Radial Power-Law Position-dependent Mass, Cylindrical Coordinates, Spectral Signatures

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We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Physics.

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ABSTRACT

By exploring the Position-dependent mass Von Roos Hamiltonian under cylindrical coordinates settings, we discuss the separation of variables of Schrödinger equation. Two radial masses of a coulomb-type and a harmonic oscillator-type are considered, and the effects of various z-dependent interaction potentials on the spectra are studied.

Keywords: Power-law potential, Position dependent-masses, cylindrically symmetric settings, exactly solvable models.

ÖZ

Silindirik simetrik problemlerde Von Roos Hamilton fonksiyonu sayesinde pozisyona bağımlı kütleli Schrödinger denkleminin değişken ayrılırlığı tartışılmıştır. Kulomb ve harmonik titreşen sistemlerde radyal uzaklığa bağımlı kütle ele alınmış olup bazı z-bağımlı etkileşim potansiyellerinin spektrumlar üzerindeki etkisi incelenmiştir.

Anahtar Kelimeler : Üstel bağımlı potansiyel, pozisyona bağımlı kütle, silindirik simetri, kesin çözünürlü modeller.

DEDICATION

I dedicate this humble effort to the prophet of this nation and its bright light Mohammad (ص), to the source of kindness my dear mother and father, to the most expensive thing in my life my lovely Sumaira, to the heart enduring my little prince Mubin, to those who reside in my heart my brothers and sisters, to the Guide my way my dear supervisor and finally to all nice people.

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Majeed Jbrail Saleem

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INTRODUCTION

One of the strongest mathematical tools for handling problems related to quantum mechanical systems in physics and is the Hamiltonian operator.

$$H = T + V(r). \tag{1.1}$$

Where *T* denotes kinetic energy operator and *V(r)* denotes potential energy. In quantum mechanics, a modified Hamiltonian operator provided by Von Roos [1] has proven to be effective in handling problems related to position- dependent mass (PDM). The Von Roos Hamiltonian is given by

$$H = -\frac{1}{4}\left(m(\vec{r})^{\gamma}\vec{\nabla}m(\vec{r})^{\beta}.\vec{\nabla}m(\vec{r})^{\alpha} + m(\vec{r})^{\alpha}\vec{\nabla}m(\vec{r})^{\beta}.\vec{\nabla}m(\vec{r})^{\gamma}\right) + V(\vec{r})\ . \tag{1.2}$$

Where $\alpha\ ,\beta,\gamma$ are called the Von Roos ordering ambiguity parameters that satisfy the relation $\alpha + \ \beta + \ \gamma = -1,$ and the PDM function takes the form $m(\vec{r}) = M(\vec{r})m_{\circ} = M(\vec{r}),$ (where $\hbar = m_{\circ} = 1$ units shall be used throughout).

As a mathematically enriched wide-range-model for solving challenging problems in quantum systems, the position-dependent mass equation provides solutions to the many-body problems, electronic properties of semiconductors and solid states physics [1].

Over the years, the ambiguity parametric set-up of the Von Roos Hamiltonian has undergone a lot of changes based on the problem at hand. For instance, Gora and Williams provided a parametric set-up as $\beta = \gamma = 0\ , \alpha = -1$, Ben Daniel and Duke

[17] suggested $=\alpha = 0\,, \beta = -1$, while, Mustafa and Mazharimousavi [9] suggested $\alpha = \gamma = -\frac{1}{4}\,, \beta = -\frac{1}{2}$. Here, we intend to discuss the radial power-Law-type position-dependent mass given by

$$M(\rho, \varphi, z) = M(\rho) = {b\rho^{2\upsilon+1}}/{2}. \tag{1.3}$$

By using $\upsilon = -\frac{3}{2}$ and $b = 2$, so that $M(\rho) = \rho^{-2}$ which gives a position-dependent mass of quantum particles.

In the second Chapter, we construct the mathematical framework for describing the above radial position-dependent mass in cylindrical coordinates and using separation the variables method where the wave function is defined us,

$$\psi(\rho, \varphi, z) = R(\rho)\phi(\varphi)Z(z). \tag{1.4}$$

In Chapter three, we look at some illustrative examples. In the first section we consider the radial cylindrical form of the coulomb potential, $\tilde{V}(\rho) = -2\rho^{-1}$, and obtain it's corresponding eigenenergies Eq. (3.9). In the second section we shall consider the harmonic oscillator potential, $\tilde{V}(\rho) = \frac{\alpha^2\rho^2}{4}$, and obtain it's eigenenergies Eq. (3.15).

In Chapter four, we intend to look at the effects of z-dependent interactions on both the radial coulomb and harmonic oscillator spectra. Namely we shall study the effects of infinite walls potential, and Morse potential.

Chapter 2

CYLINDRICAL SYMMETRY AND RADIAL POWER LAW PDM

In this Chapter, we consider a position dependent mass is the form of,

$$m(\vec{r}) \equiv M(\rho,\varphi,z) = \mathrm{g}(\rho) f(\varphi) k(z), \tag{2.1}$$

and an interaction potential

$$V(\vec{r}) = V(\rho,\varphi,z). \tag{2.2}$$

Hence

$$\vec{\nabla} m(\vec{r}) = \left(\hat{\rho}\partial_\rho + \frac{\hat{\varphi}}{\rho}\partial_\varphi + \hat{z}\partial_z\right) g(\rho) f(\varphi) k(z). \tag{2.3}$$

We now look at the kinetic energy operator of the PDM Hamiltonian Eq. (1.1), and define the following vectors,

$$\begin{aligned} \vec{A} &= \alpha M(\rho,\varphi,z)^{\alpha-1}\vec{\nabla} M(\rho,\varphi,z), \\ \vec{B} &= \beta M(\rho,\varphi,z)^{\beta-1}\vec{\nabla} M(\rho,\varphi,z), \\ \vec{C} &= \gamma M(\rho,\varphi,z)^{\gamma-1}\vec{\nabla} M(\rho,\varphi,z). \end{aligned} \tag{2.4}$$

