Vibration Analysis of Multi Degree of Freedom Self-excited Systems

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Submitted to the
Institute of Graduate Studies and Research
in the partial fulfillment of the requirements for the Degree of

Master of Science
in
Mechanical Engineering

Eastern Mediterranean University
July 2014
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

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ABSTRACT

It is known that nonlinearities in the friction-velocity curves of dry friction may lead to instability of the steady frictional sliding, namely, friction-induced vibration. Therefore, stick-slip vibration is an important issue in mechanical engineering, and understanding the dynamics of stick-slip and its possible elimination also becomes important, especially for applications requiring high-precision motion. Among the many everyday examples of friction sounds, violin music and brake noise in automobiles represent the two extremes in terms of the sounds they produce and the mechanisms by which they are generated. Of the multiple examples of friction sounds in nature, insect sounds are prominent. In this thesis, multi-degree degree of freedom self-excited vibrating systems which are excited by the friction force imposed from a moving surface will be considered. A Coulomb friction law is introduced in the equations of the contact point taking into account all the possible status: separated, sliding or sticking. The vibration response and velocity of system's components under various initial conditions will be calculated and reported. The effect of initial conditions, velocity of moving surface, mass, spring stiffness coefficient, damping and its coefficient of friction will be studied and discussed. The results are obtained for occurrence and nonoccurrence of stick slip phenomena. It is thought that this design guides may be useful in the control and reduction of the vibrations induced by dry friction. Furthermore, the result of this study provides an understandable structure for vibro-acoustical analysis of self-excited vibrating systems.

Keywords: Friction Contact, Friction-Induced Vibrations, Self-Excited systems, Brake Noise, Non-linear Time-Series Analysis, Chaotic Dynamics

ÖZ

Kuru sürtünmeli alanda sürtünme hızı eğrilerinde oluşan sapmaların sürtünmeli kaymada istikrarsızlığa yani sürtünme kaynaklı titreşime yol açtığı bilinmektedir. Bu nedenle, tutma - bırakma titreşimi makine mühendisliğinde önemli bir konudur. Tutma - bırakma ve olası eleme dinamiklerini anlamak da özellikle yüksek hassasiyetli hareket gerektiren uygulamalar için önemli hale gelir. Sürtünme seslerinin günlük örnekleri arasında, ürettikleri sesler ve üretildikleri mekanizmalar bakımından keman sesi ve otomobillerin fren gürültüsü iki uç noktayı temsil eder. Böcek sesleri doğadaki sürtünme seslerinin çoğu örneği arasından belirgin olanıdır. Bu tezde, hareketli bir yüzeyden oluşan sürtünme kuvveti tarafından uyarılan kendinden ikazlı titreşim sistemlerinin çoklu serbestlik derecesi dikkate alınacaktır. Coulomb sürtünme yasası, tüm olası durumları: ayrılmış, kayan / hareketli, sürtünme dikkate alınarak temas noktası denklemlerinde tanıtılmıştır. Titreşim tepkisi ve sistem bileşenlerinin hızı farklı başlangıç koşulları altında hesaplanacak ve rapor edilecektir. Başlangıç koşullarının etkisi, hareketli yüzey hızı, kütle, yay direngenliği katsayısı, söndürme ve sürtünme katsayısı incelenecek ve tartışılacaktır. Sonuçlar tutma – bırakma olaylarının oluşumu ve olmayan oluşumu için elde edilir. Bu tasarım kılavuzlarının kuru sürtünmeden kaynaklanan titreşimlerin azaltılması ve kontrolünde yararlı olabileceği düşünülmektedir. Ayrıca, bu çalışmanın sonucu kendinden ikazlı titreşim sistemlerinin titreşim akustiği için anlaşılır bir yapı sağlar.

Anahtar Kelimeler: Sürtünme yüzeyi, Sürtünme kaynaklı titreşim, Kendinden ikazlı titreşim sistemleri, Fren Gürültüsü, Doğrusal Olmayan Zaman Serisi Analizi, Düzensiz Dinamikler

To My Family

ACKNOWLEDGEMENT

I am deeply indebted to my advisor Asst. Prof. Dr. Mostafa Ranjbar for his constant encouragement and guidance throughout the course of this thesis. His suggestions have been of tremendous help in the preparation of this document.

My family, the most precious peoples in my life, were all the time supporting me and without their energy I could not be able to achieve this success.

My thanks go also to the jury members, Assist. Prof. Dr. Neriman Özada and Assist. Prof. Dr. Ghulam Hussain for giving useful comments on my work.

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LIST OF SYMBOLS/ ABBREVIATION

C	System viscous damping coefficient
f	Friction force
g	Gravity
K	System stiffness
M	Main mass of the system
t	Time
v_{belt}	Belt velocity
x(t)	Absolute displacement of the main mass
$\dot{x}$	Velocity of the main mass
$\ddot{x}$	Acceleration of the main mass
μ_0	Coefficient of friction
DOF	Degree of freedom

Chapter 1

INTRODUCTION

The main cause of self-excited vibration phenomena is actually the excitation imposed by friction between the mechanical parts of the system. Leonardo da Vinci was the first person who formulated the rules of friction. He hypothesized that friction might be independent of the area of contact and there is a direct connection between the applied normal load and the vibration frictional force. Although, approximately around two hundred years later Amonton in 1699 postulated da Vinci's law without dependence of his states and gives credit to it by their realization [1].

Coulomb, in 1781, realized a new concept of a limiting static friction. He believed that this friction would not prevent static bodies from movements unless the advancing force exceeds this friction. Specifically, he stated that this amount is greater than the coefficient of kinetic friction. These ideas and the concept that the frictional force is almost independent of the sliding speed are considered as Coulomb's friction law. Because of several objectives the Coulomb's friction law reminisced as a so competent pattern for friction. [1]

Recently, the ability of prediction and reduction of structural vibrations has received more attention. The origins of vibrations are either from external sources, e.g. wind, or internal ones like friction between mechanical parts of system. The main example for self-excited vibration is actually break squeal in automobiles. It can change the braking ability of car which is actually none-desirable. In fact, the reliability of braking system in cars plays an important role in the safety of passengers. Therefore, researchers put a lot of efforts on the study of self-excited vibration phenomena and its applications in the real life.

Self-excited vibrations become evident because of a mechanism of a system which will vibrate at its own natural or critical frequency spontaneously, the amplitude increasing until some nonlinear effect limits any further increase and their oscillation has been known as one of the cases of vibrations that may happen either with or without external or internal periodic forcing. Their occurrence is fundamentally depends on the special internal features of a system. From a mathematical aspect, the motion can characterize with the unstable homogeneous solution to the homogeneous equations of motion. Furthermore, in the forced or resonant vibrations, the fluctuation frequency is dependent on the frequency of a forcing function which is act as an external exciter to the system. In nonhomogeneous equations of motion the particular solution is forced vibration.

Self-excited vibrations have pervaded in different parts of physical systems that motion or time-variant factors are included. Instance of these parts are: aerothermodynamics, aerodynamics, aeromechanical systems, feedback networks, and mechanical systems [2]. Vibrations of a vehicle wheel, unwanted vibrations during machining processes, vibration of airplane wings are some classical examples of self-excited systems.

The happening of self-excited vibration in a physical system is connected with the stability of equilibrium positions of the system. If system's equilibrium disturbed, some forces emerge that lead the system to move toward or away from its equilibrium. In forced vibration system the equilibrium position is unstable therefore it may either monotonically recede from the equilibrium position until nonlinear or limiting restraints appear or oscillate with increasing amplitude. In both of the cases if the disturbed system approaches the equilibrium position either, the equilibrium is been considered as stable.

When system is disturbed and moved away from its equilibrium depending upon the displacement of the velocity the appearance of the forces change. If displacement-dependent forces become manifest and cause the system to move away from the equilibrium position, the system is said to be statically unstable. Moreover, velocity-dependent forces that lead system to withdraw from a statically stable equilibrium position results in a dynamic instability.

It is seems that the instability of the steady frictional sliding, known as friction-induced vibration, is initiated from the nonlinearities in the friction-velocity curves of dry friction. Famous examples of such friction-induced vibrations are the motion of a violin and Froude pendulum. However, the affixation inclination of the sliding components to stick and slip causes to some problems in industrial usages. For instance, in the operations such as robot joints, electric motor drives, disk brake systems, bearings, machine tools and work piece systems, wheel and rail mass transit systems, etc. it may have some negative effects. Hence, comprehension of stick-slip vibration with the hope of its elimination in industrial applications, especially in high precision motions, has recently caught more attention [3].

In the next chapter, a literature survey will be written to give the background of self-excited vibration analysis in systems.

Chapter 2

LITERATURE REVIEW

In 1938, Jarvis and Mills [4] showed the relationship between the relative velocity of brake disk and wheel speed with respect to self-excited vibration aspects. It was indicated that self-excited vibration which is induced by the friction in the brake system of car, can reduce the braking instability of car. They indicated that the alteration of the friction coefficient by sliding speed was inadequate to lead to friction-induced vibrations thus the instability was depend to coupling although the friction coefficient was constant.

A comprehensive analysis power for identification of the influences of physical parameters on the stability of system is needed in order to solve friction-induced difficulties. As an illustration, the friction coefficient has some influences on the stability of static solution. These various mechanisms should be considered to gain knowledge an appropriate mechanism to clarify friction induced oscillation in systems. It was discovered that the self-excited vibration may happen while the friction coefficient stayed constant with speed. Spurr [5] presented the sprag-slip phenomenon that doesn't rely on a friction coefficient changing by the relative rotation speed of the brake disk. Later scholars [6-11] improved a more generalized theory explaining the mechanism as a geometrically induced or kinematic constraint instability.

Although, there is a lack of uniform theory for modeling of the problem and that sprag–slip phenomenon, but there are several publications which address the mechanism of dynamic instability of brake systems by the stick-slip phenomena [12], geometric coupling of the structure involving sliding parts [4,5,7,9,13,14,15] and negative friction velocity slope [16] [17].

The fluctuation of a system that consists of a mass and spring which sliding on a moving belt can simulate the active control system. Generally, two kinds of friction-induced vibrations are known; the first one exhibits a sinusoidal wave with its frequency near the natural frequency. It is produced by the negative of friction coefficient on relative velocity which acts as a negative damping [18]. So, in order to make the relationship positive, in the design stage of practical frictional surface, some materials and lubricants have been developed to make this inversion. The second kind of friction-induced vibration has a triangular wave or a saw tooth wave that its frequency is depend on the sliding velocity. Due to the fact that this vibration is produced by the repetition of ‘‘stick’’ and ‘‘slip’’ of mating surfaces it is known as stick-slip motion [19]. The difference between the static friction coefficient and kinetic friction coefficient lead to the Stick-slip motion that is inclined to emerge in higher normal load and lower sliding velocity. [20]

Self-excited vibration has a several differences with forced vibration: in self-excited vibration the alternating force that nourish the motion is produced by the motion itself and it disappears when the motion cease. However, in the forced vibration systems the nourishing alternating force is autonomous of the motion and survives even if the vibratory motion is disappeared. [21]

There are two primary types of self-excitation, i.e., hard self-excitation characterized using an unstable limit cycle, and soft self-excitation symbolized by stable limit cycle. The trajectory in the hard self-excitement, according to the initial situation inclines to infinity or an equilibrium point. [22]

Differential equations with time delay terms or nonlinear functions of state space coordinates could be applied to exhibit the mechanism of self-excitation. It should be noted that the mathematical models of the self-excited systems instead of containing direct time elements is controlled by independent differential equations which is comprised of nonlinear terms that model the self-excitation phenomenon. [22]

In 1984 Jemielniak et al. [23] presented a procedure to analysis the efficacy of spindle speed variation on the self-excited vibration. Later, Ehrich et al. [24] represent that self-excited vibration or instability can cause a significant difficulty in high-performance turbo machinery in a style of spinning or licking at one of the rotor's natural frequencies beneath the running velocity. In addition Hagedorn [25] investigated self-excited fluctuations in electrical and mechanical systems.

Earles and Chambers [26] exposed to discussion the significance of geometrically induced instability on damped systems and figure out that the damping affects are too complex to estimate directly and couldn't be easily predicted.

The eigenvalues by positive real parts increase the self-excited motion that happens through coefficients of friction, a vast range of material pairs and sliding velocities (containing low speeds) was presented by Adams [27].The mechanism responsible for the instability was essentially one of destabilization of interfacial (slip) waves.

Brommundt [28] demonstrated a 3-DOF model which is similar with the model proposed by Hoffmann and Gaul but with an extra degree of freedom to the conveyor belt. It illustrated that even when monotonous friction is increasing, characteristic instability happens in this model. [29]

In a work by McMillan [1] a dynamical system was developed to figure out the phenomenon of squeal more than before. One type of the self-excited vibration is squeal, which can occur in violin string or railway wheels because of the frictional driving force.

In 1999, in an examination Thomsen [30] showed that how high frequency external excitation influenced the friction-induced self-excited fluctuations. Prediction of occurrence of self-excited oscillation, whether in the presence or absence of high-frequency excitation, for the traditional mass-on-moving-belt model held by simple analytical approximation. Showed that high-frequency excitation can positively stop the negative slope in the stated friction-velocity correlation and so is presenting the self-excited fluctuations.

Akay 2002 [31], in his article described the friction-induced vibrations and waves in solid and presented the other acoustic related frictional phenomena. He clarified that friction by the sliding contact of solids frequently causes different forms of waves and oscillations within solids that leads to sound radiation to the surrounding media.

Shin et al 2002 [32] [33] analyzed the noise of disc brake by using two-degree of freedom. The research that they have done cleared that additions of damping to the disc or the pad may make instability for the system and therefore noisy. They demonstrated that instability in the damped model systems which link through a sliding friction interface happens when added a damping just on one side of the sliding interface.

Hoffmann and Gaul 2002 [34] proposed a two degree of freedom model for understanding the physical mechanisms underlying the mode-coupling instability of self-excited friction induced fluctuations [29]. Finally the origin and the role of phase shifts between oscillations normal and parallel to the contact surface is clarified with respect to the mode-coupling instability.

