Wave Propagation in an Inhomogeneous Matter

Zainab Sauod Muhmmed Alhmod

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Prof. Dr. ElvanYılmaz
Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Physics.

Prof. Dr. Mustafa Halilsoy
Chair, Department of Physics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis of the degree of Master of Science in Physics.

Assoc. Prof. S. Habib Mazharimousavi
Supervisor

Examining Committee

1. Prof. Dr. Özay Gurtug
2. Prof. Dr. Mustafa Halilsoy
3. Assoc. Prof. S. Habib Mazharimousavi

ABSTRACT

We study the waves spread in the media, non-linear and non-homogeneous is basic and widespread Problem in physics, which is the subject of our research. We derive the equation of Maxwell for the wave equation based on the z axis and y and find General solutions to the wave equation. We used some of the methods and mathematical processes for the wave equation to be as (hyper-geometric differential equation), and then to be easy to find general solutions to the two parts, the first as a coefficient of permittivity function of y, and the second as a constant.

Keywords: Electromagnetics, Permittivity, Permeability

ÖZ

Doğrusal ve düzgün (homojen) olmayan bir ortamda dalgaların yayılma problemi bu tezin özünü oluşturmaktadır. Burada z ve y koordinatlarına bağımlı genel Maxwell denklemleri çözülmüştür. Matematiksel olarak hiper-geometrik difransiyel denklemi elde edilmiş olup onun elektrik geçirgenlik (permitivite) katsayının y – bağımlı ve sabit durumları için çözümler bulunmuştur.

Anahtar kelimeler: Elektromagnetizma, Elektrik ve Magnetik geçirgenlik katsayıları

DEDICATION

To my parents, my brothers and sisters

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INTRODUCTION

Wave propagation in nonlinear and inhomogeneous media is a fundamental and wide-ranging problem in physics which have been the subject of intensive research. Maxwell in 1855 in his Electromagnetic theory presented a set of four differential equations which are considered to be in an infinitesimal volume of a medium, but containing a large number of atoms [1]. These equations are given by

$$\begin{aligned} \vec{\nabla}\cdot\vec{D} &= \rho \qquad & \vec{\nabla}\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} &= 0 \qquad & \vec{\nabla}\times\vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t}, \end{aligned} \tag{1.1}$$

Here $\vec{E}$ and $\vec{B}$ are the electric and magnetic field, $\vec{D}$ and $\vec{H}$ are the displacement vector and auxiliary magnetic field, ρ is the total electric charge density and $\vec{J}$ denotes the total electric current density of the medium [4]. Then we would like to comment that the first equation is the Gauss's law in vacuum, the second equation is differential form of the Ampere's law, the third equation is the Gauss's law in magnetism and Finally the last equation is called the faraday's law of induction [9].

Equations (1.1) are used to drive the wave propagation equations in terms of either the electric field or the magnetic field. This is shown below.

For the case of free sources one has to set $J_{free} = 0, \rho_{free} = 0$ and then equations (1.1) becomes

$$\vec{\nabla} \cdot \vec{D} = 0 \tag{1.2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.3}$$

$$\vec{\nabla} . \vec{B} = 0 \tag{1.4}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}. \tag{1.5}$$

From Eq. (1.3), applying the curl operator to both sides results in

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E} \right) = -\frac{\partial}{\partial t} \left(\vec{\nabla} \times \vec{B} \right), \tag{1.6}$$

which after using the identity $\vec{A} \times \left(\vec{B} \times \vec{C} \right) = \vec{B} \left(\vec{A} \cdot \vec{C} \right) - \vec{C} \left(\vec{A} \cdot \vec{B} \right)$ it becomes

$$\vec{\nabla} \left(\vec{\nabla} \cdot \vec{E} \right) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \left(\vec{\nabla} \times \mu \vec{H} \right), \tag{1.7}$$

in which μ is the permittivity of the medium which is constant.

Using, Eq. (1.5) one finds

$$0 - \nabla^2 \vec{E} = -\frac{\partial}{\partial t} \left(\mu \vec{\nabla} \times \frac{\partial \vec{D}}{\partial t} \right), \tag{1.8}$$

which upon $\vec{D} = \varepsilon \vec{E}$ in which ε is the permeability of the medium, one finds

$$\nabla^2 \vec{E} = \varepsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2}. \tag{1.9}$$

Here $\varepsilon\mu = \varepsilon_0\varepsilon_r\mu_0\mu_r = \varepsilon_0\mu_0\varepsilon_r\mu_r = \frac{\varepsilon_r\mu_r}{c^2}$

then $\nabla^2 \vec{E} - \frac{1}{v^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0.$

Here, we used $\frac{\varepsilon_r\mu_r}{c^2} = \frac{1}{v^2}$, in which v is the speed of light in the matter and

$\varepsilon_r\mu_r = n$ is called optical index.

We note that μ_0 and ε_0 are the permittivity and the permeability of the vacuum and also c is the speed of light in the vacuum. The permittivity and permeability have many applications in physics and other related fields like Electrical engineering, optics, and field theory [5] …. In this work we shall study the wave equation in an inhomogeneous medium where $\mu = \mu_0$ and the permittivity $\varepsilon = \varepsilon(\vec{r})$.