Then, one may obtain

$$\begin{aligned} \vec{\nabla} M(\rho,\varphi,z)^{\alpha} &= \vec{A} + M(\rho,\varphi,z)^{\alpha}\vec{\nabla}, \\ \vec{\nabla} M(\rho,\varphi,z)^{\beta} &= \vec{B} + M(\rho,\varphi,z)^{\beta}\vec{\nabla}, \\ \vec{\nabla} M(\rho,\varphi,z)^{\gamma} &= \vec{C} + M(\rho,\varphi,z)^{\gamma}\vec{\nabla}. \end{aligned} \tag{2.5}$$

Using the above identities we get

$$M(\rho,\varphi,z)^{\alpha}\vec{\nabla} M(\rho,\varphi,z)^{\beta}.\vec{\nabla} M(\rho,\varphi,z)^{\gamma} =$$

$$M(\rho,\varphi,z)^{\alpha}[(\vec{B}.\vec{C}) + M(\rho,\varphi,z)^{\gamma}(\vec{B}.\vec{\nabla}) + M(\rho,\varphi,z)^{\beta}(\vec{\nabla}.\vec{C}) +$$

$$M(\rho,\varphi,z)^{\beta}\,\vec{\nabla}.M(\rho,\varphi,z)^{\gamma}\vec{\nabla}]\ . \tag{2.6}$$

Moreover, one should use

$$M(\rho,\varphi,z)^{\beta}\left(\vec{\nabla}.\vec{C}\right)=M(\rho,\varphi,z)^{\beta}\left(\vec{\nabla}.\vec{C}+\vec{C}.\vec{\nabla}\right)\ , \tag{2.7}$$

and

$$M(\rho,\varphi,z)^{\beta}\,\vec{\nabla}.M(\rho,\varphi,z)^{\gamma}\vec{\nabla}=$$

$$M(\rho,\varphi,z)^{\beta}\,\left(\gamma M(\rho,\varphi,z)^{\gamma-1}\vec{\nabla}M(\rho,\varphi,z).\vec{\nabla}\right)+M(\rho,\varphi,z)^{\beta+\gamma}\vec{\nabla}^{2}, \tag{2.8}$$

to obtain

$$M(\rho,\varphi,z)^{\alpha}\vec{\nabla}M(\rho,\varphi,z)^{\beta}.\vec{\nabla}M(\rho,\varphi,z)^{\gamma}\ =$$

$$M(\rho,\varphi,z)^{\alpha}\left(\vec{\mathrm{B}}.\vec{\mathrm{C}}\right)+M(\rho,\varphi,z)^{\alpha+\gamma}\left(\vec{\mathrm{B}}.\vec{\nabla}\right)+M(\rho,\varphi,z)^{\alpha+\beta}\left(2\vec{\nabla}.\vec{\mathrm{C}}+\vec{\mathrm{C}}.\vec{\nabla}\right)+$$

$$M(\rho,\varphi,z)^{-1}\vec{\nabla}^{2}. \tag{2.9}$$

Similarly, we get

$$M(\rho,\varphi,z)^{\gamma}\vec{\nabla}M(\rho,\varphi,z)^{\beta}.\vec{\nabla}M(\rho,\varphi,z)^{\alpha}\ =$$

$$M(\rho,\varphi,z)^{\gamma}\left(\vec{B}.\vec{A}\right)+M(\rho,\varphi,z)^{\alpha+\gamma}\left(\vec{B}.\vec{\nabla}\right)+M(\rho,\varphi,z)^{\gamma+\beta}\left(2\vec{\nabla}.\vec{A}+\vec{A}.\vec{\nabla}\right)+$$

$$M(\rho,\varphi,z)^{-1}\vec{\nabla}^{2}\ . \tag{2.10}$$

Now, applying the vectors' definitions in Eq. (2.4) to yield

$$\vec{B}.\vec{C}\ =\gamma\beta M(\rho,\varphi,z)^{\beta+\gamma-2}(\vec{\nabla}M(\rho,\varphi,z))^{2}\ , \tag{2.11}$$

$$\vec{B}.\vec{A}\ =\alpha\beta M(\rho,\varphi,z)^{\beta+\alpha-2}(\vec{\nabla}M(\rho,\varphi,z))^{2}\ , \tag{2.12}$$

$$\vec{B}.\vec{\nabla}=\beta M(\rho,\varphi,z)^{\beta-1}\vec{\nabla}M(\rho,\varphi,z).\vec{\nabla}, \tag{2.13}$$

$$\vec{C}.\vec{\nabla}=\gamma M(\rho,\varphi,z)^{\gamma-1}\vec{\nabla}M(\rho,\varphi,z).\vec{\nabla}\ , \tag{2.14}$$

$$\vec{A}.\vec{\nabla}=\alpha M(\rho,\varphi,z)^{\alpha-1}\vec{\nabla}M(\rho,\varphi,z).\vec{\nabla}, \tag{2.15}$$

$$\vec{\nabla}.\vec{C}\ =\gamma M(\rho,\varphi,z)^{\gamma-1}\vec{\nabla}^{2}M(\rho,\varphi,z)+$$

$$\gamma(\gamma-1)M(\rho,\varphi,z)^{\gamma-2}\left(\vec{\nabla}M(\rho,\varphi,z)\right)^{2}, \tag{2.16}$$

$$\vec{\nabla}.\vec{A}\ =\alpha M(\rho,\varphi,z)^{\alpha-1}\vec{\nabla}^{2}M(\rho,\varphi,z)+$$

$$\alpha(\alpha-1)M(\rho,\varphi,z)^{\alpha-2}\left(\vec{\nabla}M(\rho,\varphi,z)\right)^2. \tag{2.17}$$