Hoffmann and Gaul [35] worked on the consequence of damping on destabilizing of mode-coupling in self-excited friction induced fluctuation. Furthermore they determine that the destabilization of friction-induced fluctuations may occur by increasing damping and also noticed that they can’t relinquish the side effect simply. [14]

The “mass-on-moving-belt” model for presenting friction-induced vibrations was introduced by Thomson et al. [36] friction law described the friction forces in two phases of first decreasing and then increasing smoothly through relative interface speed. Close analytical descriptions were shown for the conditions, amplitudes and the base frequencies of friction-induced stick–slip and pure-slip oscillations. Perturbation analysis for finite time interval of the stick phase used in accomplishing the stick-slip oscillations. Which was related to the subsequent slip phase within conditions of cohesions and periodicity. The results were shown and examined by amplitude response, phase plots and time-series diagrams.

Chen et al. [37] introduced a self-excited system which depends on the time lag among the normal force oscillation and a friction oscillation that created from it. A number of simulations were performed by using the model. The consequence of his model indicates that instability vibration can be excited by the time delay.

In the field of controlling the friction that lead to self-excited vibration Chatterjee [38] presented a new method. The system model is shown by a single degree of freedom oscillator on a moving belt. The control action was set by adjusting the normal load at the frictional interface based on the state of the oscillatory system.

After performing a careful study on the history of vibration analysis of self-excited system, it has been followed that there are several open question still available in this field, e.g. behavior of system under self-excited condition and proposed noise from a self-excited vibratory system. Therefore in this thesis, a comprehensive study will be performed to understand more about the behavior of multi-dimensional self-excited systems. Chapter three discusses about the research methodology that is used in this study and related data.

Chapter 3

METHODOLOGY

3.1 General

The stick-slip phenomenon occurs while two parts non-uniform sliding in the presence of dry friction at low speed and also in many intricate mechanical systems, e.g. turbo-machinery constituent. It appears in the contact of surfaces which sticking to one another and sliding over one another by changing in the dry friction force. Mostly, the static friction coefficient is greater than the dynamic coefficient. A sudden jump in the velocity of the movement might originate from changing in the friction level to dynamic friction. This change happens when the applied force is strong enough and overcome the static friction. Den Hartog [39] was the first person who research in stick-slip oscillations. He developed an exact analytical solution for 1-DOF system with periodic oscillation and stick-slip motion. For forecasting a critical damping value to repress the stick-slip oscillations Blok [40] linearized the friction velocity curve. The explanation for the critical velocity at which stick-slip oscillation may happen between sliding surfaces improved by Derjaguin et al. [41]. Brockley et al. [42] derived a critical velocity expression for the suppression of stick-slip vibrations.

The stick-slip response of a system plays a main role in the dissipation of the energy due to the friction that act as a damping mechanism for controlling the structural response of diverse physical systems. Although, it is just in the slipping period that the friction interface dissipates energy.

Till today several researchers have presented various models for representation of stick-slip procedure in mechanical systems, e.g. Shin et al., Hoffmann and Gaul, Popp et al., Brommundt, A. J. Mcmillan and others with the aim that predict the response of the system accurately.

In this thesis, the model which presented by Mcmillan is used. A two-degree-of-freedom block on a moving belt and else a three-degree-of-freedom block on a moving belt models have been investigated.

3.2 Mechanical Model

3.2.1 Modeling of Mass-Spring system with Two-Degree-of-Freedom

For better understanding of the self-excited vibration it is sufficient to arrange a formula of a two-degree-of-freedom model and inquire into its limit-cycle dynamics and stability behavior. The governing equation of system is

$$M\ddot{X} + C\dot{X} + KX = F \tag{3.1}$$

The 2-DOF model is illustrated in Figure 1, as it shown this system consist of two masses on a moving belt and three springs that two of them fix the masses to block.

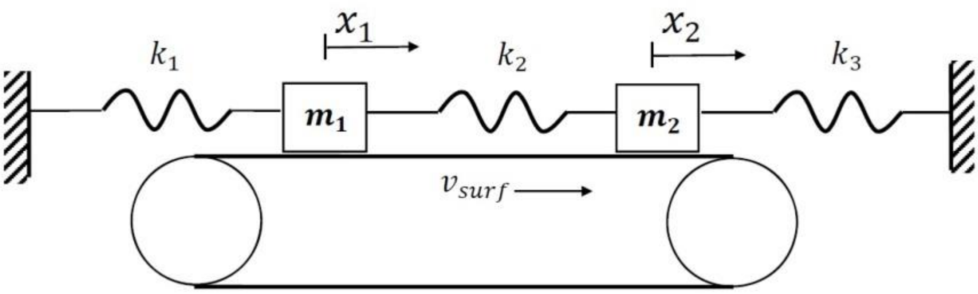


Figure 1. Self-excited (friction driven) system with 2-DOF.

Typically, for solving the vibration problems we set the motion equation in the matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad (3.2)$$

While assumed initial conditions for the system are

$$x(0) = \begin{bmatrix} 0.05 \\ 0.06 \end{bmatrix} \ (\mathrm{m}) \quad , \quad \dot{x}(0) = \begin{bmatrix} 0.04 \\ 0.03 \end{bmatrix} \ (\mathrm{m/s}) \quad (3.3)$$

The Coulomb friction model states that the frictional force is independent of the magnitude of the velocity of [MCMILLAN1997] and it can written as

$$f_i = -sign(\dot{x}_i(\mathrm{t}) - v_{belt})\,\mu_0 mg \quad , \quad i = 1,2 \quad (3.4)$$

Where $\dot{x}(t)$ is velocity of body, v_{surf} is velocity of surface and *f* is the friction force which is acting to the body, *m* is the mass, μ_0 is the dry friction coefficient, *k* is the spring stiffness coefficient, and *g* is the gravity acceleration. Although, it is indicated that the friction is independent of the magnitude of the velocity, but experiments have shown that there is velocity dependence. Lindop and Jensen [43] demonstrated this argument computationally based on a qualitative understanding of surface interactions.

To simulate the response of the spring-mass system located on a friction belt driven at some nonzero velocity, the initial value problem should be written. At this regard, we reduce the order of system by the replacing the original variable *x* with *y* as:

$$y_1(\mathrm{t}) = x_1(\mathrm{t}) \quad , \quad y_3(\mathrm{t}) = x_2(\mathrm{t})$$

$$y_2(\mathrm{t}) = \dot{x}_1(\mathrm{t}) \quad , \quad y_4(\mathrm{t}) = \dot{x}_2(t) \tag{3.5}$$

Then, we reformulate the equation 3.2 to an initial value problem as:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \dot{y}_2 \\ \dot{y}_4 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_3 \end{Bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{3.6}$$

$$\dot{y}_2 = \frac{-m_1 g \; sign(v_1 - v_{surf}) + k_2 x_2 - (k_1 + k_2) x_1}{m_1} \tag{3.7}$$

$$\dot{y}_4 = \frac{-m_2 g \; sign(v_2 - v_{surf}) + k_2 x_1 - (k_2 + k_3) x_2}{m_2} \tag{3.8}$$

Consider the masses $m_1 = m_2 = 1kg$, the dry friction constant μ_0 of 0.05, spring stiffness constants $k_1 = k_2 = k_3 = 10$ N/m, the speed of supporting surface v_{surf} as 0.01 m/s and the gravitational acceleration g is 9.81 m/s^2. The system is assumed to be initially at rest. Then by substituting of these values in equation 3.6 and changing the matrix form to a system of algebraic equations, we will have four algebraic equations as

$$\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{Bmatrix} = \begin{Bmatrix} y_2 \\ f_1 + y_3 + y_4 - 2 \times (\mathrm{y}_1 + \mathrm{y}_2) \\ \mathrm{y}_4 \\ f_2 + y_1 + y_2 - 2 \times (\mathrm{y}_2 + \mathrm{y}_3) \end{Bmatrix} \tag{3.9}$$

With the initial conditions of

$$\begin{Bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{Bmatrix} = \begin{Bmatrix} 0.05 \\ 0.04 \\ 0.06 \\ 0.03 \end{Bmatrix} \tag{3.10}$$

We can see the advertised self-excited vibrations when the system starts by carrying along the masses on the supporting surface with a specific pulling period. Then the oscillations start. Also, we may observe a bit of noise in the initial stage when the pull on the surface is not get sufficiently strong, and the mass should be carried along at a constant speed.

By solving of initial value problem indicated in the equations 3.7 and 3.8, we can calculate the vibration responses of both masses and their oscillating velocities.

3.2.2 Modeling of Mass-Spring-Damper System with Three-Degree-of-Freedom

This model considered to realize and develop the role and effect of damping. The mechanical model illustrate in Figure 2. This model comprised of three masses, four springs (stiffness *k*) and four dampers (damping coefficient *c*) on a moving belt at a constant velocity. The friction force direction assumed to be constant with a positive belt speed as a result the relative velocity becomes positive. The motion equations for the system with three degree of freedom that shown in figure 2 is given in equation 3.11 in the matrix form as:

$$
\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \end{Bmatrix} + \tag{3.11}
$$

$$
\begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{Bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}
$$

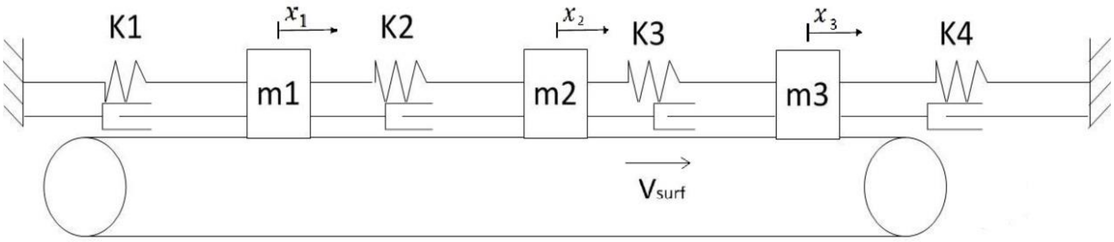


Figure 2. Self-excited (friction driven) system with 3-DOF.

While the initial conditions for the system are

$$
x(0) = \begin{bmatrix} 0.05 \\ 0.06 \\ 0.07 \end{bmatrix} \quad \text{(m)} \qquad , \qquad \dot{x}(0) = \begin{bmatrix} 0.04 \\ 0.03 \\ 0.05 \end{bmatrix} \quad \text{(m/s)} \tag{3.12}
$$

$$
\begin{Bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \\ y_5(0) \\ y_6(0) \end{Bmatrix} = \begin{Bmatrix} 0.05 \\ 0.04 \\ 0.06 \\ 0.03 \\ 0.07 \\ 0.05 \end{Bmatrix} \tag{3.13}
$$

At this regard, we reduce the order of system by the replacing the original variable *x* with *y* as:

$$y_1(\mathrm{t}) = x_1(\mathrm{t}) \quad , \quad y_3(\mathrm{t}) = x_2(\mathrm{t}) \quad , \quad y_5(\mathrm{t}) = x_3(\mathrm{t})$$

(3.14)

$$y_2(\mathrm{t}) = \dot{x}_1(\mathrm{t}) \quad , \quad y_4(\mathrm{t}) = \dot{x}_2(t) \quad , \quad y_6(\mathrm{t}) = \dot{x}_3(t)$$

Then, we reformulate the equation 3.11 to an initial value problem as:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \dot{y}_2 \\ \dot{y}_4 \\ \dot{y}_6 \end{Bmatrix} +$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \begin{Bmatrix} y_2 \\ y_4 \\ y_6 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_3 \\ y_5 \end{Bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{3.15}$$

$$\dot{y}_2 = \frac{-m_1\, g\, sign(v_1 - v_{belt}) + k_2 x_2 - (k_1 + k_2) x_1 + c_2 v_2 - (c_1 + c_2) v_1}{m_1} \tag{3.16}$$

$$\dot{y}_4 = \frac{-m_2\, g\, sign(v_2 - v_{belt}) + k_2 x_1 + k_3 x_3 - (k_2 + k_3) x_2 + c_2 v_1 + c_3 v_3 - (c_2 + c_3) v_2}{m_2} \tag{3.17}$$

$$\dot{y}_6 = \frac{-m_1\, g\, sign(v_1 - v_{belt}) + k_3 x_2 - (k_3 + k_4) x_3 + c_3 v_2 - (c_3 + c_4) v_3}{m_3} \tag{3.18}$$

A mathematical model was prepared in Matlab software to calculate the vibration responses of masses and their oscillating velocities. These results can help the designer and researchers to obtain perception to the interaction of system dynamics and friction and recognize the main parameters that influence on the system response. The next chapter reports the results of these models.

Chapter 4

RESULTS AND DISCUSSION

In the Figure 3 and Figure 4, the vibration response and oscillating velocity of masses for a system with two-degree-of-freedom are given.

Figure 3.a shows the behavior of mass m_1 when no surface friction is considered. It is clear that in this case, the system oscillate freely with a constant amplitude and period. Furthermore, Figure 3.b presents the response of system when a coefficient of moving surface friction of μ_0 equal with 0.05 is considered. It can be seen that due to effect of friction, the vibration of system is damped in 1.35 seconds and it will remain in its static position.

Table 1. Values for comparative study on the self-excited vibration analysis of 2-DOF system.

