Fractional Derivative and Integral

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ABSTRACT

In this thesis we studied fractional order derivative and integral. In Chapter1, a brief history on the foundation of fractional derivative and integration has been given. In the second chapter, some definitions and theorems have been provided. Also some needed special functions such as Gamma, Beta, Mittag-Leffler and Wright function have taken place in this chapter.

Properties of fractional derivative and integral are discussed in Chapter 3. We started to this chapter by the discussion of the Abel integral equation and it's application. In the first section of Chapter 3, fractional integral in the space of integrable functions and related properties has been given. The second section is devoted to basic definitions and properties of fractional derivative and integral. Definition of fractional integral and derivative of complex order take place in the third section together with some related theorems. Fourth section contains fractional integrals of some elementary functions. In the last section of Chapter 3, we discussed fractional differentiation and integration as reciprocal operations.

Keywords: Fractional Equation, Fractional Derivative, Fractional Integral.

ÖZ

Bu tez üç bölümden oluşmaktadır. Birinci bölüm giriş kısmına ayrılmıştır. Kesirli türev ve integralin nasıl meydana getirildiğinden bahsedilmiştir.

İkinci bölümde bazı fonksiyon tanımlarına yer verilmiştir. Ayrıca tezde kullanılacak olan bazı özel fonksiyonlar verilmiştir. Bu özel fonksiyonlar Gama fonksiyonu, Beta fonksiyonu, Mittang Leffler fonksiyonu ve Wright Fonksiyonu'dur.

Üçüncü bölümde genel olarak kesirli türev ve integrale giriş yapılmiştir , bazı özel fonksiyonlarla ilişkilendirildi ve bunların özelliklerine yer verildi. Bu bölümü inceliyelim. Öncelikle Abel integral denklemi açıklanmiş , özel fonksiyonlarla işlemler yapılmiştir. Birinci kısımda integrallenebilir fonksiyonlar uzayında kesirli integeralin çözülebilirliği bazı teoremlerle ispatlanarak açıklanmıştır. İkinci kısımda kesirli türev ve integralin tanımları verilmiş ayrıca kesirli türev ve inegralin bazı basit özelliklerinden bahsedilmiştir. Üçüncü kısımda kompleks mertebeden, kesirli türev ve integral alındı ve bunlarla ilgili teoremler ispatlanarak açıklanmiştir. Dördüncü kısımda bazı temel fonksiyonlarin kesirli integrali alınmiş ve bunlarla ilgili işlemler yapılıp istenilen temel fonksiyonlara ulaşılmiştir. Beşinci kısımda, kesirli türev ve integral karşılıklı operatör alınarak bir takım tanımlara yer verilmiş ve teoremlerle ispatlanarak açıklanmiştir. Son olarak ise, yarıgrup tanımları verilmiş, operatörlerin yarı gruplarla ilişkisi incelenmiş ve bazı uzaylarla da ilişkilendirilip ispatlar yapılmiştir.

Anahtar Kelimeler:Kesirli Denklemler, Kesirli Türev, Kesirli İntegral.

DEDICATION

My beloved mother and father for giving me the love and understanding that without their persistent encouragement, I would not have been able to complete this research.

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Chapter 1

PRELIMINARIES

In this thesis we focus on fractional integrals and derivatives. We begin with a brief historical development of the theory of fractional integral and derivative.

In 1965, L'hopital wrote a letter to Leibnitz and asked the solution of the following equation when $n = \frac{1}{2}$;

$$f(x) = x\frac{D^n}{Dx^n}$$

Leibnitz's response was "An apperent paradox, from which one day useful consequences will be drown". So the story of fractional calculus has started with the question of L'hopital.

After, L'hopital and Leibnitz and many other mathematicans like Fourier, Euler, Laplace, etc. have studied to answer L'hopital's questions. Each used their own notation and methotology and they found many concepts of a non-integer order integral or derivative.

The main part of mathematical theory of fractional calculus was developed in 20^{th} century. But engineers and scientists started using these theories 100 years later. Recently, the theory of fractional differential equation gain popularity among researchers and different studies involving solutions of linear or non-linear fractional differential equations, solutions of boundary value problems etc. been published.

Chapter 2

NOTATION AND BACKGROUND MATERIAL

2.1 Spaces of Integrable, Absolutely Continuous and Continuous Function

In this section we give some required definitions and properties that will be needed to study fractional integrals and derivatives.

Definition 1 *Let a and b be two real numbers then*

a) $[a,b] := \{x \in \mathbb{R} : \ a \le x \le b\}$ *is called the closed interval.*

b) $(a,b) := \{x \in \mathbb{R} : \ a < x < b\}$ *is called the open interval.*

c) $(a,b] := \{x \in \mathbb{R} : \ a < x \le b\}$ *and* $[a,b) := \{x \in \mathbb{R} : \ a \le x < b\}$ *are called half open intervals.*

d) $[a,\infty) := \{x \in \mathbb{R} : \ a < x \le b\}$ $(-\infty,b] := \{x \in \mathbb{R} \ : \ a < x \le b\}$ *are called closed infinite intervals.*

e) $(a,\infty) := \{x \in \mathbb{R} : \ a < x \le b\}$ $(-\infty,b) := \{x \in \mathbb{R} : \ a < x \le b\}$ *are called open infinite intervals.*

Definition 2 *Let* Λ *be a subset of real numbers then* $C(\Lambda)$ *is the set of all continuous functions on* Λ.

Definition 3 *Let* Λ *be a finite interval and* $h(x)$ *be a function defined on* Λ. *We say that the function* $h(x)$ *satisfies Hölder condition of order* λ *if*

$$|h(x_1) - h(x_2)| \le A\,|x_1 - x_2|^{\lambda} \tag{2.1}$$

for all pairs of points x_1, x_2 *of* Λ *where* A *is a constant. In this case the number* λ *is called the Hölder exponent.*

Definition 4 *For a finite interval* Λ*, the space of all complex valued functions, which satisfy the Hölder condition of order* λ *is denoted by* $H^\lambda = H^\lambda(\Lambda)$ *i.e*

$$H^\lambda(\Lambda) = \left\{h : |h(x_1) - h(x_2)| \leq A\,|x_1 - x_2|^\lambda,\ x_1, x_2 \in \Lambda\right\}.$$

For $\lambda = 1$, H^1 *is known as Lipschitz space.*

Remark 5 *It is clear that* $H^\lambda(\Lambda) \subset C(\Lambda)$*.*

Remark 6 *For* H^λ*, we are only interested in the case* $0 < \lambda \leq 1$*, because otherwise only constant functions will be contained in* H^λ*.*

Definition 7 *The space* $h^\lambda := h^\lambda(\Lambda)$ *is defined by*

$$h^\lambda := h^\lambda(\Lambda) := \left\{h : \frac{h(x) - h(x_1)}{|x - x_1|} \to 0 \ as\ x \to x_1\right\}$$

for all $x_1 \in \Lambda$*.*

Remark 8 *It is easy to see that* $h^\lambda \subset H^\lambda$*.*

In the following we will provide a space wider than H^1, which is known as the space of absolutely continuous functions.

Definition 9 *A function* h *is called absolutely continuous on an interval* Λ*, if for all* $\epsilon > 0$, $\exists\delta > 0$ *such that for any finite set of pairwise disjoint subintervals* $[a_k, b_k] \subset \Lambda$*,* $(k = 1, 2, ..., n)$*, such that,*

$$\sum_{k=1}^{n}(b_k - a_k) < \delta$$

the following inequality holds: $\sum_{k=1}^{n} |h(b_k) - h(a_k)| < \epsilon.$

The space of all absolutely continuous functions is denoted by $AC(\Lambda)$. *In other words*

$$AC(\Lambda) := \{h : h \textit{ is absolutely continuous}\}.$$

Remark 10 *(*[3],[4]*) It is easy to see that the space of primitives of Lebesgue summable functions is equivalent to* $AC(\Lambda)$, *that is;*

$$h(x) \in AC(\Lambda) \Leftrightarrow h(x) = c + \int_a^b \psi(t)\,dt, \qquad (2.2)$$

where

$$\int_a^b |\psi(t)|\,dt < \infty.$$

Remark 11 *The space* $H^1(\Lambda)$ *is included in* $AC(\Lambda)$.

The following example shows that the inverse implication does not hold in general.

Example 12 *Let* c *be a point in* Λ *then consider the function* $h(x) = (x - c)^\gamma \in AC(\Lambda)$. *The equation (2.1) does not hold at* $x = c$, *therefore* $(x - a)^\gamma \notin H^1(\Lambda)$ *for* $0 < \gamma < 1$.

Definition 13 *Let* Λ *be an interval then for each* $n \in \mathbb{N}$, *one can define the following space,*

$$AC^n(\Lambda) := \left\{ f : f^{(n-1)} \in AC(\Lambda) \textit{ and it has continuous derivatives of order } n-1 \textit{ on } \Lambda \right\}.$$

Remark 14 *It is obvious that* $AC'(\Lambda) = AC(\Lambda)$.

For the case Λ is $\mathbb{R}$ or half line. Then to define $H^\lambda(\Lambda)$ for $\Lambda = \mathbb{R}$ or Λ is half line we need to explain the Hölder property at infinity. This explanation is given below.

Definition 15 *Let Λ be $\mathbb{R}$ or half line then $H^{\lambda}(\Lambda)$ is the space of functions;*

(i) satisfying equation (2.1) for any finite subinterval of Λ.

(ii) satisfying the functions $h(x)$ Hölder property in the neighborhood of infinite

$$|h(x_1) - h(x_2)| \leq A \left| \frac{1}{x_1} - \frac{1}{x_2} \right|^{\lambda}. \tag{2.3}$$

(i.e. for all $x_1, x_2 \in \Lambda$, with sufficiently large absolute values)

Definition 16 *The set of all Lebesgue measurable functions $h(x)$ satisfying,*

$$\int_{\Lambda} |h(x)|^p \, dx < \infty, \quad 1 \leq p < \infty$$

is denoted by $L_p = L_p([a,b])$.

We shall consider the space L_p as a norm space with its usual norm which is given below.

Definition 17 ([4])*The following definition gives a norm on $L_p(\Lambda)$,*

$$\|h\|_{L_p(\Lambda)} = \left\{ \int_{\Lambda} |h(x)|^p \, dx \right\}^{\frac{1}{p}}. \tag{2.4}$$

Remark 18 ([3]) *In the case $p = \infty$, the space $L_\infty(\Lambda)$ is defined as the set of all measurable functions with a finite norm,*

$$\|h\|_{L_\infty(\Lambda)} = ess \sup_{x \in \Lambda} |h(x)| \tag{2.5}$$

For the following parts we will assume $1 \leq p \leq \infty$. Two equivalent functions in $L_p(\Lambda)$ (functions which are same except on a set with measure zero) will be considered the same element in $L_p(\Lambda)$. Therefore,

$$\|h\|_p = \|h\|_{L_p} = \|h\|_{L_p(\Lambda)}. \tag{2.6}$$

Now we shall introduce some properties of the space $L_p(\Lambda)$, which will be used in the rest of the thesis. Such properties can be found in any functional analysis text book.

Definition 19 ([1])*(Minkowski Inequality) Let h and g be any two elements of* $L_p(\Lambda)$ *then,*

$$\|h+g\|_{L_p(\Lambda)} \leq \|h\|_{L_p(\Lambda)} + \|g\|_{L_p(\Lambda)}. \tag{2.7}$$

Definition 20 ([1])*(Hölder inequality) Let h and g be any two elements of* $L_p(\Lambda)$ *and* $L_q(\Lambda)$ *respectively then,*

$$\int_\Lambda |h(x)\,g(x)|\,dx \leq \|h\|_{L_p(\Lambda)} \|g\|_{L_q(\Lambda)} \tag{2.8}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{2.9}$$

Remark 21 *Hölder inequality can be generalized as follows:*

$$\int_\Lambda |h_1(x)\ldots h_m(x)|\,dx \leq \|h\|_{L_{p_1}(\Lambda)} \cdots \|h\|_{L_{p_m}(\Lambda)}, \tag{2.10}$$

where

$$h_k(x) \in L_{p_k}(\Lambda),\ k = 1,2,\ldots,m, \text{ and } \sum_{k=1}^{\infty} \frac{1}{p_k} = 1.$$

Remark 22 *Let* Λ *be a finite interval, then using Hölder inequality one can write the following,*

$$L_{p_1}(\Lambda) \subset L_{p_2}(\Lambda), \tag{2.11}$$

and
$$\|h\|_{L_{p2}(\Lambda)} \leq \|h\|_{L_{p_1}(\Lambda)}$$

where $p_1 > p_2 \geq 1$.

Theorem 23 ([4])*(Fubini's Theorem) Assume that* $\Lambda_1 = [a,b]$, $\Lambda_2 = [c,d]$ *where* $-\infty \leq a < b \leq \infty$, $-\infty \leq c < d \leq \infty$, *for a measurable function* $h(x,y)$ *defined on* $\Lambda_1 \times \Lambda_2$, *and at least one of the following integrals*

$$\int_{\Lambda_1} dx \int_{\Lambda_2} h(x,y)\,dy,$$

$$\int_{\Lambda_2} dy \int_{\Lambda_1} h(x,y)\,dx,$$

and

$$\iint_{\Lambda_1 \times \Lambda_2} h(x,y)\,dxdy$$

are absolutely convergent, then they are all equal.

Remark 24 *The following particular case of the Fubini's Theorem is known as the*

Dirichlet formula,

$$\int_a^b dx \int_a^x h(x,y)\,dy = \int_a^b dy \int_y^b h(x,y)\,dx \tag{2.12}$$

where one of the integrals is absolutely convergent.

Remark 25 *We also have the following inequality,*

$$\left\{ \int_{\Lambda_1} dx \left| \int_{\Lambda_2} h(x,y)\,dy \right|^p \right\}^{\frac{1}{p}} \leq \int_{\Lambda_2} dy \left\{ \int_{\Lambda_1} |h(x,y)|^p\,dx \right\}^{\frac{1}{p}} \tag{2.13}$$

which is known as the generalized Minkowski inequality.