Adding Eq. (2.9) to Eq. (2.10) and using Eqs. (2.11)-(2.17) we obtain the kinetic energy operator is

$$\hat{T}=-\frac{1}{4}[\left(\gamma\beta+\alpha\beta+\gamma(\gamma-1)+\alpha(\alpha-1)\right)\frac{M(\rho,\varphi,z)^{-1}}{M(\rho,\varphi,z)^2}\left(\vec{\nabla}M(\rho,\varphi,z)\right)^2+$$

$$(2\beta+2\alpha+2\gamma)\frac{M(\rho,\varphi,z)^{-1}}{M(\rho,\varphi,z)}\left(\vec{\nabla}M(\rho,\varphi,z).\vec{\nabla}\right)+$$

$$(\gamma+\alpha)\frac{M(\rho,\varphi,z)^{-1}}{M(\rho,\varphi,z)}\vec{\nabla}^2M(\rho,\varphi,z)+2M(\rho,\varphi,z)^{-1}\vec{\nabla}^2\,], \tag{2.18}$$

and

$$\hat{T}=-\frac{1}{4}[\xi M(\rho,\varphi,z)^{-1}\left(\frac{M_\rho^2}{M(\rho,\varphi,z)^2}+\frac{1}{\rho^2}\frac{M_\varphi^2}{M(\rho,\varphi,z)^2}+\frac{M_z^2}{M(\rho,\varphi,z)^2}\right)-$$

$$2M(\rho,\varphi,z)^{-1}\left(\frac{M_\rho}{M(\rho,\varphi,z)}\partial_\rho+\frac{1}{\rho^2}\frac{M_\varphi}{M(\rho,\varphi,z)}\partial_\varphi+\frac{M_z}{M(\rho,\varphi,z)}\partial_z\right)-$$

$$(\beta+1)M(\rho,\varphi,z)^{-1}\left(\frac{\vec{\nabla}^2M(\rho,\varphi,z)}{M(\rho,\varphi,z)}\right)+2M(\rho,\varphi,z)^{-1}\vec{\nabla}^2]\,. \tag{2.19}$$

Therefore, the PDM Schrödinger equation for Hamiltonian (1.1) is written us,

$$\hat{H}\psi(\vec{r})=E\psi(\vec{r}), \tag{2.20}$$

with Eq. (1.4), and

$$\vec{\nabla}^2=\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\right). \tag{2.21}$$

Taking the first order derivatives for $\phi(\varphi)$, $Z(z)$ and $R(\rho)$, would result in

$$2M[E-V(\rho,\varphi,z)]R(\rho)\phi(\varphi)Z(z)+$$

$$\xi\left[\left(\frac{M_\rho}{M(\rho,\varphi,z)}\right)^2+\frac{1}{\rho^2}\left(\frac{M_\varphi}{M(\rho,\varphi,z)}\right)^2+\left(\frac{M_z}{M(\rho,\varphi,z)}\right)^2\right]R(\rho)\phi(\varphi)Z(z)-$$

$$\left(\frac{M_\rho}{M(\rho,\varphi,z)}R'(\rho)\phi(\varphi)Z(z)+\frac{1}{\rho^2}\frac{M_\varphi}{M(\rho,\varphi,z)}R(\rho)\phi'(\varphi)Z(z)+\right.$$

$$\left.\frac{M_z}{M(\rho,\varphi,z)}R(\rho)\phi(\varphi)Z'(z)\right)-$$

$$(\beta+1)\left(\frac{1}{2\rho}\frac{M_\rho}{M(\rho,\varphi,z)}+\frac{M_{\rho\rho}}{2M(\rho,\varphi,z)}+\frac{1}{2\rho^2}\frac{M_{\varphi\varphi}}{M(\rho,\varphi,z)}+\frac{M_{\varphi\varphi}}{2M(\rho,\varphi,z)}\right)$$

$$R(\rho)\phi(\varphi)Z(z)+$$

$$\frac{1}{\rho}\left(R'(\rho)\phi(\varphi)Z(z)+\rho R''(\rho)\phi(\varphi)Z(z)+\frac{1}{\rho}R(\rho)\phi''(\varphi)Z(z)+\right.$$

$$\left.R(\rho)\phi(\varphi)Z''(z)\right)=0. \qquad (2.22)$$

Dividing Eq. (2.22) by R(ρ)ϕ(φ)Z(z), from the left we obtain.

$$g(\rho)f(\varphi)k(z)[E-V(\rho,\varphi,z)]+$$

$$\frac{1}{2}\left[\frac{R''(\rho)}{R(\rho)}-\left(\frac{g'(\rho)}{g(\rho)}-\frac{1}{\rho}\right)\frac{R'(\rho)}{R(\rho)}+\frac{\xi}{2}\left(\frac{g'(\rho)}{g(\rho)}\right)^2-\frac{(\beta+1)}{2}\left(\frac{1}{\rho}\frac{g'(\rho)}{g(\rho)}+\frac{g''(\rho)}{g(\rho)}\right)\right]+$$

$$\frac{1}{2}\left[\frac{Z''(z)}{Z(z)}-\frac{k'(z)}{k(z)}\frac{Z'(z)}{Z(z)}+\frac{\xi}{2}\left(\frac{k'(z)}{k(z)}\right)^2-\frac{(\beta+1)}{2}\frac{k''(z)}{k(z)}\right]+$$

$$\frac{1}{2\rho^2}\left[\frac{\phi''(\varphi)}{\phi(\varphi)}-\frac{f'(\varphi)}{f(\varphi)}\frac{\phi'(\varphi)}{\phi(\varphi)}+\frac{\xi}{2}\left(\frac{f'(\varphi)}{f(\varphi)}\right)^2+\frac{(\beta+1)}{2}\frac{f''(\varphi)}{f(\varphi)}\right]=0\,, \qquad (2.23)$$

where we used

$$M_u=\frac{\partial M}{\partial u}\,,M_{uu}=\frac{\partial^2 M}{\partial u^2}\,, \qquad (2.24)$$

and

$$\xi=\alpha(\alpha-1)+\gamma(\gamma-1)-\beta(\beta+1). \qquad (2.25)$$

It is clear that the separability is awarded via various options. The easiest one is, however, suggested by the first term in Eq. (2.23), as

$$2g(\rho)f(\varphi)k(z)V(\rho,\varphi,z)=\tilde{V}(\rho)+\tilde{V}(z)+\tfrac{1}{\rho^2}\tilde{V}(\varphi). \qquad (2.26)$$

