Continuous Nowhere Differentiable Functions

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Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the Degree of

Master of Science
in
Applied Mathematics and Computer Science

Eastern Mediterranean University
May 2014
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

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ABSTRACT

In this thesis four interesting points of mathematical analysis are handled. At first, some examples of continuous nowhere differentiable functions are discussed. Secondly, the Lebesgue-Cantor singular function is considered, which is continuous but the fundamental theorem of calculus is not valid for this function. Next, space-filling functions, which are continuous surjections from the interval to the square, are considered. Finally, two examples of infinitely many times differentiable functions which are not analytic are considered.

Keywords: mathematical analysis, continuous functions, differentiable functions, series, convergence.

ÖZ

Tezde matematiksel analizin dört önemli noktası açıklanmıştır. Önce sürekli ve hiç türevi olmayan birkaç fonksiyon örneği verilmiştir. Sonra Lebesgue-Cantor singüler fonksiyonuna bakılmıştır. Bu fonksiyon sürekli olmasına rağmen analizin temel teoremi ona uygulanamamaktadır. Daha sonra uzay dolduran eğrilere bakılmıştır. Bunlar aralıktan kareye örten fonksiyonlardır. Son olarak her basamaktan türeve sahip olan fakat analitik olmayan fonksiyonlar ele alınmıştır.

Anahtar kelimeler: matematiksel analiz, sürekli fonksiyonlar, türevlenebilir fonksiyonlar, seriler, yakınsaklık.

ACKNOWLEDGMENTS

I would like to express my sincere gratitude and appreciation to my supervisor Prof. Dr. Agamirza Bashirov for his useful comments, remarks through the writing process of this master thesis and that helped me much to get task accomplished.

Furthermore I would like to thank all my professors in Mathematics Department at EMU, for their kind help and support that introduced me to a wide range of information.

I would be honored to express my gratitude to all my family and my friends for their continuous support and encouragement throughout this period.

At the end I would like express appreciation to my beloved wife Azheen who supported me through entire the process of writing.

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Chapter 1

INTRODUCTION

As it is known, continuity and differentiability are very important concepts of mathematics. Over the centuries many mathematicians have been interested in continuity and differentiability trying to find the relation between them. At the beginning, most of mathematicians believed that every continuous function is differentiable but actually this statement was not true. For the first time, in 1806 Ampere talked about continuity and differentiability trying to construct a continuous nowhere differentiable function at that time. After Ampere, Bernard Bolzano in 1830, Cellèrier in1860, and Riemann in 1861, found functions of this nature but did not publish at that time. In 1872 Weierstrass found a first continuous nowhere differentiable function and published this in 1875. Weierstrass' discovery put an end to former arguments. But the converse statement is true that if a function is differentiable then it is continuous [1]. In Chapter 2 we discuss this type of functions that are continuous everywhere but nowhere differentiable. In Chapter 3 we consider Lebesgue Cantor function, that is function based on the Cantor set. At the beginning of Chapter 3 we show how to construct the Cantor set. The Cantor set is an important set in mathematics. In Chapter 3 many significant properties of the Cantor set are proved. Then Cantor function is defined and it is proved that the Cantor function is nowhere differentiable on Cantor set. In Chapter 4 space filling curves are define. Some examples of such curves, construction and proof of the fact that they are continuous but nowhere differentiable are discussed.

In this example we showed that this function does not contain zero. In Chapter 5 there are infinitly many times differentiable but not analytic function in Chapter 5 we discussed, about analytic function while each function can be written by Taylor series is analytic. Two examples are considered in this chapter are analytic by using Taylor inequality and also we have two interesting examples that have infinitely many times differentiable but not analytic function.

Chapter 2

CONTINUOUS NOWHERE DIFFERENTIABLE FUNCTIONS

2.1 Introduction

According to the well-known relation between differentiability and continuity, if a function is differentiable at $x = c$, then it is continuous at $x = c$, but the converse of this statement does not hold. This means that a function can be continuous but not differentiable. Actually, there are many examples of continuous function which is not differentiable at one or few points. A popular example of such function is the absolute value function: $f(x) = |x|$, which is continuous at every point but not differentiable at $x = 0$ [1]. During eighteenth and early nineteenth centuries the mathematicians believed that every continuous function has derivative, but the scientist Andre Marie Ampere, did the first research about this idea in 1806, but Ampere was not successful in his effort. In 1872 Karl Theodor Wilhelm Weierstrass, showed that there exists a function that is continuous everywhere but nowhere differentiable. After that in 1903 Takagi conferred his example. In 1930 Van der Waerden published his function. After this the number of continuous nowhere differentiable functions proved rapidly. In this chapter we discuss three continuous nowhere differentiable functions. Then we consider Baire category theorem and its application to $C(a , b)$, demonstrating that continuous nowhere differentiable functions on $[a , b]$ are typical points of the Banach space $C(a , b)$ of the continuous functions on $[a , b]$.

2.2 History

At first, we give some examples supporting the idea of a continuous nowhere differentiable function.

(a) Bernard Bolzano function (1830)

The first example of continuous nowhere differentiable function found by Bolzano in 1830. Bolzano's function is build as an example of a function that is continuous on interval $[a, b]$, but not monotone on any subinterval. Later Bolzano showed that the points at which this function has no derivative [1], are everywhere dense in the interval $[a, b]$. Bolzano's function is defined as a limit of continuous functions $y_1, y_2, y_3, \ldots$ defined on an interval $[a, b]$. Here y_1 is a linear function satisfying $y_1(a) = A$ and $y_1(b) = B$.

$$y_1(x) = A + (x - a)\frac{B - A}{b - a}.$$

To define the function y_2, Bolzano divides the interval $[a, b]$ into four subintervals limited by points,

$$a \ , \quad a + \tfrac{3}{8}(b - a) \quad , \quad \tfrac{1}{2}(a + b) \ , \quad \tfrac{7}{8}(b - a) \quad , \quad b.$$

y_2 is constructed to be linear in each of these four subintervals. The function y_3 is defined analogously, if each of the four subintervals is considered instead of the interval $[a, b]$, etc… In 1922 Karel Rychlık showed that the Bolzano function is continuous and nowhere differentiable [2].

(b) Riemann function (1860)

In 1860, Riemann considered the function

$$R(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2} = \sin x + \frac{1}{4}\sin(4x) + \frac{1}{9}\sin(9x) + \cdots$$

This function continuous since the series converges uniformly and it is nowhere differentiable [3]. In 1916 Hardy showed that the Riemann function is not differentiable at all irrational multiples of π. After this in 1969 Gerver showed that this function is actually differentiable at the all the rational multiples of π of the form $\frac{\pi p}{q}$ where p and q is odd integers.

(c) Cellèrier function (1860)

In 1860 Cellèrier considered the function

$$S(x) = \sum_{n=1}^{\infty} \frac{1}{a^n} \sin(a^n x),$$

where $a > 1000$. This function is continuous but nowhere differentiable, the Cellèrier function was not published until 1890 [3].

(d) Weierstrass function (1872)

In 1872 Karl Weierstrass presented his famous Weierstrass function to the Royal Academy of Science in Berlin, Germany. The Weierstrass function is the first continuous nowhere differentiable function, published in 1875 by due Bois-Reymond,

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where $0 < a < 1$ and b is a positive odd integer greater than 1 such that $ab > 1 + \frac{3}{2}\pi$. This function is everywhere continuous but nowhere differentiable [3], [4].

(e) Darboux function (1873)

$$D(x) = \sum_{n=1}^{\infty} \frac{1}{n!} \sin\big((n+1)!\, x\big).$$

This function is published in 1875.

(f) Peano function (1890)

$$f_p : (t_1 t_2 t_3 t_4 \dots)_3 = \begin{pmatrix} (t_1 (k^{t_2} t_3)(k^{t_2+t_4}\, t_5)(k^{t_2+t_4+t_6} t_7) \quad \dots)_3 \\ ((k^1 t_2)(k^{t_1+t_3} t_4)(k^{t_1+t_3+t_5}\ ) \dots)_3 \end{pmatrix}$$

Here the operator k is defind as $kt_j = 2 - t_j$, $where\ t_j = 0{,}1{,}2$ and k^v denotes the vth iterate of k [5].

(g) Takagi function in (1903)

$$T(x) = \sum_{n=0}^{\infty} \frac{\emptyset(2^n x, \mathbb{Z})}{2^n}$$

where $\emptyset(x) = \mathrm{dist}(x, \mathbb{Z})$ the distance from x to nearest integer [6].

(h) Faber function (1907)

In 1907 the German mathematician Georg Faber found an example of everywhere continuous nowhere differentiable function in the form

$$F(x) = \sum_{n=0}^{\infty} \frac{1}{10^n} \inf_{m \in z} \mathrm{dist}(2^{n!} x, \mathbb{Z}).$$

(i) Van der Waerden function (1930)

$$V(x) = \sum_{n=0}^{\infty} \frac{1}{10^n} \mathrm{dist}(10^n x, \mathbb{Z})$$

$$= \sum_{n=0}^{\infty} \frac{1}{10^n} \inf_{m \in z} |10^n x - m|,$$

where $\inf_{m \in z} |10^n x - m|$ denotes the distance from x to the nearest integer [7].

(j) Schoenberg functions (1938)

The Schoenberg two functions $f(x)$ and $g(t)$ are defined by

$$f(x) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^{2k}} p(3^{2k}\, x) \quad ,$$

$$g(t) = \frac{1}{2}\sum_{k=o}^{\infty}\frac{1}{2^k}p(\,3^{2k+1}x\,).$$

where $p(x+2) = p(x)$ for every $x \in \mathbb{R}$ [5].

2.3 Weierstrass function

Many famous mathematicians have believed that every continuous function must be differentiable, but Karl Weierstrass shocked the mathematical community by proving the existence of a continuous nowhere differentiable function. Weierstrass was not the first mathematician who constructed a continuous nowhere differentiable function, he was the first, sharing it with the rest of the mathematical community. In 1875 he presented this result during his lecture and then in 1875 published. In fact Weierestrass simply gave a formula for such a function for any $a \in R$ with $0 < a < 1$ and for odd integer b satisfying $ab > 1 + \frac{3}{2}\pi.$ Define the function $W: \mathbb{R} \longrightarrow \mathbb{R}$ by

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x) \qquad (1)$$

$$= \cos(\pi x) + a\cos(b\pi x) + a^2\cos(b^2\pi x) + a^3\cos(b^3\pi x) + \cdots.$$

This function is called Weierstrass continuous nowhere differentiable function.

The following is the graph of Weierstrass function for $a = 0.7$ and $b = 9$. From this graph it is seen that this is a continuous function but has no tangent line anywhere, therefore $W(x)$ is nowhere differentiable.

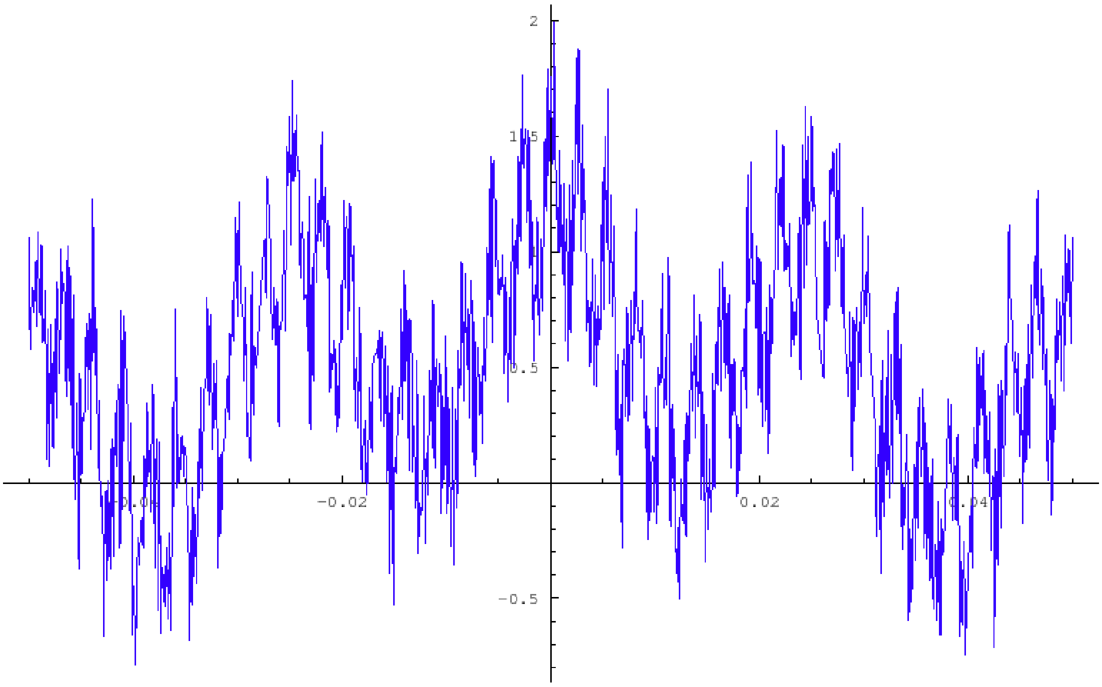

Figure 2-1: Weierstrass function where $a = 0.7$ and $b = 9$

The proof of continuity of Weierstrass function is based on Weierstrass M-test for functional series.