Lemma 26 ([4]) *Let $h(x) \in L_p(\Lambda)$, $1 \leq p < \infty$ then we have:*

$$\int_{\Lambda} |h(x+t) - h(x)|^p\,dx \to 0 \tag{2.14}$$

as $t \to 0$. We say that the function $h(x)$ is continued by zero for $x+t \notin \Lambda$.

Theorem 27 ([4])*(Lebesgue dominated convergence) Assume $h(x,t)$ and $H(x)$ satisfies condition*

$$|h(x,t)| \leq H(x)$$

where $H(x)$ does not depend on the parameter t and $H(x) \in L_1(\Lambda)$. If

$$\lim_{t \to 0} h(x,t)$$

exists for almost all x, then

$$\lim_{t\to 0}\int_{\Lambda} h(x,t)\,dx = \int_{\Lambda} \lim_{t\to 0} h(x,t)\,dx.$$

2.2 Some Special Function In Fractional Calculus

In this section, we introduce and discuss Gamma and Beta functions and their properties. Those functions plays an important role in the theory of fractional derivative and integral.

One of the basic special functions in analysis is $n!$. For non-integer values, or even complex numbers, which is called Euler's Gamma function and denoted by $\Gamma(z)$. Gamma function is simply said to be the extension of factorial for real numbers.

Definition 28 ([3]) *The gamma function* $\Gamma(z)$ *is defined as,*

$$\Gamma(z) = \int_{0}^{\infty} e^{-s} s^{z-1} ds, z \in \mathbb{R} \tag{2.15}$$

and is convergent on the plane $\mathrm{Re}(z) > 0$.

Lemma 29 *For any* $z \in \mathbb{C}$ *with* $Re(z) > 0$,

$$\Gamma(z+1) = z\Gamma(z). \tag{2.16}$$

Proof. This property can be easily proved by integration by parts.

$$\Gamma(z+1) = \int_{0}^{\infty} e^{-s} s^{z} ds = \left[-e^{-s} s^{z}\right]_{0}^{\infty} + s\int_{0}^{\infty} e^{-s} s^{z-1} ds = z\Gamma(z).$$

It's clear that $\Gamma(1) = 1$ and using (2.16) for $z = 1, 2, 3, ...,$ we have;

$$\begin{aligned}
\Gamma(2) &= 1.\Gamma(1) = 1 = 1! \\
\Gamma(3) &= 2.\Gamma(2) = 2.1! = 2! \\
\Gamma(4) &= 3.\Gamma(3) = 3.2! = 3! \\
&\dots \\
\Gamma(n+1) &= n.\Gamma(n) = n.(n-1) = n!
\end{aligned}$$

■

An other important special function which plays basic role in the theory of fractional calculus is the the Beta function which is defined as follows:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt, \qquad \mathrm{Re}\,p > 0, \mathrm{Re}\,q > 0 \tag{2.17}$$

It is well known that, Gamma and Beta functions are related to each other. In order to show the relation between Beta and Gamma fuction we will use Laplace transformation:

$$h_{p,q}(x) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt. \tag{2.18}$$

If we take $x = 1$ in (2.18) gives $h_{p,q}(1) = B(p,q)$.

Since the laplace transform of convolution of two functions is equal to the multiplication of their Laplace transformation, we get:

$$H_{p,q}(s) = \frac{\Gamma(p)}{s^p}\frac{\Gamma(q)}{s^q} = \frac{\Gamma(p)\Gamma(q)}{s^{p+q}} \tag{2.19}$$

where $H_{p,q}(s)$ is laplace transform of $h_{p,q}(x)$. Since $\Gamma(p)\Gamma(q)$ is constant, taking inverse Laplace transformation of the right-hand side of (2.19) we can get the original function

$h_{p,q}(x)$. From the uniqueness of the Laplace transformations, we have:

$$h_{p,q}(x) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} x^{p+q-1}. \tag{2.20}$$

If $x = 1$, we will get one of the most important properties of Beta function:

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \tag{2.21}$$

By the above definition it is obvious that,

$$B(p,q) = B(q,p). \tag{2.22}$$

The Mittag-Leffler function is defined by,

$$E_{\mu}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\mu k + 1)} \quad (\mu > 0). \tag{2.23}$$

The more general form which is given by Prabhakar ([2]) of Mittag-Leffler function is given by,

$$E_{\mu,\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + \nu)} \quad (\mu > 0, \quad \nu > 0). \tag{2.24}$$

Taking $\mu = 1$ and $\nu = 1$ in (2.24) gives,

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z$$

which is the well known exponential function. Similarly for $\mu = 1$ and $\nu = 2$ in (2.24) we have,

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{e^{z-1}}{z}.$$

$$\left(\sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + \sum_{k=1}^{\infty} \frac{z^k}{k!} = e^z\right)$$

$$E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)!} = \sum_{k=0}^{\infty} \frac{z^k}{(k+2)!} = \frac{1}{z^2}\sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+1)!} = \frac{e^z - 1 - z}{z^2}.$$

More generally,

$$E_{1,n}(z) = \frac{1}{z^{n-1}}\left\{e^z - \sum_{k=0}^{n-2} \frac{z^k}{k!}\right\} \tag{2.25}$$

where n is a natural number.

The Wright function which is an extension of both Bessel and exponential functions, is denoted by W and defined as;

$$W(z;\mu;\nu) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\mu k + \nu)}.$$

Using (2.24) for some valus μ and ν we can get that,

$$W(z;0;1) = e^z,$$

and

$$W\left(-z;-\frac{1}{2},-\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}}\exp\left(-\frac{z^2}{4}\right). \tag{2.26}$$

Chapter 3

FRACTIONAL INTEGRALS AND DERIVATIVES

This chapter is devoted to the theory of fractional integral and fractional derivatives. We shall start from Abel's integral equation which plays an important role in the definition of fractional integral and derivatives. After giving the idea and mathematical background of the theory of fractional integral and derivatives, we also consider some basic properties of fractional integral and derivatives. Recall that, Abel's equation is the integral equation given below for $0 < \gamma < 1$.

$$\frac{1}{\Gamma(\gamma)}\int_0^x \frac{\psi(t)}{(x-t)^{1-\gamma}}dt = h(x),\ x > 0. \tag{3.1}$$

Now apply the following process on (3.1) which is also used in ([4]). Firstly, changing $x \to t$ and $t \to p$ in (3.1) we get:

$$\frac{1}{\Gamma(\gamma)}\int_0^t \frac{\psi(p)}{(t-p)^{1-\gamma}}dp = h(t).$$

Then multiplying both sides of the equation by $\Gamma(\gamma)(x-t)^{-\gamma}$ and integrating we get that,

$$\int_a^x \frac{dt}{(x-t)^{\gamma}}\int_a^t \frac{\psi(p)}{(t-p)^{1-\gamma}}dp = \Gamma(\gamma)\int_a^x \frac{h(t)}{(x-t)^{\gamma}}dt. \tag{3.2}$$

Using Dirichlet formula in (3.2), we have:

$$\int_a^x \psi(p)\,dp\int_p^x \frac{dt}{(x-t)^{\gamma}(t-p)^{1-\gamma}} = \Gamma(\gamma)\int_a^x \frac{h(t)}{(x-t)^{\gamma}}dt\ . \tag{3.3}$$

Taking $t = p + \upsilon(x-p)$ in

$$\int_p^x \frac{dt}{(x-t)^{\gamma}(t-p)^{1-\gamma}}$$

and using the fact that,

$$\begin{aligned}
\frac{1}{(x-t)^{\gamma}(t-p)^{1-\gamma}} &= \frac{1}{(x-p-\upsilon(x-p))^{\gamma}(p+\upsilon(x-p)-p)^{1-\gamma}} \\
&= \frac{1}{[x(1-\upsilon)-p(1-\upsilon)]^{\gamma}[\upsilon(x-p)]^{1-\gamma}} \\
&= \frac{1}{[(x-p)(1-\upsilon)]^{\gamma}[\upsilon(x-p)]^{1-\gamma}} \\
&= \frac{1}{(x-p)^{\gamma}(1-\upsilon)^{\gamma}(\upsilon)^{1-\gamma}(x-p)^{1-\gamma}} \\
&= \frac{1}{(x-p)(1-\upsilon)^{\gamma}(\upsilon)^{1-\gamma}}
\end{aligned}$$

we get that,

$$\int_p^x (x-t)^{-\gamma}(t-p)^{\gamma-1}\,dt = \int_0^1 \frac{d\upsilon}{(1-\upsilon)^{\gamma}(\upsilon)^{1-\gamma}}. \tag{3.4}$$

Using the definition of Beta function on the right-hand side of (3.4) gives,

$$\begin{aligned}
\int_p^x (x-t)^{-\gamma}(t-p)^{\gamma-1}\,dt &= \int_0^1 (1-\upsilon)^{-\gamma}\upsilon^{\gamma-1}d\upsilon \\
&= B(\gamma, 1-\gamma) \\
&= \Gamma(\gamma)\Gamma(1-\gamma).
\end{aligned} \tag{3.5}$$

Substitute (3.5) and (3.4) into (3.3), we get:

$$\int_a^x \psi(p)\,dp \int_p^x \frac{1}{(x-t)^{\gamma}(t-p)^{1-\gamma}}dt = \Gamma(\gamma)\int_a^x \frac{h(t)}{(x-t)^{\gamma}}dt$$

or

$$\int_a^x \psi(p)\,dp = \frac{1}{\Gamma(1-\gamma)}\int_a^x (x-t)^{-\gamma}\, h(t)\,dt. \tag{3.6}$$

Now if we differentiate both sides, we can get that,

$$\psi(x) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_a^x \frac{h(t)}{(x-t)^{\gamma}}dt. \tag{3.7}$$

This means that if the equation given in (3.1) has a solution then it has the form given in (3.7). Moreover this solution is unique. In (3.1) we have assumed that $0 < \gamma < 1$. The case $\gamma = 1$, is clear and the case $\gamma > 1$ can be reduced to the case $0 < \gamma < 1$ by differentiating both sides of (3.1).

It should be mentioned that if we use the Abel equation,

$$\frac{1}{\Gamma(\gamma)}\int_x^b \frac{\psi(t)}{(t-x)^{1-\gamma}}dt = h(x), \qquad x \le b \tag{3.8}$$

and apply the same steps then we obtain the solution,

$$\psi(x) = -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_x^b \frac{h(t)}{(t-x)^{\gamma}}dt. \tag{3.9}$$

Example 30 *Solve the equation*

$$\frac{1}{\Gamma(\gamma)}\int_a^x \frac{\psi(t)}{(x-t)^{1-\gamma}}dt = 1, \textit{ where } 0 < \gamma < 1.$$

Let by (3.7)

$$\psi(x)=\frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_a^x 1\,(x-t)^{-\gamma}\,dt.$$

Taking $x-t=u$,

$$\begin{aligned}\psi(x) &= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\frac{x^{1-\gamma}}{1-\gamma}\\ &= \frac{1}{\Gamma(1-\gamma)}\frac{1}{1-\gamma}\frac{d}{dx}x^{1-\gamma}\\ &= \frac{1}{\Gamma(1-\gamma)}x^{-\gamma}.\end{aligned}$$

Example 31 *Solve the equation*

$$\frac{1}{\Gamma(\gamma)}\int_a^x \frac{\psi(t)}{(x-t)^{1-\gamma}}dt=t^{\beta},\ \textit{where}\ \beta>0.$$

Let from (3.7)

$$\begin{aligned}\psi(x) &= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_a^x t^{\beta}(x-t)^{-\gamma}\,dt\\ &= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}x^{-\gamma}\int_a^x t^{\beta}\left(1-\frac{t}{x}\right)^{-\gamma}dt.\end{aligned}$$

Taking $u=\frac{t}{x}$

$$\begin{aligned}\psi(x) &= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}x^{-\gamma}\int_0^1 (xu)^{\beta}(1-u)^{-\gamma}\,xdu\\ &= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}x^{\beta-\gamma+1}\int_0^1 u^{\beta}(1-u)^{-\gamma}\,du\\ &= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}x^{\beta-\gamma+1}B(\beta+1,\ -\gamma+1)\\ &= \frac{1}{\Gamma(1-\gamma)}\frac{\Gamma(\beta+1)\Gamma(1-\gamma)}{\Gamma(\beta-\gamma+2)}\frac{d}{dx}x^{\beta-\gamma+1}=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)}x^{\beta-\gamma}.\end{aligned}$$

3.1 Fractional Integral in The Space Of Integrable Functions

In this section we shall investigate conditions on $h(t) \in L_1(a,b)$ under which the Abel equation given in (3.1) has a solution.

Definition 32 *For* $0 < \gamma < 1$, *the function* $h_{1-\gamma}(x)$ *is defined by,*

$$h_{1-\gamma}(x) = \frac{1}{\Gamma(1-\gamma)} \int_a^x \frac{h(t)}{(x-t)^{\gamma}} dt. \tag{3.10}$$

Lemma 33 $h(t) \in L_1(a,b)$ *implies that* $h_{1-\gamma}(x) \in L_1(a,b)$.