Element Item	**Value**
Masses $M_{1,}$ M_2 and M_3	1 kg
Spring Stiffness Coefficient K_1, K_2, K_3 and K_4	10 N/m
Friction Coefficient μ_0	0.05
Belt Velocity V_{belt}	0.01 m/s

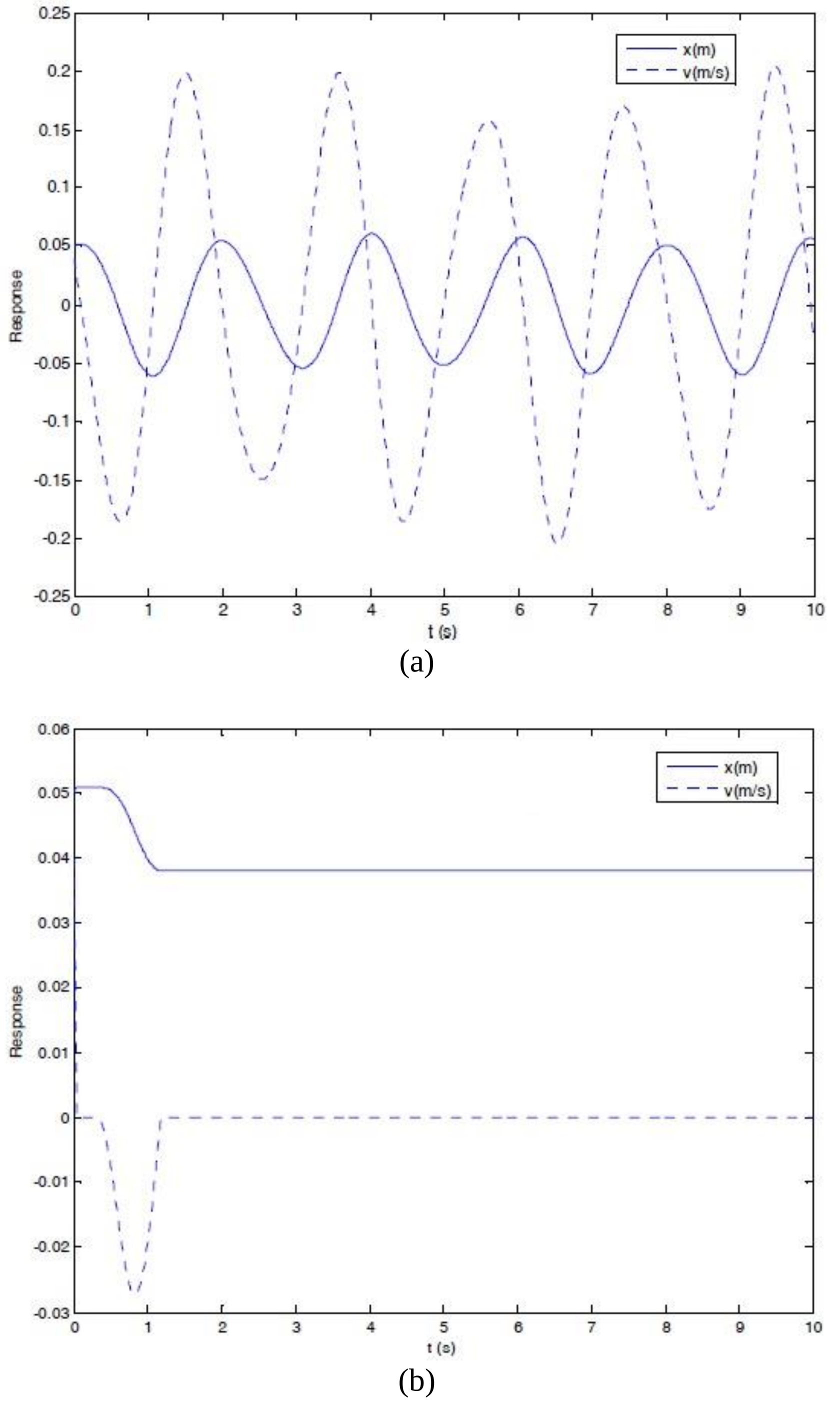
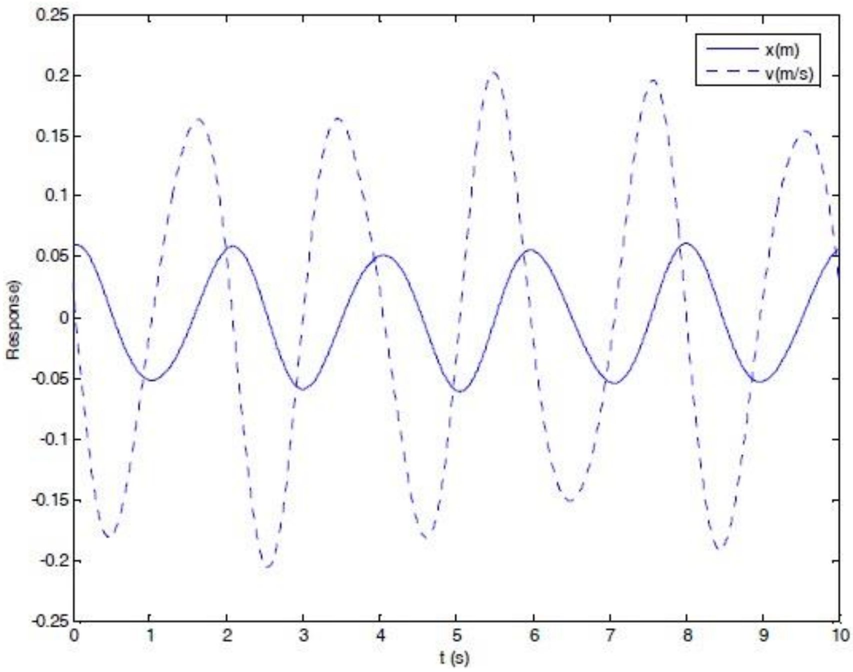


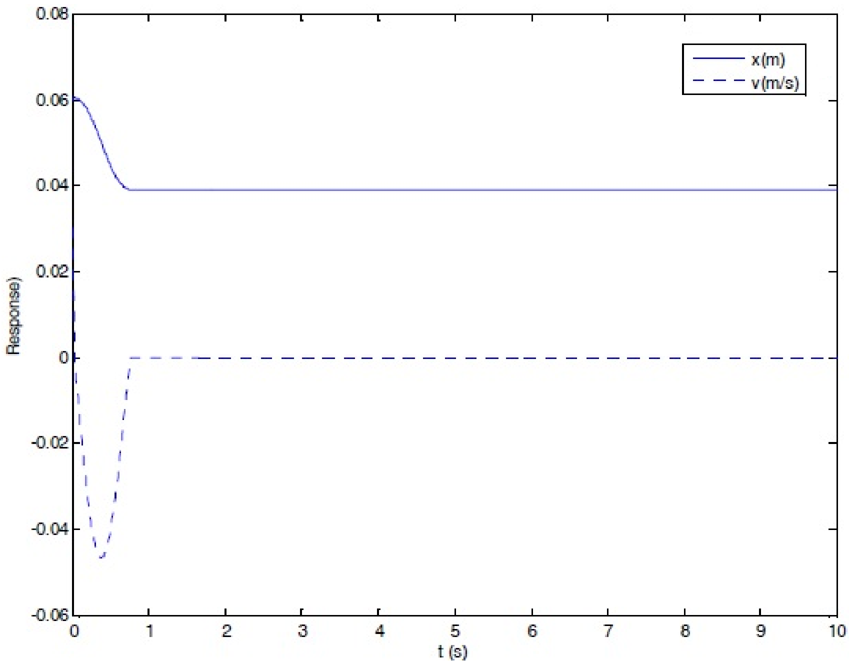
Figure 3. Vibration response and velocity of mass with given initial conditions and static supporting surface ($v_{belt} = 0$): (a) no surface friction, (b) with surface friction ($\mu_0 = 0.05$).

The amplitude of vibration for mass m_1 in Figure 3.a is 0.06 m. Also, the period of vibration in this case is 2 seconds. Furthermore, in Figure 3.b, it is shown that the mass m_1 will stick to the supporting surface and remain there after some initial movement due to initial moving condition.

In the Figure 4, Vibration response and velocity of mass m_2 with given initial conditions and static supporting surface ($v_{belt} = 0$) are shown. Although the behavior of mass m_2 in both cases of (a) and (b) are very similar with mass m_1 but obviously the amplitude of its free vibration case is smaller, i.e., 0.16 meters. Also, it vibration under the effect of friction force is stopped after 0.85 seconds (Figure 4.b).



(a)



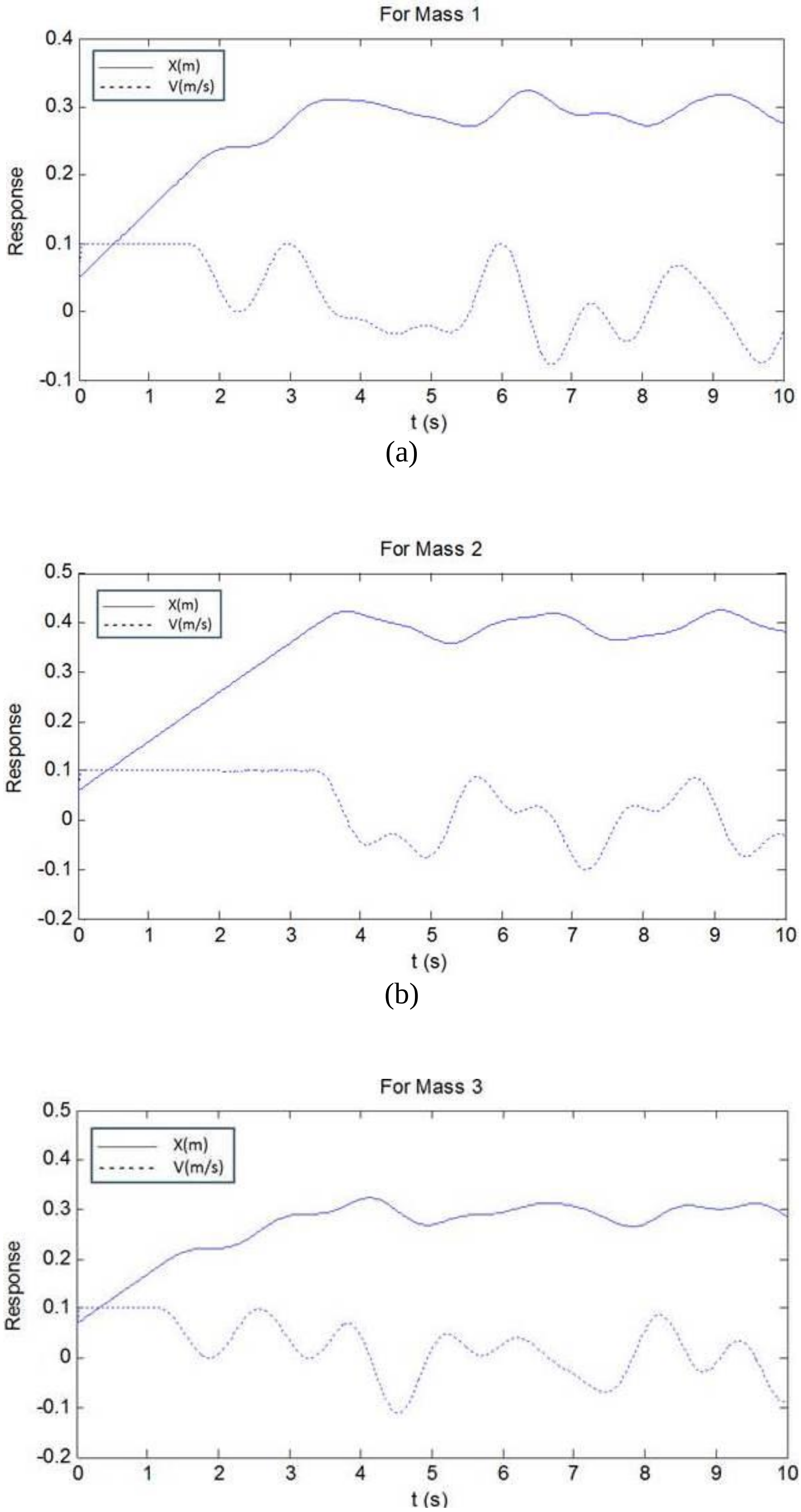
(b)

Figure 4. Vibration response and velocity of mass with given initial conditions and static supporting surface (v_{belt} = 0): (a) no surface friction, (b) with surface friction (μ_0 = 0.05).

Figure 5 illustrates a displacement (mm) and velocity (mm/sec) versus time for mass one, mass two and mass three where the values of system are

Table 2. Values for comparative study on the self-excited vibration analysis of 3-DOF system.

Element Item	**Value**
Mass M_1, M_2 and M_3	1 kg
Spring Stiffness Coefficient K_1, K_2, K_3 and K_4	10 N/m
Damping $C_{1,}$ $C_{2,}$ C_3 and C_4	0 N.m/s
Friction Coefficient μ_0	0.2
Belt Velocity V_{belt}	0.1 m/s



(a)

(b)

(c)

Figure 5. Self-excited vibration response and velocity of mass with given initial conditions: $v_{belt} = 0.1$, $\mu_0 = 0.2$, $M_1 = M_2 = M_3 = 1$ kg, $K_1 = K_2 = K_3 = 10$ N/m, $C_1 = C_2 = C_3 = 0$ N.m/s

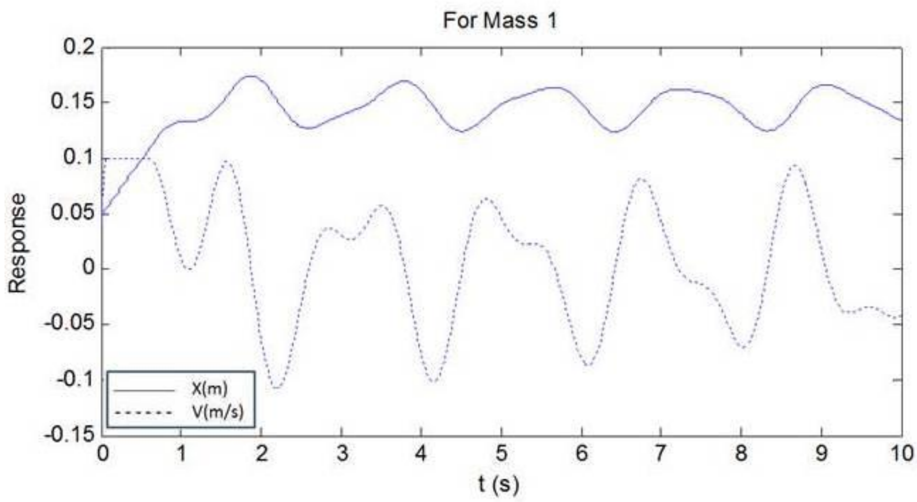
In the Figure 5.a, at first the velocity of belt increased to 0.1m/s. In first 1.85 second the friction force cause the mass to stick to the moving belt until the friction force and the spring force becomes equal. Because of friction force that is time dependent, the force of spring increases subsequently. At first the mass slip in the direction of moving belt and it stick on however the static friction changes to be kinetic friction. At specific time as the friction force and spring force being equal the mass stop immediately. After that the spring force becomes greater than friction force and pulled the mass back in the opposite direction of moving belt. Accordingly, the friction force will be greater than the spring force and lead to fluctuation again.
In the Figure 5.b and Figure 5.c, the stick time for mass two and mass three is 3.4 seconds and 1.2 second respectively and the process of fluctuation is similar to mass one.

From now on, we are going to observe the effect of changing the stick-slip parameters on the system behavior.