2.3.1 (Uniform convergence)

A sequence of a functions $\{f_n\}$, is said to be uniformly convergent to f on the set A if for each $\varepsilon > 0$ there is an integer N such that

$$|f_n(x) - f(x)| < \varepsilon$$

for all $n \geq N$ and $x \in A$. A series

$$\sum_{n=0}^{\infty} f_n(x)$$

converges uniformly on A if the sequence $\{S_n\}$, of partial sums defined by

$$S_n(x) = \sum_{j=1}^{n} f_j(x),$$

converges uniformly on A [8] .

Theorem 2.3.2 The limit of a uniformly convergent sequence or series of continuous function is continuous [8].

Theorem 2.3.3 (Weierstrass M-test) Suppose that f_n is a sequence of real valued functions, defined on a set $X \subseteq \mathbb{R}$. Suppose further that for every $n \in \mathbb{N}$ there exist a real constant M_n such that $|f_n(x)| \leq M_n$ for every $x \in X$. If the numerical series

$$\sum_{n=1}^{\infty} M_n$$

converges, then the series

$$\sum_{n=1}^{\infty} f_n(x)$$

is uniformly convergent on X [9].

Theorem 2.3.4 The Weierstrass function $W(x)$ defined by (1) is continous.

Proof. Since $0 < a < 1$ then

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} < \infty.$$

Because $\sup_{x\in\mathbb{R}}|a^n\cos(b^n\pi x)| \leq 1$, we have that $\sup_{x\in\mathbb{R}}|a^n\cos(b^n\pi x)| \leq a^n$. Thus, by the Weierstrass M-Test, and by Theorem (2.3.2) the Weierstrass function is the uniform limit of continuous functions. Therefore, W from (1) is continuous [9],[10].

Theorem 2.3.5 The Weierstrass function $W(x)$ defined by (1) is nowhere differentible.

Proof. Let x_0 be a fixed arbitrary real number. We will show that $W(x)$ is not differentiable at x_0, by constracting two sequences y_m and z_m, such that $z_m \to x_0$ from the right and $y_m \to x_0$ from the left, such that the difference quotients

$$\frac{W(y_m) - W(x_0)}{y_m - x_0}, \qquad \frac{W(z_m) - W(x_0)}{z_m - x_0}$$

do not have the same limit. In fact, the absolute value of the left and right quotients diverge to ∞ as $m \to \infty$, and have opposite signs.

For each $m \in \mathbb{N}$, choose an integer $\beta_m \in \mathbb{Z}$ such that

$$\frac{-1}{2} < b^m x_0 - \beta_m \le \frac{1}{2}$$

Define

$$x_{m+1} = b^m x_0 - \beta_m.$$

Let y_m and z_m be

$$y_m = \frac{\beta_m - 1}{b^m} \quad , \quad z_m = \frac{\beta_m + 1}{b^m}.$$

This gives

$$y_m - x_0 = -\frac{1 + x_{m+1}}{b^m} \quad , \quad z_m - x_0 = \frac{1 - x_{m+1}}{b^m}.$$

Therefore,

$$y_m < x_0 < z_m .$$

As $m \to \infty$, $y_m \to x_0$ from the left and $z_m \to x_0$ from the right. We first calculate the left-hand difference quotient:

$$\begin{aligned}\frac{W(y_m) - W(x_0)}{y_m - x_0} &= \sum_{n=0}^{\infty} a^n \frac{\cos(b^n\pi\, y_m) - \cos(b^n\pi x_0)}{y_m - x_0} \\ &= \sum_{n=0}^{m-1} (ab)^n \frac{\cos(b^n\pi y_m) - \cos(b^n\pi y_m)}{b^n(y_m - x_0)} \\ &+ \sum_{n=m}^{\infty} \left(a^{m+n} \frac{\cos(b^{m+n}\pi y_m) - \cos(b^{m+n}\pi x_0)}{y_m - x_0} \right) \qquad (2) \\ &= S_1 + S_2 ,\end{aligned}$$

where S_1 and S_2 refer to respective sums. We will consider each of these sums separately by first rearranged S_1:

$$S_1 = \sum_{n=0}^{m-1}\left((ab)^n \frac{\cos(b^n\pi y_m) - \cos(b^n\pi x_0)}{b^n(y_m - x_0)}\right)$$

$$= \sum_{n=0}^{m-1}(-\pi\ )(ab)^n \sin\left(\frac{b^n\pi(y_m + x_0)}{2}\right)\frac{\sin\left(\frac{b^n\pi(y_m - x_0)}{2}\right)}{\frac{b^n\pi(y_m - x_0)}{2}}.$$

Here we used the trigonometric identity

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}.$$

We have

$$\left|\frac{\sin\left(b^n\pi\frac{(y_m - x_0)}{2}\right)}{b^n\pi\frac{(y_m - x_0)}{2}}\right| \le 1$$

since $\frac{\sin x}{x} \le 1$. The absolute value of the first sum in (2) can be estimated as follows

$$|S_1| = \left|\sum_{n=0}^{m-1}(-\pi\ )(ab)^n \sin\left(\frac{b^n\pi(y_m + x_0)}{2}\right)\frac{\sin\left(\frac{b^n\pi(y_m - x_0)}{2}\right)}{\frac{b^n\pi(y_m - x_0)}{2}}\right|$$

$$\le \pi\sum_{n=0}^{m-1}(ab)^n = \frac{\pi}{ab-1}\ [(ab)^m - 1]\ < \frac{\pi}{ab-1}(ab)^m. \qquad (3)$$

Since b is odd integer and $\beta_m \in \mathbb{Z}$, the terms of the second sum in (2) can be rearranged as

$$\cos(b^{m+n}\pi y_m) = \cos\left(b^{m+n}\pi\left(\frac{\beta_m - 1}{b^m}\right)\right)$$

$$= \cos(b^n\pi(\ \beta_m - 1))$$

$$= [(-1)^{b^n}]^{\beta_m - 1} = -(-1)^{\beta_m}$$

and

$$\cos(b^{m+n}\pi x_0) = \cos(b^{m+n}\,\pi\left(\frac{\beta_m + x_{m+1}}{b^m}\right) = \cos(b^n\pi\beta_m + b^n\pi x_{m+1}).$$

By the summation formula for cosine

$$\cos(\mathrm{A}+\mathrm{B}) = \cos\mathrm{A}\cos\mathrm{B} - \sin\mathrm{A}\sin\mathrm{B}\,,$$

we have

$$\begin{aligned}\cos(b^{m+n}\pi x_0) &= \cos(b^n\pi\beta_m)\cos(b^n\pi x_{m+1}) - \sin(b^n\pi\beta_m)\sin\,(b^n\pi x_{m+1})\\ &= [(-1)^{b^n}]^{\beta_m}\cos(b^n\pi x_{m+1}) - 0\\ &= (-1)^{\beta_m}\cos(b^n\pi x_{m+1}).\end{aligned}$$

This means that we can express S_2 as

$$\begin{aligned}S_2 &= \sum_{n=0}^{\infty}\left(a^{m+n}\frac{\cos(b^{m+n}\pi y_m) - \cos(b^{m+n}\pi x_0)}{y_m - x_0}\right)\\ &= \sum_{n=0}^{\infty}\left(a^{m+n}\frac{-(-1)^{\beta_m} - (-1)^{\beta_m}\cos(b^n\pi x_{m+1})}{-\frac{1+x_{m+1}}{b^m}}\right)\\ &= (-1)^{\beta_m}(ab)^m\sum_{n=0}^{\infty}a^n\frac{1+\cos(b^n\pi x_{m+1})}{1+x_{m+1}}\ .\end{aligned}$$

By assumption $a \in (0,1)$, each term in the series

$$\sum_{n=0}^{\infty}\frac{1+\cos(b^n\pi x_{m+1})}{1+x_{m+1}}$$

is nonnegative and $x_{m+1} \in (-\frac{1}{2}\,,\frac{1}{2}]$. Then we can find the lower bound of this series

$$\sum_{n=0}^{\infty}a^n\frac{1+\cos(b^n\pi x_{m+1})}{1+x_{m+1}} \geq \frac{1+\cos(\pi x_{m+1})}{1+x_{m+1}} \geq \frac{1}{1+\frac{1}{2}} = \frac{2}{3}\,. \qquad (4)$$

The inequalities in (3) and (4) imply the existence of ε_1 and γ_1 with $\gamma_1 > 1$ and $-1 \leq \varepsilon_1 \leq 1$ such that

$$\frac{W(y_m) - W(x_0)}{y_m - x_0} = (-1)^{\beta_m}(ab)^m\gamma_1\left(\frac{2}{3} + \varepsilon_1\frac{\pi}{ab-1}\right).$$

If we consider the right-hand difference quotient, then the process is much the same. We express the difference quotient as a two partial sums, shown as

$$\frac{W(z_m) - W(x_0)}{z_m - x_0} = S_1{}' + S_2{}'$$

Similar to what we showed before, it can be deduce that

$$|S_1{}'| \leq \frac{\pi}{ab-1}(ab)^m. \qquad (5)$$

Considering the cosine term containing $S_2{}'$, we arrive at, because b is odd integer and $\beta_m \in \mathbb{Z}$,

$$\cos(b^{m+n}\pi z_m) = \cos\left(b^{m+n}\pi\left(\frac{\beta_m + 1}{b^m}\right)\right)$$

$$= \cos(b^n\pi(\beta_m + 1))$$

$$= [(-1)^{b^n}]^{\beta_m+1} = -(-1)^{\beta_m}$$

$$\text{and } S'_2 = \sum_{n=0}^{\infty}\left(a^{m+n}\frac{\cos(b^{m+n}\pi z_m - \cos(b^{m+n}\pi x_0}{z_m - x_0}\right)$$

$$= \sum_{n=0}^{\infty}\left(a^{m+n}\frac{-(-1)^{\beta_m} - (-1)^{\beta_m}\cos(b^n\pi x_{m+1})}{\frac{1 - x_{m+1}}{b^m}}\right)$$

$$= -(-1)^{\beta_m}(ab)^m\sum_{n=0}^{\infty}\frac{1 + \cos(b^n\pi x_{m+1})}{1 - x_{m+1}}.$$

Similar to above, we can find a lower bound for the series

$$\sum_{n=0}^{\infty}a^n\frac{1 + \cos(b^n\pi x_{m+1})}{1 - x_{m+1}} \geq \frac{1 + \cos(\pi x_{m+1})}{1 - x_{m+1}} \geq \frac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{2}{3}. \qquad (6)$$

Hence, as before there exist $\varepsilon_2 \in [-1,1]$ and $\gamma_2 > 1$ such that

$$\frac{W(z_m) - W(x_0)}{z_m - x_0} = -(-1)^{\beta_m}(ab)^m\gamma_2\left(\frac{2}{3} + \varepsilon_2\frac{\pi}{ab-1}\right).$$

Since $ab > 1 + \frac{3}{2}\pi$, which is equivalent to $\frac{\pi}{ab-1} < \frac{3}{2}$, the left-hand in (4) and right-hand in (6) difference quotients have differet signs. Also, since

$$\lim_{m\to\infty} (ab)^m = \infty \text{ as } m \to 0,$$

we see that the Weierstrass function $W(x)$ has no derivative at x_0. Since x_0 was arbitrary real number, then $W(x)$ is nowhere differentiable [10], [9] ,[11], [12].