Proof. Assume that $h(t)$ is any element of $L_1(a,b)$ we have to show that $h_{1-\gamma}(x) \in L_1(a,b)$. Now consider the integral

$$\begin{aligned} \int_a^b \left|h_{1-\gamma}(x)\right| dx &= \frac{1}{\Gamma(1-\gamma)} \int_a^b \left| \int_a^x \psi(t)(x-t)^{-\gamma} dt \right| dx \\ &\leq \frac{1}{\Gamma(1-\gamma)} \int_a^b \int_a^x |\psi(t)|(x-t)^{-\gamma} dt dx \\ &= \frac{1}{\Gamma(1-\gamma)} \int_a^b |\psi(t)| dt \int_t^b (x-t)^{-\gamma} dx \end{aligned} \tag{3.11}$$

On the other hand, the second integral on right hand side is

$$\int_t^b (x-t)^{-\gamma} dx = \left[\frac{(x-t)^{-\gamma+1}}{(-\gamma+1)} \right]_t^b = \frac{(b-t)^{1-\gamma}}{1-\gamma}. \tag{3.12}$$

Substituting (3.12) in (3.11) we have,

$$\int_a^b \left|h_{1-\gamma}(x)\right| dx \leq \frac{1}{\Gamma(1-\gamma)(1-\gamma)} \int_a^b |\psi(t)|(b-t)^{1-\gamma} dt$$

We have;

$$\int_a^b \left|h_{1-\gamma}(x)\right| dx \leq \frac{1}{\Gamma(2-\gamma)} \int_a^b |\psi(t)| (b-t)^{1-\gamma} dt \,. \tag{3.13}$$

Since $(b-t)^{1-\gamma}$ is bounded on $[a,b]$ we have $h_{1-\gamma} \in L_1(a.b)$. ■

Theorem 34 ([4],[3])*The equation (3.1) defined on $\gamma \in (0,1)$ is solvable in $L_1(a,b)$ if and only if*

$$h_{1-\gamma}(x) \in AC([a,b]) \text{ and } h_{1-\gamma}(a) = 0 \tag{3.14}$$

In this case, the equation (3.14) has a unique solution in the form of (3.7).

Proof. Assume that the equation (3.1) is solvable in $L_1(a,b)$. Applying the same steps as in previous section we can obtained that

$$\int_a^x \psi(p)\, dp = h_{1-\gamma}(x). \tag{3.15}$$

As a consequence of (3.7) and (2.2), we have,

$$h_{1-\gamma}(x) \in AC([a,b]) \text{ and } h_{1-\gamma}(a) = 0.$$

Conversely assume that $h_{1-\gamma}(x) \in AC([a,b])$. Then

$$h'_{1-\gamma}(x) = \frac{d}{dx} h_{1-\gamma}(x) \in L_1(a,b)$$

Therefore (3.7) exists a.e and belongs to $L_1(a,b)$. We must show that (3.7) is a solution of (3.1). By substituting (3.7) in (3.1) we get,

$$\frac{1}{\Gamma(\gamma)}\int_a^x \frac{\frac{d}{dx}h_{1-\gamma}(x)}{(x-t)^{1-\gamma}}dt = g(x),$$

or

$$\frac{1}{\Gamma(\gamma)}\int_a^x \frac{h'_{1-\gamma}(t)}{(x-t)^{1-\gamma}}dt = g(x). \tag{3.16}$$

Now, it suffices to show that $g(x) = h(x)$. Since (3.16) is an equation similar to (3.1) with respect to $h'_{1-\gamma}(x)$, using (3.7) we have,

$$h'_{1-\gamma}(x) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_a^x \frac{g(x)}{(x-t)^{\gamma}}dt = g'_{1-\gamma}(x)$$

or equivalently,

$$h'_{1-\gamma}(x) = g'_{1-\gamma}(x).$$

Functions $h_{1-\gamma}(x)$ and $g_{1-\gamma}(x)$ are elements of $AC([a,b])$. The first one by hypothesis and the second one by virtue of (3.6) with $g(t)$ on the right hand side. Hence,

$$h_{1-\gamma}(x) - g_{1-\gamma}(x)$$

is a constant function. On the other hand, $h_{1-\gamma}(0) = 0$ and $g_{1-\gamma}(0) = 0$, since (3.16) is solvable. Hence,

$$h_{1-\gamma}(x) - g_{1-\gamma}(x) = 0.$$

So

$$\int_a^x \frac{h(t)-g(t)}{(x-t)^\gamma}dt = 0.$$

This is an equation of the form (3.1). Since the solution is unique from (3.7), we get $h(x)-g(x)=0$ or equivalently $h(x)=g(x)$. ■

Lemma 35 ([4]) *If $h(x)$ is an absolutely continuous function on $[a,b]$, then $h_{1-\gamma}(x)$ is also an absolutely continuous function on $[a,b]$ and*

$$h_{1-\gamma}(x) = \frac{1}{\Gamma(2-\gamma)}\left[h(a)(x-a)^{1-\gamma} + \int_a^x h^{'}(t)(x-t)^{1-\gamma}dt\right].$$

Proof. From (2.2) we have:

$$h(t) = h(a) + \int_a^t h^{'}(p)dp. \tag{3.17}$$

Now substitute (3.17) into (3.10) to get,

$$\begin{aligned} h_{1-\gamma} &= \frac{1}{\Gamma(1-\gamma)}\int_a^x \left[\frac{h(a)+\int_a^t h'(p)}{(x-t)^\gamma}\right]dt \\ &= \frac{1}{\Gamma(1-\gamma)}\left[\int_a^x \frac{h(a)}{(x-t)^\gamma}dt + \int_a^x\int_a^t \frac{h^{'}(p)}{(x-t)^\gamma}dt\right]. \end{aligned} \tag{3.18}$$

For the first integral apply the change variable, we get;

$$\frac{1}{\Gamma(1-\gamma)}h(a)\left[-\int_{x-a}^{0}\frac{1}{u^{\gamma}}du\right]=\frac{1}{\Gamma(1-\gamma)}h(a)\left[-\frac{u^{-\gamma+1}}{1-\gamma}\right]_{x-a}^{0}$$
$$=\frac{1}{\Gamma(1-\gamma)}h(a)\left[\frac{(x-a)^{1-\gamma}}{1-\gamma}\right]$$
$$=\frac{1}{(1-\gamma)\Gamma(1-\gamma)}h(a)(x-a)^{1-\gamma}$$
$$=\frac{1}{\Gamma(2-\gamma)}h(a)(x-a)^{1-\gamma}.$$

Substiting this in (3.18) gives,

$$h_{1-\gamma}(x)=\frac{1}{\Gamma(2-\gamma)}h(a)(x-a)^{1-\gamma}+\int_{a}^{x}\frac{1}{(x-t)^{\gamma}}dt\int_{a}^{t}h^{'}(p)\,dp. \tag{3.19}$$

Then first term is absolutely continuous function because

$$(x-a)^{1-\gamma}=(1-\gamma)\int_{a}^{x}(t-a)^{-\gamma}\,dt,$$

then we change the variable $u=t-a$, to obtain,

$$\int_{0}^{x-a}u^{-\gamma}du=\left[\frac{u^{-\gamma+1}}{-\gamma+1}\right]_{0}^{x-a}=\frac{(x-a)^{1-\gamma}}{1-\gamma}.$$

The second term is also a primitive of summable function and it is absolutely continuous,

$$\int_{a}^{x}\frac{1}{(x-t)^{\gamma}}dt\int_{a}^{t}h^{'}(p)\,dp=\int_{a}^{x}\left(\int_{a}^{t}\frac{h^{'}(p)}{(t-p)^{\gamma}}dp\right)dt. \tag{3.20}$$

Abel's solution is unique. ■

Corollary 36 ([4])*: If $h(x) \in AC([a,b])$, the Abel's equation (3.1) with $0 < \gamma < 1$ is solvable in $L_1(a,b)$ and its solution (3.7) may be written in the form*

$$\psi(x) = \frac{1}{\Gamma(1-\gamma)}\left[h(0)x^{\gamma} + \int_a^x h'(s)(x-s)^{\gamma}\,ds\right]. \tag{3.21}$$

Proof. *Using Lemma (3.0.34), (3.19) and (3.20), the solvability conditions (3.14) are satisfied. Using the fact that,*

$$\psi(x) = \frac{d}{dx}h_{1-\gamma}(x)$$

and differentiating (3.35) we can obtain (3.21). ■

Corollary 37 *Similar to Theorem 3.0.33, we can show that (3.8) is solvable in $L_1(a,b)$ if and only if $\tilde{h}_{1-\gamma}(x) \in AC([a,b])$ and $\tilde{h}_{1-\gamma}(b) = 0$, where,*

$$\tilde{h}_{1-\gamma}(x) = \frac{1}{\Gamma(1-\gamma)}\int_x^b \frac{h(t)}{(t-x)^{\gamma}}dt, \quad 0 < \gamma < 1.$$

Proof. The proof can be done in a way paralle to the proof of above corollary. The solution (3.9) of (3.8) where $h(x)$ is absolutely continuous on $[a,b]$, may be written as,

$$\psi(t) = \frac{1}{\Gamma(1-\gamma)}\left[h(b)(b-t)^{-\gamma} + \int_t^b h'(s)(s-t)^{-\gamma}\,ds\right]. \tag{3.22}$$

■

3.2 Basic Definitions and Properties of Fractional Integral and Derivatives

Lemma 38 *We shall start with the formula for n-fold integrals ;*

$$\int_a^{x_1}\int_a^{x_2}...\int_a^{x_n}\psi(x_n)\,dx_n...dx_2dx_1 = \frac{1}{(n-1)!}\int_a^{x}\psi(t)(x-t)^{n-1}\,dt. \tag{3.23}$$

Proof. We shall prove by induction, for $n = 1$, obviously

$$\int_a^{x}\psi(x_1)\,dx_1 = \int_a^{x}\psi(x)\,dx.$$

Assume that (3.23) is true for $n-1$, that is

$$\int_a^{x_1}\int_a^{x_2}\int_a^{x_3}...\int_a^{x_{n-1}}\psi(x_n)\,dx_n...dx_3dx_2 = \frac{1}{(n-2)!}\int_a^{x_1}(x_1-t)^{n-2}\psi(t)\,dt.$$

Integrating both sides from a to x, to get:

$$\begin{aligned}
\int_a^{x}\int_a^{x_1}\int_a^{x_2}...\int_a^{x_{n-1}}\psi(x_n)\,dx_n...dx_2dx_1 &= \frac{1}{(n-2)!}\int_a^{x}\left\{\int_a^{x_1}(x_1-t)^{n-2}\psi(t)\,dt\right\} \\
&= \frac{1}{(n-2)!}\int_a^{x}\psi(t)\left\{\int_t^{x}(x_1-t)^{n-2}\,dx_1\right\}dt \\
&= \frac{1}{(n-1)!}\int_a^{x}\psi(t)(x-t)^{n-1}\,dt\ .
\end{aligned}$$

(3.23) may be written as,

$$(I^{\gamma}_{a+}\psi) = \frac{1}{\Gamma(\gamma)}\int_a^{x}\psi(t)(x-t)^{n-1}\,dt.$$

On the other hand using $\Gamma(n) = (n-1)!$, we get the desired result. ■

Now, we are ready to give Riemann-Liouville fractional integral see also ([4],[3]).

Definition 39 ([2],[3],[4]) *Let $\psi(x) \in L_1(a,b)$ then the left-sided and right-sided Riemann Liouville fractional integrals of order γ are defined respectively as follows,*

$$(I_{a+}^{\gamma}\psi)(x) = \frac{1}{\Gamma(\gamma)}\int_{a}^{x} \frac{\psi(t)}{(x-t)^{1-\gamma}}dt \qquad x > a \tag{3.24}$$

$$(I_{b-}^{\gamma}\psi)(x) = \frac{1}{\Gamma(\gamma)}\int_{x}^{b} \frac{\psi(t)}{(t-x)^{1-\gamma}}dt \qquad x < b \tag{3.25}$$

where $\gamma > 0$.

Lemma 40 *Let Q be the reflection operator with $(Q\psi)(x) = \psi(a+b-x)$ then*

$$QI_{a+}^{\gamma} = I_{b-}^{\gamma}Q \text{ and } QI_{b-}^{\gamma} = I_{a+}^{\gamma}Q\ . \tag{3.26}$$

Proof. Take any $\psi \in L_1(a,b)$. We want to show that $Q\left(I_{a+}^{\gamma}\psi\right)(x) = I_{b-}^{\gamma}(Q\psi)(x)$.

$$Q\left(I_{a+}^{\gamma}\psi\right)(x) = \frac{1}{\Gamma(\gamma)}\int_{a}^{a+b-x} \frac{\psi(t)}{(t-a-b+x)^{1-\gamma}}dt.$$

Now take $t = a+b-u$, we get,

$$\begin{aligned} Q\left(I^{\gamma}_{a+}\psi\right)(x) &= \frac{1}{\Gamma(\gamma)}\int_{a}^{a+b-x}\frac{\psi(t)}{(t-a-b+x)^{1-\gamma}}dt \\ &= \frac{-1}{\Gamma(\gamma)}\int_{b}^{x}\frac{\psi(a+b-u)}{(-u+x)^{1-\gamma}}du \\ &= \frac{1}{\Gamma(\gamma)}\int_{x}^{b}\frac{Q(\psi)(u)}{(x-u)^{1-\gamma}}du \\ &= I^{\gamma}_{b-}(Q\psi)(x). \end{aligned}$$

In a parallel way, one can prove that

$$Q\left(I^{\gamma}_{b-}\psi\right)(x) = \frac{1}{\Gamma(\gamma)}\int_{a+b-x}^{b}\frac{\psi(t)}{(t-a-b+x)^{1-\gamma}}dt.$$

Taking $t = a+b-u$ we have,

$$\begin{aligned} Q\left(I^{\gamma}_{b-}\psi\right)(x) &= \frac{-1}{\Gamma(\gamma)}\int_{x}^{a}\frac{\psi(a+b-u)}{(x-u)^{1-\gamma}}du \\ &= \frac{1}{\Gamma(\gamma)}\int_{a}^{x}\frac{\psi(a+b-u)}{(x-u)^{1-\gamma}}du \\ &= I^{\gamma}_{a+}Q(\psi)(x). \end{aligned}$$

Therefore

$$QI^{\gamma}_{b-} = I^{\gamma}_{a+}Q\ .$$

■

Lemma 41 ([4]) *For any pair of functions $\psi, \varphi \in L_1(a,b)$, we have,*

$$\int_{a}^{b}\psi(x)\left(I^{\gamma}_{a+}\varphi\right)(x)\,dx = \int_{a}^{b}\varphi(x)\left(I^{\gamma}_{b-}\psi\right)(x)\,dx\ . \tag{3.27}$$

Proof. Let $\psi, \varphi \in L_1(a,b)$ then,

$$\int_a^b \psi(x)\left(I_{a+}^{\gamma}\varphi\right)(x)\,dx = \frac{1}{\Gamma(\gamma)}\int_a^b \psi(x)\int_a^x \frac{\varphi(x)}{(x-t)^{1-\gamma}}dtdx$$

By using Dirichlet formula we get;

$$\begin{aligned}\int_a^b \psi(x)\left(I_{a+}^{\gamma}\varphi\right)(x)\,dx &= \frac{1}{\Gamma(\gamma)}\int_a^b \left\{\int_t^b \frac{\psi(x)}{(x-t)^{1-\gamma}}dx\right\}\varphi(t)\,dt \\ &= \int_a^b \varphi(t)\left(I_{b-}^{\gamma}\psi\right)(t)\,dt.\end{aligned}$$

Changing variable t by x we get the result which satisfies (3.27). ■

Remark 42 *The equation (3.27) is valid for any pair of functions $\psi(x) \in L_p$, $\varphi(x) \in L_q$; where,*

i) $\frac{1}{p} + \frac{1}{q} \leq 1 + \gamma$,

ii) $\frac{1}{p} + \frac{1}{q} = \gamma + 1$, *if* $p \neq 1$ *and* $q \neq 1$.