Let us now introduce new functions in order to remove the first order derivatives *Z'(z), R'(ρ)* and $\phi'(\varphi)$ in Eq. (2.23) and cast

$$R(\rho) = \rho^{\vartheta} U(\rho) \ , \tag{2.27}$$

$$Z(z) = K^{1/2}(z)\tilde{Z}(z), \tag{2.28}$$

$$\phi(\varphi) = f^{1/2}(\varphi)\tilde{\phi}(\varphi) \ , \tag{2.29}$$

and consider

$$g(\rho) = \frac{b}{2}\rho^{2\vartheta+1} \ ; \vartheta, b \in \mathbb{R} \, . \tag{2.30}$$

Herein b and ϑ are non-zero constants and both are said to be real. From Eq. (2.27)

$$R'(\rho) = \vartheta\rho^{\vartheta-1}U(\rho) + \rho^{\vartheta}U'(\rho) \ , \tag{2.31}$$

$$R''(\rho) = \vartheta(\vartheta-1)\rho^{\vartheta-2}U(\rho) + 2\vartheta\rho^{\vartheta-1}U'(\rho) + \rho^{\vartheta}U''(\rho) \ . \tag{2.32}$$

From Eq. (2.30)

$$g'(\rho) = \frac{b}{2}(2\vartheta+1)\rho^{2\vartheta} \ , \tag{2.33}$$

$$g''(\rho) = \vartheta(2\vartheta+1)b\rho^{2\vartheta-1} \ . \tag{2.34}$$

From Eq. (2.28)

$$Z'(z) = \frac{1}{2}K^{-1/2}(z)\tilde{Z}(z) + \tilde{Z}'(z)K^{1/2}(z) \ , \tag{2.35}$$

$$Z''(z) = -\frac{1}{4}K^{-3/2}(z)\tilde{Z}(z) + K^{-1/2}(z)\tilde{Z}'(z) + \tilde{Z}''(z)K^{1/2}(z) \ . \tag{2.36}$$

From Eq. (2.29)

$$\phi'(\varphi) = \frac{1}{2}f^{-1/2}(\varphi)\tilde{\phi}(\varphi) + \tilde{\phi}'(\varphi)f^{1/2}(\varphi) \ , \tag{2.37}$$

$$\phi''(\varphi) = -\frac{1}{4}f^{-3/2}(\varphi)\tilde{\phi}(\varphi) + f^{-1/2}(\varphi)\tilde{\phi}(\varphi) + \tilde{\phi}''(\varphi)f^{1/2}(\varphi) \ . \tag{2.38}$$

In this case, we have, the z-dependent term of Eq. (2.23) become

$$\left[-\frac{3}{4}\left(\frac{k'(z)}{k(z)}\right)^2 + \frac{1}{2}\frac{k''(z)}{k(z)} + \frac{\tilde{Z}''(z)}{\tilde{Z}(z)} + \frac{\xi}{2}\left(\frac{k'(z)}{k(z)}\right)^2 - \frac{\beta}{2}\frac{k''(z)}{k(z)} - \frac{1}{2}\frac{k''(z)}{k(z)}\right] =$$

$$\frac{\tilde{Z}''(z)}{\tilde{Z}(z)} + \frac{(2\xi-3)}{4}\left(\frac{k'(z)}{k(z)}\right)^2 - \frac{\beta}{2}\frac{k''(z)}{k(z)} \ . \tag{2.39}$$

For the Radial part of Eq. (2.23) we have

$$\frac{R''(\rho)}{R(\rho)} - \left(\frac{g'(\rho)}{g(\rho)} - \frac{1}{\rho}\right)\frac{R'(\rho)}{R(\rho)} = \frac{U''(\rho)}{U(\rho)} - \frac{\upsilon(\upsilon+1)}{\rho^2},$$

to imply, from Eq. (2.23), that

$$\left[\frac{U''(\rho)}{U(\rho)} - \frac{\upsilon(u+1)}{\rho^2} + \frac{\xi}{2}\left(\frac{g'(\rho)}{g(\rho)}\right)^2 - \frac{(\beta+1)}{2}\left(\frac{1}{\rho}\frac{g'(\rho)}{g(\rho)} + \frac{g''(\rho)}{g(\rho)}\right)\right] =$$

$$\frac{U''(\rho)}{U(\rho)} - \frac{2\upsilon(\upsilon+1) + (2\upsilon+1)^2[\xi-\beta-1]}{2\rho^2}. \qquad (2.40)$$

Next, for the φ dependent term, from Eq. (2.23)

$$-\frac{3}{4}\left(\frac{f'(\varphi)}{f(\varphi)}\right)^2 + \frac{1}{2}\frac{f''(\varphi)}{f(\varphi)} + \frac{\tilde{\phi}''(\varphi)}{\tilde{\phi}(\varphi)} + \frac{\xi}{2}\left(\frac{f'(\varphi)}{f(\varphi)}\right)^2 + \frac{(\beta+1)}{2}\frac{f''(\varphi)}{f(\varphi)} =$$

$$\frac{\tilde{\phi}''(\varphi)}{\tilde{\phi}(\varphi)} + \frac{(2\xi-3)}{4}\left(\frac{f'(\varphi)}{f(\varphi)}\right)^2 + \frac{\beta}{2}\frac{f''(\varphi)}{f(\varphi)}. \qquad (2.41)$$