2.4 Takagi function

In 1903 Takagi discovered a continuous nowhere differentiable function that is simpler than Weierstrass function. The Takagi function T is defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{\emptyset(2^n x)}{2^n}$$

$$= \emptyset(x) + \frac{1}{2}\emptyset(2x) + \frac{1}{4}\emptyset(4x) + \frac{1}{8}\emptyset(8x) + \cdots,$$

where $\emptyset(x) = \text{dist}(x, \mathbb{Z})$ the distance from x to nearest integer the following is the graph of Takagi function [6].

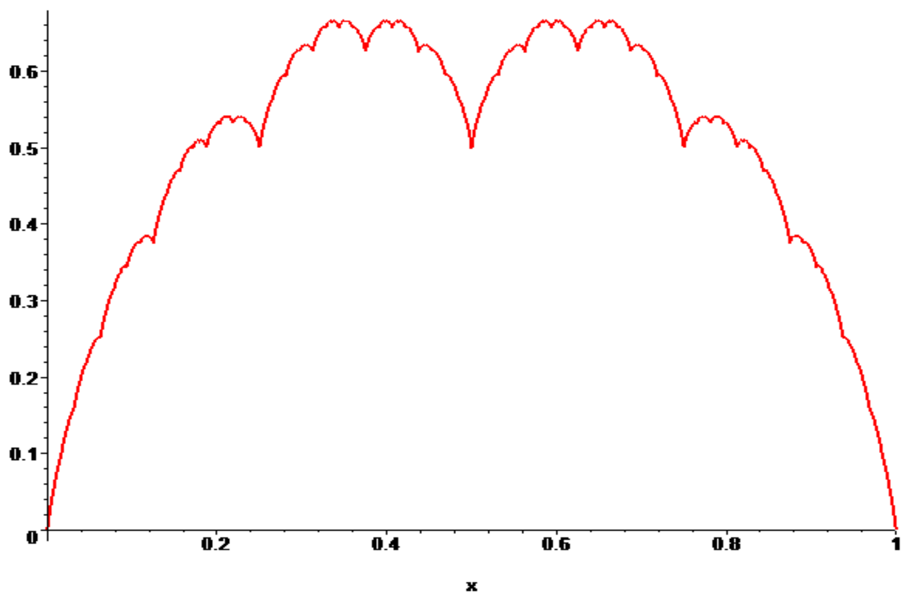

Figure 2-2: Takagi function

In order to show that Takagi function is nowhere differentiable we use the following lemma.

Lemma 2.4.1 Let f defined on the open interval (a, b) and differentiable at the point $c \in (a, b)$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in (a, b) converging to c such that $x_n \leq c \leq y_n$ and $x_n \leq y_n$, for $n = 0,1,2,3, \ldots$ [6].

Then

$$f'(c) = \lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} \ .$$

Proof We must prove that $\forall \varepsilon > 0$, there exist an integer N such that

$$\left| \frac{f(y_n) - f(x_n)}{y_n - x_n} - f'(x) \right| < \varepsilon, \text{ for all } n > N. \qquad (1)$$

Since f is differentiable at c, then for each positive number ε there exist a positive number δ such that

$$\left| \frac{f(y_n) - f(x_n)}{y_n - x_n} - f'(x) \right| < \frac{1}{4}\varepsilon \ , \text{ for all } x \text{ satisfying } 0 < |x - c| < \delta \ ,$$

so that

$$|f(x) - f(c) - f'(x)(x - c)| \leq \frac{1}{4}\varepsilon \, |x - c|, \text{for all } x \text{ satisfying } 0 < |x - c| < \delta.$$

Since $x_n \to \infty$ and $y_n \to \infty$ as n $\to \infty$, it follows that there are numbers N_1 *and* N_2 such that

$$|x_n - c| < \delta \ \text{ for all } n \geq N_1, \text{ and } |y_n - c| < \delta \text{ for all } n > N_2 \ ,$$

so if we choose $N = \max\{ N_1, N_2 \}$, then we have

$$|x_n - c| < \delta \ \text{ and } |y_n - c| < \delta \ \text{ for all } n > N \ .$$

From (1) that, $\forall\, n > N$, we have

$$|f(x_n) - f(c) - f'(c)(x_n - c)| \leq \frac{1}{4}\varepsilon \, |x_n - c|$$

and

$$|f(y_n) - f(c) - f'(c)(y_n - c)| \le \frac{1}{4}\varepsilon\,|y_n - c|\,.$$

By using the triangle inequality we obtain

$$\begin{aligned}
&|f(y_n) - f(c) - f'(c)(y_n - c)| \\
&= |\{f(y_n) - f(c) - f'(c)(y_n - c)\} - \{f(x_n) - f(c) - f'(c)(x_n - c)\}| \\
&\le |f(y_n) - f(c) - f'(c)(y_n - c)| + |f(x_n) - f(c) - f'(c)(x_n - c)| \\
&\le \frac{1}{4}\varepsilon\,|y_n - c| + \frac{1}{4}\varepsilon\,|x_n - c| \\
&\le \frac{1}{4}\varepsilon\,|y_n - x_n| + \frac{1}{4}\varepsilon\,|y_n - x_n| \\
&= \frac{1}{2}\varepsilon\,|x_n - y_n|, \text{ for all } n > N\,.
\end{aligned}$$

Since $x_n \neq y_n$ divided this inequality by non-zero term $x_n - y_n$ then for $n > N$ we get

$$\left|\frac{f(y_n) - f(x_n)}{y_n - x_n} - f'(x)\right| < \frac{1}{2}\varepsilon < \varepsilon\,,$$

then

$$f'(c) = \lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{y_n - x_n}\,.$$

Theorem 2.4.2 The Takagi function is continuous but nowhere differentiable.

Proof: First, demonstrate that $T(x)$ is continuous by using Weierstrass M-test and definition of uniform convergence. Let $M_n = \frac{1}{2^n}$, n is positive integer. Then

$$\left|\frac{\emptyset(2^n x)}{2^n}\right| \le \frac{1}{2^n}\,,$$

by geometric sereis test

$$\sum_{n=1}^{\infty} \frac{1}{2^n} < \infty\,.$$

Then $T(x)$ is uniformly convergent. This implies that $T(x)$ is uniformly continuous [13] .

Show that Takagi function is nowhere differentiable. Let x be a fixed arbitrary real number, and for each $n \in \mathbb{N}$, let v_n and u_n be any two sequences such that $u_n = \frac{i}{2^n}$ and $v_n = \frac{i+1}{2^n}$, where $i \in \mathbb{N}$. By contrary, assume that $T'(x)$ exists. Then by lemma (2.4.1) $u_n \leq x \leq v_n$ where $u_n < v_n$ and $v_n - u_n = \frac{i+1}{2^n} - \frac{i}{2^n} = 2^{-n} \to 0$. Then

$$\frac{T(v_n) - T(u_n)}{v_n - u_n} \to T'(x).$$

$$\frac{T(v_n) - T(u_n)}{v_n - u_n} = \sum_{k=0}^{n-1} \frac{\emptyset_k(v_n) - \emptyset_k(u_n)}{v_n - u_n},$$

where $\emptyset_k(x) = 2^{-n}\emptyset(2^k x)$ for $k = 0,1,2, \ldots.$ In the case $k \geq n$, we have $\emptyset(u_n) = \emptyset(v_n) = 0$. But in case $k < n$, $\emptyset_k$ is linear on $[u_n, v_n]$ with the slope ${\emptyset_k}^+(x)$, which is the right-hand derivative of $\emptyset_k$ at x. Thus,

$$\frac{T(v_n) - T(u_n)}{v_n - u_n} = \sum_{k=0}^{n-1} {\emptyset_k}^+(x).$$

Since ${\emptyset_k}^+(x) = \pm 1$ then as $n \to \infty$ the series does not converge form the right. This is contradiction with the assumption that $T'(x)$ exists. Therefore the Takagi function is nowhere differentiable [6] , [12] , [13].

2.5 Van der Waerden function

Another example of continuous nowhere differentiable function is the Van der Waerden. The construction of Takagi and Van der Waerden functions are very similar. Van der Waerden published his function in 1930. This function is defined by

$$V(x) = \sum_{n=0}^{\infty} \frac{1}{10^n} \operatorname{dist}(10^n x, \mathbb{Z}) = \sum_{n=0}^{\infty} \frac{1}{10^n} \inf_{m \in \mathbb{Z}} |10^n x - m|$$

Where $\inf_{m\in\mathbb{Z}}|10^n - m|$ denotes the distance from x to nearest integer [7]. The following is the graph Van der Waerden function.

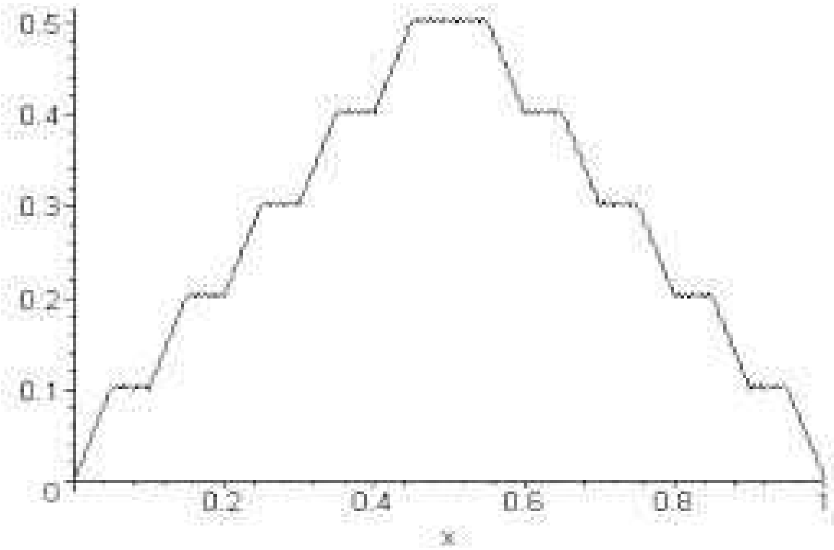

Figure 2-3: Van der Waerden function

Theorem 2.5.1 The Van der Waerden function is continuous but nowhere differentiable.

Proof: First, we prove that it is continuous. This follows from that fact that the infinite sum of continuous functions, which converges uniformly, is itself continuous. For this we use the Weierstrass M-test we have

$$V_n(x) = \frac{1}{10^n}\,\mathrm{dist}(10^n x, \mathbb{Z}),$$

then

$$|V_n(x)| \leq \frac{1}{10^n} \text{ for all } x \in \mathbb{R}\,.$$

Since $$\sum_{n=0}^{\infty} \frac{1}{10^n} \text{ converges,}$$

and $$\sum_{n=0}^{\infty} V_n \text{ converges uniformly,}$$

then

$$V(x) = \sum_{n=0}^{\infty} V_n = \sum_{n=0}^{\infty} \frac{1}{10^n} \operatorname{dist}(10^n x, \mathbb{Z}).$$

So, the Van der Waerden function is continuous.