Lemma 43 ([3],[4]) *Let $\psi(t) \in C(a,b)$, then*

$$I_{a+}^{\gamma}I_{a+}^{\beta}\psi = I_{a+}^{\gamma+\beta}\psi,\ I_{b-}^{\gamma}I_{b-}^{\beta}\psi = I_{b-}^{\gamma+\beta}\psi,\ \text{where } \gamma > 0, \beta > 0. \tag{3.28}$$

Proof. Now let $\psi(t) \in C(a,b)$ then,

$$I_{a+}^{\gamma}I_{a+}^{\beta}\psi = \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_a^x \left\{\int_a^t \frac{\psi(\upsilon)}{(x-\upsilon)^{1-\beta}}d\upsilon\right\}\frac{dt}{(x-t)^{1-\gamma}}.$$

Using the Dirichlet formula we have,

$$\begin{aligned}
&= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_{\upsilon}^{x}\frac{1}{(x-t)^{1-\gamma}}dt\int_{a}^{x}\frac{\psi(\upsilon)}{(t-\upsilon)^{1-\beta}}d\upsilon \qquad (3.29)\\
&= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_{a}^{x}\psi(\upsilon)\,d\upsilon\int_{\upsilon}^{x}\frac{1}{(t-\upsilon)^{1-\beta}(x-t)^{1-\gamma}}dt.
\end{aligned}$$

In the the second integral, take $t=\upsilon+p(x-\upsilon)$, we have:

$$\begin{aligned}
\int_{\upsilon}^{x}\frac{1}{(t-\upsilon)^{1-\beta}(x-t)^{1-\gamma}}dt &= \int_{\upsilon}^{x}\frac{1}{(\upsilon+p(x-\upsilon)-\upsilon)^{1-\beta}(x-\upsilon-p(x-\upsilon))^{1-\gamma}}(x-\upsilon)\,dp\\
&= \int_{\upsilon}^{x}\frac{(x-\upsilon)}{[p(x-\upsilon)]^{1-\beta}(x-\upsilon)^{1-\gamma}(1-p)^{1-\gamma}}dp\\
&= \int_{\upsilon}^{x}\frac{(x-\upsilon)}{(x-\upsilon)^{2-\beta-\gamma}p^{1-\beta}(1-p)^{1-\gamma}}dp\\
&= \int_{\upsilon}^{x}\frac{1}{(x-\upsilon)^{1-\beta-\gamma}p^{1-\beta}(1-p)^{1-\gamma}}dp\\
&= \frac{1}{(x-\upsilon)^{1-\gamma-\beta}}\int_{\upsilon}^{x}\frac{1}{p^{1-\beta}(1-p)^{1-\gamma}}dp\\
&= \frac{1}{(x-\upsilon)^{1-\gamma-\beta}}B(\gamma,\beta). \qquad (3.30)
\end{aligned}$$

Writing (3.30) in (3.29) we have,

$$\begin{aligned}
I_{a+}^{\gamma}I_{a+}^{\beta}\psi &= \frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_{a}^{x}\psi(\upsilon)\frac{1}{(x-\upsilon)^{1-\gamma-\beta}}B(\gamma,\beta)\,d\upsilon\\
&= \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)}\frac{1}{\Gamma(\gamma)\Gamma(\beta)}\int_{a}^{x}\frac{\psi(\upsilon)}{(x-\upsilon)^{1-\gamma-\beta}}d\upsilon\\
&= \frac{1}{\Gamma(\gamma+\beta)}\int_{a}^{x}\frac{\psi(\upsilon)}{(x-\upsilon)^{1-\gamma-\beta}}d\upsilon.
\end{aligned}$$

proving (3.28).

The equations in (3.28) are called "a semigroup property of fractional integration". It is natural to show that fractional differentiation is an operation inverse to fractional integration. For this consider the definition below: ■

Remark 44 *Equation (3.28) holds almost all* $\psi(t) \in L_1(a,b)$ *when* $\gamma+\beta \geq 1$.

Definition 45 ([3],[4],[2]) *The left and right-handed Riemann-Liouville fractional derivatives of order* γ, *for a functions* $h(x)$ *on interval* $[a,b]$ *are defined as follows:*

$$\left(D_{a+}^{\gamma}h\right)(x) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_{a}^{x}\frac{h(t)}{(x-t)^{\gamma}}dt \tag{3.31}$$

$$\left(D_{b-}^{\gamma}h\right)(x) = -\frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_{x}^{b}\frac{h(t)}{(t-x)^{\gamma}}dt. \tag{3.32}$$

In the next lemma, we will provide a sufficient condition for the existence of fractional derivatives.

Lemma 46 ([4]) *Let* $h(x)$ *be an absolutely continuous function on* $[a,b]$ *then* $D_{a+}^{\gamma}h$ *and* $D_{b-}^{\gamma}h$ *exist almost everywhere for* $\gamma \in (0,1)$. *Moreover* $D_{a+}^{\gamma}h$, $D_{b-}^{\gamma}h \in L_{\upsilon}(a,b)$ *for* $1 \leq \upsilon < \frac{1}{\gamma}$, *and*

$$D_{a+}^{\gamma}h = \frac{1}{\Gamma(1-\gamma)}\left[\frac{h(a)}{(x-a)^{\gamma}} + \int_{a}^{x}\frac{h'(t)}{(x-t)^{\gamma}}dt\right] \tag{3.33}$$

$$D_{b-}^{\gamma}h = \frac{1}{\Gamma(1-\gamma)}\left[\frac{h(b)}{(b-x)^{\gamma}} - \int_{x}^{b}\frac{h'(t)}{(t-x)^{\gamma}}dt\right]. \tag{3.34}$$

Proof. Using conditions given in the statement of the Lemma and the definition of fractional derivative we get,

$$\begin{aligned}
\left(D_{a+}^{\gamma}h\right)(x) &= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_{a}^{x}\frac{h(t)}{(x-t)^{\gamma}}dt \\
&= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_{a}^{x}\left(h(a)+\int_{a}^{t}h'(u)\,du\right)(x-t)^{-\gamma}\,dt \\
&= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\left(h(a)\int_{a}^{x}(x-t)^{-\gamma}\,dt+\int_{a}^{x}\int_{a}^{t}\frac{h'(u)}{(x-t)^{\gamma}}dudt\right) \\
&= \frac{1}{\Gamma(1-\gamma)}\left(\frac{h(a)}{(x-a)^{\gamma}}+\frac{d}{dx}\int_{a}^{x}\int_{a}^{t}\frac{h'(u)}{(x-t)^{\gamma}}dudt\right) \\
&= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_{a}^{x}\left(h(a)+\int_{a}^{t}h'(u)\,du\right)(x-t)^{-\gamma}\,dt \\
&= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\left(h(a)\int_{a}^{x}(x-t)^{-\gamma}\,dt+\int_{a}^{x}\int_{a}^{t}\frac{h'(u)}{(x-t)^{\gamma}}dudt\right) \\
&= \frac{1}{\Gamma(1-\gamma)}\left(\frac{h(a)}{(x-a)^{\gamma}}+\frac{d}{dx}\int_{a}^{x}\int_{a}^{t}\frac{h'(u)}{(x-t)^{\gamma}}dudt\right).
\end{aligned}$$

We use the Dirichlet formula in second term and we get:

$$\begin{aligned}
\left(D_{a+}^{\gamma}h\right)(x) &= \frac{1}{\Gamma(1-\gamma)}\left[\frac{h(a)}{(x-a)^{\gamma}}+\frac{d}{dx}\left[\int_a^x h'(u)\,du\int_u^x (x-t)^{-\gamma}\,dt\right]\right] \\
&= \frac{1}{\Gamma(1-\gamma)}\left[\frac{h(a)}{(x-a)^{\gamma}}+\frac{d}{dx}\left[\int_a^x h'(u)\,du\frac{(x-u)^{-\gamma+1}}{-\gamma+1}\right]\right] \\
&= \frac{1}{\Gamma(1-\gamma)}\left[\frac{h(a)}{(x-a)^{\gamma}}+\int_a^x h'(u)\,du\frac{d}{dx}\frac{(x-u)^{-\gamma+1}}{-\gamma+1}\right] \\
&= \frac{1}{\Gamma(1-\gamma)}\left[\frac{h(a)}{(x-a)^{\gamma}}+\int_a^x \frac{h'(u)}{(x-u)^{\gamma}}du\right].
\end{aligned}$$

■

Example 47 ([4]) *Consider the function* $h(x)=(x-a)^{-\eta}$, $0<\eta<1$, *then*

$$\begin{aligned}
\left(D_{a+}^{\gamma}h\right)(x) &= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_a^x \frac{h(t)}{(x-t)^{\gamma}}dt \\
&= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_a^x \frac{(x-t)^{-\eta}}{(x-t)^{\gamma}}dt \\
&= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_a^x (x-t)^{-\eta-\gamma}\,dt. \qquad (3.35)
\end{aligned}$$

Changing the variable t by $a+p(x-a)$ *in (3.35) we have,*

$$\int_a^x (x-t)^{-\eta-\gamma}\,dt=(x-a)^{1-\eta-\gamma}\int_0^1 (1-p)^{-\gamma}p^{-\eta}dp.$$

Substitute into (3.35) we have,

$$\begin{aligned}
\left(D_{a+}^{\gamma}h\right)(x) &= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_0^1 \frac{p^{-\eta}(x-a)^{1-\eta}}{(x-a)^{\gamma}(1-p)^{\gamma}}dp \\
&= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\left[(x-a)^{1-\eta-\gamma}\right]\int_0^1 (1-p)^{-\gamma}p^{-\eta}dp \\
&= \frac{1}{\Gamma(1-\gamma)}(1-\eta-\gamma)(x-a)^{1-\eta-\gamma-1}\frac{\Gamma(1-\gamma)\Gamma(1-\eta)}{\Gamma(2-\gamma-\eta)} \\
&= \frac{\Gamma(1-\eta)}{\Gamma(1-\gamma-\eta)}\frac{1}{(x-a)^{\eta+\gamma}}.
\end{aligned} \tag{3.36}$$

Example 48 ([4]) *Consider the function*

$$h(x) = \frac{1}{(x-a)^{1-\gamma}}$$

where $0 < \gamma < 1$, *then*

$$\begin{aligned}
\left(D_{a+}^{\gamma}h\right)(x) &= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_a^x (x-a)^{\gamma-1}(x-t)^{-\gamma}dt \\
&= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}(x-a)^{\gamma-1}\int_a^x (x-t)^{-\gamma}dt \\
&= \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}(x-a)^{\gamma-1}\left[\frac{-(x-t)^{-\gamma+1}}{-\gamma+1}\right]_a^x \\
&= 0.
\end{aligned}$$

We have

$$\left(D_{a+}^{\gamma}h\right)(x) \equiv 0 \textit{ where } h(x) = \frac{1}{(x-a)^{1-\gamma}}. \tag{3.37}$$

Now let us assume that $\gamma \geq 1$. In such cases we will use the notation $[\gamma]$ to represent the integer part of a number γ and $\{\gamma\}$ to represent the fractional part of γ. It is obvious that for any real number γ, $0 \leq \{\gamma\} < 1$ and

$$\gamma = [\gamma] + \{\gamma\}. \tag{3.38}$$

Definition 49 *If γ is an integer then*

$$D_{a+}^{\gamma} = \left(\frac{d}{dx}\right)^{\gamma} \tag{3.39}$$

and

$$D_{b-}^{\gamma} = \left(-\frac{d}{dx}\right)^{\gamma}.$$

Definition 50 *If γ is not an integer then*

$$\begin{aligned} D_{a+}^{\gamma} h &= \left(\frac{d}{dx}\right)^{[\gamma]} D_{a+}^{\{\gamma\}} h \\ &= \left(\frac{d}{dx}\right)^{[\gamma]+1} I_{a+}^{1-\{\gamma\}} h \\ &= \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dx}\right)^{n} \int_{a}^{x} \frac{h(t)}{(x-t)^{\gamma-n+1}} dt, \textit{ where } n = [\gamma] + 1 . \end{aligned} \tag{3.40}$$

and

$$D^{\gamma}_{b-}h = \left(-\frac{d}{dx}\right)^{[\gamma]} D^{\{\gamma\}}_{b-}h = \left(-\frac{d}{dx}\right)^{[\gamma]+1} I^{1-\{\gamma\}}_{b-}h = \frac{(-1)^n}{\Gamma(n-\gamma)}\left(\frac{d}{dx}\right)^n \int_x^b \frac{h(t)}{(t-x)^{\gamma-n+1}}dt, \tag{3.41}$$

where $n = [\gamma] + 1$.