Substituting Eqs. (2.39)- (2.41) and (2.26) into Eq. (2.23) gives

$$\left[\frac{U''(\rho)}{U(\rho)} + \frac{(2\upsilon+1)^2[\xi-\beta-1] - 2\upsilon(\upsilon+1)}{2\rho^2}\right] +$$

$$\left[\frac{\tilde{Z}''(z)}{\tilde{Z}(z)} + \frac{(2\xi-3)}{4}\left(\frac{k'(z)}{k(z)}\right)^2 - \frac{\beta}{2}\frac{k''(z)}{k(z)}\right] +$$

$$\frac{1}{\rho^2}\left[\frac{\tilde{\phi}''(\varphi)}{\tilde{\phi}''(\varphi)} + \frac{(2\xi-3)}{4}\left(\frac{f'(\varphi)}{f(\varphi)}\right)^2 - \frac{\beta}{2}\frac{f''(\varphi)}{f(\varphi)}\right] +$$

$$b\rho^{2\upsilon+1}f(\varphi)k(z) - \tilde{V}(\rho) - \tilde{V}(z) - \frac{1}{\rho^2}\tilde{V}(\varphi) = 0, \qquad (2.42)$$

and collecting like terms together we obtain

$$b\rho^{2\upsilon+1}f(\varphi)k(z)E +$$

$$\left[\frac{U''(\rho)}{U(\rho)} + \frac{(2\upsilon+1)^2[\xi-\beta-1] - 2\upsilon(\upsilon+1)}{2\rho^2} - \tilde{V}(\rho)\right] +$$

$$\left[\frac{\tilde{Z}''(z)}{\tilde{Z}''(z)} + \frac{(2\xi-3)}{4}\left(\frac{k'(z)}{k(z)}\right)^2 - \frac{\beta}{2}\frac{k''(z)}{k(z)} + \tilde{V}(z)\right] +$$

$$\frac{1}{\rho^2}\left[\frac{\tilde{\phi}''(\varphi)}{\tilde{\phi}''(\varphi)}+\frac{(2\xi-3)}{4}\left(\frac{f'(\varphi)}{f(\varphi)}\right)^2-\frac{\beta}{2}\frac{f''(\varphi)}{f(\varphi)}-\tilde{V}(\varphi)\right]=0. \tag{2.43}$$

Now we will consider the PDM function to be only an explicit function of ρ. Namely, we choose $f(\varphi)=1=k(z)$ and $g(\rho)=\rho^{-2}$ so that $M(\rho,\varphi,z)=M(\rho)=\rho^{-2}$, $\vartheta=-3/2$. Hence, Eq. (2.43) takes the form

$$\left[\frac{\phi''(\varphi)}{\phi(\varphi)}+2E-\tilde{V}(\varphi)+2(\xi-\beta-1)\right]+$$

$$\rho^2\left[\frac{R''(\rho)}{R(\rho)}+\frac{3}{\rho}\frac{R'(\rho)}{R(\rho)}-\tilde{V}(\rho)+\frac{Z''(z)}{Z(z)}-\tilde{V}(z)\right]=0\,. \tag{2.44}$$

Eq. (2.44) along with $\tilde{V}(\varphi)=0$ would, right away, imply

$$\frac{\phi''(\varphi)}{\phi(\varphi)}+2E+2(\xi-\beta-1)=\lambda_\varphi^2\,. \tag{2.45}$$

In Eq. (2.43) to save azimuthal symmetrization, we substitute $\tilde{V}(\varphi)=0$ and $f(\varphi)=1$, also, we choose $k(z)=1$. Then, one obtain

$$\frac{\phi''(\varphi)}{\phi(\varphi)}=-m^2 : |m|=0,1,2\ldots|\ell|\,. \tag{2.46}$$

In due course, the solution of Eq. (2.45) gives us

$$\lambda_\varphi^2=2E+2(\xi-\beta-1)-m^2\,, \tag{2.47}$$

where m is the magnetic quantum number.

Eq. (2.44) becomes, moreover

$$\left[\frac{R''(\rho)}{R(\rho)}+\frac{3}{\rho}\frac{R'(\rho)}{R(\rho)}+\frac{\lambda_\varphi^2}{\rho^2}-\tilde{V}(\rho)\right]+\left[\frac{Z''(z)}{Z(z)}-\tilde{V}(z)\right]=0\,. \tag{2.48}$$

As a result, one may consider that

$$\frac{Z''(z)}{Z(z)}-\tilde{V}(z)=-\lambda_Z^2\,. \tag{2.49}$$

Hence

$$\frac{R''(\rho)}{R(\rho)}+\frac{3}{\rho}\frac{R'(\rho)}{R(\rho)}+\frac{\lambda_\varphi^2}{\rho^2}-\tilde{V}(\rho)=\lambda_Z^2. \tag{2.50}$$

Now, we remove the first order derivative in the radial part Eq. (2.50) and redefine, Eq. (2.27), where $\upsilon = -\frac{3}{2}$, to obtain

$$\frac{R''(\rho)}{R(\rho)} + \frac{3}{\rho}\frac{R'(\rho)}{R(\rho)} = \frac{U''(\rho)}{U(\rho)} - \frac{3}{4\rho^2} \,. \tag{2.51}$$

Substituting Eq. (2.51) into Eq. (2.50) we get

$$\frac{U''(\rho)}{U(\rho)} - \frac{3}{4\rho^2} + \frac{\lambda_\varphi^2}{\rho^2} - \tilde{V}(\rho) = \lambda_Z^2 \,,$$

which after multiplying by $U(\rho)$ we get

$$-U''(\rho) + \left(\frac{\frac{3}{4}-\lambda_\varphi^2}{\rho^2} + \tilde{V}(\rho)\right)U(\rho) = -\lambda_Z^2 U(\rho) \,. \tag{2.52}$$

This equation is to be used and solved (in Chapter 3) $\tilde{V}(\rho) = -\frac{2}{\rho}$ and $\tilde{V}(\rho) = \frac{\alpha^2\rho^2}{4}$.

Then we shall see the effect (in Chapter 4) of having some $\tilde{V}(z)$ interactions on the spectra.