Chapter 2

WAVE EQUATION IN NONHOMOGENEOUS MEDIUM

2.1 Introduction

In this chapter, we shall consider both the electric current density $\vec{J}$ and electric charge density ρ to be zero. let's write Maxwell's equations,

as:

$$\vec{\nabla} \cdot \vec{D} = 0 \tag{2.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{2.3}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \tag{2.4}$$

Herein $\vec{D} = \varepsilon \vec{E}$ is the displacement vector, where $\vec{E}$ is the electric field, $\vec{H}$ is the auxiliary magnetic field which can be represented as $\vec{H} = \frac{\vec{B}}{\mu}$, and $\vec{B}$ is the magnetic field. Also, as we mentioned before, ε and μ are function of $\vec{r}$.

2.2 Maxwell's Equations in Inhomogeneous Medium

In electromagnetism Maxwell's equations are the basic equations which must be satisfied by any electric and magnetic field. These equations are differential equations and over all they couple electric and magnetic field to their sources such as electric charge density or electric current density.

Maxwell's equations can be expressed in either differential form or integral form and in addition one may use directly the vector fields of electric $\vec{E}$ or magnetic $\vec{B}$ or their electric potential ϕ and vector potential $\vec{A}$ As we have already shown, In this thesis we use the differential form of the Maxwell's equations and the vector form of the fields directly $[1,7]$.

Let's start again from Eq. (2.3) by applying the curl operator to the both side. This yields

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)=-\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right), \tag{2.5}$$

and upon using the identity $\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\vec{B}\left(\vec{A}\cdot\vec{C}\right)-\vec{C}\left(\vec{A}\cdot\vec{B}\right)$ the latter equation become

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right)-\nabla^2 E=-\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{B}\right). \tag{2.6}$$

Nevertheless Eq. (2.1) implies

$$\vec{\nabla}\cdot\vec{D}=0\Rightarrow\vec{\nabla}\cdot\left(\varepsilon\vec{E}\right)=0, \tag{2.7}$$

or after expansion, we find $\left(\vec{\nabla}\varepsilon\right)\cdot\vec{E}+\varepsilon\left(\vec{\nabla}\cdot\vec{E}\right)=0$

and finally

$$\vec{\nabla}\cdot\vec{E}=-\frac{\left(\vec{\nabla}\varepsilon\right)\cdot\vec{E}}{\varepsilon}. \tag{2.8}$$

In the other hand Eq. (2.4) upon considering $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$ admits,

$$\vec{\nabla}\left(\frac{\vec{\nabla}\varepsilon}{\varepsilon}\cdot\vec{E}\right)+\nabla^2\vec{E} = \frac{\partial}{\partial t}\vec{\nabla}\times\left(\mu\vec{H}\right). \tag{2.9}$$

One may use the identity

$$\vec{\nabla}\times\left(f\,\vec{F}\right)=\vec{\nabla}f\times F+f\,\vec{\nabla}\times\vec{F}, \tag{2.10}$$

and write

$$\vec{\nabla}\times\left(\mu\vec{H}\right)=\left(\vec{\nabla}\mu\right)\times\vec{H}+\mu\vec{\nabla}\times\vec{H}\,. \tag{2.11}$$

Imposing this latter equation in Eq. (2.9) we find

$$\vec{\nabla}\left(\frac{\vec{\nabla}\varepsilon}{\varepsilon}\cdot\vec{E}\right)+\nabla^2\vec{E} = \frac{\partial}{\partial t}\left[\left(\vec{\nabla}\mu\right)\times\vec{H}+\mu\vec{\nabla}\times\vec{H}\right], \tag{2.12}$$

or after some simplification

$$\vec{\nabla}\left(\frac{\vec{\nabla}\varepsilon}{\varepsilon}\cdot\vec{E}\right)+\nabla^2\vec{E} = \vec{\nabla}\mu\times\frac{\partial\vec{H}}{\partial t}+\mu\frac{\partial}{\partial t}\left(\frac{\partial\vec{D}}{\partial t}\right). \tag{2.13}$$

Using $\vec{H} = \dfrac{\vec{B}}{\mu}$ and $\vec{D} = \varepsilon\vec{E}$ one gets

$$\vec{\nabla}\left(\frac{\vec{\nabla}\varepsilon}{\varepsilon}\cdot\vec{E}\right)+\nabla^2\vec{E} = \vec{\nabla}\mu\times\frac{1}{\mu}\frac{\partial\vec{B}}{\partial t}+\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}, \tag{2.14}$$

which after Eq. (2.3) it may be written as

$$\vec{\nabla}\left(\frac{\vec{\nabla}\varepsilon}{\varepsilon}\cdot\vec{E}\right)+\nabla^2\vec{E} = -\left(\frac{\vec{\nabla}\mu}{\mu}\right)\times\left(\vec{\nabla}\times\vec{E}\right)+\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}. \tag{2.15}$$

Now, we define $\varepsilon = \ln \varepsilon$ and $\tilde{\mu} = \ln \mu$

then

$$\frac{\vec{\nabla}\varepsilon}{\varepsilon} = \vec{\nabla}\varepsilon \ \text{ and } \frac{\vec{\nabla}\mu}{\mu} = \vec{\nabla}\tilde{\mu}. \tag{2.16}$$