Remark 51 *Using definitions we see that,*

$$D^{\gamma}_{a+}h = I^{-\gamma}_{a+}h = (I^{\gamma}_{a+})^{-1}h$$

and

$$D^{\gamma}_{b-}h = I^{-\gamma}_{b-}h = (I^{\gamma}_{b-})^{-1}h.$$

Remark 52 *The fractional derivatives formula (3.31) an (3.32), are exist if*

$$\int_a^x \frac{h(t)}{(x-t)^{\{\gamma\}}} \in AC^{[\gamma]}([a,b])$$

or equivalently

$$h(x) \in AC^{[\gamma]}([a,b]).$$

Lemma 53 *If* $h(x) = (x-a)^{\gamma-k}$, $k = 1, 2, ..., [\gamma] + 1$ *then* $\left(D^{\gamma}_{a+}h\right)(x) \equiv 0$.

Proof. It is not difficult to verify that (3.36) is true for any $\gamma > 0$ and similarly for (3.37). Recall that,

$$D^{\gamma}_{a+}(x-a)^{\gamma-k} = \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{(t-a)^{\gamma-k}}{(x-k)^{\gamma-n+1}}dt, \qquad n = [\gamma]+1.$$

Changing t by $a + p(x-a)$ we have:

$$\begin{aligned} D^{\gamma}_{a+}(x-a)^{\gamma-k} &= \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{dx}\right)^n \int_0^1 \frac{p^{\gamma-k}(x-a)^{\gamma-k+1}}{(x-a)^{\gamma-n+1}(1-p)^{\gamma-n+1}}dp \\ &= \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{dx}\right)^n \left[(x-a)^{n-k}\int_0^1 p^{\gamma-k}(1-p)^{\gamma-n+1}\,dp\right] \\ &\equiv 0. \end{aligned}$$

$n = [\gamma]+1, n-k < n, k = 1,2,\ldots$ ■

3.3 Fractional Integrals and Derivatives of Complex Order

In this section we focus on fractional integral and derivatives of complex order. Recall that, for a complex number $\gamma = \gamma_0 + i\theta$, If $\gamma_0 = 0$ then $\gamma = i\theta$ is called purely imaginary complex number.

Definition 54 ([3],[4]) *Let $\gamma = i\theta$ then the formula*

$$\left(D^{\gamma}_{a+}h\right)(x) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_a^x \frac{h(t)}{(x-t)^{\gamma}}dt.$$

below make sense. Thus, replace γ by $i\theta$ we have,

$$\left(D^{i\theta}_{a+}h\right)(x) = \frac{1}{\Gamma(1-i\theta)}\frac{d}{dx}\int_a^x h(t)(x-t)^{i\theta}\,dt\,. \tag{3.42}$$

The formula given in (3.24) does not work for the fractional integral of purely imaginary order since the integral is divergent for $\gamma = i\theta$. Therefore, in this case we need a different definition which is given below.

Definition 55 ([3],[4]) *Let* $\gamma = i\theta$*, then*

$$\begin{aligned} I_{a+}^{i\theta}h &= \frac{d}{dx}I_{a+}^{1+i\theta}h \\ &= \frac{1}{\Gamma(1+i\theta)}\frac{d}{dx}\int_a^x h(t)(x-t)^{i\theta}dt, \quad x>a \end{aligned} \tag{3.43}$$

and

$$I_{b-}^{i\theta}h = \frac{1}{\Gamma(1+i\theta)}\frac{d}{dx}\int_x^b (t-x)^{i\theta}h(t)dt. \tag{3.44}$$

In order to extend above definition to all complex number we need to define the identity operator, $D_{a+}^{0}\psi$.

Definition 56 *The identity operation,* D_{a+}^{0}*, acts on* ψ *as follows;*

$$D_{a+}^{0}\psi = I_{a+}^{0}\psi = \psi \tag{3.45}$$

and in (3.42), take $\gamma = 0$

$$\left(D_{a+}^{0}\psi\right)(x) = \frac{1}{\Gamma(1)}\frac{d}{dx}\int_a^x \psi(t)dt = \frac{1}{\Gamma(1)}\frac{d}{dx}\int_a^x \psi(t)dt = \left(I_{a+}^{0}\psi\right).$$

Lemma 57 ([4]) *Let $h(x)$ be an absolutely continuous function on $[a,b]$ then $D^{i\theta}_{a+}h$ exists for all x and it may be represented in the form*

$$D^{\gamma}_{a+}h = \frac{1}{\Gamma(1-\gamma)}\left[h(a)(x-a)^{-\gamma} + \int_a^x h'(t)(x-t)^{-\gamma}\,dt\right]$$

(3.33) with $\gamma = i\theta$.

Proof. Assume, $h(x) \in AC[a,b]$ then $h_{1-\gamma}(x) \in AC([a,b])$ where

$$h(t) = h(a) + \int_a^t h'(p)dp \tag{3.46}$$

and

$$h_{1-\gamma}(x) = \frac{1}{\Gamma(1-i\theta)}\int_a^x h(t)(x-t)^{-i\theta}\,dt. \tag{3.47}$$

Now use (3.46) in (3.47) we have,

$$
\begin{aligned}
h_{1-\gamma}(x) &= \frac{1}{\Gamma(1-i\theta)}\int_a^x \left[h(a)+\int_a^t h'(p)\,dp\right](x-t)^{-i\theta}\,dt \\
&= \frac{1}{\Gamma(1-i\theta)}\left[h(a)\int_a^x (x-t)^{-i\theta}\,dt+\int_a^x\int_a^t h'(p)(x-t)^{-i\theta}\,dpdt\right] \\
&= \frac{1}{\Gamma(1-i\theta)}\left[h(a)\int_a^x (x-t)^{-i\theta}\,dt+\int_a^x (x-t)^{-i\theta}\,dt\int_a^t h'(p)\,dp\right] \\
&= \frac{1}{\Gamma(1-i\theta)}\left[h(a)\frac{(x-a)^{1-i\theta}}{(1-i\theta)}+\int_a^x h'(p)\,dp\int_p^x (x-t)^{-i\theta}\,dt\right] \\
&= \frac{1}{\Gamma(1-i\theta)}\left[h(a)\frac{(x-a)^{1-i\theta}}{(1-i\theta)}+\int_a^x h'(p)\frac{(x-p)^{1-i\theta}}{(1-i\theta)}\,dp\right] \\
&= \frac{1}{\Gamma(2-i\theta)}\left[h(a)(x-a)^{1-i\theta}+\int_a^x h'(p)(x-p)^{1-i\theta}\,dp\right].
\end{aligned}
$$

But,

$$
\begin{aligned}
h'_{1-\gamma}(x) &= \psi(x) \\
&= \frac{1}{\Gamma(2-i\theta)}\left[h(a)(1-i\theta)(x-a)^{-i\theta}+\int_a^x h'(t)(1-i\theta)(x-p)^{-i\theta}\,dp\right] \\
&= \frac{1}{\Gamma(1-i\theta)}\left[h(a)(x-a)^{-i\theta}+\int_a^x h'(t)(x-t)^{-i\theta}\,dp\right]
\end{aligned}
$$

which is (3.33) where $\gamma = i\theta$. ■

Lemma 58 ([4]) *The space $AC^n[a,b]$ consists of those and only those functions $h(x)$, which are represented in the form:*

$$
h(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}\psi(t)\,dt+\sum_{k=0}^{n-1} c_k(x-a)^k
$$

where $\psi(t) \in L_1(a,b)$ and c_k is constant.

Proof. Assume that $\psi(t) = h^{(n)}(t)$ and $c_k = \frac{h^{(k)}(a)}{k!}$ then

$$\begin{aligned} h^{(n-1)}(x) &= c + \int_a^x \psi(t)\,dt \\ &= c + \int_a^x h^{(n)}(t)\,dt \end{aligned}$$

which implies that

$$\begin{aligned} \int_a^x h^{(n-1)}(x)\,dx &= c(x-a) + \int_a^x \int_a^x h^{(n)}(t)\,dt \\ &= c(x-a) + \int_a^x \int_a^x \psi(t)\,dt. \end{aligned}$$

On the other hand,

$$h^{(n-2)}(x) - h^{(n-2)}(a) = c(x-a) + \int_a^x \int_a^x \psi(t)\,dt.$$

Continuing in this way we obtain that,

$$h(x) = \sum_{k=0}^{n-1} (x-a)^k c_k + \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} \psi(t)\,dt.$$

■

Theorem 59 ([4]) *Let $Re\gamma \geq 0$ and $h(x) \in AC^n[a,b]$, $n = [\mathrm{Re}\,\gamma] + 1$. Then $D_{a+}^{\gamma} h$ exists almost everywhere and may be represented in the form.*

$$D_{a+}^{\gamma}h = \sum_{k=0}^{n-1}\frac{h^{(k)}(a)}{\Gamma(1+k-\gamma)}(x-a)^{k-\gamma} + \frac{1}{\Gamma(n-\gamma)}\int_a^x \frac{h^{(n)}(t)}{(x-t)^{\gamma-n+1}}dt. \quad (3.48)$$

Lemma 60 ([4]) *Let $\psi(t) \in L_1(a,b)$. The homogenous Abel integral equation $I_{a+}^{\gamma}\psi \equiv 0$ has only trivial solution $\psi(x) \equiv 0$ for any γ with* $\operatorname{Re}\gamma > 0$.

Proof. Let $m = [\operatorname{Re}\gamma]$, and let $\operatorname{Re}\gamma \neq 1, 2, \ldots$. Differentiating m times the equality $I_{a+}^{\gamma}\psi = 0$, we have,

$$I_{a+}^{\gamma-m}\psi = 0.$$

It is obvious that, $0 < \operatorname{Re}(\gamma - m) < 1$, so ψ in view of Theorem3.0.33, which is valid for complex exponent. If $\gamma = m - i\theta$, differentiating $(m-1)$ times, the result $I_{a+}^{\gamma}\psi = 0$ and

$$\int_a^x \psi(t)(x-t)^{-i\theta}dt = 0.$$

If $\theta = 0$, clearly $\psi(x) = 0$, a.e.

If $\theta \neq 0$, then replace x to t, t to s multiply both sides by $\frac{1}{(x-t)^{1+i\theta}}$ and integrate both sides over $[a, x-\varepsilon]$, to get

$$\int_a^{x-\varepsilon}\left[\int_a^t (t-s)^{-i\theta}\psi(s)\right]\frac{1}{(x-t)^{1+i\theta}}dt = 0$$

$$\int_a^{x-\varepsilon}\psi(s)\left[\int_s^{x-\varepsilon}(t-s)^{-i\theta}(x-t)^{-1-i\theta}dt\right]ds = 0.$$

Change the variable, $\varepsilon = \frac{t-s}{x-s}$,

$$\int_{a}^{x-\varepsilon} \psi(s) \int_{a}^{1-\frac{\varepsilon}{x-s}} \varepsilon^{i\theta} (1-\varepsilon)^{-1-i\theta} d\varepsilon ds = 0. \tag{3.49}$$

Since $\psi(s) \in L_1$, the passage to the limit is possible under the first integral sign if the inner integral converges as $\varepsilon \to 0$. To show this, we need some facts, for the imaginary order Beta function. It is known that the Beta function is defined by (2.17).(2.17) make sense when $\operatorname{Re} p = 0$ or $\operatorname{Re} q = 0\,(p \neq 0, q \neq 0)$. In this case, it is understood to be conditionally convergent.In particular, there exists the limit

$$B(p, i\theta) = \lim_{\varepsilon \to 0} \int_{0}^{1-\varepsilon} t^{p-1} (1-t)^{i\theta-1} dt, \qquad \operatorname{Re} p > 0, \theta \neq 0 \tag{3.50}$$

which coincides with the analytic continuation of $B(p,q)$ with respect to the values $\operatorname{Re} q = 0, q \neq 0$. The inner integral in (3.49) converges as $\varepsilon \to 0$. So letting $\varepsilon \to 0$ in (3.49), we have by (3.50) that

$$B(1-i\theta, i\theta) \int_{a}^{s} \psi(s) ds = 0$$

$\psi(s) = 0$ a.e which completes the proof. ■

3.4 Fractional Integrals of Some Elementary Functions

In this section we shall evaluate fractional integral of some well known functions.