Chapter 3

RADIALLY CYLINDRICAL COULOMB AND HARMONIC OSCILLATOR POTENTIAL

In this Chapter we consider, $\tilde{V}(\rho)$ to represent a Coulomb and a harmonic oscillator potentials each at a time.

3.1 The Radial Cylindrical Coulomb Potential

We take a coulomb model

$$\tilde{V}(\rho) = -2\rho^{-1} \,. \tag{3.1}$$

So Eq. (2.52) becomes

$$-U''(\rho) + \left(\frac{\ell^2 - \frac{1}{4}}{\rho^2} - \frac{2}{\rho}\right) U(\rho) = -\lambda_Z^2 U(\rho) \,, \tag{3.2}$$

where

$$\ell^2 - \frac{1}{4} = \frac{3}{4} - \lambda_Z^2, \tag{3.3}$$

$$\lambda_Z^2 = 1 - \ell^2 \,, \tag{3.4}$$

and

$$\ell = \left(1 - \lambda_\varphi^2\right)^{1/2} . \tag{3.5}$$

Eq. (3.2) has exact eigenvalue given by

$$\lambda_z = \left(n_\rho + \ell + 1\right)^{-1} \tag{3.6}$$

where $n_\rho = 0,1,2, \ldots$, is the radial quantum number.

Eq. (3.6) becomes

$$\lambda_z = \frac{1}{n_\rho + \sqrt{1-\lambda_\varphi^2} + 1} \,, \tag{3.7}$$

which implies

$$\lambda_\varphi^2 = 1 - \left(\frac{1}{\lambda_z^2} - n_\rho - 1\right)^2. \tag{3.8}$$

Substituting Eq. (3.8) into Eq. (2.47) we obtain

$$1 - \left(\frac{1}{\lambda_z} - n_\rho - 1\right)^2 = 2E + 2(\xi - \beta) - 2 - m^2 \,,$$

which gives

$$E = \left(\frac{m^2+3}{2}\right) - (\xi - \beta) - \frac{1}{2}\left(\frac{1}{\lambda_z} - n_\rho - 1\right)^2. \tag{3.9}$$

This is energy for the radial coulomb potential.

3.2 The Radial Harmonic-oscillator Potential

Now we consider the radial harmonic oscillator model

$$\tilde{V}(\rho) = \frac{\alpha^2 \rho^2}{4} \,. \tag{3.10}$$

Again with Eq. (3.5), and Eq. (2.52) becomes

$$-U''(\rho) + \left(\frac{\ell^2 - \frac{1}{4}}{\rho^2} + \frac{\alpha^2\rho^2}{4}\right) U(\rho) = -\lambda_Z^2 U(\rho), \tag{3.11}$$

which has exact eigenvalue given by

$$\lambda_Z^2 = \alpha\left(\ell - 2n_\rho - 1\right) . \tag{3.12}$$

Let

$$\lambda_Z^2 = \alpha\left(\sqrt{1-\lambda_\varphi^2} - 2n_\rho - 1\right), \tag{2.13}$$

which implies

$$\lambda_\varphi^2 = 1 - \left(\frac{\lambda_Z^2}{\alpha} - 2n_\rho - 1\right)^2 . \tag{3.14}$$

Putting Eq. (3.14) into Eq. (2.47) we obtain

$$E = \left(\frac{m^2+3}{2}\right) - (\xi - \beta) - \frac{1}{2}\left(\frac{\lambda_Z^2}{\alpha} - 2n_\rho - 1\right)^2. \tag{3.15}$$

This is energy for the radial harmonic oscillator potential.

Chapter 4

EFFECTS OF Z-DEPENDENT INTERACTIONS ON THE SPECTRA

In this Chapter we shall study the effects of some z-dependent interaction potentials ($\tilde{V}(z) \neq o$) on the radial coulomb and. harmonic oscillator spectra.

4.1 Effect of Infinite Walls on the Radial Coulomb and Harmonic Oscillator Spectra

Let us consider PDM-particle trapped to move between tow impenetrable walls at $z = 0$ and $z = L$ under the influence of a potential

$$\tilde{V}(z) = \begin{cases} 0, & 0 < z < L \\ \infty, & \text{elsewhere} \end{cases} . \tag{4.1}$$

Using the Schrödinger Eq. (2.49)

$$\frac{d^2 Z(z)}{dz^2} + \lambda_Z^2 Z(z) = 0 \, , \tag{4.2}$$

with a solution

$$Z(z) = Acos(\lambda_z z) + Bsin(\lambda_z z). \tag{4.3}$$

The boundary condition imply

$$Z(z)\big|_{z=0} = 0 \Rightarrow A = 0 \, , \tag{4.4}$$

and

$$Z(z)\big|_{z=L} = 0 \Rightarrow Bsin(\lambda_z z) = 0 \, , \tag{4.5}$$

which results in

$$\lambda_z = \frac{n_z \pi}{L} \, , \tag{4.6}$$

and finally

$$\lambda_z^2 = \frac{{n_z}^2\pi^2}{L^2} \ , \tag{4.7}$$

where $n_z = 1,2,3, \dots$. Therefor, a quantum particle with $M(\rho,\varphi,z) = M(\rho) = \rho^{-2}$ subjected to an interaction potential of the form

$$V(\rho,\varphi,z) = -2\rho^{-1} + \rho^2\tilde{V}(z), \tag{4.8}$$

with $\tilde{V}(z)$ defined in Eq. (4. 1), will admit exact energy eigenvalues given by

$$E = \left(\frac{m^2+3}{2}\right) - (\xi - \beta) - \frac{1}{2}\left(\frac{L}{n_z\pi} - n_\rho - 1\right)^2. \tag{4.9}$$