First, this simplifies Eq. (2.15) as follows.

$$\frac{\vec{\nabla}\mu}{\mu} \times \left(\vec{\nabla} \times \vec{E}\right) = \left(\vec{\nabla}\tilde{\mu}\right) \times \left(\vec{\nabla} \times \vec{E}\right), \tag{2.17}$$

then

$$\Rightarrow \vec{\nabla}\left(\vec{\nabla}\tilde{\mu} \cdot \vec{E}\right) - \left(\vec{\nabla}\tilde{\mu} \cdot \vec{\nabla}\right)\vec{E}, \tag{2.18}$$

and after a substitution in to Eq. (2.15), it become

$$\vec{\nabla}\left(\vec{\nabla}\varepsilon \cdot \vec{E}\right) + \nabla^2 \vec{E} \ = -\vec{\nabla}\left(\vec{\nabla}\tilde{\mu} \cdot \vec{E}\right) + \left(\vec{\nabla}\tilde{\mu} \cdot \vec{\nabla}\right)\vec{E} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}. \tag{2.19}$$

Let's rearrange this equation, to get

$$\vec{\nabla}\left[\left(\vec{\nabla}\varepsilon + \vec{\nabla}\tilde{\mu}\right) \cdot \vec{E}\right] + \nabla^2 \vec{E} = \left(\vec{\nabla}\tilde{\mu} \cdot \vec{\nabla}\right)\vec{\mathrm{E}} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}, \tag{2.20}$$

which after considering $\vec{M} = \vec{\nabla}\varepsilon$ and $\vec{S} = \vec{\nabla}\tilde{\mu}$ we find

$$\vec{\nabla}\left[\left(\vec{M} + \vec{S}\right) \cdot \vec{E}\right] + \nabla^2 \vec{E} \ = \left(\vec{S} \cdot \vec{\nabla}\right)\vec{E} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}. \tag{2.21}$$

Finally, the wave equation, in an inhomogeneous medium get the following form,

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \left(\vec{S} \cdot \vec{\nabla}\right)\vec{E} - \vec{\nabla}\left[\left(\vec{M} + \vec{S}\right) \cdot \vec{E}\right]. \tag{2.22}$$

With the knowledge that the permittivity and permeability are $\vec{r}$ -dependent, while $\vec{E}$ and $\vec{B}$ are function of time and space the latter is equivalent with the following three equations

$$\nabla^2 E_i - \mu\varepsilon \frac{\partial^2 E_i}{\partial t^2} = \left(\vec{S} \cdot \vec{\nabla}\right) E_i - \frac{\partial}{\partial x_i}\left[\left(\vec{M} + \vec{S}\right) \cdot \vec{E}\right], \quad \text{i=1, 2, 3.} \qquad (2.23)$$

Let's consider that the medium is uniform in x and z direction and the corresponding ε is only a function of y while $\mu = \mu_0$ is a constant, and $\vec{S}$ =0. The wave equation (2.23), then reads as

$$\nabla^2 E_1 - \mu_0 \varepsilon(y) \frac{\partial^2 E_1}{\partial t^2} = -\frac{\partial}{\partial x}\left(\vec{M} \cdot \vec{E}\right), \qquad (2.24)$$

$$\nabla^2 E_2 - \mu_0 \varepsilon(y) \frac{\partial^2 E_2}{\partial t^2} = -\frac{\partial}{\partial y}\left(\vec{M} \cdot \vec{E}\right), \qquad (2.25)$$

and

$$\nabla^2 E_3 - \mu_0 \varepsilon(y) \frac{\partial^2 E_3}{\partial t^2} = -\frac{\partial}{\partial z}\left(\vec{M} \cdot \vec{E}\right). \qquad (2.26)$$

Not that, since $\varepsilon = \varepsilon(y)$ then $\vec{M} = \vec{\nabla} \ln \varepsilon = \frac{\varepsilon'}{\varepsilon}\hat{j}$

and upon a substitution in (2.24), (2.26) one finds

$$\nabla^2 E_1 - \mu_0 \varepsilon(y) \frac{\partial^2 E_1}{\partial t^2} = -\frac{\varepsilon'}{\varepsilon}\frac{\partial E_2}{\partial x}, \qquad (2.27)$$

$$\nabla^2 E_2 - \mu_0 \varepsilon(y) \frac{\partial^2 E_2}{\partial t^2} = -\frac{\partial}{\partial y}\left(\frac{\varepsilon'}{\varepsilon} E_2\right), \qquad (2.28)$$

and

$$\nabla^2 E_3 - \mu_0 \varepsilon(y) \frac{\partial^2 E_3}{\partial t^2} = -\frac{\varepsilon'}{\varepsilon} \frac{\partial E_2}{\partial z}. \tag{2.29}$$

Eq. (2.28) can be expanded as

$$\nabla^2 E_2 - \mu_0 \varepsilon \frac{\partial^2 \vec{E}_2}{\partial t^2} = -\frac{\partial}{\partial y}\left(\frac{\varepsilon'}{\varepsilon}\right) E_2 - \frac{\varepsilon'}{\varepsilon} \frac{\partial \vec{E}_2}{\partial y}. \tag{2.30}$$

Next, we assume that the only non–zero component of the electric field is its x-component i.e. $E_1 \neq 0$ and the other components are zero i.e. $E_2 = E_3 = 0$. This in turn implies, that the only equation left is (2.27) which becomes

$$\nabla^2 E_1 - \mu_0 \varepsilon(y) \frac{\partial^2 E_1}{\partial t^2} = 0. \tag{2.31}$$

Now, also let's consider $E_1 = \psi\,(y, z, t) = e^{i\omega t}\varphi(z, y)$, which means, the electric field is only function of y and z but not x.