Lemma 61 *Consider the power function*

$$\psi(x) = (x-a)^{\beta-1}$$

and assume that $\mathrm{Re}\beta > 0$*, then*

$$I_{a+}^{\gamma}\psi = (x-a)^{\beta+\gamma-1}\frac{\Gamma(\beta)}{\Gamma(\beta+\gamma)} \tag{3.51}$$

where $a\epsilon\mathbb{C}$*.*

Proof. *Let we use (3.24) and taking* $t = a + (x-a)\,p$*,*

$$\begin{aligned} I_{a+}^{\gamma}\psi(x) &= \frac{1}{\Gamma(\gamma)}\int_{a}^{x}\frac{(x-a)^{\beta-1}}{(x-t)^{1-\gamma}}dt \\ &= \frac{(x-a)^{\beta+\gamma-1}}{\Gamma(\gamma)}\int_{0}^{1}p^{\beta-1}(1-p)^{\gamma-1} \\ &= (x-a)^{\beta+\gamma-1}\frac{\Gamma(\beta)}{\Gamma(\beta+\gamma)}\;\; a\epsilon C. \end{aligned}$$

■

Lemma 62 *Consider the power function*

$$\psi(x) = (b-x)^{\beta-1}$$

and assume that $\mathrm{Re}\beta > 0$*, then*

$$I_{b-}^{\gamma}\psi = (b-x)^{\beta+\gamma-1}\frac{\Gamma(\beta)}{\Gamma(\beta+\gamma)}. \tag{3.52}$$

Proof. *Let we use (3.25) and taking $t = a + (b - x)\,p$,*

$$\begin{aligned} I_{b-}^{\gamma}\psi(x) &= \frac{1}{\Gamma(\gamma)}\int_x^b \frac{(b-x)^{\beta-1}}{(t-x)^{1-\gamma}}dt \\ &= \frac{(b-x)^{\beta+\gamma-1}}{\Gamma(\gamma)}\int_0^1 p^{\beta-1}(1-p)^{\gamma-1} \\ &= (b-x)^{\beta+\gamma-1}\frac{\Gamma(\beta)}{\Gamma(\beta+\gamma)}. \end{aligned}$$

■

Lemma 63 *Consider the function*

$$I_{a+}^{\gamma}\left[\frac{(x-a)^{\beta-1}}{(b-x)^{\gamma+\beta}}\right] = \frac{1}{(b-a)^{\gamma}}\frac{\Gamma(\beta)}{\Gamma(\beta+\gamma)}\frac{(x-a)^{\gamma+\beta-1}}{(b-x)^{\beta}}, \quad a < x < b.$$

Proof. We use, (3.24) we get:

$$I_{a+}^{\gamma}\psi = \frac{1}{\Gamma(\gamma)}\int_a^x \frac{(x-a)^{\beta-1}(b-x)^{\gamma-1}}{(x-t)^{1-\gamma}}dt$$

Changing to variable; $t = a + p(x-a)$, we have:

$$\begin{aligned} I_{a+}^{\gamma}\psi &= \frac{1}{\Gamma(\gamma)}(x-a)^{\beta+\gamma-1}\int_0^1 p^{\beta-1}(1-p)^{\gamma-1}\left(1-\left(\frac{x-a}{b-a}\right)p\right)^{\gamma-1}dp \\ &= \frac{(x-a)^{\beta+\gamma-1}}{\Gamma(\gamma)}B(\beta,\gamma)_2F_1\left(1-\gamma,\beta,\gamma+\beta;\frac{x-a}{b-a}\right) \end{aligned}$$

$$I_{a+}^{\gamma}\psi = (b-x)^{\gamma-1}\frac{\Gamma(\beta)}{\Gamma(\beta+\gamma)}(x-a)^{\beta+\gamma-1}\,{}_2F_1\left(1-\gamma,\beta,\gamma+\beta;\frac{x-a}{b-a}\right). \tag{3.53}$$

Useful particular:

$$I^{\gamma}_{a+}\left[\frac{(x-a)^{\beta-1}}{(b-x)^{\gamma+\beta}}\right] = \frac{1}{(b-a)^{\gamma}}\frac{\Gamma(\beta)}{\Gamma(\beta+\gamma)}\frac{(x-a)^{\gamma+\beta-1}}{(b-x)^{\beta}} \qquad a<x<b\,. \tag{3.54}$$

■

3.5 Fractional Integration and Differentiation as Reciprocal Operations

As we know differentiation and integration has the following property

$$\left(\frac{d}{dx}\right)\int_a^x \psi(t)\,dt = \psi(x)$$

but in general

$$\int_a^x \psi'(t)\,dt \neq \psi(x)$$

because of the constant value $\psi(a)$. Similarly

$$\left(\frac{d}{dx}\right)^n I^n_{a+}\psi \equiv \psi,$$

but

$$I^n_{a+}\psi^{(n)} \neq \psi.$$

Therefore we can state

$$D^{\gamma}_{a+}I^{\gamma}_{a+}\psi = \psi, \tag{3.55}$$

but in general $I^{\gamma}_{a+}D^{\gamma}_{a+}\psi$ is not equal to $\psi(x)$.

Definition 64 ([4]) *Let* $I^{\gamma}_{a+}\left(L_p\right)$*,* $\operatorname{Re}\gamma > 0$*, denote the space of functions* $h(x)$*, which can be represented by the left-sided fractional integral of order* γ *of a summable function i.e.*

$$h \in I^{\gamma}_{a+}\left(L_p\right)$$

means

$$h = I^{\gamma}_{a+}\psi, \ \textit{for some } \psi \in L_p(a,b), 1 \leq p \leq \infty.$$

Theorem 65 ([4])*In order to have,* $h(x) \in I^{\gamma}_{a+}(L_1)$, $\operatorname{Re}\gamma > 0$*, it is neccessary and sufficient that*

$$h_{n-\gamma}(x) = I^{n-\gamma}_{a+}h \in AC^n([a,b]) \tag{3.56}$$

where $n = [\operatorname{Re}\gamma] + 1$ and that

$$h^{(k)}_{n-\gamma}(a) = 0, \ \ k = 1,2,...,n-1. \tag{3.57}$$

Proof. Let $h = I^{\gamma}_{a+}(\psi)$ and $\psi \in L_1(a,b)$. Because of the semigroup property we have $I^{n-\gamma}_{a+}h = I^{\gamma}_{a+}(\psi)$, where $\psi \in L_1(a,b)$ and $I^{n-\gamma}_{a+}h \in AC^n([a,b])$. Therefore we get (3.56) and then (3.57) is satisfied. Conversly, let (3.56) and (3.57) be satisfied, we can write $h_{n-\gamma}(x) = I^{n}_{a+}h$ where $\psi \in L_1(a,b)$. Consequently, by semigroup property;

$$I^{n-\gamma}_{a+}h = I^{n}_{a+}\psi = I^{n-\gamma}I^{\gamma}\psi.$$

Hence

$$I_{a+}^{n-\gamma}\left[h-I_{a+}^{\gamma}\psi\right]=0.$$

Since $\operatorname{Re}(n-\gamma)>0$, by Lemma 3.0.59 we have that $h-I_{a+}^{\gamma}\psi=0$ a.e therefore the proof is completed. ■

Definition 66 *Let* $\operatorname{Re}\gamma>0$*, a function* $h(x)\in L_1(a,b)$ *is said to have a summable fractional derivative* $D_{a+}^{\gamma}h$*, if* $I_{a+}^{n-\gamma}h\in AC^n([a,b])$*,* $n=[\operatorname{Re}\gamma]+1$.

Remark 67 *If* $I_{a+}^{n-\gamma}h$ *is n times differentiable at every point i.e* $D_{a+}^{\gamma}h=\left(\frac{d}{dx}\right)^n I_{a+}^{n-\gamma}h$ *exist then* $h(x)$ *has a summable fractional derivatives.*

Theorem 68 ([4]) *Let* $\operatorname{Re}\gamma>0$*. Then the equality*

$$D_{a+}^{\gamma}I_{a+}^{\gamma}\psi=\psi(x) \tag{3.58}$$

is valid for any summable function $\psi(x)$ *while*

$$I_{a+}^{\gamma}D_{a+}^{\gamma}h=h(x) \tag{3.59}$$

is satisfied for

$$h(x)\in I_{a+}^{\gamma}(L_1). \tag{3.60}$$

If we assume that instead of (3.60) a function $h(x)\in L_1(a,b)$ *has a summable derivative*

$D^{\gamma}_{a+}h$ then (3.59) is not true in general and is to be replaced by the result

$$I^{\gamma}_{a+}D^{\gamma}_{a+}h = h(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\gamma-k-1}}{\Gamma(\gamma-k)} h^{(n-k-1)}_{n-\gamma}(a), \tag{3.61}$$

where $n = [\operatorname{Re}\gamma] + 1$ and $h_{n-\gamma}(x) = I^{n-\gamma}_{a+}h$. In particular we have:

$$I^{\gamma}_{a+}D^{\gamma}_{a+}h = h(x) - \frac{h_{1-\gamma}(x)}{\Gamma(\gamma)}(x-a)^{\gamma-1}, \tag{3.62}$$

for $0 < \operatorname{Re}\gamma < 1$.

Proof. *By the definitions we have,*

$$D^{\gamma}_{a+}I^{\gamma}_{a+}\psi = \frac{1}{\Gamma(\gamma)\Gamma(n-\gamma)} \left(\frac{d}{dx}\right)^{n} \int_{a}^{x} \left[\int_{a}^{t} \frac{\psi(p)}{(t-p)^{1-\gamma}} dp \right] \frac{dt}{(x-t)^{\gamma-n+1}}.$$

Interchanging the order of integration and evaluating the inner integral we get:

$$D^{\gamma}_{a+}I^{\gamma}_{a+}\psi = \frac{1}{\Gamma(n)} \left(\frac{d}{dx}\right)^{n} \int_{a}^{x} \psi(p)(x-p)^{n-1}.$$

Therefore (3.58) follows by (3.1) and (2.17). To prove (3.59); with the assumption (3.60) immediately follows from (3.58). ■

Corollary 69 ([4]) *Assume that $h(x)$ have a summable derivative $D^{\gamma+n}_{a+}h$ in the sense of the above definition. Then,*

$$h(x) = \sum_{j=-n}^{n-1} \frac{\left(D^{\gamma+j}_{a+}h\right)(a)}{\Gamma(\gamma+j+1)}(x-a)^{\gamma+j} + R_n(x), \ (\operatorname{Re}\gamma > 0) \tag{3.63}$$

is valid for where $R_n(x) = \left(I^{\gamma+n}_{a+}D^{\gamma+n}_{a+}h\right)(x)$.

Corollary 70 ([4]) *Assume that* $h(x) \in I^{\gamma}_{b-}\left(L_p\right)$, $g(x) \in I^{\gamma}_{a+}\left(L_q\right)$ *;* $\frac{1}{p}+\frac{1}{q} \le 1+\gamma$, *then*

$$\int_a^b h(x)\left(D^{\gamma}_{a+}g\right)(x)\,dx = \int_a^b g(x)\left(D^{\gamma}_{b-}h\right)(x)\,dx \quad (0 < \mathrm{Re}\,\gamma < 1). \tag{3.64}$$

Simple sufficiency conditions for functions$h(x), g(x)$ to satisfy (3.64) is that $h(x), g(x)$ should be continuous and $\left(D^{\gamma}_{a+}g\right)(x)$ and $\left(D^{\gamma}_{b-}h\right)(x)$ exists at every point $x \in [a,b]$ and they are also continuous.

In the following part section we will use the notations of (**??**) to represent fractional integral and fractional derivatives. Consedering $I^{\gamma}_{a+} = D^{\gamma}_{a+}$ for $\mathrm{Re}\,\gamma < 0$.

Theorem 71 ([4]) *Assume that* $\mathrm{Re}\beta > 0$, $\mathrm{Re}(\gamma+\beta) > 0$ *and* $\psi(x) \in L_1(a,b)$ *then,*

$$I^{\gamma}_{a+}I^{\beta}_{a+}\psi = I^{\gamma+\beta}_{a+}\psi. \tag{3.65}$$

Proof. In the case $\mathrm{Re}(\gamma) > 0$ and $\mathrm{Re}(\beta) > 0$, the semigroup property (3.65) is already established in (3.28). Let's consider the case $\mathrm{Re}(\gamma) = 0$, $\mathrm{Re}\beta > 0$, letting $\gamma = i\theta$. Then,

$$\begin{aligned}
I^{i\theta}_{a+}I^{\beta}_{a+}\psi &= \frac{1}{\Gamma(\beta)\Gamma(1+i\theta)}\frac{d}{dx}\int_a^x \psi(s)\,ds\int_s^x (x-t)^{i\theta}(t-s)^{\beta-1}\,dt \\
&= \frac{B(1+i\theta,\beta)}{\Gamma(\beta)\Gamma(1+i\theta)}\frac{d}{dx}\int_a^x \psi(s)(t-s)^{i\theta+\beta}\,ds \\
&= \frac{1}{\Gamma(1+i\theta+\beta)}\frac{d}{dx}\int_a^x \psi(s)(t-s)^{i\theta+\beta}\,ds
\end{aligned}$$

$$= \frac{d}{dx} I_{a+}^{i\theta+\beta+1}\psi. \tag{3.66}$$

Since $\mathrm{Re}(1+i\theta+\beta) = \mathrm{Re}(\beta)+1 > 1$, and (3.65) is already proved in the case $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\beta) > 0$, we have;

$$I_{a+}^{i\theta+\beta+1}\psi = I_{a+}^{1}\left(I_{a+}^{i\theta+\beta}\psi\right) = \int_a^x \left(I_{a+}^{i\theta+\beta}\psi\right)(t)\,dt$$

so by (3.66) we get,

$$\begin{aligned} I_{a+}^{i\theta} I_{a+}^{\beta}\psi &= \frac{d}{dx}\left[\int_a^x \left(I_{a+}^{i\theta+\beta}\psi\right)(t)\,dt\right] \\ &= I_{a+}^{i\theta+\beta}\psi \end{aligned}$$

which is (3.65), when $\gamma = i\theta$. It remains to consider the case $\mathrm{Re}(\gamma) < 0$, then use (3.65)

$$\begin{aligned} I_{a+}^{\gamma} I_{a+}^{\beta}\psi &= D_{a+}^{-\gamma} I_{a+}^{-\gamma+\beta+\gamma}\psi \\ &= D_{a+}^{-\gamma} I_{a+}^{-\gamma} I_{a+}^{\beta+\gamma}\psi \end{aligned} \tag{3.67}$$

from (3.65), because $\mathrm{Re}(-\gamma) > 0$, $\mathrm{Re}(\gamma+\beta) > 0$. By (3.58), since

$$I_{a+}^{\gamma} I_{a+}^{\beta}\psi = I_{a+}^{\gamma+\beta}\psi$$

which is (3.65). ■

Theorem 72 ([4]) *Assume that* $\mathrm{Re}\,\gamma < 0$, $\mathrm{Re}(\gamma+\beta) < 0$ *and* $\psi(x) \in I_{a+}^{-\gamma-\beta}(L_1)$*, then,*

$$I_{a+}^{\gamma} I_{a+}^{\beta}\psi = I_{a+}^{\gamma+\beta}\psi.$$

Proof. Now consider the case $\mathrm{Re}(\beta) < 0$, $\mathrm{Re}(\gamma) > 0$. Since $\psi(x) \in I_{a+}^{-\beta}(L_1)$, we have

$$\psi = I_{a+}^{-\beta}\varphi,$$

where $\varphi \in L_1(a,b)$. Thus

$$I_{a+}^{\gamma+\beta}\psi = I_{a+}^{\gamma+\beta}I_{a+}^{-\beta}\varphi.$$

Since $\mathrm{Re}(\gamma+\beta-\beta) > 0$, by the case1 that

$$\begin{aligned} I_{a+}^{\gamma+\beta}\psi &= I_{a+}^{\gamma+\beta-\beta}\varphi \\ &= I_{a+}^{\gamma}\varphi \\ &= I_{a+}^{\gamma}D_{a+}^{-\beta}\psi \\ &= I_{a+}^{\gamma}I_{a+}^{\beta}\psi. \end{aligned}$$

