interested in

$$\psi_R(r_0) = \int_0^1 f(y, r_{ps}) dy + O\left(r_0 - r_{ps}\right) = n_R. \tag{5.23}$$

Hence the deflection angle is obtained as

$$\alpha(r_0) = -m\, ln\left(\frac{r_0}{r_{ps}} - 1\right) + n + O\left(r_0 - r_{ps}\right), \tag{5.23}$$

where

$$n = n_D + n_R - \pi. \tag{5.23}$$

Here n_R can be found either analytically or numerically. The minimum of b is at $r = r_{ps}$ so

$$b_{ps} = \sqrt{\frac{C(r_{ps})}{A(r_{ps})}}. \tag{5.23}$$

Furthermore the Taylor expanssion of (5.3) gives us

$$b - b_{ps} = \frac{A(r_{ps})C''(r_{ps}) - A''(r_{ps})C(r_{ps})}{4\sqrt{A^3(r_{ps})C(r_{ps})}} (r_0 - r_{ps})^2 = a\,(r_0 - r_{ps})^2\,, \tag{5.23}$$

in which

$$a = \frac{\beta_{ps} C'^2(r_{ps})}{2\,(1 - A(r_{ps}))^2} \sqrt{\frac{A(r_{ps})}{C^3(r_{ps})}}. \tag{5.23}$$

Thus the bending angle in terms of θ and impact parameter is given by

$$\alpha(\theta) = -\frac{m}{2}\left[ln\left(\frac{b}{b_{ps}} - 1\right) - ln\left(\frac{a r_{ps}^2}{b_{ps}}\right)\right] + n,$$

which by subtituting (5.36), (5.37), (5.39) and (5.2), reduces to

$$\alpha(\theta) = -\frac{m}{2}\, ln\left(\frac{b}{b_{ps}} - 1\right) - \frac{m}{2}\, ln\left(\frac{2\beta_{ps}}{A(r_{ps})}\right) + n_R - \pi.$$

Finally we can write the deflection angle as

$$\alpha(\theta) = -\bar{m}\, ln\left(\frac{b}{b_{ps}} - 1\right) + \bar{n}, \tag{5.21}$$

where

$$\bar{m} = \frac{m}{2} = \frac{I_1(0, r_{ps})}{2\sqrt{\beta_{ps}}}, \tag{5.21}$$

and

$$\bar{n} = n_R - \pi + \bar{m}\ ln\left(\frac{2\beta_{ps}}{A(r_{ps})}\right). \tag{5.21}$$

5.2 Regular Bardeen Black Hole

A *regular black hole* does have an event horizon but there is no singularity in it. Bardeen black hole is the first regular black hole, introduced by J. Bardeen in 1968. This section is devoted to finding bending angle in strong deflection limit for Bardeen spacetime implementing the method discussed in previous section. Since Bardeen spacetime is spherically symmetric it can be described by the line element (5.1) [2], in which

$$A(r) = \frac{1}{B(r)} = 1 - \frac{2Mr^2}{(r^2+q^2)^{3/2}}, \tag{5.21}$$

and

$$C(r) = r^2. \tag{5.21}$$

Here in M is the mass of the black hole and q is a constant which can be considered as the charge of the black hole with natural units $G = c = 1$. If $q = 0$ our metric reduces to the Schwarzschild metric. For more simplicity we can devide our line element by $2M$.

This does not have any effect on the form of line element but affects metric functions, i.e.,

$$A(r) = \frac{1}{B(r)} = 1 - \frac{r^2}{\left(r^2+q^2\right)^{3/2}}, \tag{5.21}$$

and

$$C(r) = r^2. \tag{5.21}$$

According to [11] the photon sphere radius is the largest positive solution to (5.2) i.e.,

$$4\left(r^2+q^2\right)^5 - 9r^8 = 0. \tag{5.21}$$

For small values of q we can find r in terms of q

$$r = \sqrt{\frac{9-20q^2}{4}}. \tag{5.21}$$

Now, all we have to do is to find $\bar{m}$ and $\bar{n}$ from (5.41) and (5.42), respectively. We recall (5.17) and (5.21) to write

$$I_1(0, r_{ps}) = 2\frac{r^2+q^2}{r^2-2q^2}, \tag{5.21}$$

and

$$\beta = \frac{-3r^6 + \sqrt{r^2+q^2}\left(r^6+9r^4q^2-8q^6\right)}{\sqrt{r^2+q^2}\left(r^2-2q^2\right)^3}, \tag{5.21}$$