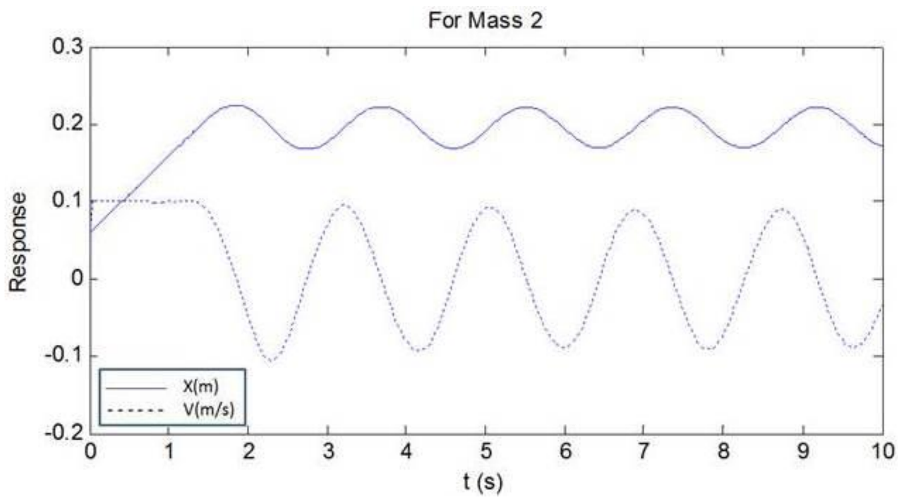
Figure 6 and Figure 7 illustrates the system with different spring stiffness. As it shown the amplitude decreased by increasing the stiffness. It can state also the stick-slip phenomenon decreased while the stiffness increased.

Table 3. Spring-Stiffness value for the self-excited vibration analysis of 3-DOF system.

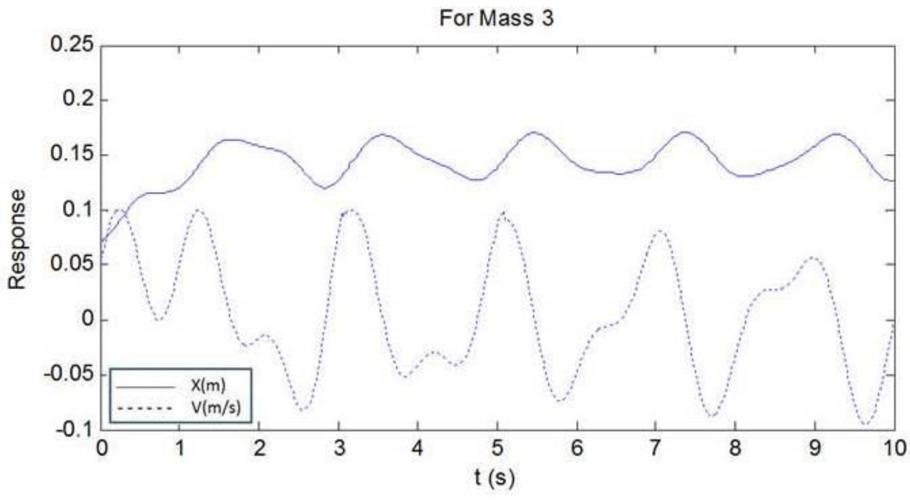
Element Item	**Case I**	**Case II**
Mass M_1, M_2, M_3	1 kg	1 kg
Spring Stiffness Coefficient K_1, K_2, K_3, K_4	20 N/m	30 N/m
Damping C_1, C_2, C_3, C_4	0 N.m/s	0 N.m/s
Friction Coefficient μ_0	0.2	0.2
Belt Velocity V_{belt}	0.1 m/s	0.1 m/s



(a)

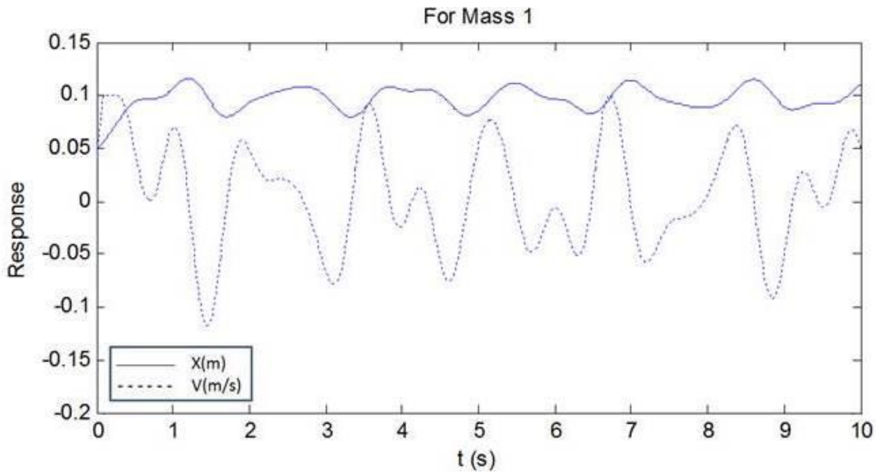


(b)

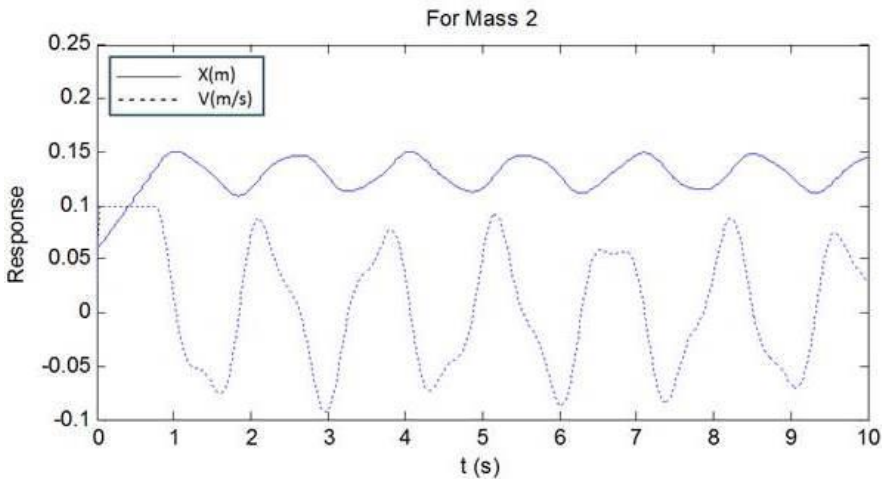


(c)

Figure 6. Stick-slip for $K_1 = K_2 = K_3 = 20$ N/m, v_{belt} = 0.1, μ_0 = 0.2, $M_1 = M_2 = M_3 = 1$ kg, $C_1 = C_2 = C_3 = 0$ N.m/s



(a)



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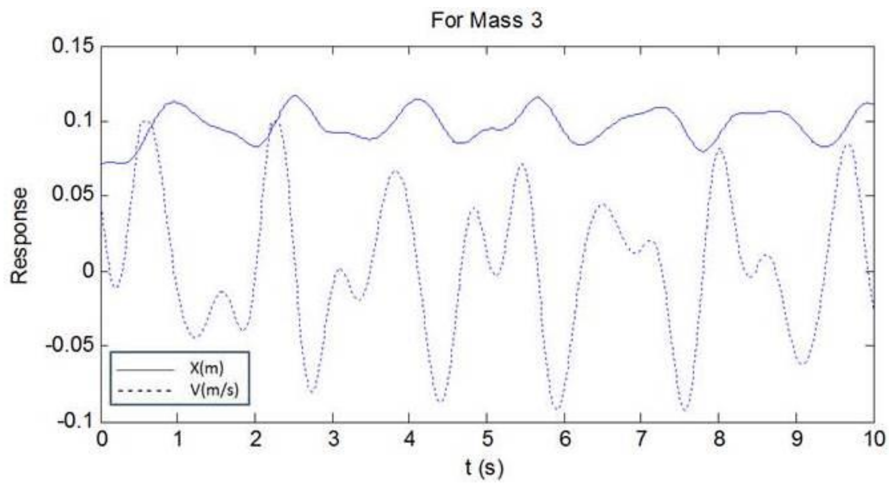


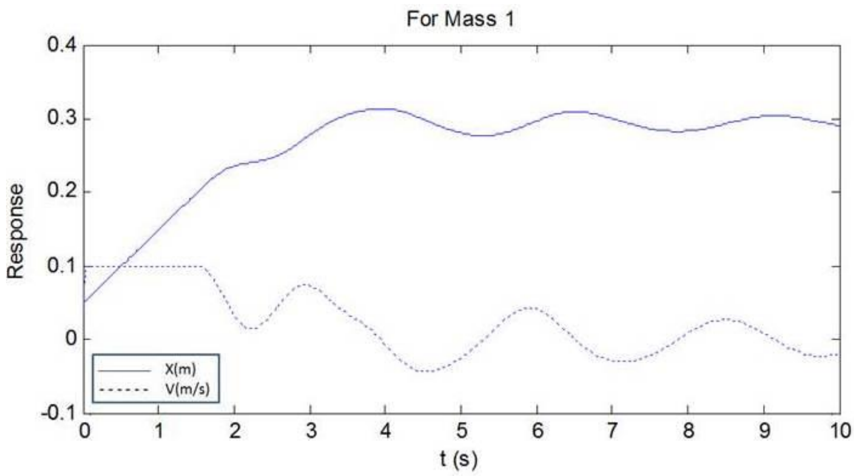
(c)

Figure 7. Stick-slip for $K_1 = K_2 = K_3 = 30$ N/m, v_{belt} = 0.1, μ_0 = 0.2, $M_1 = M_2 = M_3 = 1$ kg, $C_1 = C_2 = C_3 = 0$ N.m/s

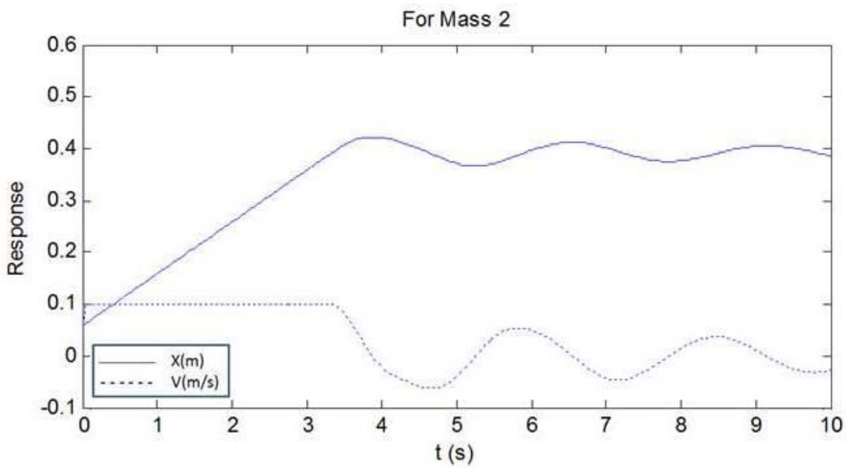
In the Figure 8 and Figure 9 the result shows that while the damping value increase from 0.5 N.m/s to 2 N.m/s the number of fluctuation decreased also the stick phenomenon. It is obvious in Figure 9 that after a while the system is going to be under-damped.

Table 4. Damping value for the self-excited vibration analysis of 3-DOF system.

Element Item	**Case I**	**Case II**
Mass M_1, M_2, M_3	1 kg	1 kg
Spring Stiffness Coefficient K_1, K_2, K_3, K_4	10 N/m	10 N/m
Damping C_1, C_2, C_3, C_4	0.5 N.m/s	2 N.m/s
Friction Coefficient μ_0	0.2	0.2
Belt Velocity V_{belt}	0.1 m/s	0.1 m/s



(a)



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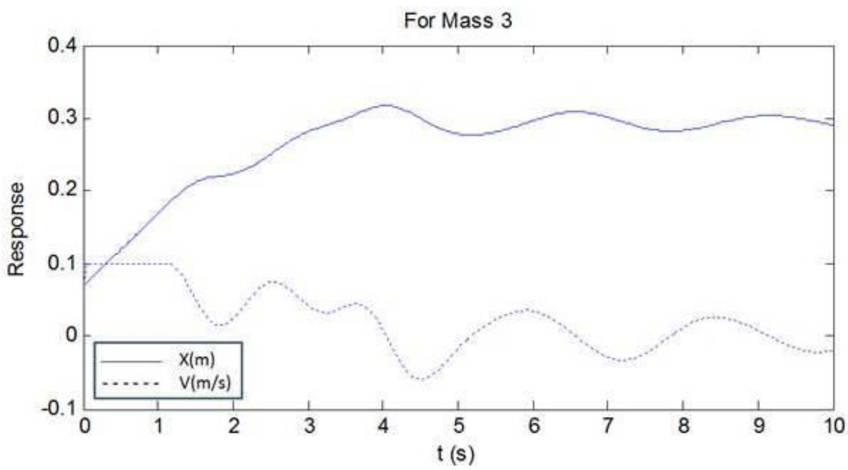
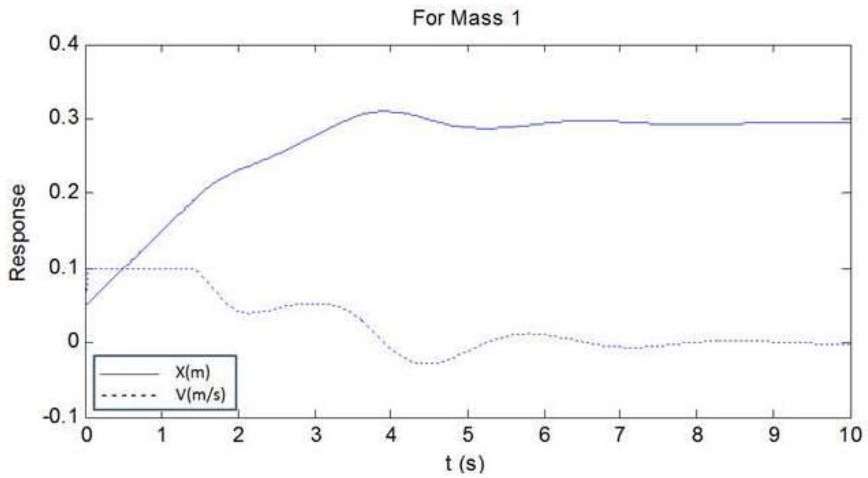
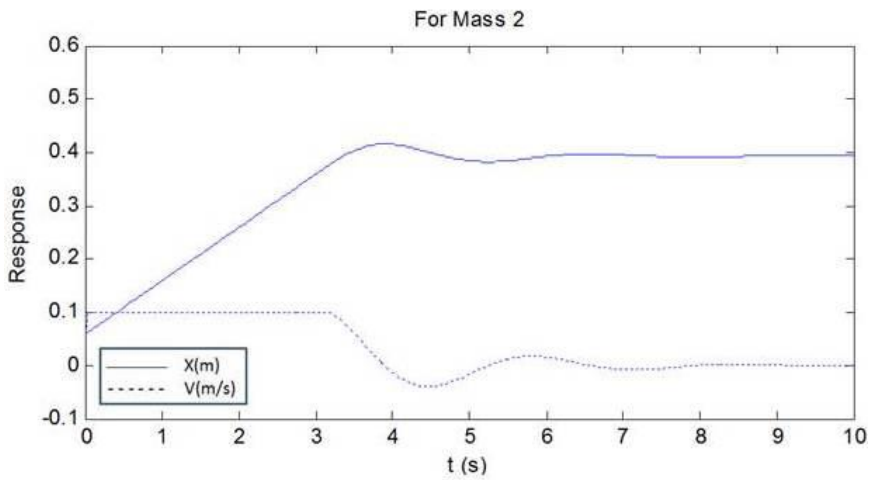