■

Theorem 73 *Assume that* $\mathrm{Re}\,\gamma < 0$, $\mathrm{Re}(\gamma+\beta) < 0$ *and* $\psi(x) \in I_{a+}^{-\gamma-\beta}(L_1)$ *then,*

$$I_{a+}^{\gamma}I_{a+}^{\beta}\psi = I_{a+}^{\gamma+\beta}\psi.$$

Proof. Let $\mathrm{Re}(\gamma) < 0$, $\mathrm{Re}(\gamma+\beta) < 0$. By the assumption $\psi(x) \in I_{a+}^{-\gamma-\beta}(L_1)$, then

$$\psi(x) = I_{a+}^{-\gamma-\beta}\varphi,\ \varphi \in L_1(a,b)$$

By case1,

$$\begin{aligned} I^{\gamma}_{a+}I^{\beta}_{a+}\psi &= I^{\gamma}_{a+}I^{\beta}_{a+}I^{-\gamma-\beta}_{a+}\varphi \\ &= I^{\gamma}_{a+}I^{\beta-\gamma-\beta}_{a+}\varphi \\ &= I^{\gamma}_{a+}I^{-\gamma}_{a+}\varphi \\ &= D^{-\gamma}_{a+}I^{-\gamma}_{a+}\varphi. \end{aligned}$$

So, by (3.58),

$$I^{\gamma}_{a+}I^{\beta}_{a+}\psi = \varphi = I^{\gamma+\beta}_{a+}\psi.$$

Finally, note that the cases $\gamma = 0, \beta = 0$ are trivial, while the case $\gamma+\beta = 0$ coincides with (3.58) and (3.59), which completes the proof. ■

Remark 74 *The cases $\gamma = 0$, $\beta = 0$ and $\gamma+\beta = 0$ being also admissible for real γ and β.*

Remark 75 *Theorem does not incluede the fallowing cases*

i) $\mathrm{Re}\beta = 0, \mathrm{Re}\gamma > 0,$

ii) $\mathrm{Re}(\gamma+\beta) = 0, \mathrm{Re}\beta > 0,$

iii) $\mathrm{Re}\gamma = 0, \mathrm{Re}\beta < 0.$

Theorem 76 *Assume that*

i) $\mathrm{Re}\beta = 0, \mathrm{Re}\gamma > 0$ *and there exists a summable derivative* $D^{-\beta}_{a+}\psi$ *of purely imaginary order.*

ii) $\mathrm{Re}(\gamma+\beta) = 0, \mathrm{Re}\beta > 0$ *and there exists a summable derivative* $D^{-\beta-\gamma}_{a+}\psi$ *of purely imag-*

inary order.

iii) $\mathrm{Re}\,\gamma = 0, \mathrm{Re}\beta < 0$,*and there exists a summable derivative* $D_{a+}^{-\beta}\psi$ *and* $D_{a+}^{-\beta-\gamma}\psi$*,*

then

$$I_{a+}^{\gamma} I_{a+}^{\beta}\psi = \varphi = I_{a+}^{\gamma+\beta}\psi$$

holds.

Theorem 77 *Assume that* $\mathrm{Re}\,\gamma < 0$*,* $\mathrm{Re}\,(\gamma+\beta) < 0$ *and* $\psi(x)$ *has a summable fractional derivative then*

$$I_{a+}^{\gamma} I_{a+}^{\beta}\psi = I_{a+}^{\gamma+\beta}\psi - \sum_{k=0}^{n-1} \frac{\psi_{n+\beta}^{(n-k-1)}}{\Gamma(\gamma-k)} (x-a)^{\gamma-k-1}, \qquad (3.68)$$

where $n = [-\mathrm{Re}\beta] + 1$ *and* $\psi_{n+\beta}(x) = I_{a+}^{n+\beta}\psi$*.*

Definition 78 ([4]) *Let X be a Banach space and* T_γ *be a linear bounded operator in X for* $\gamma \geq 0$*, a one parameter family of* T_γ *is called a semigroup if*

$$T_\gamma T_\beta = T_{\gamma+\beta}, \quad \gamma \geq 0, \beta \geq 0 \qquad (3.69)$$

and

$$T_0\psi = \psi, \quad \psi \in X.$$

Definition 79 *A semigroup is called strongly continuous if for any $\psi \in X$,*

$$\lim_{\gamma \to \gamma_0} \left\| T_\gamma \psi - T_{\gamma_0} \psi \right\|_x = 0, \quad 0 \le \gamma_0 < \infty. \tag{3.70}$$

Definition 80 *A semigroup is called continuous in uniform topology if the limit above (3.70) exists in the operator topology in other words if*

$$\lim \left\| T_\gamma - T_{\gamma_0} \right\| = 0$$

when $\gamma \to \gamma_0$.

Lemma 81 *If the semigroup mentioned in (3.69) is strongly continuous for $\gamma = 0$ then it is strongly continuous for all $\gamma \ge 0$.*

Lemma 82 *The operator I^γ_{a+} and I^γ_{b-} are bounded in $L_p(a,b)$.*

Proof. By using simple operations and the generalized Minkowski inequality one can show that

$$\left\| I^\gamma_{a+} \psi \right\|_{L_p(a,b)} \le \frac{(b-a)^{\mathrm{Re}\,\gamma}}{\mathrm{Re}\,\gamma \, |\Gamma(\gamma)|} \, \|\psi\|_{L_p(a,b)}, \quad \mathrm{Re}\,\gamma > 0, \tag{3.71}$$

$$\left\| I^\gamma_{b-} \psi \right\|_{L_p(a,b)} \le \frac{(b-a)^{\mathrm{Re}\,\gamma}}{\mathrm{Re}\,\gamma \, |\Gamma(\gamma)|} \, \|\psi\|_{L_p(a,b)}, \quad \mathrm{Re}\,\gamma > 0. \tag{3.72}$$

Indeed,

$$
\begin{aligned}
\left\|I_{a+}^{\gamma}\psi(x)\right\|_{L_p(a,b)} &= \left\|\frac{1}{\Gamma(\gamma)}\int_a^x \frac{\psi(t)}{(x-t)^{1-\gamma}}dt\right\|_{L_p(a,b)} \\
&= \left\{\int_a^b \left|\frac{1}{\Gamma(\gamma)}\int_a^x \frac{\psi(t)}{(x-t)^{1-\gamma}}dt\right|^p dx\right\}^{\frac{1}{p}} \\
&= \frac{1}{|\Gamma(\gamma)|}\left\{\int_a^b dx\left|\int_a^x \frac{\psi(t)}{(x-t)^{1-\gamma}}dt\right|^p\right\}^{\frac{1}{p}}
\end{aligned}
$$

we use generalized Minkowski inequality in (2.13),

$$
\begin{aligned}
&\leq \frac{1}{|\Gamma(\gamma)|}\int_a^b \left\{\int_t^b \left|\frac{\psi(t)}{(x-t)^{1-\gamma}}dt\right|^p dx\right\}^{\frac{1}{p}} dt \\
&= \frac{1}{|\Gamma(\gamma)|}\int_a^b |\psi(t)|\left\{\int_t^b (x-t)^{(\gamma-1)p}dx\right\}^{\frac{1}{p}} dt \\
&= \frac{1}{|\Gamma(\gamma)|}\int_a^b |\psi(t)|\left\{\frac{(b-t)^{(\gamma-1)p+1}}{(\gamma-1)p+1}\right\}^{\frac{1}{p}} dt \\
&= \frac{1}{|\Gamma(\gamma)|\left[(\gamma-1)p+1\right]^{\frac{1}{p}}}\int_a^b |\psi(t)|(b-t)^{(\gamma-1)+\frac{1}{p}}dt
\end{aligned}
$$

and use Hölder's inequality in (2.8) and (2.9) gives

$$
\begin{aligned}
&\leq \frac{1}{|\Gamma(\gamma)|\left[(\gamma-1)p+1\right]^{\frac{1}{p}}}\left\{\int_a^b |\psi(t)|^p dt\right\}^{\frac{1}{p}}\left\{\int_a^b (b-t)^{(\gamma-1)q+\frac{q}{p}}dt\right\}^{\frac{1}{q}} \\
&= \frac{\|\psi\|_{L_p(a,b)}}{|\Gamma(\gamma)|\left[(\gamma-1)p+1\right]^{\frac{1}{p}}}\left[\frac{(b-a)^{(\gamma-1)q+\frac{q}{p}+1}}{(\gamma-1)q+\frac{p}{q}+1}\right]^{\frac{1}{q}} \\
&= \frac{(b-a)^{\gamma}\|\psi\|_{L_p(a,b)}}{|\Gamma(\gamma)|\left[(\gamma-1)p+1\right]^{\frac{1}{p}}(\gamma q)^{\frac{1}{q}}} \\
&= \frac{(b-a)^{\mathrm{Re}\,\gamma}}{\mathrm{Re}\,\gamma|\Gamma(\gamma)|}\|\psi\|_{L_p(a,b)}.
\end{aligned}
$$

■

Theorem 83 ([4]) *Operators of fractional integration form a semigroup in $L_p(a,b)$, $p \geq 1$, which is continuous in uniform topology for $\gamma > 0$ and strongly continuous for all $\gamma \geq 0$.*

Proof. *It is obvious that*

$$T_\gamma T_\beta = T_{\gamma+\beta}, \quad \gamma \geq 0, \beta \geq 0.$$

Now we have to show the continutiy of the semigroup . For $\gamma_0 > 0$, we have:

$$\begin{aligned}
& I_{a+}^{\gamma_0}\psi - I_{a+}^{\gamma}\psi \\
= & \left[\frac{1}{\Gamma(\gamma_0)} - \frac{1}{\Gamma(\gamma)}\right]\int_a^x \frac{\psi(t)}{(x-t)^{1-\gamma_0}}dt + \frac{1}{\Gamma(\gamma)}\int_a^x \left[(x-t)^{\gamma_0-1} - (x-t)^{\gamma-1}\right]\psi(t)\,dt \\
= & A\psi + B\psi.
\end{aligned}$$

On the other hand we have:

$$\|A\psi\|_{L_p} \leq \left|1 - \frac{\Gamma(\gamma_0)}{\Gamma(\gamma)}\right| \frac{(b-a)^{\gamma_0}}{\gamma_0\Gamma(\gamma_0)} \|\psi\|_{L_p}. \tag{3.73}$$

Let $\psi(x)$ to be zero outside $[a,b]$. Then,

$$\begin{aligned}
|B\psi| &= \left|\frac{1}{\Gamma(\gamma)}\int_a^x \left[(x-t)^{\gamma_0-1} - (x-t)^{\gamma-1}\right]\psi(t)\,dt\right| \\
&\leq \frac{1}{\Gamma(\gamma)}\left|\int_a^x \left[(x-t)^{\gamma_0-1} - (x-t)^{\gamma-1}\right]\psi(t)\,dt\right|
\end{aligned}$$

taking $t = x - t$,

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\gamma)}\left|-\int_{x-a}^{0}\left[(x-(x-t))^{\gamma_0-1}-(x-(x-t))^{\gamma-1}\right]\psi(x-t)\,dt\right| \\
&= \frac{1}{\Gamma(\gamma)}\left|\int_{0}^{x-a}\left[t^{\gamma_0-1}-t^{\gamma-1}\right]\psi(x-t)\,dt\right| \\
&\le \frac{1}{\Gamma(\gamma)}\int_{0}^{b-a}\left|t^{\gamma_0-1}-t^{\gamma-1}\right|\psi(x-t)\,dt \\
&= \frac{1}{\Gamma(\gamma)}\int_{0}^{b-a}t^{\gamma_0-1}\left|1-\frac{t^{\gamma-1}}{t^{\gamma_0-1}}\right|\psi(x-t)\,dt \\
&= \frac{1}{\Gamma(\gamma)}\int_{0}^{b-a}t^{\gamma_0-1}\left|1-t^{\gamma-\gamma_0}\right|\psi(x-t)\,dt \\
&= \frac{1}{\Gamma(\gamma)}\int_{0}^{b-a}\frac{\left|1-t^{\gamma-\gamma_0}\right|}{t^{1-\gamma_0}}\,|\psi(x-t)\,dt|\,.
\end{aligned}
$$

Applying Minkowski's inequality

$$
\|B\psi\|_{L_p} \le \frac{1}{\Gamma(\gamma)}\int_{0}^{b-a}\frac{\left|1-t^{\gamma-\gamma_0}\right|}{t^{1-\gamma_0}}dt\left(\int_{a}^{b}|\psi(x-t)|^p\,dx\right)^{\frac{1}{p}}
$$

$$
\le \frac{1}{\Gamma(\gamma)}\int_{0}^{b-a}\frac{\left|1-t^{\gamma-\gamma_0}\right|}{t^{1-\gamma_0}}dt\,\|\psi\|_{L_p}\,. \tag{3.74}
$$