Furthermore

$$\bar{m} = \frac{\left(r^2+q^2\right)\sqrt{r^2-2q^2}}{\sqrt{8q^2-9r^4q^2+r^6\left(1-\frac{3}{\sqrt{r^2+q^2}}\right)}}. \tag{5.21}$$

Now, we use (5.21) and (5.45) to obtain

$$\gamma = ln\left(\frac{2\beta}{A(r_{ps})}\right) = ln\left[\frac{2\left(8q^2-9r^4q^2+r^6\left(1-\frac{3}{\sqrt{r^2+q^2}}\right)\right)}{(r^2-2q^2)^3\left(1-\frac{r^2}{\left(r^2+q^2\right)^{3/2}}\right)}\right], \tag{5.21}$$

in which

$$\bar{n} = \bar{m}\gamma + n_R - \pi. \tag{5.21}$$

The only remaining task is to calculate n_R. For small values of q we can use power expanssion of second order

$$n_R = n_{R,0} + n_{R,2}q^2 + O\left(q^4\right), \tag{5.21}$$

Calculation of $n_{R,0}$ and $n_{R,2}$ requires finding $f(y, r_{ps})$, finding its second order expansion for q and then inegrating the result. Finally we get

$$n_{R,0} = ln\left[36\left(7-4\sqrt{3}\right)\right] \approx 0.949603, \tag{5.21}$$

and

$$n_{R,2} = \frac{4}{9}\left[-18+4\sqrt{3}-5ln\left(\frac{2+\sqrt{3}}{6}\right)\right] \approx -3.86568. \tag{5.21}$$

We should note here that this expanssion is not precise for large values of q. Subtituting these results into (5.40) gives us our deflection angle.

Chapter 6

CONCLUSION

In this thesis we have studied ”Bending of Light in General Relativity”. After a short introduction we have started with one of the simplest solution of the Einstein's equation, the Schwarzschild black hole, which is also the first solution of the Einstein's equation which has been found by a German physicist Karl Schwarzschild in 1916 [11]. (A little history: This solution was found a month after the publication of the Einstein's famous paper on general relativity [1] where the Einstein's equation was given there. Although Karl Schwarzschild died shortly after his solution was published on May 11, 1916 his solution has kept him alive so far and will keep forever.) Schwarzschild black hole solution which is sometimes called Schwarzschild static / vacuum solution is a one parameter solution of the Einstein solution which can be accepted for many interesting spherical objects; for instance Sun, that either does not rotates or rotate slowly. The only parameter which must be given in this solution is the mass of the central object M. The central object may be a black hole or just a cosmological object depend on the magnitude of the mass and the radius of the object. As it is well known if the radius of a spherical object is smaller than Schwarzschild radius $r_s = \frac{2GM}{c^2}$ then the object is a black hole, otherwise it is not. For instance to make the Sun a Schwarzschild black hole, its radius must be less than $3km$.

Being the first, the simplest and the most applicable and interesting solution of the Einstein's equation, forced us to study first the lensing effect in this space-time. Next, we developed our analysis toward the Schwarzschild-de Sitter spacetime. This is the solution of Einstein's equation with a cosmological constant and without other matter sources. After considering the first solution in general relativity we consider a very recent cosmological model at large distances introduced by Grumiller [4]. Although this model is proposed almost a century after Schwarzschild solution, surprisingly they are very similar to each other. The recent model has just a linear term in addition to the former one. This was the reason that we decided to investigate the light bending in this model of gravity. In continuation, we have studied also the light bending by a regular black hole. Regular black holes are in contrast with the singular black holes like Schwarzschild. (Schwarzschild black hole and most of the other spherical symmetry black hole solutions admit a singularity at $r = 0$.) The regular black hole that we have studied is called Bardeen black hole which has been found during the 70s [7]. Finally we studied the bending of light by a wormhole, particularly the wormhole solution given by Ellis [3].

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