(c)

Figure 8. System response for $C_1 = C_2 = C_3 = 0.5$ N.m/s, v_{belt} = 0.1, μ_0 = 0.2, $M_1 = M_2 = M_3 = 1$ kg, $K_1 = K_2 = K_3 = 10$ N/m



(a)



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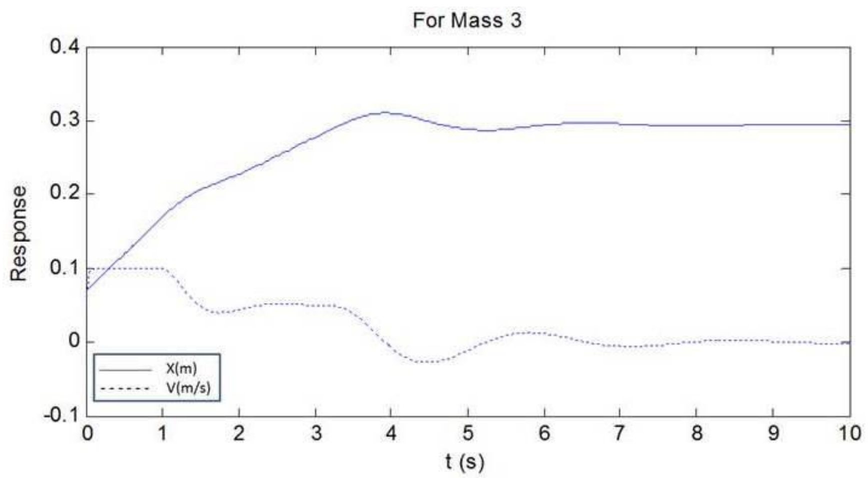


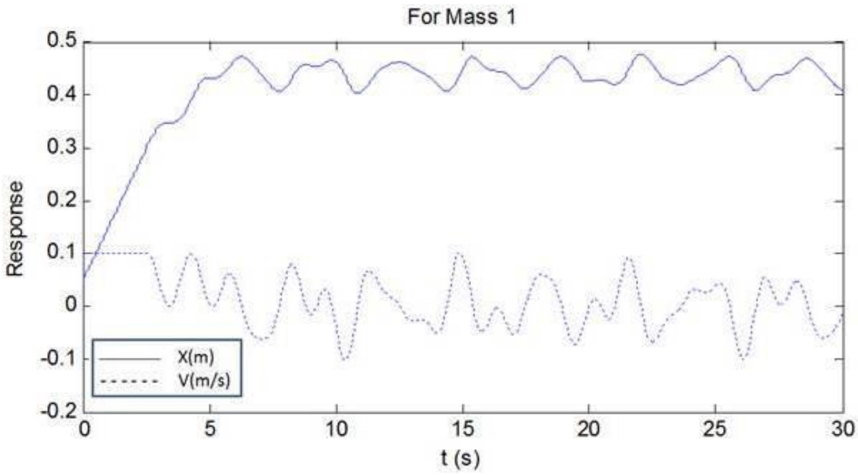
(c)

Figure 9. System response for $C_1 = C_2 = C_3 = 2$ N.m/s, v_{belt} = 0.1, μ_0 = 0.2, $M_1 = M_2 = M_3 = 1$ kg, $K_1 = K_2 = K_3 = 10$ N/m

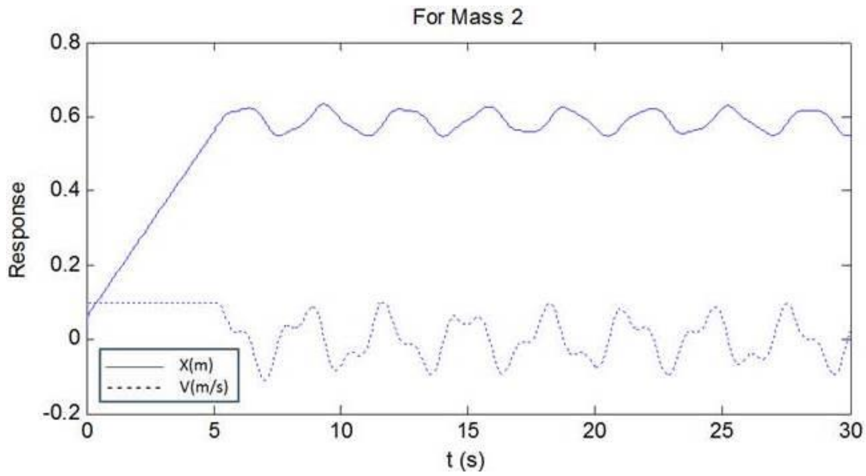
Figure 10 and figure 11 demonstrates that the amplitude has no considerable change by increasing the mass. A point is, with increasing a mass the potential energy that makes the mass pull back becomes greater so the number of oscillations decreased and the stick period becomes larger.

Table 5. Mass value for the self-excited vibration analysis of 3-DOF system.

Element Item	**Case I**	**Case II**
Mass M_1, M_2, M_3	1.5 kg	2.5 kg
Spring Stiffness Coefficient K_1, K_2, K_3, K_4	10 N/m	10 N/m
Damping C_1, C_2, C_3, C_4	0 N.m/s	0 N.m/s
Friction Coefficient μ_0	0.2	0.2
Belt Velocity V_{belt}	0.1 m/s	0.1 m/s



(a)



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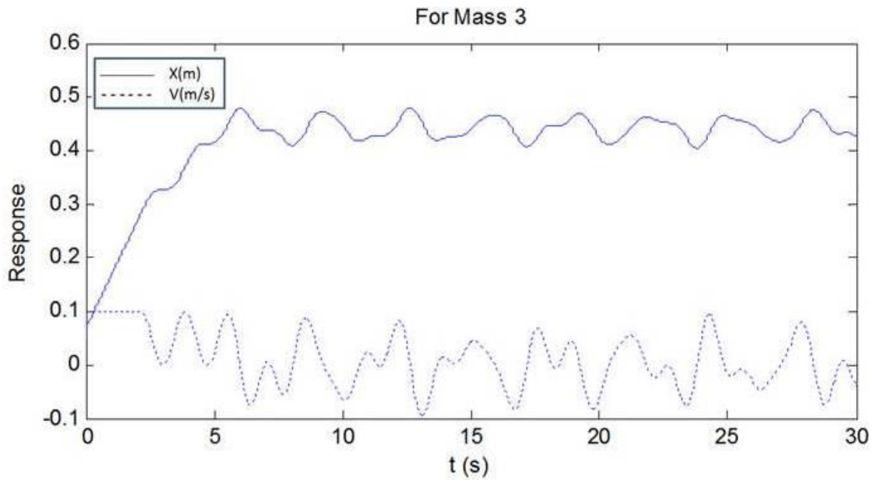
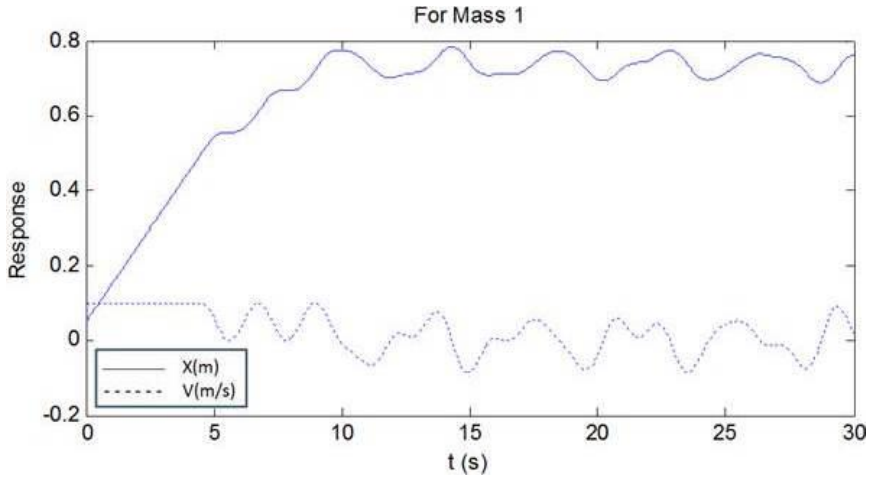
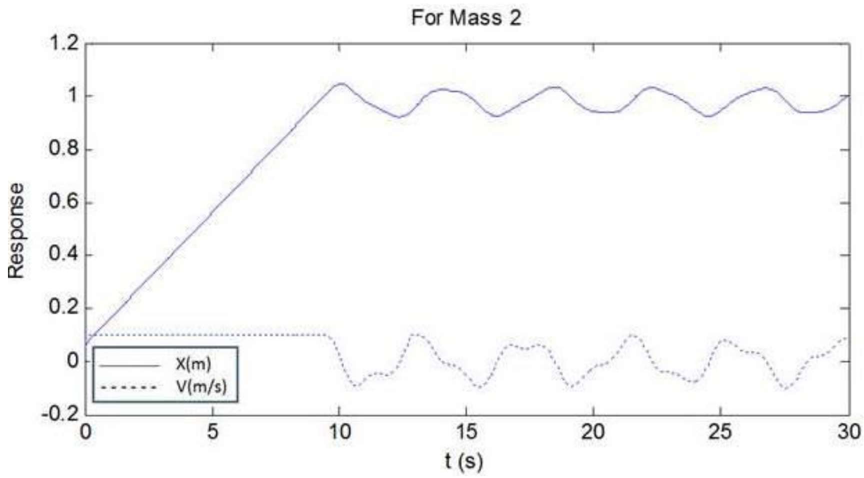


(c)

Figure 10. System response for $M_1 = M_2 = M_3 = 1.5$ kg, v_{belt} = 0.1, μ_0 = 0.2, $C_1 = C_2 = C_3 = 0$ N.m/s, $K_1 = K_2 = K_3 = 10$ N/m



(a)



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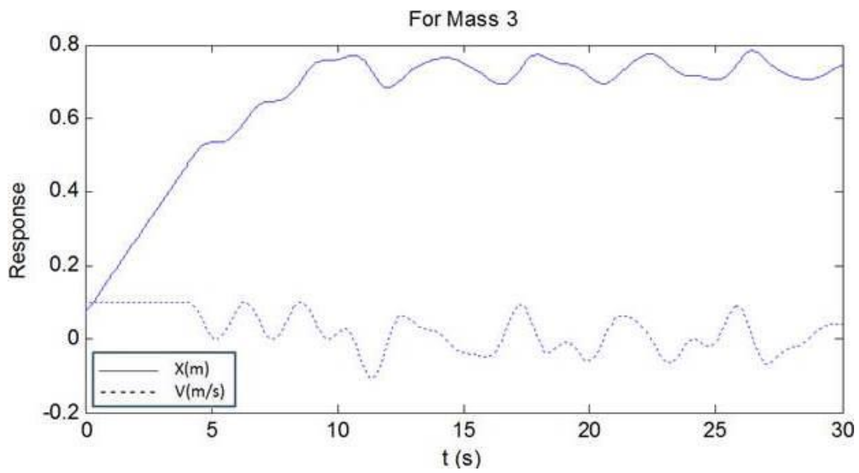


(c)

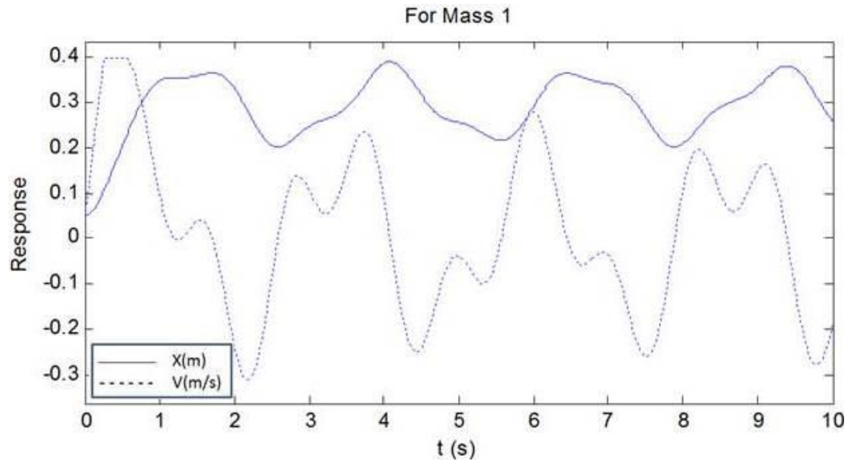
Figure 11. System response for $M_1 = M_2 = M_3 = 2.5\ \text{kg}$, v_{belt} = 0.1, μ_0 = 0.2, $C_1 = C_2 = C_3 = 0\ \text{N.m/s}$, $K_1 = K_2 = K_3 = 10\ \text{N/m}$

The velocity of belt simulates a disk speed in brake system therefore understanding the effect of increasing the belt speed is important. It will permit to know the relation between the belt velocity and a stick-slip phenomenon and a noise that radiated from the disk.