Combining the inequalities (3.73) and (3.74), we have;

$$
\frac{\left\|\left(I_{a+}^{\gamma}-I_{a+}^{\gamma_0}\right)\psi\right\|}{\|\psi\|} \le \left|1-\frac{\Gamma(\gamma_0)}{\Gamma(\gamma)}\right|\frac{(b-a)^{\gamma_0}}{\Gamma(\gamma_0+1)}+\frac{1}{\Gamma(\gamma)}\int_{0}^{b-a}\frac{\left|1-t^{\gamma-\gamma_0}\right|}{t^{1-\gamma_0}}dt.
$$

Taking limit as $\gamma \to \gamma_0$ in the integral in right hand side of above inequality and also having in mind that for $\gamma > 0$, $\Gamma(\gamma)$ is continuous and nonzero, we have the following result

$$\lim_{\gamma \to \gamma_0} \left\| I_{a+}^{\gamma} - I_{a+}^{\gamma_0} \right\| = 0.$$

If $\gamma_0 = 0$ then

$$\lim_{\gamma \to 0} \left\| I_{a+}^{\gamma} \psi - \psi \right\|_{L_p} = 0.$$

Consider,

$$I_{a+}^{\gamma} \psi = \frac{1}{\Gamma(\gamma)} \int_a^x (x-t)^{\gamma-1} \psi(t)\, dt$$

and replacing t by $x-t$, gives

$$= \frac{1}{\Gamma(\gamma)} \int_0^{x-a} t^{\gamma-1} \psi(x-t)\, dt.$$

On the other hand,

$$
\begin{aligned}
I_{a+}^{\gamma}\psi-\psi &= \frac{1}{\Gamma(\gamma)}\int_0^{x-a} t^{\gamma-1}\psi(x-t)\,dt-\psi(x)\\
&= \frac{1}{\Gamma(\gamma)}\int_0^{x-a}\frac{\psi(x-t)-\psi(x)}{t^{1-\gamma}}dt+\frac{1}{\Gamma(\gamma)}\int_0^{x-a}\frac{\psi(x)}{t^{1-\gamma}}dt-\psi(x)\\
&= \frac{1}{\Gamma(\gamma)}\int_0^{x-a}\frac{\psi(x-t)-\psi(x)}{t^{1-\gamma}}dt+\frac{1}{\Gamma(\gamma)}\psi(x)\left[\frac{t^{\gamma}}{\gamma}\right]_0^{x-a}-\psi(x)\\
&= \frac{1}{\Gamma(\gamma)}\int_0^{x-a}\frac{\psi(x-t)-\psi(x)}{t^{1-\gamma}}dt+\frac{1}{\gamma\Gamma(\gamma)}\psi(x)(x-a)^{\gamma}-\psi(x)\\
&= \frac{1}{\Gamma(\gamma)}\int_0^{x-a}\frac{\psi(x-t)-\psi(x)}{t^{1-\gamma}}dt+\frac{\psi(x)(x-a)^{\gamma}}{\Gamma(\gamma+1)}-\psi(x)\\
&= \frac{\gamma}{\Gamma(\gamma+1)}\int_0^{x-a}\frac{\psi(x-t)-\psi(x)}{t^{1-\gamma}}dt+\psi(x)\left[\frac{(x-a)^{\gamma}}{\Gamma(\gamma+1)}-1\right]\\
&= U\psi+V\psi.
\end{aligned}
$$

So

$$\left\|I_{a+}^{\gamma}\psi-\psi\right\|_{L_p}\leq\|U\psi\|_{L_p}+\|V\psi\|_{L_p}.$$

It is clear that,

$$\|V\psi\|_{L_p}^{p}\leq\int_a^b|\psi(x)|^p\left|\frac{(x-a)^{\gamma}}{\Gamma(\gamma+1)}-1\right|^p dx.$$

Appliying Lebesgue Dominated Convergence Theorem,we have

$$\lim_{\gamma\to0+}\|V\psi\|_{L_p}=0.$$

Furthermore, we approximate $\psi(x)$ by a polynomial $P(x)$ in L_p-space, then

$$\|U\psi\|_{L_p} \le \|U(\psi - P)\|_{L_p} + \|UP\|_{L_p}. \tag{3.75}$$

Using Minkowski's inequality on the first term $\psi(x) = 0$ outside $[a,b]$, we get;

$$U\psi = \frac{\gamma}{\Gamma(\gamma+1)} \int_0^{x-a} \frac{\psi(x-t) - \psi(x)}{t^{1-\gamma}} dt$$

$$\begin{aligned}
U(\psi - P) &= \frac{\gamma}{\Gamma(\gamma+1)} \int_0^{x-a} \frac{(\psi - P)(x-t) - (\psi - P)(x)}{t^{1-\gamma}} dt \\
&= \frac{\gamma}{\Gamma(\gamma+1)} \int_0^{x-a} \frac{(\psi - P)(x) - (\psi - P)(t) - (\psi - P)(x)}{t^{1-\gamma}} dt \\
&= -\frac{\gamma}{\Gamma(\gamma+1)} \int_0^{x-a} (\psi - P)(t)\, t^{\gamma-1} dt
\end{aligned}$$

$$\begin{aligned}
\|U(\psi - P)\|_{L_p} &= \left\| \frac{\gamma}{\Gamma(\gamma+1)} \int_0^{x-a} \frac{(\psi - P)(x-t) - (\psi - P)(x)}{t^{1-\gamma}} dt \right\| \\
&\le \left\| \frac{\gamma}{\Gamma(\gamma+1)} \left[\int_0^{x-a} \frac{(\psi - P)(x-t)}{t^{1-\gamma}} dt - \int_0^{x-a} \frac{(\psi - P)(x)}{t^{1-\gamma}} dt \right] \right\| \\
&\le \left[\left\| \frac{1}{\Gamma(\gamma)} \int_0^{x-a} (\psi - P)(x-t)\, t^{\gamma-1} dt \right\|_{L_p} + \left\| \frac{1}{\Gamma(\gamma)} \int_0^{x-a} (\psi - P)(x)\, t^{\gamma-1} dt \right\|_{L_p} \right] \\
&= \left\| I^{\gamma}_{a+}(\psi - P) \right\|_{L_p} + \left\| \frac{1}{\Gamma(\gamma)} \int_0^{x-a} (\psi - P)(x-t)\, t^{\gamma-1} dt \right\|_{L_p} \\
&= \left\| I^{\gamma}_{a+}(\psi - P) \right\|_{L_p} + \left\| I^{\gamma}_{a+}(\psi - P) \right\|_{L_p} \\
&\le \frac{(b-a)^{\gamma}}{\Gamma(\gamma)\gamma} \|(\psi - P)\|_{L_p} + \frac{(b-a)^{\gamma}}{\Gamma(\gamma)\gamma} \|(\psi - P)\|_{L_p}
\end{aligned}$$

$$\|U(\psi - P)\|_{L_p} \leq \frac{2(b-a)^{\gamma}}{\Gamma(\gamma+1)} \|(\psi - P)\|_{L_p} < const\epsilon.$$

The second term in (3.75);

$$|UP| = \left| \frac{\gamma}{\Gamma(\gamma+1)} \int_0^{x-a} \frac{P(x-t) - P(x)}{t^{1-\gamma}} dt \right| \leq \frac{\gamma}{\Gamma(\gamma+1)} \int_0^{b-a} t^{\gamma} \frac{P(x-t) - P(x)}{t} dt,$$

and we have;

$$|UP| \leq \frac{\gamma}{\Gamma(\gamma+1)} \int_0^{b-a} t^{\gamma} \max \left| P'(t) \right| dt \underset{\gamma \to 0}{\to} 0.$$

Therefore proof is complete. ∎

Definition 84 ([4]) *A Lebesgue point x_0 of a function $\psi(x) \in L_1(a,b)$ is a point which satisfies the following equation*

$$\lim_{t \to 0} \frac{1}{t} \int_0^t [\psi(x_0 - s) - \psi(x_0)] \, ds = 0. \tag{3.76}$$

Remark 85 *For the function $\psi(x) \in L_1(a,b)$ almost all points $x_0 \in [a,b]$ are Lebesgue point.*

Theorem 86 ([4]) *Let $\psi(x) \in L_1(a,b)$ then for any Lebesgue point of a function $\psi(x)$,*

$$\lim_{a \to 0} \left(I_{a+}^{\gamma} \psi \right)(x) = \psi(x). \tag{3.77}$$

Proof. For a Lebesgue point x_0 of a function $\psi(x)$ we will have the following notation

$$\Psi(t) = \int_{x_0-t}^{x_0} \psi(s)\,ds = \int_0^t \psi(x_0-s)\,ds. \tag{3.78}$$

Taking $s = x_0 - s$, we have

$$\Psi(t) = \int_{x_0-t}^{x_0} \psi(s)\,ds = \int_0^t \psi(x_0-s)\,ds$$

and use the second inequality in (3.78), we have:

$$\begin{aligned}
\frac{\Psi(t)}{t} - \psi(x_0) &= \frac{1}{t}\int_0^t \psi(x_0-s)\,ds - \psi(x_0) \\
&= \frac{1}{t}\int_0^t \psi(x_0-s)\,ds - \frac{1}{t}\int_0^t \psi(x_0)\,ds \\
&= \frac{1}{t}\int_0^t [\psi(x_0-s) - \psi(x_0)]\,ds.
\end{aligned}$$

We get:

$$\frac{\Psi(t)}{t} - \psi(x_0) = \frac{1}{t}\int_0^t [\psi(x_0-s) - \psi(x_0)]\,ds \to 0$$

Thus we can write $\Psi(t)$ as

$$\Psi(t) = t\,[\psi(x_0) + b(t)]$$

where $b(t)$ is a bounded function such that $0 < t < \tau = \tau(\epsilon)$. Therefore;

$$\begin{aligned} I_{a+}^{\gamma}\psi &= \frac{1}{\Gamma(\gamma)}\int_{0}^{x_0-a} t^{\gamma-1}\psi(x_0-t)\,dt \\ &= \frac{1}{\Gamma(\gamma)}\int_{0}^{x_0-a} t^{\gamma-1}d\Psi \end{aligned}$$

taking $u = t^{\gamma-1}$, $v = \Psi(t)$ and using integration by parts,

$$\begin{aligned} I_{a+}^{\gamma}\psi &= \frac{1}{\Gamma(\gamma)}\left(\left[t^{\gamma-1}\Psi(t)\right]_{0}^{x_0-a} - \int_{0}^{x_0-a}(\gamma-1)\,t^{\gamma-2}\Psi(t)\,dt\right) \\ &= \frac{1}{\Gamma(\gamma)}\left((x_0-a)^{\gamma-1}\,\Psi(x_0-a) - \left[t^{\gamma-1}\Psi(t)\right]_{0} - \int_{0}^{x_0-a}(\gamma-1)\,t^{\gamma-2}\Psi(t)\,dt\right) \\ &= \frac{\Psi(x_0-a)}{\Gamma(\gamma)(x_0-a)^{1-\gamma}} - \frac{1}{\Gamma(\gamma)}\frac{\Psi(t)}{t^{1-\gamma}}\,|_{t=0} + \frac{1-\gamma}{\Gamma(\gamma)}\int_{0}^{x_0-a}\frac{\Psi(t)}{t^{2-\gamma}}dt \\ &= \frac{1-\gamma}{\Gamma(\gamma)}\int_{0}^{x_0-a} t^{\gamma-1}b(t)\,dt + \frac{\Psi(x_0-a)}{\Gamma(\gamma)(x_0-a)^{1-\gamma}} \\ &\quad + \frac{1-\gamma}{\Gamma(\gamma)}\psi(x_0)\int_{0}^{x_0-a} t^{\gamma-1}dt + \frac{1-\gamma}{\Gamma(\gamma)}\int_{0}^{\tau} t^{\gamma-1}b(t)\,dt. \end{aligned}$$

So;

$$\begin{aligned} \left(I_{a+}^{\gamma}\psi\right)(x_0) - \psi(x_0) &= \frac{\Psi(x_0-a)}{\Gamma(\gamma)(x_0-a)^{1-\gamma}} + \psi(x_0)\left[\frac{1-\gamma}{\gamma\Gamma(\gamma)}(x_0-a)^{\gamma} - 1\right] \\ &\quad + \frac{1-\gamma}{\Gamma(\gamma)}\int_{0}^{\tau} t^{\gamma-1}b(t)\,dt + \frac{1-\gamma}{\Gamma(\gamma)}\int_{0}^{x_0-a} t^{\gamma-1}b(t)\,dt. \end{aligned}$$

By interchanging the limit and integral sign we get:

$$\begin{aligned}
\lim_{\gamma\to 0+}\left|\left(I_{a+}^{\gamma}\psi\right)(x_0)-\psi(x_0)\right| &\leq |\psi(x_0)| \lim_{\gamma\to 0+}\left[\frac{1-\gamma}{\Gamma(\gamma+1)}(x_0-a)^{\gamma}-1\right] \\
&\quad + \lim_{\gamma\to 0+}\frac{1-\gamma}{\Gamma(\gamma)}\left|\int_0^{\tau} t^{\gamma-1}b(t)\,dt\right| \\
&\leq \lim_{\gamma\to 0+}\frac{1-\gamma}{\Gamma(\gamma+1)}\tau^{\gamma}\epsilon = \epsilon.
\end{aligned}$$

The equation (3.77) is obtained since ϵ is arbitrary. ■

Chapter 4

CONCLUSION

As a result, we can take derivative and integral easily with integer but if we try to take fractional we will have some problem. Furthermore I tired some methods and operations for solving this problem. I solved some equation, theorem etc. with some special functions and properties.

The main problem is $0 < t < 1$ and it may main purpose for making this research.

REFERENCES

[1] R.G. Bartle, The elements of integration and Lebesque Measure, A wiley-Interscience Publication, 1995.

[2] K. Diethelm, The analysis of Fractional Differential equations. Springer-Verlag, Berlin Heidelberg 2010.

[3] A.A. Kilbas, H. M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential equations, Elsevier, 2006.

[4] S.G. Samko, A.A. Kilbas and O.I. Marichev, Fractional Integrals and Derivatives Theory and Applications, Gordon and Breach Science Publishers, 1993.