We now prove that for all x, $V(x)$ is not differentiable at x. To show that $V(x)$ is not differentiable, we will construct a sequence $s_m \to 0$ such that

$$\lim_{m\to\infty} \frac{T(x+s_m) - T(x)}{s_m},$$

does not exist. Consider $0 \le x \le 1$ write x in decimal expansion $x = 0.a_1a_2 \dots a_n \dots$. Let

$$s_m = \begin{cases} -10^{-m} & \text{if } m = 4 \text{ or } 9\,, \\ 10^{-m} & \text{otherwise}\,. \end{cases}$$

Note that as $m \to \infty$, $s_m \to 0$ and $x + s_m \to x$. We calculate

$$\frac{T(x+s_m) - T(x)}{s_m} = \sum_{n=0}^{\infty} \frac{1}{10^n} \frac{\operatorname{dist}(10^n(x+s_m), \mathbb{Z}) - \operatorname{dist}(10^n x, \mathbb{Z})}{\mp 10^{-m}},$$

$$= \sum_{n=0}^{\infty} \pm 10^{n-m} (\operatorname{dist}(10^n(x+s_m), \mathbb{Z}) - \operatorname{dist}(10^n x, \mathbb{Z})).$$

This infinite series has actually no finite sum. Consider two cases. First for $n \ge m$ the terms in this sum are equal to zero, because

$$\operatorname{dist}(10^n(x+s_m), \mathbb{Z}) - \operatorname{dist}(10^n x, \mathbb{Z})$$

$$= \operatorname{dist}(10^n x, \mathbb{Z}) + \operatorname{dist}(10^n s_m, \mathbb{Z}) - \operatorname{dist}(10^n x, \mathbb{Z}) = 0\,.$$

On the other hand, in the second case when $n < m$, we can write

$$10^n x = M + 0.a_{n+1}a_{n+2}a_{n+3} \dots a_m \dots$$

$$10^n(x+s_m) = M + 0.a_{n+1}a_{n+2}a_{n+3} \dots (a_{m\pm 1}) \dots.$$

where $M \in \mathbb{Z}$. Suppose now that

$$0.a_{n+1}a_{n+2}a_{n+3} \dots a_m \dots \le \frac{1}{2}.$$

Then we also have

$$0.a_{n+1}a_{n+2}a_{n+3}\dots(a_{m\pm1})\dots\leq\frac{1}{2}.$$

This means that

$$\mathrm{dist}(10^n(x+s_m),\mathbb{Z})-\mathrm{dist}(10^n x,\mathbb{Z})=\pm10^{n-m}.$$

The same equation can be derived when $0.a_{n+1}a_{n+2}a_{n+3}\dots>\frac{1}{2}$. Therefore , for $n<m$ we have

$$10^{m-n}\mathrm{dist}(10^n(x+s_m),\mathbb{Z})-\mathrm{dist}(10^n x,\mathbb{Z})=\pm1\,.$$

In other word

$$\frac{T(x+s_m)-T(x)}{s_m}=\sum_{n=0}^{\infty}\pm1$$

does not exists. Therefore $V(x)$ is nowhere differentiable [7] , [14].

2.6 Baire category theorem

The Baire category classifies the points of a given metric space to be the typical and non-typical. To explain the issue give the following definitions.

2.6.1 A metric space is a pair (X,d), where X is a nonempty set and d is a metric on X, that is a function defined on $X\times X$ such that for all $x,y,z\in X$ we have [15]:

(i) d is real-valued, nonnegative ,

(ii) $d(x,y)=0$ *if and only if* $x=y$,

(iii) $d(x,y)=d(y,x)$,

(iv) $d(x,y)\leq d(x,z)+d(z,y)$.

2.6.2 The sequence $\{x_n\}$ is Cauchy sequence in a metric space (X,d) if $\forall\,\varepsilon>0$, $\exists N\in\mathbb{N}$ such that $\forall\,n,m>N$, $d(x_n,x_m)<\varepsilon$. The metric space (X,d) is said to be complete if every Cauchy sequence $\{x_n\}$ in X is convergent, that is there is $x_0\in x$ such that $d(x_n,x_0)\to0$ [3].

2.6.3 Let X be a matric space and $A \subset X$. We say that A is dense in X if $\overline{A} = X$, where $\overline{A}$ is closure of A.

2.6.4 A is said to be nowhere dense in X if every neighborhood S in A contains another neighborhood S' such that $S' \cap A = \emptyset$ [15].

2.6.5 We say that the set A is of the first category if it is a union of countable number of nowhere dense sets in X. It is said to be the second category if it is not of the first category [15].

Theorem 2.6.6 Baire category theorem Let X be a complete metric space and let $A \subseteq X$ be a set of the first category in X. Then A^c is dense in X.

Proof: Let V be arbitrary neighborhood in X. It suffices to prove that $A^c \cap V \neq \emptyset$. For this let

$$A = \bigcup_{n=1}^{\infty} A_n ,$$

where every A_n is nowhere dense in X. Construct a nested sequence of closed balls $\{B_n\}$ in the following way. Let T_1 be any neighborhood of V and radius less than one. It contains a neighborhood V_1 such that

$$V_1 \cap A_1 = \emptyset ,$$

since A_1 is nowhere dense in X. Then we take a closed ball B_1 contained V_1. After that we consider any neighborhood T_2 in B_1 of radius less than $\frac{1}{2}$. Since A_2 is nowhere dense in X, T_2 contains a neighborhood V_2 such that $V_2 \cap A_2 = \emptyset$. Take a closed ball B_2 contained in V_2. Continuing in this way we construct nested sequence of closed ball $\{B_n\}$. Hence there exists

$$p \in \bigcap_{n=1}^{\infty} B_n \subseteq V.$$

Additionally, from $B_n \cap A_n = \emptyset$ we get $p \notin A_n$ for every positive integer n. Hence,

$$p \notin \bigcup_{n=1}^{\infty} A_n = A\,,$$

that is $p \in A^c$. We conclude that $p \in A^c \cup V$ [15] , [16].

According to this theorem the points of a set of the first category are non-typical while the points of the set of the second category are typical. Application of this theorem to ℝ shows that ℚ is a set of the first category while the set II of irrational numbers is of the second category. Respectively, irrational numbers are typical points of ℝ, while rational numbers are non-typical.

Banach, Mazurkiewicz theorem considers this issue for the space $C(a, b)$ of continuous functions on $[a, b]$.

Theorem 2.6.7 (Banach-Mazurkiewicz) The set of all continuous nowhere differentiable function on $[a, b]$ is of the second category in $C(a, b)$, where $C(a, b)$ is the space of continuous functions [12].

This theorem shows that the main body of $C(a, b)$ consists of continuous nowhere differentiable functions. Differentiable functions form a small part of $C(a, b)$.

Chapter 3

SINGULAR FUNCTIONS

3.1 Introduction

By fundamental theorem of calculus

$$\int_a^b f'(x)dx = f(b) - f(a)$$

if f continuously differentiable. More generally, this equality holds if f has at most countable number of discontinuities. In the case when the number of discontinuities are uncountable then this equality does not hold. The example of such singular function is Lebesgue-Cantor function. Definition of Lebesgue-Cantor function is based on Cantor ternary set, or shortly Cantor set. Therefore, in this chapter we first consider Cantor set, derive properties of this set, and then define Lebesgue-Cantor function.

3.2 Cantor set

The Cantor set is a very interesting set, constructed by Georg Cantor in 1883. It is simply a subset of the interval $[0,1]$ which is defined in the following way. Let $C_0 = [0,1]$. Define C_1 to be the set that results when the open middle third is removed; that is

$$C_1 = C_0 \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Using the way of definition of C_1, construct C_2 by removing the open middle third of each of the components of C_1:

$$C_2 = \left(\left[0,\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{1}{3}\right]\right) \bigcup \left(\left[\frac{2}{3},\frac{7}{9}\right] \cup \left[\frac{8}{9},1\right]\right).$$

Continuing in this way n times we obtain a set C_n which contains 2^n closed intervals each having length $\frac{1}{3^n}$. Finally, we define the Cantor set C to be the intersection of all C_n:

$$C = \bigcap_{n=0}^{\infty} C_n.$$

In other words C is the remainder of the interval $[0,1]$ after the iterative process of removing open middle thirds taken to infinity:

$$C = [0,1] \setminus \left[\left(\frac{1}{3},\frac{2}{3}\right) \cup \left(\frac{1}{9},\frac{2}{9}\right) \cup \dots\right].$$

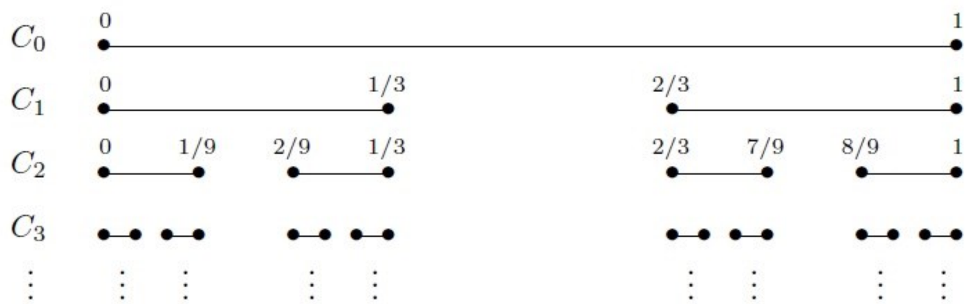

Figure 3-1: The Cantor set.

In fact $0 \in C$, since we are always removing open middle thirds, then for any $n \in \mathbb{N}$, $0 \in C_n$. Also $1 \in C$ in the same manner. Moreover, if x is an endpoint of some closed interval of some particular set C_n, then it will be an endpoint of one of the intervals of C_{n+1}. Since at each stage endpoints are never removed, then $x \in C_n$ for all n. Thus, C at least includes the endpoints of all of the intervals that construct each of the sets C_n. Sometimes, the Cantor set is called as Cantor ternary set, because the numbers from this set can be written as

$$\sum_{i=1}^{\infty} \frac{x_i}{3^i} \quad ,$$

where $x_i \in \{0, 2\}$ for each i. Generally every point $x \in [0, 1]$ has a ternary expansion of the form

$$\sum_{i=1}^{\infty} \frac{x_i}{3^i} \quad ,$$

where $x_i \in \{0,1,2\}$ for each i, and this expansion is unique for x except x has a finite expansion of the form

$$x = \sum_{i=1}^{n} \frac{x_i}{3^i} \ ,$$

where $x_n \in \{1,2\}$. In this case we let the expansion of x be as

$$x = \sum_{i=1}^{n} \frac{x_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{0}{3^i}$$

if $x_n = 2$, and

$$x = \sum_{i=1}^{n} \frac{x_i}{3^i} + \sum_{i=n+1}^{\infty} \frac{2}{3^i}$$

if $x_n = 1$. Then each $x \in [0,1]$ has a unique ternary expansion. In addition, the intervals that construct C_{n+1} are obtained by removing the middle thirds from the intervals that construct C_n, therefore,

$$C_n = \{x \in [0,1]: x = \sum_{i=1}^{\infty} \frac{x_i}{3^i} \ , \qquad x_i \in \{0,2\} \text{ for } 1 \le i \le n\}.$$

Hence,

$$x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} \in C$$

if and only if $x_n \in \{0,2\}$ for each positive integer n [10].

The Cantor set has properties:

1) The Cantor set is non empty.

2) The Cantor set is closed.

3) The Cantor set nowhere dense.

4) The Cantor set is compact.

5) The Cantor set is uncountable.

6) The Cantor set has measure zero.

Proof 1) Since $0 \in C$ then C is non empty.

Proof 2) Every C_n is a finite union of closed intervals. Since, all C_n are closed, C is closed because it is an intersection of closed sets C_n , $n = 1,2,3, \dots$

Proof 3) Every element of C is a limit point of a sequence of elements of the complement of C . This shows that every neighborhood of a point in C intersects with the complement, this means that there does not exist an open subset of C so $\operatorname{int} C = \emptyset$.

Proof 4) In part 2) it was proved that the Cantor set C is closed. It is also bounded since $C \subseteq [0,1]$. Then, C is compact since it is closed and bounded.

Proof 5 Assume the contrary, C is countable. Definitely, C is not finite. So , it should be denumerable set like $\{a_1, a_2, a_3, \dots\}$. We can write the ternary expansion of $a_1, a_2, a_3, \dots$ as follows

$$a_1 = 0.\,a_{11}a_{12}a_{13} \dots$$
$$a_2 = 0.\,a_{21}a_{22}a_{23} \dots$$
$$a_3 = 0.\,a_{31}a_{32}a_{33} \dots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

where $a_{ij} \in \{0,1\}$. Define a new number a which has ternary expansion

$$a = \alpha_1, \alpha_2, \alpha_3, \dots \ .$$

with $\alpha_i = 2$ if $a_{ii} = 0$ and $\alpha_i = 0$ if $a_{ii} = 2$. This number is obviously in C , but it is not inside in the set $\{a_1, a_2, a_3, \dots\}$. This contradiction proves that C is uncountable.

Proof 6) Note that the measure of the interval $[a,b]$ is $b-a$. If

$$[a,b] \cap [c,d] = \emptyset\,,$$

then the measure of $[a,b] \cup [c,d]$ is $(a-b)+(c-d)$. If $[c,d] \subseteq [a,b]$, then the measure of $[a,b] \setminus [c,d]$ is $(b-a)-(d-c)$. Apply thus measure operations to the Cantor set. Then the measure of $[0,a] \setminus C$ equals

$$1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{27} + \cdots + 2^n \cdot \frac{1}{3^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1-\frac{2}{3}} = 1\,.$$

So the measure of the Cantor set equals to $1-1=0$.