Upon that Eq. (2.31) becomes

$$\nabla^2 \varphi - \mu_0 \varepsilon (i\,\omega)^2 \varphi^2 = 0, \tag{2.32}$$

or simply

$$\nabla^2 \varphi + \mu_0 \varepsilon \omega^2 \varphi^2 = 0, \tag{2.33}$$

which is very similar to the wave equation in a homogeneous medium but here ε is not a constant parameter but a function of y.

Let's use the separation method

$$\varphi = Y\,(y)\,Z\,(z). \tag{2.34}$$

By considering Eq. (2.34) in Eq. (2.33) which yields the following

$$\frac{Y''}{Y}+\frac{Z''}{Z}+\mu_0\varepsilon(y)\omega^2=0. \tag{2.35}$$

Now, we take the final step to separate the equation by considering

$$\frac{Z''}{Z}=-\alpha^2. \tag{2.36}$$

Which is equivalent with

$$Z''+\alpha^2 Z=0. \tag{2.37}$$

Following that the y-component becomes

$$Y''+\left[\mu_0\varepsilon(y)\omega^2-\alpha^2\right]Y=0, \tag{2.38}$$

in which α is just a constant parameter .

The solution to Eq. (2.37) is given by

$$Z(z)=C_1e^{i\alpha z}+C_2e^{-i\alpha z}. \tag{2.39}$$

In which C_1 and C_2 are two integration constants.

To finalize this chapter we note that only equation left to be solved is Eq. (2.38) and its solution depends on the form of $\varepsilon(y)$.

To have an estimation about the constant $\alpha,$ let's move to the case of the vacuum . In such case $\varepsilon=\varepsilon_0$ and $\mu=\mu_0$ which implies $Z(z)=C_1e^{i\alpha z}+C_2e^{-i\alpha z}$.

This shows that $\alpha=\kappa$ for the vacuum in which κ is the wave number such that $c=\dfrac{\omega}{\kappa}$. Here, c is the speed of light and ω is the frequency of the wave. We have therefore $\lim\limits_{\varepsilon\to\varepsilon_0}\alpha=\kappa=\dfrac{\omega}{c}.$

In the following chapter we shall consider a specific model for ε and then we will try to solve the last equation.

Chapter 3

A SMOOTH MOVE DIELECTRIC CONSTANT

In the present chapter we will analyze the solutions to the smooth step dielectric constant. First section will be the generalization of the wave equation, while in the second section we will solve the wave equation by using hyper geometric function. However, we may consider equation (2.38). In the previous chapter we obtain from the wave equation in the majority general form for a medium with y coordinate position-dependent characteristics.

3.1 The Wave Equation

Previously, in chapter 2, we have found the wave equation in an inhomogeneous medium whose y-component is given by

$$Y'' + \left[\mu_0 \varepsilon(y)\omega^2 - \alpha^2\right] Y = 0, \tag{3.1}$$

in which ω is the wave frequency, and α is the wave number. Now to go further to introduce

$$\varepsilon(y) = K(y)\varepsilon_0, \tag{3.2}$$

in which $K(y)$ is defined as

$$K(y) = K_2 - \frac{\Delta K}{4}(1 - \tanh(ay))^2. \tag{3.3}$$

Herein $\Delta K = K_2 - K_1$ where K_2, K_1 are given as

$$K_1 = \lim_{a \to -\infty} K(y),$$

and

$$K_2 = \lim_{a \to +\infty} K(y). \tag{3.4}$$

In Fig. 3.1 we plot $K(y)$ with respect to y for the specific values of $K_1 = 2,\ K_2 = 1$ and $a = 0.1, 0.5, 1, 5, 10$