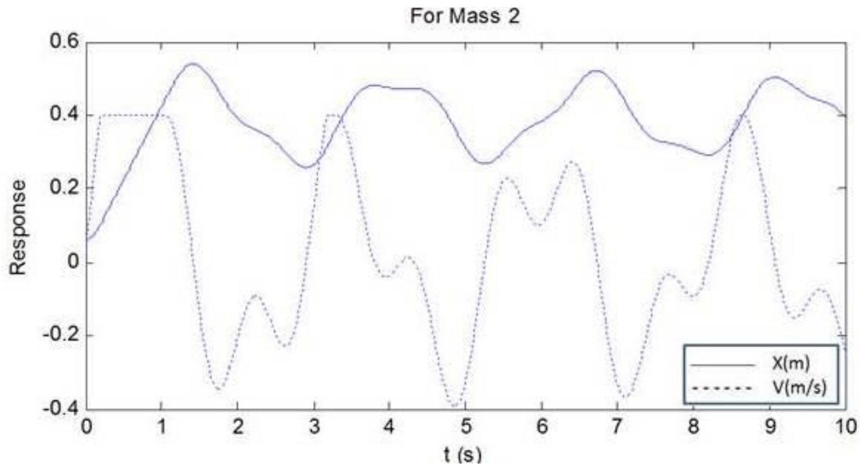
In the Figures 12 and 13 a belt velocity increased to 0.4 m/s and 0.7 m/s respectively. As it shows in Figure 12 by increasing a belt velocity the stick-slip phenomenon start to wipe. At velocity 0.7 m/s it is clear that the system tends to quasi-harmonic oscillation with a constant amplitude and period. It illustrates that increasing a belt velocity cause an increasing in amplitude. Also it increases the number of oscillations. It can be conclude that in a disk system, increasing a disk speed lead to more oscillations.

Table 6. Belt velocity value for the self-excited vibration analysis of 3-DOF system.

Element Item	**Case I**	**Case II**
Mass M_1, M_2, M_3	1 kg	1 kg
Spring Stiffness Coefficient K_1, K_2, K_3, K_4	10 N/m	10 N/m
Damping C_1, C_2, C_3, C_4	0 N.m/s	0 N.m/s
Friction Coefficient μ_0	0.2	0.2
Belt Velocity V_{belt}	0.4 m/s	0.7 m/s



(a)



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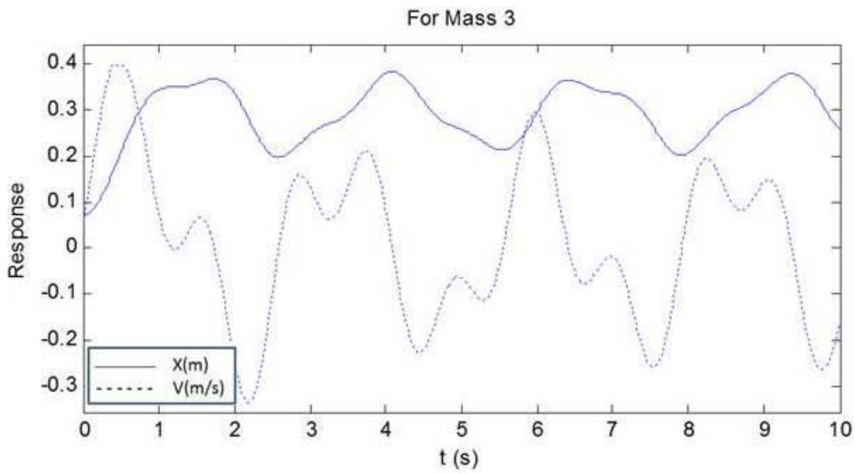
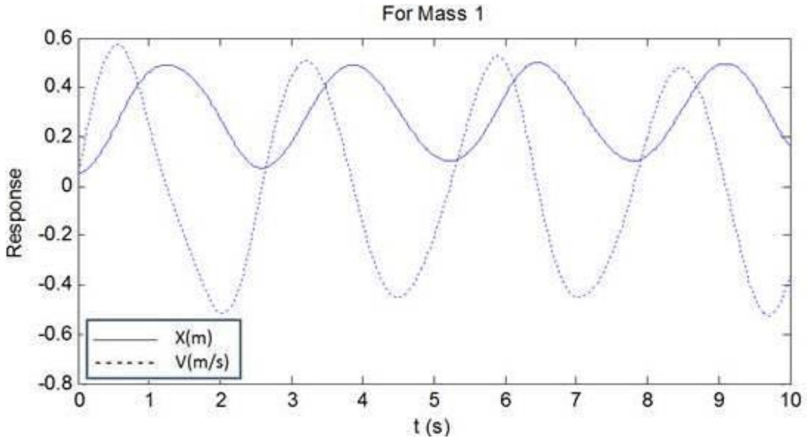


(c)

Figure 12. System response for v_{belt} = 0.4, $M_1 = M_2 = M_3 = 1$ kg, μ_0 = 0.2, $C_1 = C_2 = C_3 = 0$ N.m/s, $K_1 = K_2 = K_3 = 10$ N/m



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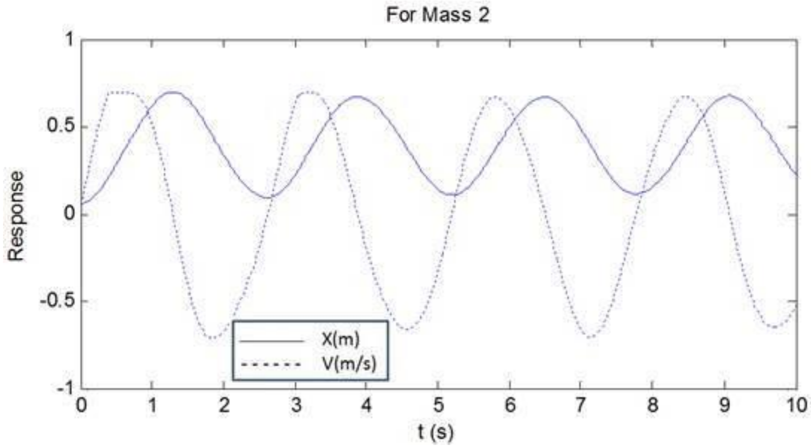

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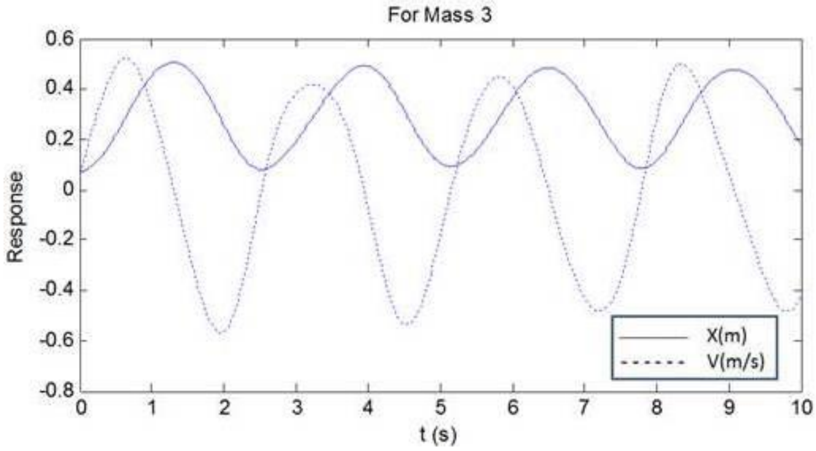

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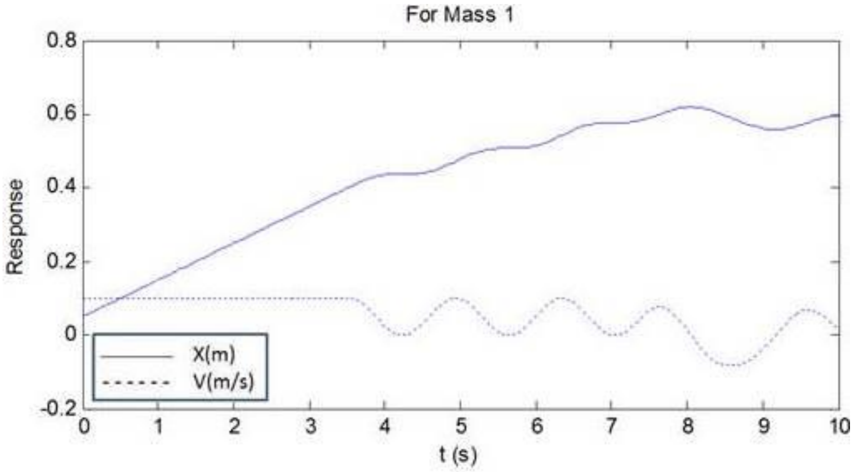
Figure 13. System response for v_{belt} = 0.7, $M_1 = M_2 = M_3 = 1$ kg, μ_0 = 0.2, $C_1 = C_2 = C_3 = 0$ N.m/s, $K_1 = K_2 = K_3 = 10$ N/m

A long stick interval at outset represented in the Figure 14 because of increasing a static friction coefficient from 0.2 to 0.4 and lead the spring force to pull back the mass hardly.

Table 7. Friction coefficient value for the self-excited vibration analysis of 3-DOF system.

Element Item	**Value**
Mass M_1, M_2, M_3	1 kg
Spring Stiffness Coefficient K_1, K_2, K_3, K_4	10 N/m
Damping C_1, C_2, C_3, C_4	0 N.m/s
Friction Coefficient μ_0	0.4
Belt Velocity V_{belt}	0.1 m/s



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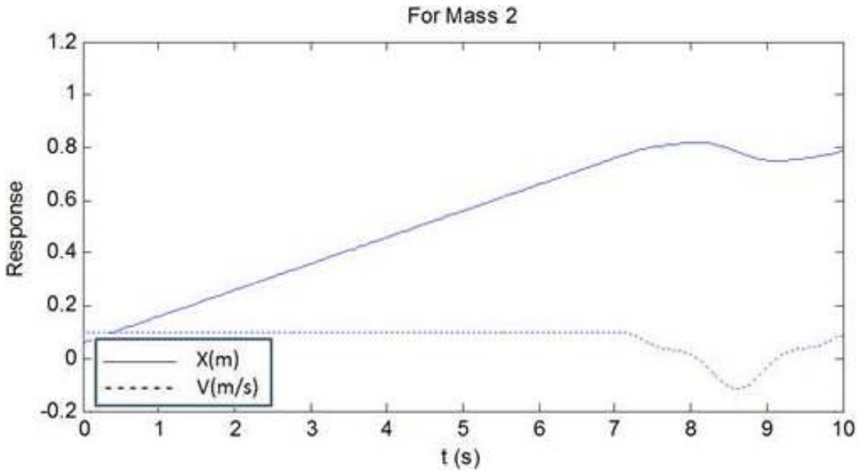

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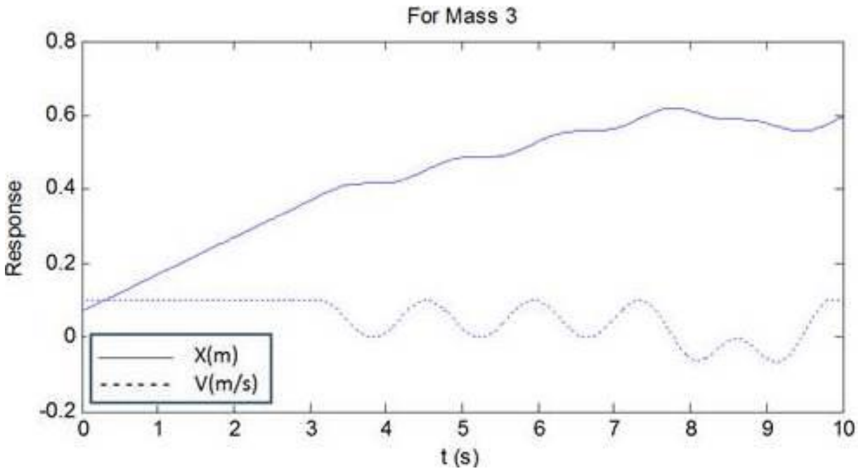

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Figure 14. System response for μ_0 = 0.4, v_{belt} = 0.1, $M_1 = M_2 = M_3 = 1\,\text{kg}$, $C_1 = C_2 = C_3 = 0$ N.m/s, $K_1 = K_2 = K_3 = 10$ N/m

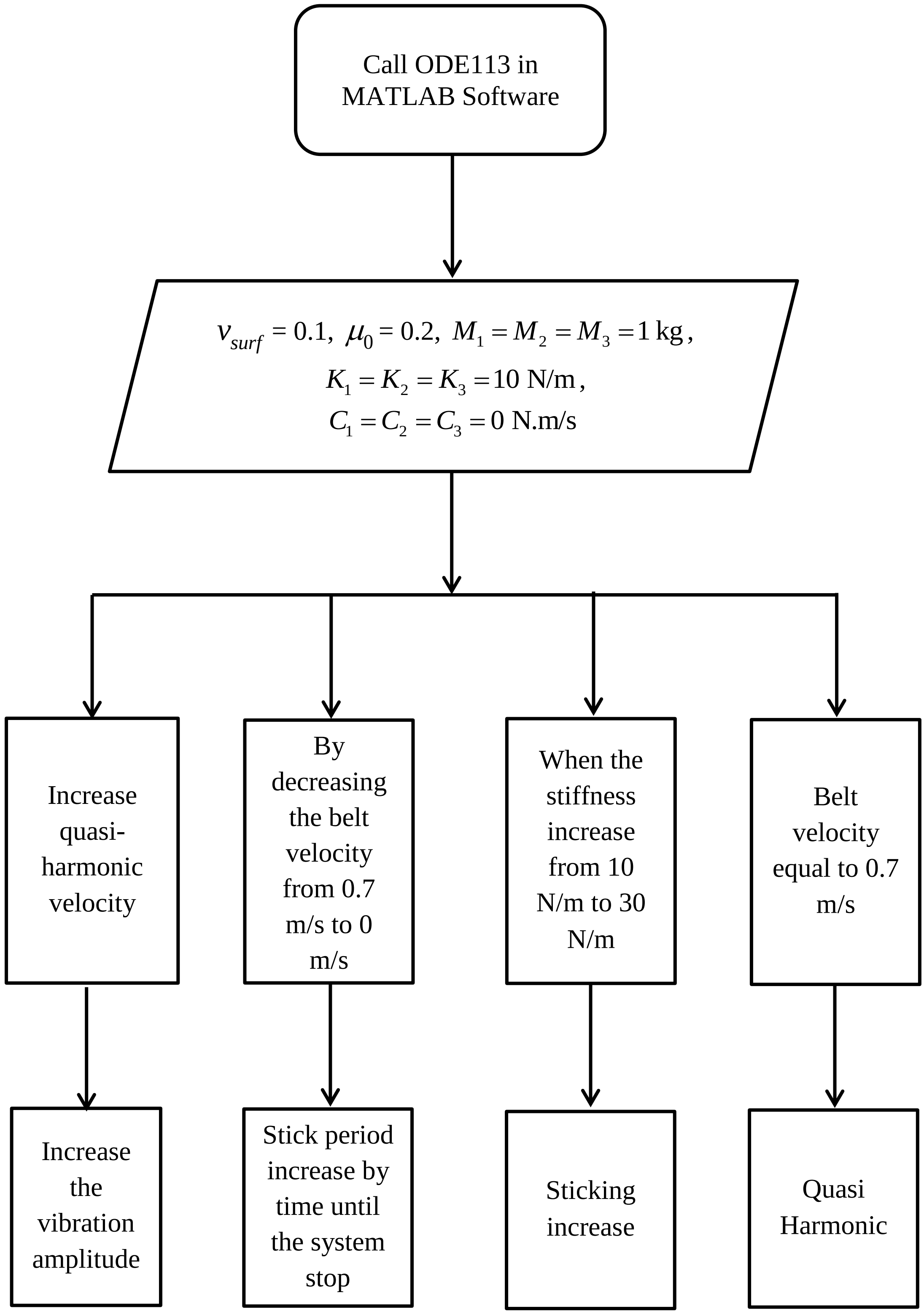

Figure 15. General numerical procedure of self-excited vibration analysis of mechanical systems.