3.3 Lebesgue-Cantor function

A Lebesgue-Cantor function is an example of singular function for which the fundamental theorem of calculus does not hold. Define the Cantor function on the Cantor set C. By construction, $x \in C$ if and only if x has the ternary expansion

$$x = \frac{m_1}{3} + \frac{m_2}{9} + \frac{m_3}{27} + \cdots,$$

where $m_1, m_2, m_3, \ldots \in \{0,2\}$. Define the Lebesgue-Cantor function f on C by

$$f(x) = \frac{m_1/2}{2} + \frac{m_2/2}{4} + \frac{m_3/2}{8} + \cdots.$$

Clearly, $x_1, x_2 \in C$ and $x_1 < x_2$ imply that $f(x_1) < f(x_2)$. Moreover, $f(C) = [0,1]$ since every $y \in [0,1]$ which has the binary expansion

$$y = \frac{k_1}{2} + \frac{k_2}{2^2} + \frac{k_3}{2^3} + \cdots,$$

where $k_1, k_2, k_3, \ldots \in \{0,1\}$, corresponds to

$$x = \frac{2k_1}{3} + \frac{2k_2}{9} + \frac{2k_3}{27} + \cdots \in C\,.$$

Extend f to [0,1] in the following way. Let $I_{n,k}$, $n \in \mathbb{N}$, $k = 1, \dots, 2^{n-1}$ be the open intervals which are removed from [0,1] to obtain the Cantor set C. The left boundary point $c_{n,k} = \inf I_{n,k}$ of $I_{n,k}$ belongs to C. Let $f(x) = f(c_{n,k})$ if $x \in I_{n,k}$. Then f is defined on [0,1] so that f increases at points of C and is constant on each $I_{n,k}$. The following is the graph of Lebesgue-Cantor function.

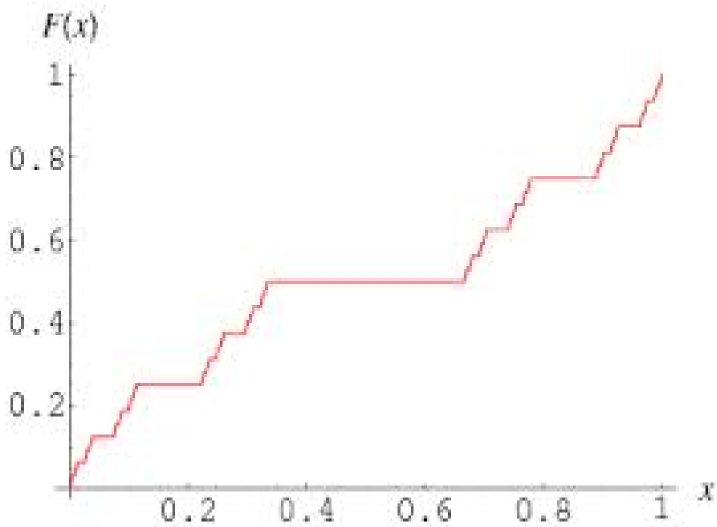

Figure 3-2: The Lebesgue-Cantor function

This extension of f is called the Lebesgue-Cantor function. By definition, this function is an increasing function from [0,1] to [0,1]. It has no jump discontinuity since its rang equal to [0,1]. Hence, the Cantor function is continuous. Let us show that the Lebesgue-Cantor function is non-differentiable at every point of the Cantor set C. Take any $c \in C$. Let $J_{n,k}$ be the closed intervals as defined above. Introduce the numbers $a_{n,k}$ and $b_{n,k}$ by letting $J_{n,k}$ =$[a_{n,k}, b_{n,k}]$. Then there exist a sequence $\{k_n\}$ such that

$$a_{n,k_n} \leq c \leq b_{n,k_n}.$$

By construction, $a_{n,k}$, $b_{n,k} \in C$ with

$$a_{n,k_n} = \frac{m_1}{3} + \cdots + \frac{m_n}{3^n} + \frac{0}{3^{n+1}} + \frac{0}{3^{n+2}} \cdots$$

and

$$b_{n,k_n} = \frac{m_1}{3} + \cdots + \frac{m_n}{3^n} + \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} \cdots \ ,$$

where $m_1, \dots, m_2 \in \{0,2\}$. We have

$$b_{n,k_n} - a_{n,k_n} = \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + \cdots$$

$$= \frac{2}{3^{n+1}} \sum_{n=0}^{\infty} \frac{1}{3^i} .$$

$$= \frac{1}{3^n} .$$

Then

$$b_{n,k_n} \ = a_{n,k_n} + \frac{1}{3^n} ,$$

and

$$f(b_{n,k_n}) - f(a_{n,k_n}) = \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots$$

$$= \frac{1}{2^{n+1}} \sum_{n=0}^{\infty} \frac{1}{2^i}$$

$$= \frac{1}{2^n}.$$

Then

$$f(b_{n,k_n}) \ = \ f(a_{n,k_n}) + \frac{1}{2^n} .$$

So

$$\lim_{n=\infty} \frac{f(b_{n,k_n}) - f(a_{n,k_n})}{b_{n,k_n} - a_{n,k_n}} = \frac{\frac{1}{2^n}}{\frac{1}{3^n}} = \infty .$$

If $c = a_{n,k_n}$ for some n, then $c = a_{n,k_n} = a_{n+1,k_{n+1}} = \cdots$ implying that the right derivative of f at c does not exist. If $c = b_{n,k_n}$ for some n, then $c = b_{n,k_n} = b_{n+1,k_{n+1}} = \cdots$, implying that the left derivative of f at c does not exist. If $a_{n,k_n} < c < b_{n,k_n}$ for every n, then from

$$\frac{f(b_{n,k_n}) - f(c)}{b_{n,k_n} - c} + \frac{f(c) - f(a_{n,k_n})}{c - a_{n,k_n}} \geq \frac{f(b_{n,k_n}) - f(a_{n,k_n})}{b_{n,k_n} - a_{n,k_n}},$$

we get that the derivative of f at c does not exist. Then f is non differentiable at every point of c of the Cantor set [4] , [17] .

Chapter 4

SPACE FILLING CURVE

4.1 Introduction

In 1878 the German mathematician George Cantor made a shocking discovery by finding a remarkable bijective function from [0,1] to [0,1] × [0,1]. But in 1879 Netto proved that the Cantor's map is not continuous. After Netto's result, some mathematicians began to look for continuous surjective mappings of this sort . In 1890 Giuseppe Peano found one, continuous function that maps the unit interval surjectively to the unit square. Such a map is called space filling curve. After this the other space filling curves were followed by Hilbert in 1891, H.Lebesgue in 1904, Sierpinski in 1912, K. Knopp in 1917… etc [17].

4.1.1 A space filling curve is a surjective continuous function from the unit interval onto $\mathbb{R}^d$ assuming that $d > 1$.

4.1.2 A function $f: [a, b] \to \mathbb{R}^d$ is continuous if all its components are continuous.

4.1.3: A function $f: [a, b] \to \mathbb{R}^d$ is differentiable if all its components are differentiable.

4.2 Peano function

The Peano curve or Peano function is maps $[0,1]$ onto $[0,1] \times [0,1]$. It is based on the ternary expansion of the real numbers. Let $t \in [0,1]$ has the ternary expansion $t = 0_3 t_1 t_2 t_3 \ldots$. This means that

$$t = \frac{t_1}{3} + \frac{t_2}{3^2} + \frac{t_3}{3^3} + \cdots,$$

where $t_1, t_2, t_3 \cdots \in \{0,1,2\}$. Then the Peano function $f(t)$ is defined as

$$f(t) = \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix} \qquad (1)$$

where $\varphi(t)$ and $\psi(t)$ are the first and the second components of $f(t)$ and they are defined as ternary expansions

$$\varphi(t) = 0_{\dot{3}} t_1 (k^{t_2} t_3)(k^{t_2+t_4}\, t_5) \dots, \qquad (2)$$

$$\psi(t) = 0_{\dot{3}} (k^{t_1} t_2)(k^{t_1+t_3} t_4) \cdots. \qquad (3)$$

Here, k is operator $\mathrm{k}_{t_j} = 2 - t_j$, for $t_j \in \{0,1,2\}$ and v_k is the v^{th} iterate of k.

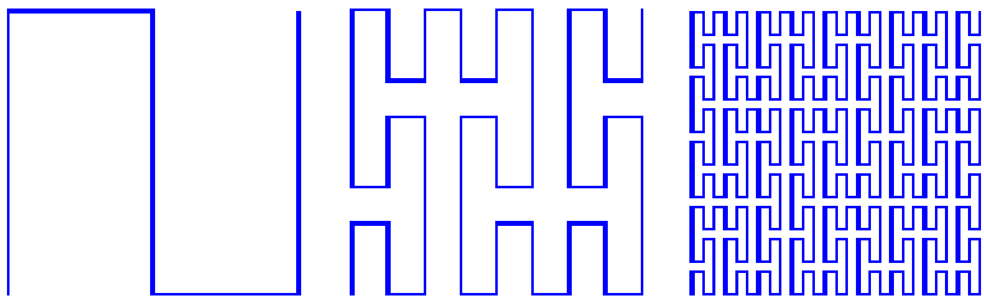

Figure 4-1: First three iteration of Peano curve

Theorem 4.2.1 The Peano function $f(t)$ is continuous but nowhere differentiable.

Proof. First, we prove that both components $\varphi(t)$ and $\psi(t)$ are continuous. Let as show that the first component φ of Peano function, defined by (2), is continuous from the right at all $t_0 \in [0,1)$.

Let $t_0 = 0_{\dot{3}} t_1 t_2 t_3 \dots t_{2n} t_{2n+1} \dots$ be the ternary representation of t_0 that does not have infinitely many trailing $2's$, and let $\delta = 3^{-2n} - 0_{\dot{3}}000 \dots t_{2n+1} t_{2n+2} \dots$. We have $t_0 + \delta = 0_{\dot{3}} t_1 t_2 t_3 \dots t_{2n} t_{2n+1} \dots + 3^{-2n} - 0_{\dot{3}}000 \dots t_{2n+1} t_{2n+2} \dots$

$$= 0_{\dot{3}} t_1 t_2 t_3 \dots t_{2n} \bar{2}\,.$$

So for any $t \in [\, t_0 \,, t_0 + \delta\,)$, the first $2n$ digits after the ternary point are equal

$$t = 0_{\dot{3}} t_1 t_2 t_3 \dots t_{2n} \tau_{2n+1} \tau_{2n+2} \dots.$$

Let $\epsilon = t_2 + t_4 + t_6 + \cdots + t_{2n}$.

We have

$$| \varphi(t) - \varphi(t_\circ)| = |0_3 t_1 (k^{t_2} t_3) \dots (k^{\epsilon} \tau_{2n+1}) \dots$$

$$- 0_3 t_1 (k^{t_2} t_3) \dots (k^{\epsilon} t_{2n+1}) \dots |$$

$$\leq \frac{|k^{\epsilon}\tau_{2n+1} - k^{\epsilon} t_{2n+1}|}{3^{n+1}} + \frac{|k^{\epsilon+\tau_{2n+2}} \tau_{2n+3 -} k^{\epsilon+t_{2n+2}} t_{2n+3}|}{3^{n+2}} + \cdots$$

$$\leq \left(\frac{2}{3^{n+1}}\right)\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\right) = \frac{1}{3^n} \to 0 \ as \ n \to \infty .$$

Hence φ is continuous from the right.