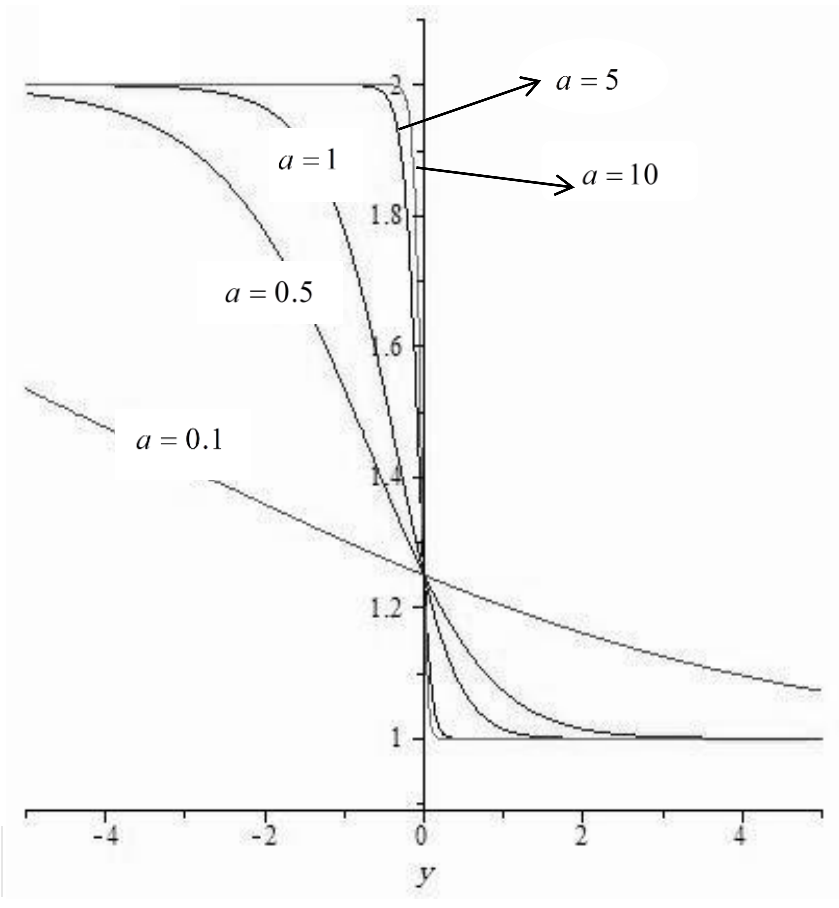

Figure 3.1. A plot of the relative permittivity function $K(y)$ in term of y for $a = 0.1, 0.5, 1, 5$ and 10 and $K_1 = 2$ and $K_2 = 1$.

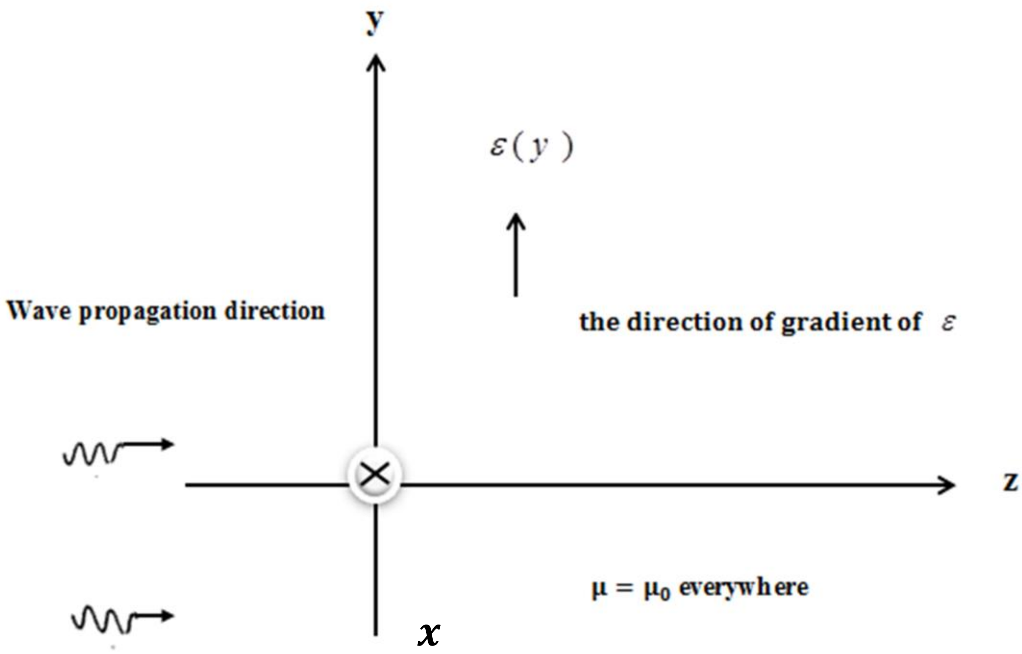

Figure 3.2 Shows the relation between wave propagation direction and the direction of gradient.

Next, Eq. (3.1) may be written as

$$\left(\frac{\partial^2}{\partial y^2}+\mu_0\varepsilon_0 K(y)\omega^2-\alpha^2\right)Y(y)=0, \tag{3.5}$$

and upon considering $\mu_0\varepsilon_0\,\omega^2=\kappa^2$, we write it as

$$\left(\frac{\partial^2}{\partial y^2}+\kappa^2 K(y)-\alpha^2\right)Y(y)=0. \tag{3.6}$$

Considering the explicit form of $K(y)$ given by Eq. (3.3) the latter equation becomes

$$\left(\frac{\partial^2}{\partial y^2}+\kappa^2\left(K_2-\frac{\Delta K}{4}\left(1-\tanh(ay)\right)^2\right)-\alpha^2\right)Y(y)=0. \tag{3.7}$$