Figure 15 illustrates a brief look to the results of stick-slip phenomena and a general numerical procedure of self-excited vibration analysis for the mechanical systems by using ODE113 in Matlab software.

Chapter 5

CONCLUSION AND FUTURE WORK

In this thesis, a comprehensive study on self-excited vibration of systems with 2- and 3-DOF were performed and presented. The friction coefficient has a significant effect in the stability of a system. There are factors that changing the values of them, affect the behavior of a system and also stick-slip phenomenon. Hence these results should be informed to cause a reduction or generation in a brake noise.

In the stick period there is a static force which controls the motion and in the slip period the velocity dependent on the dynamic friction. In the stick-slip phenomenon, stick occurs because the surfaces static friction is great and the slip happens caused by the low dynamic friction within sliding. By increasing the belt velocity at a specific velocity the stick-slip phenomenon ceases to exist. Meanwhile, decreasing of surface velocity without considering an appropriate amount for coefficient of friction can cause the masses to stick onto surface of moving belt.

It is necessary to perform a stability analysis for the system to understand in which values of system parameter, it is oscillating or sticking. Increasing the friction coefficient reduce the number of oscillations and decreasing it lead a decrease in stick-slip oscillation (noise) but the problem is it lessen the brake efficiency by the way. It was shown that by decreasing the damping coefficient the stick-slip oscillation decrease and the system tend to damped. We also demonstrated that increasing the mass value makes the sticking period longer and the system tends to quasi-harmonic oscillations while increasing the spring stiffness cause to decrease the amplitude and stick-slip oscillation. Last but not least, the role of boundary condition is significant and we can’t disregard the effect of it on producing the stick-slip phenomenon.

As the future work, other suggested model for the approximation of friction force should be investigated. Also approximation of friction force for complex system, where there is no analytical formulation available for the models of friction force, should be studied. At this regard, such models like friction of two discs i.e. squeal phenomenon and other shapes can be considered. Friction induced vibration can be studied in various real life applications like rail-wheel noise, the motion of violin string under the action of a bow, break squeal in automobiles, aircraft and automotive brake squeals. Also it is suggested to do research on flow-induced vibration like wind turbines, machining, pumps and pipelines.

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APPENDICES

Appendix A: 2-DOF System Program

```
%This  program calculates the response of a 2DOF system.
function
varargout = ode113(ode,tspan,y0,options,varargin)

solver_name = 'ode113';

if nargin < 4
  options = [];
  if nargin < 3
    y0 = [];
    if nargin < 2
      tspan = [];
      if nargin < 1
        error(message('MATLAB:ode113:NotEnoughInputs'));
      end
    end
  end
end

% Stats
nsteps  = 0;
nfailed = 0;
nfevals = 0;

% Output
FcnHandlesUsed  = isa(ode,'function_handle');
output_sol = (FcnHandlesUsed && (nargout==1));      % sol =
odeXX(...)
output_ty  = (~output_sol && (nargout > 0));  % [t,y,...] =
odeXX(...)
% There might be no output requested...

sol = []; klastvec = []; phi3d = []; psi2d = [];
if output_sol
  sol.solver = solver_name;
  sol.extdata.odefun = ode;
  sol.extdata.options = options;
  sol.extdata.varargin = varargin;
end

% Handle solver arguments
[neq, tspan, ntspan, next, t0, tfinal, tdir, y0, f0, odeArgs,
odeFcn, ...
 options, threshold, rtol, normcontrol, normy, hmax, htry, htspan,
dataType] = ...
    odearguments(FcnHandlesUsed, solver_name, ode, tspan, y0,
options, varargin);
nfevals = nfevals + 1;

% Handle the output
if nargout > 0
  outputFcn = odeget(options,'OutputFcn',[],'fast');
else
  outputFcn = odeget(options,'OutputFcn',@odeplot,'fast');
end
```

```
outputArgs = {};
if isempty(outputFcn)
  haveOutputFcn = false;
else
  haveOutputFcn = true;
  outputs = odeget(options,'OutputSel',1:neq,'fast');
  if isa(outputFcn,'function_handle')
    % With MATLAB 6 syntax pass additional input arguments to
outputFcn.
    outputArgs = varargin;
  end
end
refine = max(1,odeget(options,'Refine',1,'fast'));
if ntspan > 2
  outputAt = 'RequestedPoints';         % output only at tspan
points
elseif refine <= 1
  outputAt = 'SolverSteps';             % computed points, no
refinement
else
  outputAt = 'RefinedSteps';            % computed points, with
refinement
  S = (1:refine-1) / refine;
end
printstats = strcmp(odeget(options,'Stats','off','fast'),'on');

% Handle the event function
[haveEventFcn,eventFcn,eventArgs,valt,teout,yeout,ieout] = ...
    odeevents(FcnHandlesUsed,odeFcn,t0,y0,options,varargin);

% Handle the mass matrix
[Mtype, M, Mfun] =
odemass(FcnHandlesUsed,odeFcn,t0,y0,options,varargin);
if Mtype > 0
  Msingular = odeget(options,'MassSingular','no','fast');
  if strcmp(Msingular,'maybe')
    warning(message('MATLAB:ode113:MassSingularAssumedNo'));
  elseif strcmp(Msingular,'yes')
    error(message('MATLAB:ode113:MassSingularYes'));
  end
  % Incorporate the mass matrix into odeFcn and odeArgs.
  [odeFcn,odeArgs] =
odemassexplicit(FcnHandlesUsed,Mtype,odeFcn,odeArgs,Mfun,M);
  f0 = feval(odeFcn,t0,y0,odeArgs{:});
  nfevals = nfevals + 1;
end

% Non-negative solution components
idxNonNegative = odeget(options,'NonNegative',[],'fast');
nonNegative = ~isempty(idxNonNegative);
if nonNegative  % modify the derivative function
  [odeFcn,thresholdNonNegative] =
odenonnegative(odeFcn,y0,threshold,idxNonNegative);
  f0 = feval(odeFcn,t0,y0,odeArgs{:});
  nfevals = nfevals + 1;
end

t = t0;
y = y0;
yp = f0;
```

```
% Allocate memory if we're generating output.
nout = 0;
tout = []; yout = [];
if nargout > 0
  if output_sol
    chunk = min(max(100,50*refine), refine+floor((2^10)/neq));
    tout = zeros(1,chunk,dataType);
    yout = zeros(neq,chunk,dataType);
    klastvec = zeros(1,chunk);           % order of the method --
integers
    phi3d = zeros(neq,14,chunk,dataType);
    psi2d = zeros(12,chunk,dataType);
  else
    if ntspan > 2                        % output only at tspan
points
      tout = zeros(1,ntspan,dataType);
      yout = zeros(neq,ntspan,dataType);
    else                                 % alloc in chunks
      chunk = min(max(100,50*refine), refine+floor((2^13)/neq));
      tout = zeros(1,chunk,dataType);
      yout = zeros(neq,chunk,dataType);
    end
  end
  nout = 1;
  tout(nout) = t;
  yout(:,nout) = y;
end

% Initialize method parameters.
maxk = 12;
two = 2 .^ (1:13)';
gstar = [ 0.5000;  0.0833;  0.0417;  0.0264;  ...
          0.0188;  0.0143;  0.0114;  0.00936; ...
          0.00789;  0.00679; 0.00592; 0.00524; 0.00468];

hmin = 16*eps(t);
if isempty(htry)
  % Compute an initial step size h using y'(t).
  absh = min(hmax, htspan);
  if normcontrol
    rh = (norm(yp) / max(normy,threshold)) / (0.25 * sqrt(rtol));
  else
    rh = norm(yp ./ max(abs(y),threshold),inf) / (0.25 *
sqrt(rtol));
  end
  if absh * rh > 1
    absh = 1 / rh;
  end
  absh = max(absh, hmin);
else
  absh = min(hmax, max(hmin, htry));
end

% Initialize.
k = 1;
K = 1;
phi = zeros(neq,14,dataType);
phi(:,1) = yp;
psi = zeros(12,1,dataType);
```

```
alpha = zeros(12,1,dataType);
beta = zeros(12,1,dataType);
sig = zeros(13,1,dataType);
sig(1) = 1;
w = zeros(12,1,dataType);
v = zeros(12,1,dataType);
g = zeros(13,1,dataType);
g(1) = 1;
g(2) = 0.5;

hlast = 0;
klast = 0;
phase1 = true;

% Initialize the output function.
if haveOutputFcn
  feval(outputFcn,[t tfinal],y(outputs),'init',outputArgs{:});
end

% THE MAIN LOOP

done = false;
while ~done

  % By default, hmin is a small number such that t+hmin is only 
slightly
  % different than t.  It might be 0 if t is 0.
  hmin = 16*eps(t);
  absh = min(hmax, max(hmin, absh));    % couldn't limit absh until 
new hmin
  h = tdir * absh;

  % Stretch the step if within 10% of tfinal-t.
  if 1.1*absh >= abs(tfinal - t)
    h = tfinal - t;
    absh = abs(h);
    done = true;
  end

  if haveEventFcn
    % Cache for adjusting the interplant in case of terminal event.
    phi_start = phi;
    psi_start = psi;
  end

  % LOOP FOR ADVANCING ONE STEP.
  failed = 0;
  if normcontrol
    invwt = 1 / max(norm(y),threshold);
  else
    invwt = 1 ./ max(abs(y),threshold);
  end
  while true

    % Compute coefficients of formulas for this step.  Avoid 
computing
    % those quantities not changed when step size is not changed.

    % ns is the number of steps taken with h, including the
```

```
% current one.  When k < ns, no coefficients change
if h ~= hlast
  ns = 0;
end
if ns <= klast
  ns = ns + 1;
end
if k >= ns
  beta(ns) = 1;
  alpha(ns) = 1 / ns;
  temp1 = h * ns;
  sig(ns+1) = 1;
  for i = ns+1:k
    temp2 = psi(i-1);
    psi(i-1) = temp1;
    temp1 = temp2 + h;

    beta(i) = beta(i-1) * psi(i-1) / temp2;
    alpha(i) = h / temp1;
    sig(i+1) = i * alpha(i) * sig(i);
  end
  psi(k) = temp1;

  % Compute coefficients g.
  if ns == 1                           % Initialize v and set w
    v = 1 ./ (K .* (K + 1));
    w = v;
  else
    % If order was raised, update diagonal part of v.
    if k > klast
      v(k) = 1 / (k * (k+1));
      for j = 1:ns-2
        v(k-j) = v(k-j) - alpha(j+1) * v(k-j+1);
      end
    end
    % Update v and set w.
    for iq = 1:k+1-ns
      v(iq) = v(iq) - alpha(ns) * v(iq+1);
      w(iq) = v(iq);
    end
    g(ns+1) = w(1);
  end

  % Compute g in the work vector w.
  for i = ns+2:k+1
    for iq = 1:k+2-i
      w(iq) = w(iq) - alpha(i-1) * w(iq+1);
    end
    g(i) = w(1);
  end
end

% Change phi to phi star.
i = ns+1:k;
phi(:,i) = phi(:,i) * diag(beta(i));

% Predict solution and differences.
phi(:,k+2) = phi(:,k+1);
phi(:,k+1) = zeros(neq,1,dataType);
p = zeros(neq,1,dataType);
```

```
for i = k:-1:1
  p = p + g(i) * phi(:,i);
  phi(:,i) = phi(:,i) + phi(:,i+1);
end

p = y + h * p;
tlast = t;
t = tlast + h;
if done
  t = tfinal;   % Hit end point exactly.
end

yp = feval(odeFcn,t,p,odeArgs{:});
nfevals = nfevals + 1;

% Estimate errors at orders k, k-1, k-2.
phikp1 = yp - phi(:,1);
if normcontrol
  temp3 = norm(phikp1) * invwt;
  err = absh * (g(k) - g(k+1)) * temp3;
  erk = absh * sig(k+1) * gstar(k) * temp3;
  if k >= 2
    erkm1 = absh * sig(k) * gstar(k-1) * ...
        (norm(phi(:,k)+phikp1) * invwt);
  else
    erkm1 = 0.0;
  end
  if k >= 3
    erkm2 = absh * sig(k-1) * gstar(k-2) * ...
        (norm(phi(:,k-1)+phikp1) * invwt);
  else
    erkm2 = 0.0;
  end
else
  temp3 = norm(phikp1 .* invwt,inf);
  err = absh * (g(k) - g(k+1)) * temp3;
  erk = absh * sig(k+1) * gstar(k) * temp3;
  if k >= 2
    erkm1 = absh * sig(k) * gstar(k-1) * ...
        norm((phi(:,k)+phikp1) .* invwt,inf);
  else
    erkm1 = 0.0;
  end
  if k >= 3
    erkm2 = absh * sig(k-1) * gstar(k-2) * ...
        norm((phi(:,k-1)+phikp1) .* invwt,inf);
  else
    erkm2 = 0.0;
  end
end

% Test if order should be lowered
knew = k;
if (k == 2) && (erkm1 <= 0.5*erk)
  knew = k - 1;
end
if (k > 2) && (max(erkm1,erkm2) <= erk)
  knew = k - 1;
end
```

```
    if nonNegative  && (err <= rtol) && any(y(idxNonNegative)<0)
      if normcontrol
        errNN = norm( max(0,-y(idxNonNegative)) ) * invwt;
      else
        errNN = norm( max(0,-y(idxNonNegative)) ./
thresholdNonNegative, inf);
      end
      if errNN > rtol
        err = errNN;
      end
    end

    % Test if step successful
    if err > rtol                           % Failed step
      nfailed = nfailed + 1;
      if absh <= hmin
        warning(message('MATLAB:ode113:IntegrationTolNotMet',
sprintf( '%e', t ), sprintf( '%e', hmin )));
        solver_output = odefinalize(solver_name, sol,...
                                    outputFcn, outputArgs,...
                                    printstats, [nsteps, nfailed,
nfevals],...
                                    nout, tout, yout,...
                                    haveEventFcn, teout, yeout,
ieout,...

{klastvec,phi3d,psi2d,idxNonNegative});
        if nargout > 0
          varargout = solver_output;
        end
        return;
      end

      % Restore t, phi, and psi.
      phase1 = false;
      t = tlast;
      for i = K
        phi(:,i) = (phi(:,i) - phi(:,i+1)) / beta(i);
      end
      for i = 2:k
        psi(i-1) = psi(i) - h;
      end

      failed = failed + 1;
      reduce = 0.5;
      if failed == 3
        knew = 1;
      elseif failed > 3
        reduce = min(0.5, sqrt(0.5*rtol/erk));
      end
      absh = max(reduce * absh, hmin);
      h = tdir * absh;
      k = knew;
      K = 1:k;
      done = false;

    else                                    % Successful step
      break;

    end
```

```
  end
  nsteps = nsteps + 1;

  klast = k;
  hlast = h;

  % Correct and evaluate.
  ylast = y;
  y = p + h * g(k+1) * phikp1;
  yp = feval(odeFcn,t,y,odeArgs{:});
  nfevals = nfevals + 1;

  % Update differences for next step.
  phi(:,k+1) = yp - phi(:,1);
  phi(:,k+2) = phi(:,k+1) - phi(:,k+2);
  for i = K
    phi(:,i) = phi(:,i) + phi(:,k+1);
  end

  if (knew == k-1) || (k == maxk)
    phase1 = false;
  end

  % Select a new order.
  kold = k;
  if phase1                               % Always raise the order in
phase1
    k = k + 1;
  elseif knew == k-1                      % Already decided to lower
the order
    k = k - 1;
    erk = erkm1;
  elseif k+1 <= ns                        % Estimate error at higher
order
    if normcontrol
      erkp1 = absh * gstar(k+1) * (norm(phi(:,k+2)) * invwt);
    else
      erkp1 = absh * gstar(k+1) * norm(phi(:,k+2) .* invwt,inf);
    end
    if k == 1
      if erkp1 < 0.5*erk
        k = k + 1;
        erk = erkp1;
      end
    else
      if erkm1 <= min(erk,erkp1)
        k = k - 1;
        erk = erkm1;
      elseif (k < maxk) && (erkp1 < erk)
        k = k + 1;
        erk = erkp1;
      end
    end
  end
  if k ~= kold
    K = 1:k;
  end

  NNreset_phi = false;
  if nonNegative && any(y(idxNonNegative) < 0)
```

```
    NNidx = idxNonNegative(y(idxNonNegative) < 0); % logical
indexing
    y(NNidx) = 0;
    NNreset_phi = true;
  end

  if haveEventFcn
    [te,ye,ie,valt,stop] =
odezero(@ntrp113,eventFcn,eventArgs,valt,...

tlast,ylast,t,y,t0,klast,phi,psi,idxNonNegative);
    if ~isempty(te)
      if output_sol || (nargout > 2)
        teout = [teout, te];
        yeout = [yeout, ye];
        ieout = [ieout, ie];
      end
      if stop               % Stop on a terminal event.
        % Adjust the interpolation data to [t te(end)].

        % Update the derivative at tzc using the interpolating
polynomial.
        tzc = te(end);
        [~,ypzc] =
ntrp113(tzc,[],[],t,y,klast,phi,psi,idxNonNegative);

        % Update psi and phi using hzc and ypzc.
        psi = psi_start;
        hzc = tzc - tlast;
        beta(1) = 1;
        temp1 = hzc;
        for i = 2:klast
          temp2 = psi(i-1);
          psi(i-1) = temp1;
          temp1 = temp2 + hzc;
          beta(i) = beta(i-1) * psi(i-1) / temp2;
        end
        psi(klast) = temp1;

        phi = phi_start;
        phi(:,2:klast) = phi(:,2:klast) * diag(beta(2:klast));
        phi(:,1:klast+2) = cumsum([ypzc, -phi(:,1:klast+1)],2);

        t = te(end);
        y = ye(:,end);
        done = true;
      end
    end
  end

  if output_sol
    nout = nout + 1;
    if nout > length(tout)
      tout = [tout, zeros(1,chunk,dataType)];  % requires chunk >=
refine
      yout = [yout, zeros(neq,chunk,dataType)];
      klastvec = [klastvec, zeros(1,chunk)];   % order of the method
-- integers
      phi3d = cat(3,phi3d,zeros(neq,14,chunk,dataType));
      psi2d = [psi2d, zeros(12,chunk,dataType)];
```

```
    end
    tout(nout) = t;
    yout(:,nout) = y;
    klastvec(nout) = klast;
    phi3d(:,:,nout) = phi;
    psi2d(:,nout) = psi;
  end

  if output_ty || haveOutputFcn
    switch outputAt
     case 'SolverSteps'        % computed points, no refinement
      nout_new = 1;
      tout_new = t;
      yout_new = y;
     case 'RefinedSteps'       % computed points, with refinement
      tref = tlast + (t-tlast)*S;
      nout_new = refine;
      tout_new = [tref, t];
      yout_new =
[ntrp113(tref,[],[],t,y,klast,phi,psi,idxNonNegative), y];
     case 'RequestedPoints'    % output only at tspan points
      nout_new =  0;
      tout_new = [];
      yout_new = [];
      while next <= ntspan
        if tdir * (t - tspan(next)) < 0
          if haveEventFcn && stop     % output tstop,ystop
            nout_new = nout_new + 1;
            tout_new = [tout_new, t];
            yout_new = [yout_new, y];
          end
          break;
        end
        nout_new = nout_new + 1;
        tout_new = [tout_new, tspan(next)];
        if tspan(next) == t
          yout_new = [yout_new, y];
        else
          yout_new = [yout_new,
ntrp113(tspan(next),[],[],t,y,klast,phi,psi,...
                      idxNonNegative)];
        end
        next = next + 1;
      end
    end

    if nout_new > 0
      if output_ty
        oldnout = nout;
        nout = nout + nout_new;
        if nout > length(tout)
          tout = [tout, zeros(1,chunk,dataType)];  % requires chunk
>= refine
          yout = [yout, zeros(neq,chunk,dataType)];
        end
        idx = oldnout+1:nout;
        tout(idx) = tout_new;
        yout(:,idx) = yout_new;
      end
      if haveOutputFcn
```

```
        stop =
feval(outputFcn,tout_new,yout_new(outputs,:),'',outputArgs{:});
        if stop
          done = true;
        end
      end
    end
  end

  if done
    break
  end

  % Select a new step size.
  if phase1
    absh = 2 * absh;
  elseif 0.5*rtol >= erk*two(k+1)
    absh = 2 * absh;
  elseif 0.5*rtol < erk
    reduce = (0.5 * rtol / erk)^(1 / (k+1));
    absh = absh * max(0.5, min(0.9, reduce));
  end

  if NNreset_phi
    % Used phi for unperturbed solution to select order and
interpolate.
    % In perturbing y, defined NNidx.  Use now to reset phi to move
along
    % constraint.
    phi(NNidx,:) = 0;
  end

end

solver_output = odefinalize(solver_name, sol,...
                            outputFcn, outputArgs,...
                            printstats, [nsteps, nfailed,
nfevals],...
                            nout, tout, yout,...
                            haveEventFcn, teout, yeout, ieout,...
                            {klastvec,phi3d,psi2d,idxNonNegative});
if nargout > 0
  varargout = solver_output;
end


function test
% Ex5_20.m
clc
clear all
tspan = [0: 0.01: 10];
y0 = [0.050; 0.040; 0.06; 0.03];
[t,y] = ode113(@f1, tspan, y0);

plot (t,y (:,1));
hold on
plot(t,y(:,2),':')
%grid on
xlabel ('t (s)');
ylabel ('Response');
```

```
legend({'x(m)','v(m/s)'})
hold off

%[t,y] = ode113(@f2, tspan, y0);

%plot (t,y (:,1));
%hold on
%plot(t,y(:,2),':')
%xlabel ('t (s)');
%ylabel ('x1 (m)');
%grid on
%legend({'x','v'})
%hold off

function out = f1(t,y)
g=9.81;
vsurf=0.01;%m/s
mu0=0.05;
m1=1;%kg
m2=1;
k1=10;%N/M
k2=10;
k3=10;
c1=0;
c2=0;
c3=0;

out = [y(2,:)
       (-m1*g*sign(y(2,:)-vsurf)*mu0 + c2*y(4,:)+ k2*y(3,:)-
(c1+c2)*y(2,:)- (k1+k2)* y(1,:))/m1
       y(4,:)
       (-m2*g*sign(y(4,:)-vsurf)*mu0 + c2*y(2,:)+ k2*y(1,:)-
(c2+c3)*y(4,:)- (k2+k3)* y(3,:))/m2];

%With Firiction
function out1 = f2(t,y)
g=9.81;
vsurf=0.0;%m/s
mu0=0.0;
m1=1;%kg
m2=1;
k1=10;%N/M
k2=10;
k3=10;
c1=0;
c2=0;
c3=0;

out1 = [y(2,:)
       (-m1*g*sign(y(2,:)-vsurf)*mu0 + c2*y(4,:)+ k2*y(3,:)-
(c1+c2)*y(2,:)- (k1+k2)* y(1,:))/m1
       y(4,:)
       (-m2*g*sign(y(4,:)-vsurf)*mu0 + c2*y(2,:)+ k2*y(1,:)-
(c2+c3)*y(4,:)- (k2+k3)* y(3,:))/m2];
```