To show that φ continuous from the left in $(0,1]$, assume t_0 has the ternary representation $t_0 = 0_3 t_1 t_2 t_3 \dots t_{2n} t_{2n+1} \dots$

and let

$$\delta = 0_3 000 \dots 0 t_{2n+1} t_{2n+2} \dots .$$

Then

$$t_0 - \delta = 0_3 t_1 t_2 t_3 \dots t_{2n} 000 \dots$$

Hence, for $t \in (t_0 - \delta, t_0]$, which has a ternary representation with the same first $2n$ digits as t_0, we obtain

$$\left| \varphi_p (t) - \varphi_p (t_0) \right| = |0_3 t_1 (k^{t_2} t_3) \dots (k^{\epsilon} \tau_{2n+1}) \dots - 0_3 t_1 (k^{t_2} t_3) \dots (k^{\epsilon} t_{2n+1}) 00 \dots |$$

$$\leq \frac{2}{3^{n+1}}\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} \dots\right) = \frac{1}{3^n} \to 0 \ as \ n \to \infty .$$

So φ is continuous from the left in (0,1]. Then φ is continuous in [0,1]. The continuity of second component of $\psi(t)$ follows from

$$\psi(t) = 3\varphi\left(\frac{t}{3}\right).$$

Now let as show that φ is nowhere differentiable on [0,1]. For any $t = 0_3 t_1 t_2 t_3 \dots t_{2n} t_{2n+1} t_{2n+2} \dots \in [0,1]$, we define the sequence $\{t_n\}$ by $t_n = 0_3 t_1 t_2 t_3 \dots t_{2n} \tau_{2n+1} t_{2n+2} \dots$, where $\tau_{2n+1} = t_{2n+1} + 1 \ (mod\ 2)$. This implies that $|t - t_n| = \frac{1}{3^{2n+1}}$. By (2), $\varphi(t)$ and $\varphi(t_n)$ differ only at position $n+1$ in the ternary representations. So, we have

$$|\varphi(t) - \varphi(t_n)| = \frac{|k^{t_2+\cdots+t_{2n}} t_{2n+1} - k^{t_2+\cdots+t_{2n}} \tau_{2n+1}|}{3^{n+1}} = \frac{1}{3^{n+1}},$$

and hence ,

$$\frac{|\varphi(t) - \varphi(t_n)|}{|t - t_n|} = \frac{1}{3^{n+1}} \times \frac{3^{2n+1}}{1} = 3^n \to \infty \text{ as } n \to \infty.$$

Hence φ is not differentiable at t. Since $t \in [0,1]$ is arbitrary φ is nowhere differentiable on $[0,1]$. Since $\psi(t) = 3\varphi\left(\frac{t}{3}\right)$, $\psi(t)$ is also nowhere differentiable on [0,1]. This proves the theorem [5] , [18].

4.3 Hilbert's space filling curve

After Giuseppe Peano, in 1891 David Hilbert found another space filling curve . Let $I = [0,1]$ and $\tau = [0,1]^d = [0,1] \times [0,1] \times \cdots \times [0,1]$, where $d = 2,3,4,\cdots$ Hilbert divided I and τ into the same number of subsets and define the mapping between them. Its easy to proceed this in case $d = 2$. So we let $d = 2$.

Define mapping from $[0,1]$ to $[0,1] \times [0,1]$ as shown in Picture 4.2 (left). Then divide each subset into the same number of subsets define a mapping between them as in Picture 4.2 (medium). Continuing in this way for third and other iterations, in the unit square we obtain a curve which is called the Hilbert's curve [5].

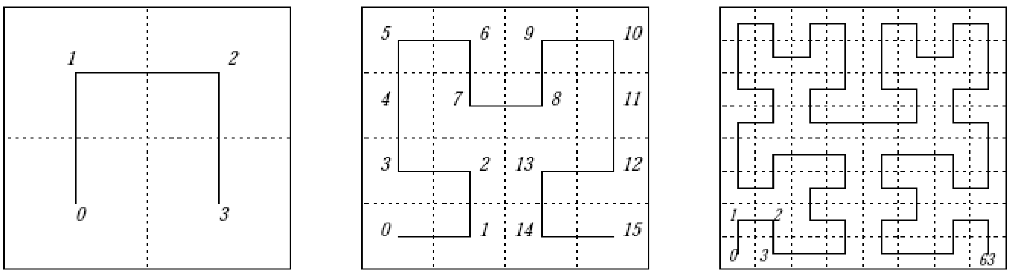

Figure 4-2: Hilbert curve

Theorem 4.3.1 The Hilbert's curve is continuous but nowhere differentiable.

Proof: First we show that the Hilbert curve is continuous. Since the curve at the n^{th} iteration is obtained by division of I in to 2^{2n} subintervals, the length of each subintervals is $\frac{1}{2^{2n}}$. therefore taking $t_1, t_2 \in [0,1]$ so that

$$| t_1 - t_2| < \frac{1}{2^{2n}},$$

we obtain

$$\|f_n(t_1) - f_n(t_2)\| \leq \frac{\sqrt{5}}{2^n}.$$

Thus, $f_n: [0,1] \to [0,1] \times [0,1]$ is continuous.

To show that the Hilbert function nowhere differentiable, let $n \geq 3$. Then for any $t \in [0,1]$, choose a $t_n \in [0,1]$ such that

$$|t - t_n| < \frac{10}{2^{2n}},$$

the components φ and ψ of the images of t and t_n are separated by at least a square side length of $\frac{1}{2^n}$. So

$$\left|\frac{\varphi(t) - \varphi(t_n)}{t - t_n}\right| \geq \frac{2^n}{10}.$$

This proves the theorem [5] .

4.4 Sierpiński curve

In 1921 Sierpiński introduced another example of space filling curve. The Sierpiński function is defined

$$\begin{cases} x = S(t), \\ y = S(t-1)/4. \end{cases} \qquad 0 \leq t \leq 1,$$

where $S(t)$ is continuous bounded even function

$$S(t) = \frac{\Phi(t)}{2} - \frac{\Phi(t)(\tau_1(t))}{4} + \frac{\Phi(t)\Phi(\tau_1(t))\Phi(\tau_2(t))}{8} - \cdots ,$$

where both $\Phi,\ \tau$ are periodic functions with the period 1 and defined by

$$\Phi(t) = \begin{cases} -1 & \text{if} \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right) , \\ 1 & \text{if} \quad t \in \left[0, \frac{1}{4}\right) \cup \left[\frac{3}{4}, 1\right) , \end{cases}$$

$$\tau_k(t) = \begin{cases} \frac{1}{8+4t} & \text{if} \quad t \in \left[0, \frac{1}{4}\right) \cup \left[\frac{1}{2}, \frac{3}{4}\right) , \\ \frac{1}{8-4t} & \text{if} \quad t \in \left[\frac{1}{4}, \frac{1}{2}\right) \cup \left[\frac{3}{4}, 1\right) , \end{cases}$$

where $\tau_{k+1}(t) = \tau_k(\tau_1(t))$, for every $k \in \mathbb{N}$. The following is the graph of the second, third and fourth iterations of Sierpiński curve [5].

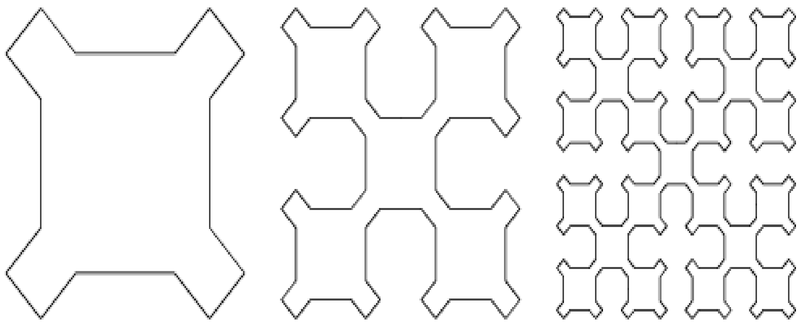

Figure 4-3: Sierpiński curve

4.5 Schoenberg curve

Another example of space filling curve is Schoenberg function. Schoenberg published this example in 1938. Schoenberg showed that his function is continuous space filling curve. After many years in 1981, J.Alsina proved that it is nowhere differentiable.

Define the first and second components of Schonberg function as

$$f(x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{2^k} p(3^{2k} x) ,$$

$$g(x) = \frac{1}{2}\sum_{k=o}^{\infty} \frac{1}{2^k} p(\,3^{2k+1}x\,),$$

where

$$p(x) = \begin{cases} 0 & \text{for} \quad 0 \le x \le 1/3\ , \\ 3x - 1 & \text{for} \quad 1/3 \le x \le 2/3\ , \\ 1 & \text{for} \quad 2/3 \le\ x \le 4/3\, , \\ 5 - 3x & \text{for} \quad 4/3 \le x \le 5/3\ , \\ 0 & \text{for} \quad 5/3 \le x \le 2\, , \end{cases}$$

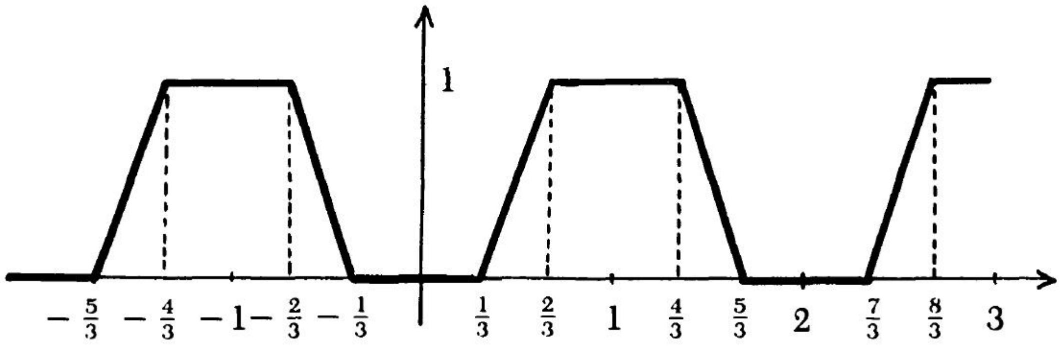

Figure 4-4: Schoenberg’s p function.

It is seen that $p(x)$ is even and periodic function with the period 2.

Theorem 4.5.1: The Schoenberg function is continuous but nowhere differentiable.

Proof: First, we show that f and g are continuous. We know

$$\sup\left|\frac{1}{2^k}p(x)\right| \le \frac{1}{2^k}.$$

For any $k \in \mathbb{N}$ we also note that

$$\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k < \infty\,.$$

By Weierstrass M-test $f(x)$ and $g(x)$ are both uniformly converges since a uniformly convergent series of continuous functions represents a continuous function. Then $f(x)$ and $g(x)$ are continuous. To show that both are nowhere differentiable on (0 , 1), we distinguish three cases

(i) First let $t = 0$ choose $b_n = \frac{1}{9^n}$ and consider $f(0) = 0$. Then

$$f(1/_{9^n}) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(\,9^{k-n}) \,.$$

Since

$$p(9^{k-n}) = \begin{cases} 0 & \text{for } k < n \,, \\ 1 & \text{for } k \geq n \,, \end{cases}$$

we get

$$f\left(\frac{1}{9^n}\right) = \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k} = \frac{1}{2^n} \;.$$

Hence

$$\frac{f(b_n) - f(0)}{b_n} = \left(\frac{9}{2}\right)^n \;\to \infty.$$

which diverges to ∞ as $n \to \infty$. This means $f'(0)$ does not exist. As $g(t) = f(3t)$ we also have that $g'(0)$ does not exist.

(ii) Let $t = 1$. Then we see that

$$f(1) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k} = \frac{1}{2}\times 2 = 1.$$

Choose $a_n = 1 - \frac{1}{9^n}$, note that $a_n \to 1$ as $n \to \infty$. Then we have

$$f(a_n) = f\left(1 - \frac{1}{9^n}\right) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(\,9^k - 9^{k-n}) \;.$$

When $k < n$, $p(\,9^k - 9^{k-n}) = 0$, and when $k \geq n$, $p(\,9^k - 9^{k-n}) = 1$. Then we get

$$f(a_n) = \frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k} = \left(\frac{1}{2}\right)^n \,.$$

This gives us

$$\frac{f(1)-f(a_n)}{1-\ a_n}=\frac{1-\left(\frac{1}{2}\right)^n}{1-(1-\frac{1}{9^n})}=\frac{1-\left(\frac{1}{2}\right)^n}{\frac{1}{9^n}}\ =\ 9^n-\left(\frac{9}{2}\right)^n\ \to\infty,$$

as $n\to\infty$ which diverges to ∞. This means that $f'(1)$ does not exist. As $g(t)=f(3t)$ we also have $g'(1)$ does not exist.