A rearrangement implies from Eq. (3.7)

$$\left(\frac{\partial^2}{\partial y^2}+(\kappa^2 K_2-\alpha^2)-\kappa^2\frac{\Delta K}{4}\left(1-\tanh(ay)\right)^2\right)Y(y)=0, \tag{3.8}$$

which upon introducing $\beta^2=\left(\kappa^2 K_2-\alpha^2\right)$ and $\gamma^2=\kappa^2\Delta K$, Eq. (3.8) can be written as

$$\left(\frac{\partial^2}{\partial y^2}+(\beta^2-\frac{1}{4}\gamma^2\left(1-\tanh(ay)\right)^2\right)Y(y)=0. \tag{3.9}$$

To solve this differential equation we use a change of variable as

$$Y(y)=(\xi-1)^{\frac{i\tilde{\beta}}{2}}\xi^{\frac{\tilde{\mu}}{2}}H(\xi), \tag{3.10}$$

in which

$\tilde{\mu}=\sqrt{2\tilde{\gamma}^2-\tilde{\beta}^2}$ with $\tilde{\beta}^2=\dfrac{\beta^2}{a^2}$ and $\tilde{\gamma}^2=\dfrac{\gamma^2}{a^2}$.

We note that, $\xi(y)$ is a dimensionless variable given by

$$\xi(y)=\frac{1}{2}\left(\tanh(ay)+1\right). \tag{3.11}$$

To transform the differential Eq. (3.9) into our new variable we use the chain rule, to get

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial y}\Rightarrow \frac{\partial}{\partial y}=\frac{1}{2}a\ \mathrm{sech}^2\left(ay\right)\frac{\partial}{\partial \xi} \tag{3.12}$$

And once more, we find

$$\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2+\frac{\partial^2 \xi}{\partial y^2}\frac{\partial}{\partial \xi} \tag{3.13}$$

or after simplification

$$\frac{\partial^2}{\partial y^2}=4a^2\xi^2\left(\xi-1\right)^2\frac{\partial^2}{\partial \xi^2}+\left(4a^2\xi\left(\xi-1\right)\left(2\xi-1\right)\frac{\partial}{\partial \xi}\right). \tag{3.14}$$

A substitution into Eq. (3.9) one finds

$$\begin{gathered}4a^2\xi^2\left(\xi-1\right)^2\frac{\partial^2}{\partial \xi^2}+\left(4a^2\xi\left(\xi-1\right)\left(2\xi-1\right)\frac{\partial}{\partial \xi}\right)+\\ \left(\beta^2-\gamma^2\left(4\left(\xi-1\right)^2\right)\right)\left(\xi-1\right)^{\frac{i\tilde{\beta}}{2}}\xi^{-\frac{\tilde{\mu}}{2}}H\left(\xi\right)=0,\end{gathered} \tag{3.15}$$

and after some manipulations, it becomes

$$\begin{gathered}\xi\left(1-\xi\right)H''+\left(1-\tilde{\mu}-\left(i\tilde{\beta}-\tilde{\mu}+2\right)\xi\right)H'-\\ \frac{1}{4}\left(\tilde{\mu}^2-\tilde{\beta}^2\right)+\frac{1}{2}\left(\tilde{\mu}^2-\tilde{\beta}^2+\tilde{\mu}i\tilde{\beta}\right)H=0.\end{gathered} \tag{3.16}$$

This is a hypergeometric differential equation $\left[2,3,8\right]$.

3.2 Solution to the Wave Equation

The general form of the hypergeometric differential equation is given by $\left[10,11,12\right]$

$$x\left(1-x\right)Y''\left(x\right)+\left(c-\left(a+b+1\right)x\right)Y'\left(x\right)-a\,bY\left(x\right)=0, \tag{3.17}$$

in which a,b,c are real parameters.

The general solution to this equation, is given by $\left[12,13,14\right]$.

$$Y(x)=C_{1\,2}F_1(a,b,c;x)+C_2x^{(1-c)}{}_2F_1(a-c+1,b-c+1,2-c;x), \qquad (3.18)$$

in which C_1 and C_2 are two integration constants and $c\neq 0,-1,-2,\ldots$. We note that ${}_2F_1(a,b,c;x)$ is second kind Hypergeometric Function

$$\xi(1-\xi)H''(\xi)+(\delta-(\lambda+P+1)\xi)H'(\xi)-\lambda PH(\xi)=0, \qquad (3.19)$$

in which

$$\lambda=\frac{1}{2}\left(i\tilde{\beta}-\tilde{\mu}\right), \qquad (3.20)$$

$$p=\frac{1}{2}\left(i\tilde{\beta}-\tilde{\mu}\right)+1, \qquad (3.21)$$

and

$$\delta=1-\tilde{\mu}. \qquad (3.22)$$

As we mentioned before, the general solution is given by

$$H=C_{1\,2}F_1(\lambda,p,\delta;\xi)+C_2\xi^{1-\delta}{}_2F_1(\lambda-\delta+1,p-\delta+1,2-\delta;\xi), \qquad (3.23)$$

$$\delta\neq 0,-1,-2,-3\ldots$$

in which C_1 and C_2 are integration constants.