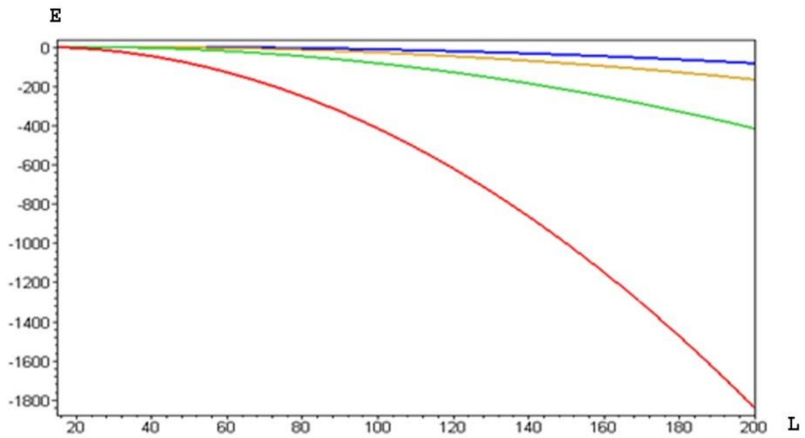

Figure 4.1: The plot of E versus L, using Eq. (4.9) and taking $\left(m = 0,\ n_\rho = 2\ ,\ \alpha = \gamma = -\frac{1}{4} \text{ and } \beta = -\frac{1}{2}\ , \text{for } n_z = 1,2,3,4\right)$.

From Fig. 4.1, for the radial coulomb case, it is clear that $E \propto -L$. It shows that when *L=0*, the energy of the system remains constant and the spectra gives a straight line. However, as *L* increases, the space between the line spectra also increases.

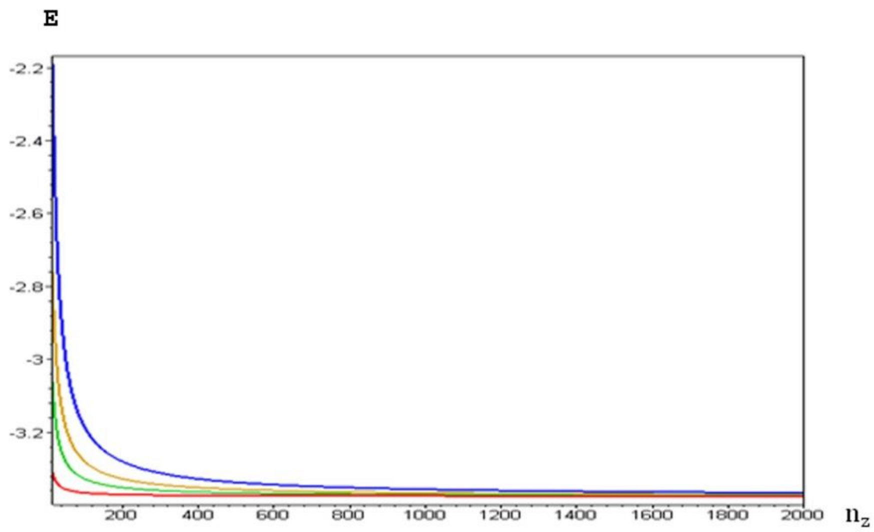

Figure 4.2: The plot of E versus n_z, using Eq. (4.9) and taking $\left(m = 0,\ n_\rho = 2\ ,\ \alpha = \gamma = -\frac{1}{4}\right.$ and $\beta = -\frac{1}{2}$, for $\left. L = 1{,}5{,}10{,}20 \right)$.

From Fig. 4.2, for the radial coulomb case, it is clear that $E \propto -\frac{1}{n_z}$. It shows that as $n_z \to 0$ the energy of the system diverges towards negative infinity. However, as n_z increases, the energy of the system decreases and the space between the line spectra narrow down to a straight line.

On the other hand, a quantum particle with $M(\rho, \varphi, z) = M(\rho) = \rho^{-2}$ subjected to an interaction potential

$$V(\rho, \varphi, z) = \frac{\mathrm{a}^2 \rho^2}{4} + \rho^2 \tilde{V}(z), \tag{4.10}$$

by substituting Eq. (4.6) into Eq. (3.15) will have exact energy eigenvalues of the form

$$E = \left(\frac{m^2 + 3}{2}\right) - (\xi - \beta) - \frac{1}{2}\left(\frac{{n_z}^2 \pi^2}{\mathrm{a} L^2} - 2n_\rho - 1\right)^2 . \tag{4.11}$$

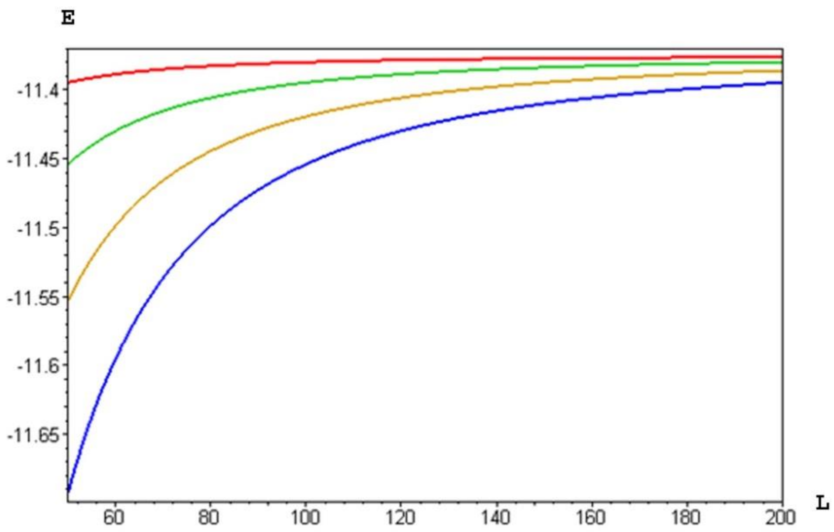

Figure 4.3: The plot of E versus L, using Eq. (4.11) and taking $\left(m = 0,\ n_\rho = 2\ ,\ \alpha = \gamma = -\frac{1}{4} \text{ and } \beta = -\frac{1}{2}\ , \text{for } n_z = 1,2,3,4 \right)$.