(iii) Let $t\in(\,0\,,1\,)$. We can find for every such t two sequence $\{\,a_n\}\,,\{b_n\}$, that satisfy the requirement of the Lemma 2.4.1 so that the limit does not exist. Let $k_n=[9^n t]$, where $[x]$ denotes the integer part of x. And let $a_n=k_n 9^{-n}$, and $b_n=k_n 9^{-n}+9^{-n}$, it is easy to show that, for sufficiently large n, $a_n\le t\le b_n$ and that $a_n\to t\,,\ b_n\to t$. These satisfy the conditions of Lemma 2.4.1 so that the sequence k_n can have infinitely many even values or infinitely many odd values or both. If there are infinitely many even values, denote the subsequence of even integers again by k_n. Then we have

$$f(a_n)=\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(9^{k-n}k_n)\,,$$

$$f(b_n)=\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(9^{k-n}k_n+9^{k-n}).$$

Hence

$$f(b_n)-f(a_n)=\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(9^{k-n}k_n+9^{k-n})-\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^k}p(9^{k-n}k_n)$$

$$=\frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{2^k}\left(p(9^{k-n}k_n+9^{k-n})-p(9^{k-n}k_n)\right)$$

$$+\frac{1}{2}\sum_{k=n}^{\infty}\frac{1}{2^k}\left(p(9^{k-n}k_n+9^{k-n})-p(9^{k-n}k_n)\right)$$

$$=M_1+M_2\,.$$

For the first summation if $k < n$ then $9^{k-n} \leq \frac{1}{9}$ and from a definition of $P(x)$ we can get the lower bound

$$p(9^{k-n}k_n + 9^{k-n}) - p(9^{k-n}k_n) \geq -3 \times 9^{k-n}.$$

From this we can get a lower bound form M_1 by

$$M_1 \geq -\frac{3}{2}\sum_{k=0}^{n-1} \frac{1}{2^k} 9^{k-n}$$

$$= -\frac{3}{2 \times 9^n}\sum_{k=0}^{n-1} \left(\frac{9}{2}\right)^k$$

$$= -\frac{3}{2 \times 9^n}\left(\left(\frac{9}{2}\right)^n - 1\right).$$

For the second summation as $k \geq n$, $9^{k-n} \geq 1$ is odd. Recall that each k_n is even in this case so let $u_k = 9^{k-n}k_n$ and $v_k = 9^{k-n}k_n + 9^{k-n}$ and we know that u_k is even and v_k is odd (u_k is the product of an even and odd v_k is the sum of an even and odd integers). This means that $p(u_k) = 0$ and $p(v_k) = 1$ then

$$M_2 = \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k} \left(p(v_k) - p(u_k)\right)$$

$$= \frac{1}{2}\sum_{k=n}^{\infty} \frac{1}{2^k}(1 - 0) = \frac{1}{2^n}.$$

We note that $b_n - a_n = 9^{-n}$ and putting it all together we have

$$\frac{f(b_n) - f(a_n)}{b_n - a_n} = 9^n(M_1 + M_2)$$

$$\geq 9^n \left(\frac{1}{2^n} - \frac{3}{7 \times 9^n}\left(\left(\frac{9}{2}\right)^n - 1\right)\right)$$

$$= \frac{4}{7}\left(\frac{9}{2}\right)^n + \frac{3}{7} \to \infty \text{ as } n \to \infty.$$

Which is diverges to ∞ as $n \to \infty$. This means that $f'(t)$ does not exist. If there are infinitly many odd, then we can define k_n, a_n and b_n as before with the odd subsequence of k_n. We want to get an upper bound for $M_1 + M_2$ such as before. We estimate M_1 by

$$M_1 \leq \frac{3}{2}\sum_{k=0}^{n-1} \frac{1}{2^k}\, 9^{k-n} = \frac{3}{7 \times 9^n}\left(\left(\frac{9}{2}\right)^n - 1\right),$$

For M_2 as $k \geq n$ and k_n is odd, $u_k = 9^{k-n}k_n$ odd and $v_k = 9^{k-n}k_n + 9^{k-n}$ is even this means that $p(u_k) = 1$ and $p(v_k) = 0$, then we get

$$M_2 = \frac{1}{2}\sum_{k=0}^{n-1} \frac{1}{2^k}\left(p(v_k) - p(u_k)\right) = \frac{1}{2}\sum_{k=0}^{n-1} \frac{1}{2^k}\,(0-1) = -\frac{1}{2^n}\,.$$

Putting it all together we have

$$\begin{aligned} \frac{f(b_n) - f(a_n)}{b_n - a_n} &= 9^n(M_1 + M_2) \\ &\leq 9^n\left(\frac{3}{7 \times 9^n}\left(\left(\frac{9}{2}\right)^n - 1\right) - \frac{1}{2^n}\right) \\ &= -\frac{4}{7}\left(\frac{9}{2}\right)^n - \frac{3}{7} \to \infty \text{ as } n \to \infty\,, \end{aligned}$$

which diverges to ∞ as $n \to \infty$. This means that $f'(t)$ does not exist. Then in both cases $f'(t)$ does not exist, since $t \in (0,1)$ was arbitrary, then $f(t)$ is nowhere differentiable on (0,1). And since $g(t) = f(3t)$ we obtain that $g(t)$ is nowhere differentiable on $(0,1)$ [5].

Example 4.5.4 A space filling curve is continuous function which does not have content zero. Let us demonstrate this in the example of Schoenberg function. Let

$$f(t) = \begin{cases} 0 & 0 \leq t \leq {}^1/_3, \\ 3t-1 & {}^1/_3 < t \leq {}^2/_3, \\ 1 & {}^2/_3 < t \leq 1. \end{cases}$$

Extend f to [-1,1] as an even function with the period 2 and let

$$g_1(t) = \sum_{n=1}^{\infty} \frac{f(3^{2(n-1)}t)}{2^n},$$

$$g_2(t) = \sum_{n=1}^{\infty} \frac{f(3^{2n-1}t)}{2^n}.$$

where $0 \le t \le 1$. Since g_1 and g_2 are well defined on [0,1] and both g_1 and g_2 are continuous we consider $g = (g_1, g_2) \in (0,1; \mathbb{R}^2)$, which describes the curve $C = \{ g_1(t), g_2(t)\}$ on $\mathbb{R}$. We confirm that $C = [0,1] \times [0,1]$. Indeed, we have

$$0 < \sum_{n=1}^{\infty} \frac{f(3^{2(n-1)}t)}{2^n} \le \sum_{n=1}^{\infty} \frac{1}{2^n} = 1, \qquad 0 \le t \le 1,$$

since, $0 \le f(t) \le 1,\ t \in \mathbb{R}$. Similarly, $0 \le g_2(t) \le 1,\ \ 0 \le t \le 1$. This implies the inclusion $C \subseteq [0,1] \times [0,1]$.

For the reverse inclusion $[0,1] \times [0,1] \subseteq C$, take any $(x_0, y_0) \in [0,1] \times [0,1]$ and write the binary expansion of $x_0$and y_0 as

$$x_0 = \sum_{n=1}^{\infty} \frac{a_n}{2^n},$$

and

$$y_0 = \sum_{n=1}^{\infty} \frac{b_n}{2^n},$$

where a_n and b_n are either 0 or 1. Then $x_0 = g_1(t_0)$ and $y_0 = g_2(t_0)$, if t_0 has the ternary expansion. Then

$$t_0 = \frac{2a_1}{3} + \frac{2b_1}{3^2} + \frac{2a_2}{3^3} + \frac{2b_2}{3^4} + \cdots,$$

since $0 \le t_0 \le 1$. Then we calculate that

$$3^{2(n-1)}t_0 = 3^{2n-2}\left(\frac{2a_1}{3} + \frac{2b_1}{3^2} + \cdots + \frac{2a_{n-1}}{3^{2n-3}} + \frac{2b_{n-1}}{3^{2n-2}} + \frac{2a_n}{3^{2n-1}} + \frac{2b_n}{3^{2n}} + \cdots\right)$$

$$= 2(3^{2n-3}a_1 + 3^{2n-4}b_1 + \cdots + 3a_{n-1} + b_n) + \frac{2a_n}{3} + \frac{2b_n}{3^2} + \cdots$$

Letting

$$\omega_n = \frac{2a_n}{3} + \frac{2b_n}{3^2} + \frac{2a_{n+1}}{3^3} + \frac{2b_{n-1}}{3^4} + \cdots,$$

we get that $f(3^{2(n-1)}t_0) = f(\omega_n)$ since f has period 2. If $a_n = 0$, then, $0 \leq \omega_n \leq 1/3$. This implies that $f(\omega_n) = 0$. Also if $a_n = 1$ then $1/3 \leq \omega_n \leq 1$. This implies that $f(\omega_n) = 1$. Also, $f(3^{2(n-1)}t_0) = f(\omega_n) = a_n$. This proves that

$$g_1(t_0) = \sum_{n=1}^{\infty} \frac{f(3^{2(n-1)}t_0)}{2^n} = \sum_{n=1}^{\infty} \frac{a_n}{2^n} = x_0.$$

In the same analogue, we can prove that $g_2(t_0) = y_0$. This $[0,1] \times [0,1] \subseteq C$ and, hence, $C = [0,1] \times [0,1]$, this implies that C does not have content zero [10].

Chapter 5

INFINITELY MANY TIMES DIFFERENTIABLE BUT NOT ANALYTIC FUNCTIONS

5.1 Introduction

Since the Taylor series of $f(x)$ includes all order derivative of $f(x)$, every analytic function has all order derivative. The converse of this statement does not hold: there are infinitely many times differentiable functions on the interval I which are not analytic. In this chapter we consider two such interesting functions.

5.1.1 Assume that $f(x)$ has all order derivatives at x_0. The series of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

or

$$f(x_0) + f(x_0)'(x-x_0) + \frac{f(x_0)''}{2!} + (x-x_0)^2 + \frac{f(x_0)^3}{3!}(x-x_0)^3 + \cdots$$

is called the Taylor series of $f(x)$ about x_0 [19].

Remark 5.1.2 Taylor series of $f(x)$ about x_0 always converges to $f(x)$ if $x = x_0$ but it may not converges to $f(x)$ for $x \neq x_0$.

5.1.3 If a Taylor series of $f(x)$ about x_0 converges to $f(x)$ for all x in same neighborhood of x_0, then f is said to be analytic at x_0. If $x_0 = 0$ then the Taylor series is called Maclaurin series [19] .

5.2 Analytic functions

According to definition, a function is analytic if it is equal to its Taylor series. How can we show that the function is analytic? For this, we split Taylor series into the sum of nth Taylor polynomial $P_n(x)$ and the remainder $R_n(x)$ as

$$\sum_{k=1}^{\infty} \frac{f^k(x_0)}{k!}(x-x_0)^k = P_n(x) + R_n(x).$$

Then $f(x)$ equal to its Taylor series about x_0 on the same neighborhood of x_0, if and only if $R_n(x) \to 0$ as $n \to \infty$ for all x in this neighborhood [20].

Theorem 5.2.1 (Taylor inequality) Let f has continuous $(n+1)$ order derivatives for $|x-x_0| < d$. If $|f^{n+1}(x)| \le M_n$ then the reminder term satisfies

$$|R_n(x)| \le \frac{M_n}{(n+1)!}|x-x_0|^{n+1}.$$

Proof: We consider the case $n = 1$. The higher values of n, the proof can be done by repeating the proof for the case $n = 1$ many times. By fundamental theorem of calculus

$$\begin{aligned} f(x) &= f(x_0) + \int_{x_0}^{x} f'(t)\,dt \\ &= f(x_0) + \int_{x_0}^{x}\left(f'(x_0) + \int_{x_0}^{t} f''(s)\,ds\right)dt \\ &= f(x_0) + f'(x_0)(x-x_0) + \int_{x_0}^{x}\int_{x_0}^{t} f''(s)\,dsdt. \end{aligned}$$

Here

$$P_1(x) = f(x_0) + f'(x_0)(x-x_0),$$

and

$$R_1(x) = \int_{x_0}^{x}\int_{x_0}^{t} f''(s)\,dsdt.$$

Therefore,

$$|R_1(x)| \le \int_{x_0}^{x}\int_{x_0}^{t} f''(s)\,dsdt$$

$$\leq M_1 \int_{x_0}^{x} \int_{x_0}^{t} ds dt = M_1 \int_{x_0}^{x} (t - x_0) dt$$

$$= M_1 \frac{(x - x_0)^2}{2!}.$$

This proves the theorem.