Having λ,p and δ explicitly in (3.20) –(3.22) we find

$$\begin{aligned}H(\xi)=&C_{1\,2}F_1\left[\left(\frac{1}{2}\left(i\tilde{\beta}-\tilde{\mu}\right)\right),\left(\frac{1}{2}\left(i\tilde{\beta}-\tilde{\mu}\right)+1\right),(1-\tilde{\mu});\xi\right]+\\&C_2\xi^{\tilde{\mu}}{}_2F_1\left[\frac{1}{2}\left(i\tilde{\beta}-\tilde{\mu}\right)-(1-\tilde{\mu})+1,\left(\frac{1}{2}\left(i\tilde{\beta}-\tilde{\mu}\right)+1\right)-\right.\\&\left.(1-\tilde{\mu})+1,2-(1-\tilde{\mu});\xi)\right].\end{aligned} \qquad (3.24)$$

We note that, as we have mentioned before, when $\varepsilon = \varepsilon_0$ i.e. $K(y) = 1$, then the solution is the plane–wave propagating in z–direction and oscillating in x –direction. This means $\vec{E} = E_1\hat{x} = e^{i(\omega t - \kappa z)}\hat{x}$, in which $\kappa = \frac{\omega}{c}$.

In our case when $K(y) \neq 1$ and $\xi \to +\infty$ then $\vec{E}$ must be the same as,

$\vec{E} = E_1\hat{x} = e^{i(\omega t - \kappa z)}\hat{x}$

Let's go to the main equation back to Eq. (3.1) where we have

$$Y'' + \left(\mu_0 \varepsilon(y)\omega^2 - \alpha^2\right)Y = 0, \tag{3.25}$$

and consider $\alpha = \frac{\omega}{c}$ which implies

$$Y'' + \left(\mu_0 \varepsilon(y)\omega^2 - \frac{\omega^2}{c^2}\right)Y = 0, \tag{3.26}$$

we recall that $\varepsilon(y)$ is given by

$$\varepsilon(y) = K(y)\varepsilon_0 , \tag{3.27}$$

which after a substitution in Eq. (3.26) we find

$$Y'' + \left(\mu_0 K(y)\varepsilon_0\omega^2 - \frac{\omega^2}{c^2}\right)Y = 0. \tag{3.28}$$

Also, the explicit form of $K(y)$

$$K(y) = K_2 - \frac{\Delta K}{4}\left(1 - \tanh(ay)\right)^2 , \tag{3.29}$$

at the limit when $a \to 0$, yields

$$\lim_{a\to 0} K(y) = K_2 - \frac{\Delta K}{4}. \tag{3.30}$$

Therefore the equation at this limit becomes

$$Y'' + \kappa^2\left(\left(K_2 - \frac{\Delta K}{4}\right) - 1\right)Y\,(y) = 0. \qquad (3.\,31)$$

In which $\kappa^2 = \dfrac{\omega^2}{c^2}$

Introducing, $\beta^2 = \kappa^2\left(K_2 - 1\right)$ and $\gamma^2 = \kappa^2 \Delta K$ one finds

$$\left(\frac{\partial^2}{\partial y^2} + \left(\beta^2 - \frac{1}{4}\gamma^2\right)\right)Y = 0. \qquad (3.32)$$

The solution to Eq. (3.32)is given by

$$Y = C_1 e^{i\tilde{\omega} y} + C_2 e^{-i\tilde{\omega} y}. \qquad (3.33)$$

In the solution found in Eq. (3.24) let’s consider $K_2 = 1$ which implies $\tilde{\beta} = 0$ and consequently

$$H\left(\xi\right) = C_1 F\left(\lambda, p, \delta; \xi\right) + C_2 \xi^{\tilde{\mu}} F\left(\lambda - \delta + 1, p - \delta + 1, 2 - \delta; \xi\right) \qquad (3.34)$$

Now, when we consider $K_2 = 1$ then $\tilde{\beta} = 0$, and λ, p, δ become,

$$\lambda = -\frac{1}{2}\tilde{\kappa}\sqrt{1 - 2K_1}, \qquad (3.35)$$

$$p = -\frac{1}{2}\tilde{\kappa}\sqrt{1 - 2K_1} + 1, \qquad (3.36)$$

and

$$\delta = 1 - \tilde{\kappa}\sqrt{1-2K_1}. \tag{3.37}$$

In which $\tilde{\kappa} = \dfrac{\kappa}{a}$.

The solution to the main equation, i.e. Y (y) is given by

$$\begin{aligned} Y\left(y\right) &= C_1(\xi)^{-\frac{\tilde{\mu}}{2}}(1-\xi)^{\frac{i\tilde{\beta}}{2}} F\left(\lambda, p, \delta; \xi\right) \\ &+ C_2\left(\xi\right)^{1-\delta-\tilde{\mu}/2}\left(1-\xi\right)^{i\tilde{\beta}/2} F\left(\lambda-\delta+1, p-\delta+1, 2-\delta; \xi\right), \end{aligned} \tag{3.38}$$

and upon this the form of the electric field becomes

$$\begin{aligned} E\left(y,z,t\right) &= \hat{\imath} e^{\pm i\kappa z} e^{i\omega t}\left[C_1\left(\left(\frac{1}{2}\left(\tanh(\mathrm{ay})+1\right)\right)^{-\frac{1}{2}\tilde{\kappa}\sqrt{1-2K_1}} F\left(\lambda, p, \delta; \xi\right)\right)+\right. \\ &\left. C_2\left(\left(\frac{1}{2}\left(\tanh\left(ay\right)+1\right)\right)^{\frac{1}{2}\tilde{\kappa}\sqrt{1-2K_1}} F\left(\lambda-\delta+1, p-\delta+1, 2-\delta; \xi\right)\right)\right] \end{aligned} \tag{3.39}$$