Appendix B: 3-DOF System Program

```
%This  program calculates the response of a 3DOF system.
function test
clc
clear all
tspan = [0: 0.01: 10];
y0 = [0;0 ; 0;0 ; 0;0];
[t,y] = ode113(@f1, tspan, y0);
[t,y] = ode113(@f2, tspan, y0);
[t,y] = ode113(@f3, tspan, y0);

subplot(2,2,1);plot (y(:,2),y (:,1));title('For Mass 1');
hold on
xlabel ('t (s)');
ylabel ('Response');
subplot(2,2,2);plot (y(:,4),y (:,3));title('For Mass 2');
hold on
xlabel ('t (s)');
ylabel ('Response');
subplot (2,2,3);plot (y(:,6),y (:,5));title('For Mass 3');
hold on
xlabel ('t (s)');
ylabel ('Response');

%legend({'x(m)','v(m/s)'})
hold off

function out1 = f1(t,y)
g=9.81;
vsurf=0.1;%m/s
mu0=0.2;
m1=1;%kg
m2=1;
m3=1;
k1=10;%N/M
k2=10;
k3=10;
k4=10;
c1=0;
c2=0;
c3=0;
c4=0;
out1 = [y(2,:)
     (-m1*g*sign(y(2,:)-vsurf)*mu0 + c2*y(4,:) + k2*y(3,:) -
(c1+c2)*y(2,:) - (k1+k2)*y(1,:))/m1
     y(4,:)
     (-m2*g*sign(y(4,:)-vsurf)*mu0 + c2*y(2,:) + k2*y(1,:) +
c3*y(6,:) + k3*y(5,:) - (c2+c3)*y(4,:) - (k2+k3)*y(3,:))/m2
     y(6,:)
     (-m3*g*sign(y(6,:)-vsurf)*mu0 + c3*y(4,:) + k3*y(3,:) -
(c3+c4)*y(6,:) - (k3+k4)*y(5,:))/m3] ;

function out2 = f2(t,y)
g=9.81;
vsurf=0.1;%m/s
mu0=0.2;
```

```
m1=1;%kg
m2=1;
m3=1;
k1=10;%N/M
k2=10;
k3=10;
k4=10;
c1=0;
c2=0;
c3=0;
c4=0;

out2 = [y(2,:)
     (-m1*g*sign(y(2,:)-vsurf)*mu0 + c2*y(4,:) + k2*y(3,:) -
(c1+c2)*y(2,:) - (k1+k2)*y(1,:))/m1
     y(4,:)
     (-m2*g*sign(y(4,:)-vsurf)*mu0 + c2*y(2,:) + k2*y(1,:) +
c3*y(6,:) + k3*y(5,:) - (c2+c3)*y(4,:) - (k2+k3)*y(3,:))/m2
     y(6,:)
     (-m3*g*sign(y(6,:)-vsurf)*mu0 + c3*y(4,:) + k3*y(3,:) -
(c3+c4)*y(6,:) - (k3+k4)*y(5,:))/m3] ;

function out3 = f3(t,y)
g=9.81;
vsurf=0.1;%m/s
mu0=0.2;
m1=1;%kg
m2=1;
m3=1;
k1=10;%N/M
k2=10;
k3=10;
k4=10;
c1=0;
c2=0;
c3=0;
c4=0;

out3 = [y(2,:)
     (-m1*g*sign(y(2,:)-vsurf)*mu0 + c2*y(4,:) + k2*y(3,:) -
(c1+c2)*y(2,:) - (k1+k2)*y(1,:))/m1
     y(4,:)
     (-m2*g*sign(y(4,:)-vsurf)*mu0 + c2*y(2,:) + k2*y(1,:) +
c3*y(6,:) + k3*y(5,:) - (c2+c3)*y(4,:) - (k2+k3)*y(3,:))/m2
     y(6,:)
     (-m3*g*sign(y(6,:)-vsurf)*mu0 + c3*y(4,:) + k3*y(3,:) -
(c3+c4)*y(6,:) - (k3+k4)*y(5,:))/m3] ;
```