Example 5.2.2 The exponential function $f(x) = e^x$ is analytic function on $\mathbb{R}$. To prove we use Taylor inequality . We have

$$f^{(n+1)}(x) = e^x.$$

So,

For all $x \in \mathbb{R}$ with $|x - x_0| < d$.

We have

$$\left|f^{(n+1)}(x)\right| \leq e^{x_0+d}.$$

Then by Taylor's inequality

$$|R_n(x)| \leq \frac{e^{x_0+d}}{(n+1)!} |x - x_0|^{n+1}$$

$$\leq e^{x_0+d} \frac{|x - x_0|^{n+1}}{(n+1)!}.$$

This tends to 0 as $n \to \infty$. So for each $x \in \mathbb{R}$ with $|x - x_0| \leq d$,

$$e^x = \sum_{n=0}^{\infty} \frac{e^{x_0}}{n!} (x - x_0)^n .$$

This means that e^x is analytic function on $\mathbb{R}$.

Example 5.2.3 The function $f(x) = \sin x$ is analytic on $\mathbb{R}$. To prove note that $f^{n+1}(x)$ equals to one of the function $\sin x, -\sin x, \cos x, -\cos x$. Therefore,

$$|R_n(x)| \leq \frac{|x - x_0|^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty.$$

Thus $\sin x$ is analytic on $\mathbb{R}$. Its Taylor series about $x_0 = 0$ equal to.

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \ \frac{x^{2k+1}}{(2k+1)!}.$$

5.3 Elements of multiplicative differentiation.

To consider examples of infinitely many times differentiable but not analytic function we will use methods of multiplicative calculus. If $f(x)$ is a positive function, then its multiplicative derivative is defined by

$$f^*(x) = \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}}.$$

Comparing $f^*(x)$ with

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h},$$

we see that the difference $f(x+h)-f(x)$ is replaced by the ratio $f(x+h)\,/\,f(x)$ and the division by h is replaced by the raising to the reciprocal power $1/h$. As it follows from the above, the multiplicative derivative is denoted by $f^*(x)$. The multiplicative derivative of f^* is called the second multiplicative of f, denoted by f^{**}. In a similar way the nth multiplicative derivative of f can be defined. We use the notion $f^{*(n)},\ n = 1,2,3,\ldots$, where $f^{*(0)} = f$. If f is a positive function on E and the derivative of f at x exists, then one can calculate

$$\begin{aligned} f^*(x) &= \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}} \\ &= \lim_{h\to 0}\left(1+\frac{f(x+h)-f(x)}{f(x)}\right)^{\frac{f(x)}{f(x+h)-f(x)}\cdot\frac{f(x+h)-f(x)}{h}\cdot\frac{1}{f(x)}} \\ &= e^{\frac{f'(x)}{f(x)}} = e^{(\ln\circ f)'(x)}, \end{aligned}$$

where $(\ln\circ f)(x) = \ln f(x)$. If the second derivative of f at x exists, then by substitution we obtain

$$f^{**}(x) = e^{(\ln \circ f^*)'}(x) = e^{(\ln \circ f)''}(x)\,.$$

Repeating this n times, we conclude that if f is positive function and nth derivative of f at x exists, then

$$f^{*(n)}(x) = e^{(\ln \circ f)^{(n)}}(x), \qquad n = 0,1,2, \dots.$$

A formula similar to Newton's binomial formula can be derived to express $f^{(n)}(x)$ in terms of multiplicative derivatives:

$$f^{(n)}(x) = \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-k-1)!} f^{(k)}(x)(\ln\circ f^{*(n-k)})(x). \tag{5.3.1}$$

5.4 First example

The function

$$f(x)=\begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

is infinitely many times differentiable at $x_0 = 0$ but is not analytic at $x_0 = 0$. We claim that f is infinitely many times differentiable on $\mathbb{R}$. A verification of this statement is easy at every $x \neq 0$ but more difficult at $x = 0$. We use multiplicative calculus to prove that $f^n(0) = 0$ for every $n = 0,1, \dots$ For this, consider $x > 0$ and show that

$$f^{*(n)}(x) = e^{\frac{(-1)^{n+1}(n+1)!}{x^{n+2}}},\ n=0,\,1,\,2, \dots \tag{5.4.1}$$

This is true for $n = 0$ in the form $f^{*(0)}(x) = f(x) = e^{-\frac{1}{x^2}}$. Assume that its true for n and calculate for $(n+1)$:

$$f^{*(n+1)}(x) = \lim_{h\to 0} e^{\left(\frac{(-1)^{n+1}(n+1)!}{(x+h)^{n+2}} - \frac{(-1)^{n+1}(n+1)!}{x^{n+2}}\right)\cdot\frac{1}{h}}\,.$$

By binomial formula,

$$\ln f^{*(n+1)}(x) = \lim_{h\to 0}\left(\frac{(-1)^{n+2}(n+1)!}{(x+h)^{n+2}} - \frac{(-1)^{n+2}(n+1)!}{x^{n+2}}\right)\frac{1}{h}$$

$$= \lim_{h\to 0}\frac{(-1)^{n+2}(n+1)!\,((x+h)^{n+2} - x^{n+2})}{hx^{n+2}(x+h)^{n+2}}$$

$$= \lim_{h\to 0}\frac{(-1)^{n+2}(n+1)!\,((n+2)hx^{n+1} + \cdots + h^{n+2})}{hx^{n+2}(x+h)^{n+2}}$$

$$= \lim_{h\to 0}\frac{(-1)^{n+2}(n+1)!\,((n+2)x^{n+1} + \cdots + h^{n+1})}{x^{n+2}(x+h)^{n+2}}$$

$$= \frac{(-1)^{n+2}(n+2)!\,x^{n+1}}{x^{2n+4}}$$

$$= \frac{(-1)^{n+2}(n+2)!}{x^{n+3}}\ .$$

Hence,

$$f^{*(n+1)}(x) = e^{\frac{(-1)^{n+2}(n+2)!}{x^{n+3}}}\ .$$

By induction, (5.4.1) holds for every n=0, 1, …. Next, we use the formula (5.3.1) and obtain

$$f^{(n)}(x) = \sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}(n-1)!\,(n-k+1)!.\,f^{(k)}(x)}{k!\,(n-k-1)!.\,x^{n-k+2}}, \qquad n = 1, 2, \dots .$$

Multiple application of this formula yields

$$\frac{f^{(n)}(x)}{x} = f(x)\sum_{m=4}^{N_n}\frac{M_{n,m}}{x^m}, \qquad n = 1, 2, \dots ,$$

where $M_{n,m}$ is integer and N_n is positive integer $N_n \geq 4$. We need not the exact value of these integers. By multiply application, we have

$$\lim_{x\to 0+}\frac{f(x)}{x^m} = \lim_{x\to 0+}\frac{e^{\frac{-1}{x^2}}}{x^m} = \lim_{x\to 0+}\frac{\frac{1}{x^m}}{e^{\frac{1}{x^2}}} = \lim_{z\to\infty}\frac{z^{\frac{m}{2}}}{e^z} = 0.$$

Thus,

$$\lim_{x\to 0+}\frac{f^{(n)}(x)}{x}=0, \qquad n=0,1,\dots,$$

Implying $f^{(n+1)}(0+)=0$ whenever $f^{(n)}(0+)=0$. Since $f(0)=0$. By induction, we conclude that $f^{(n)}(0+)=0$ for every $n=0,1,\dots$. Since f in an even function, we easily deduce $f^{(n)}(0-)=0$ for every $n=0,1,\dots$ Thus $f^{(n)}(0)=0$ for every $n=0,1,\dots$. So, the Taylor series of $f(x)$ about 0 is

$$\sum_{n=0}^{\infty}\frac{f^{n}(0)}{n!}x^{n},$$

which converges for every $x\in\mathbb{R}$ and it is sum is the zero function while $f(x)=0$ only at $x=0$. In other words f is not analytic at $c=0$ and on every interval containing $c=0$ while it is infinitely many times differentiable on $\mathbb{R}$ [20].

5.5 Second example

The function

$$f(x)=\begin{cases} e^{-\frac{1}{x}} & \text{if } x\neq 0\,, \\ 0 & \text{if } x=0\,. \end{cases}$$

is infinitely many times differentiable at $x_0=0$ but is not analytic at $x_0=0$. To show that f is infinitely many times differentiable on $\mathbb{R}$ we use the same technique as in the previous example. Clearly $f(x)$ infinitely many times differentiable at $x\neq 0$. We claim that the same holds at $x=0$. Let as show that $f^{n}(0)=0$ for every $n=0,1,\dots$. For this, consider $x>0$ and show that

$$f^{*(n)}(x)=e^{\frac{(-1)^{n+1}(n)!}{x^{n+1}}},\ n=0,1,2,\dots \qquad (5.5.1)$$

This is true for $n=0$ in the form $f^{*(0)}(x)=f(x)=e^{-\frac{1}{x}}$. Assume that it is true for n and calculate for $(n+1)$:

$$f^{*(n+1)}(x)=\lim_{h\to 0}e^{\left(\frac{(-1)^{n+1}(n)!}{(x+h)^{n+1}}-\frac{(-1)^{n+1}(n)!}{x^{n+1}}\right)\cdot\frac{1}{h}}\ .$$

By binomial formula,

$$\ln f^{*(n+1)}(x) = \lim_{h\to 0}\left(\frac{(-1)^{n+1}(n)!}{(x+h)^{n+1}} - \frac{(-1)^{n+1}(n)!}{x^{n+1}}\right)\frac{1}{h}$$

$$= \lim_{h\to 0}\frac{(-1)^{n+2}(n)!\,((x+h)^{n+1} - x^{n+1})}{hx^{n+1}(x+h)^{n+1}}$$

$$= \lim_{h\to 0}\frac{(-1)^{n+2}(n)!\,((n+1)hx^{n} + \cdots + h^{n+1})}{hx^{n+1}(x+h)^{n+1}}$$

$$= \lim_{h\to 0}\frac{(-1)^{n+2}(n)!\,((n+1)x^{n} + \cdots + h^{n})}{x^{n+1}(x+h)^{n+1}}$$

$$= \frac{(-1)^{n+2}(n+1)!\,x^{n}}{x^{2n+2}}$$

$$= \frac{(-1)^{n+2}(n+1)!}{x^{n+2}}.$$

Hence,

$$f^{*(n+1)}(x) = e^{\frac{(-1)^{n+2}(n+1)!}{x^{n+2}}}.$$

By induction, (5.5.1) holds for every n=0, 1, ... Next, we use the formula (5.3.1) and obtain

$$f^{(n)}(x) = \sum_{k=0}^{n-1}\frac{(-1)^{n-k+1}(n-1)!\,(n-k)!.\,f^{(k)}(x)}{k!\,(n-k-1)!\,x^{n-k+1}}, n = 1, 2, \ldots.$$

Multiple application of this formula yields

$$\frac{f^{(n)}(x)}{x} = f(x)\sum_{m=4}^{N_n}\frac{M_{n,m}}{x^{m}}, \qquad n = 1, 2, \ldots,$$

where $M_{n,m}$ is an integer and N_n is positive integer $N_n \geq 4$. We need not the exact value of these integers. By multiply application, we have

$$\lim_{x\to 0+}\frac{f(x)}{x^{m}} = \lim_{x\to 0+}\frac{e^{\frac{-1}{x}}}{x^{m}} = \lim_{x\to 0+}\frac{\frac{1}{x^{m}}}{e^{\frac{1}{x}}} = \lim_{z\to\infty}\frac{z^{m}}{e^{z}} = 0.$$

Thus,

$$\lim_{x \to 0+} \frac{f^{(n)}(x)}{x} = 0, \qquad n = 0, 1, \dots,$$

implying $f^{(n+1)}(0+) = 0$ whenever $f^{(n)}(0+) = 0$. Since $f(0) = 0$. By induction, we conclude that $f^{(n)}(0+) = 0$ for every $n = 0, 1, \dots$. We easily deduce $f^{(n)}(0-) = 0$ for every $n = 0, 1, \dots$ Thus $f^{(n)}(0) = 0$ for every $n = 0, 1, \dots$. So, the Taylor series of $f(x)$ about 0 is

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

which converges for every $x \in \mathbb{R}$ and it is sum is the zero function while $f(x) = 0$ only at $x = 0$. In other words f is not analytic at $c = 0$ and on every interval containing $c = 0$ while it is infinitely many times differentiable on $\mathbb{R}$ [20].

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