In which λ, p and δ are defined above while

$$\xi = \frac{1}{2}\left(\tanh(ay)+1\right) \tag{3.40}$$

In the limit, when $a \to 0$ one finds

$$\lim_{a\to 0} \xi = \frac{1}{2} \tag{3.41}$$

and after imposing to the general solution Eq. (3.39) we find

$$\lim_{a\to 0} E\left(y,z,t\right) = \hat{i}e^{i\kappa z \pm i\omega t}\left[\tilde{C}_1 F\left(\lambda, p, \delta; \frac{1}{2}\right) + \right.$$
$$\left. \tilde{C}_2 F\left(\lambda - \delta + 1, p - \delta + 1, 2 - \delta; \frac{1}{2}\right)\right], \tag{3.42}$$

in which we have defined $\tilde{C}_1 = C_1 \left(\frac{1}{2}\right)^{-\frac{1}{2}\tilde{\kappa}\sqrt{1-2K_1}}$ and $\tilde{C}_2 = C_2 \left(\frac{1}{2}\right)^{\frac{1}{2}\tilde{\kappa}\sqrt{1-2K_1}}$.

From the identity

$$F\left(\lambda, p, \delta; \xi\right) = \left(1-\xi\right)^{\delta-\lambda-P} F\left(\delta-\lambda, \delta-p, \delta; \xi\right), \tag{3.43}$$

and setting $\delta - \lambda - p = 0$ we find

$$F\left(\lambda, p, \delta; \xi\right) = F\left(\delta-\lambda, \delta-p, \delta; \xi\right), \tag{3.44}$$

or consequently

$$\tilde{C}_1 \ F\left(\lambda, p, \delta; \frac{1}{2}\right) + \tilde{C}_2 \ F\left(\lambda - \delta + 1, p - \delta + 1, 2 - \delta; \frac{1}{2}\right) = 1 \tag{3.45}$$

Since in this limit the electromagnetic wave propagates only in positive z direction, the second term vanishes. Therefore, we can set $\tilde{C}_2 = 0,$ then

$$\lim_{a\to 0} \vec{E}\left(y,z,t\right) = \hat{i}e^{i\kappa z \pm i\omega t}\tilde{C}_1 \ F\left(\lambda, p, \delta; \frac{1}{2}\right), \tag{3.46}$$

and

$$\lim_{a\to 0} \tilde{C}_1 F\left(\lambda, p, \delta; \frac{1}{2}\right) = 1, \tag{3.47}$$

upon that Eq. (3.47) becomes

$$\vec{E}\left(y,z,t\right) = \hat{i}e^{i\kappa z \pm i\omega t}. \tag{3.48}$$

To complete this chapter we write the complete form of electric field as

$$\vec{E}\left(y,z,t\right)=\hat{i}E_0\left(\frac{1}{2}(\tanh\left(ay\right)+1)\right)^{-\frac{\tilde{\mu}}{2}}\left(1-(\frac{1}{2}(\tanh\left(ay\right)+1)\right)^{\frac{i\tilde{\beta}}{2}}F\left(\lambda,p,\delta;\xi\right)e^{i\kappa z\pm i\omega t}.$$

(3.49)

In which we set $C_1=E_0$.

Chapter 4

CONCLUSION

In this thesis we considered Maxwell's equations inside an inhomogeneous medium. An inhomogeneous matter possesses position dependent electric permittivity and magnetic permeability or at least one of them. Having $\epsilon\ and\ \mu$ functions of position, causes the Maxwell's equations to be much more complicated than the case they are constant. In this context we have combined the field equations in order to find the wave propagation equation inside such kind of medium. The general wave equation in the limit of constant permittivity and permeability reduces to the standard wave equation which simply admits plane waves as the solution. In the rest of the thesis we considered $\mu = \mu_0$ while $\varepsilon = \varepsilon(y)$ and the direction of the propagation in z direction. The explicit form of the electric permittivity has been chosen in a manner that by varying a parameter in it, the case of sharp or smooth change of the permittivity function is achieved. This is shown in Fig. 1 very clearly. In this configuration we solved the wave equation and a solution of the form of Hypergeometric function was determined. Matching the boundary conditions in the general solution was the last part of the thesis. The subject of the thesis is of highly importance especially in electronic and electrical engineering. There are more yet to be done but those are beyond the scope of this thesis. For instance one may consider the propagation of the electromagnetic wave within an inhomogeneous matter whose electric permittivity and magnetic permeability both are variable but not function of

y but all three coordinates. Going through spherical or cylindrical configuration are some natural extension of the theory.

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