From Fig. 4.3, for radial harmonic oscillator, it is clear that $E \propto -\frac{1}{L^2}$. It shows that when $L=0$, energy diverges towards negative infinity and the spacing between the line spectra also widens-up. However, as L increase, the Energy of the system decreases to a constant (steady) as the spacing between the line spectra narrows down to a straight line.

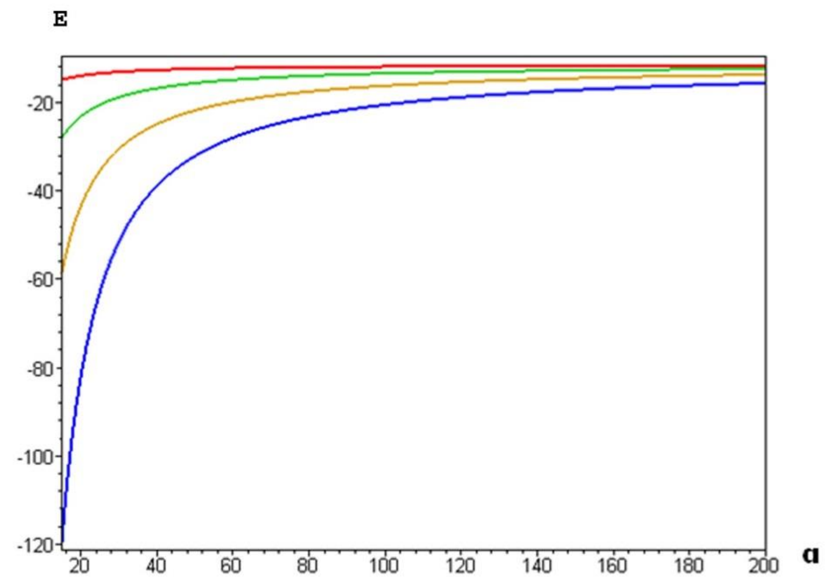

Figure 4.4: The plot of E versus ɑ, using Eq. (4.11) and taking $\left(m = 0,\ n_{\rho} = 2\ ,\ \alpha = \gamma = -\frac{1}{4} \text{ and } \beta = -\frac{1}{2}\ , \text{for } n_z = 1{,}2{,}3{,}4 \right)$.

From Fig. 4.4, for the radial harmonic oscillator, it is clear that $E \propto -\frac{1}{ɑ}$, it is shows that when ɑ → *0*, energy diverges towards negative infinity and the spacing between the line spectra also widens-up. However, as ɑ increases, the energy of the system increases to a constant and the spacing between the line spectra narrows down to a straight line.

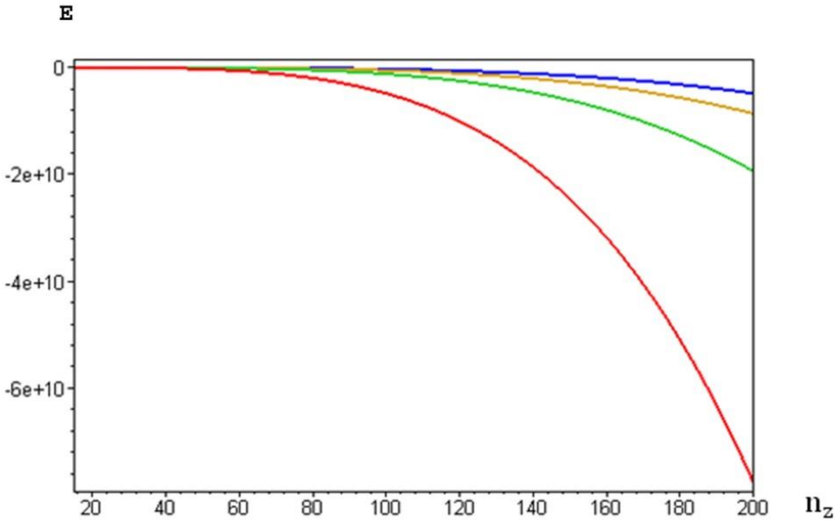

Figure 4.5: The plot of E versus n_z, using Eq. (4.11) and taking $\left(m = 0,\ n_\rho = 2\ , L = 1,\ \alpha = \gamma = -\frac{1}{4} \text{ and } \beta = -\frac{1}{2}\ , \text{for } \alpha = 1,2,3,4 \right)$.

From Fig. 4.5, for the radial harmonic oscillator, it is clear that $E \propto -n_z^2$. It shows that when n_z^2=*0* the energy of the system remains constant and the spectra gives a straight line. However, as n_z^2 increases, the spacing between the line spectra increases.

4.2 Effect of a $\tilde{V}(z)$ Morse Model on the Radial Coulomb and Harmonic-oscillator Spectra.

Let us consider a z-depndent Morse type interaction potential

$$\tilde{V}(z) = D(e^{-2\epsilon z} - 2e^{-\epsilon z})\ ; D > 0\ . \tag{4.12}$$

In Eq. (2.49)

$$\frac{Z''(z)}{Z(z)} - \tilde{V}(z) = -\lambda_Z^2\ ,$$

then, it has a well-known solution of the form

$$\lambda_Z^2 = \left(\frac{\sqrt{D}}{\epsilon} - \tilde{n}_z - \frac{1}{2}\right)\ ;\ \ \tilde{n}_z = 0,1,2, \ldots\ . \tag{4.13}$$

In this case, a position-dependent mass defined in Eq. (2.1) moving in a potential function,

$$V(\rho,\varphi,z) = -2\rho + \rho^2 D(e^{-2\epsilon z} - 2e^{-\epsilon z}), \tag{4.14}$$

will have exact eigenenergies given by

$$E = \left(\frac{m^2+3}{2}\right) - (\xi - \beta) - \frac{1}{2}\left(\frac{1}{\sqrt{\frac{\sqrt{D}}{\epsilon} - \tilde{n}_z - \frac{1}{2}}} - 2n_\rho - 1\right)^2. \tag{4.15}$$

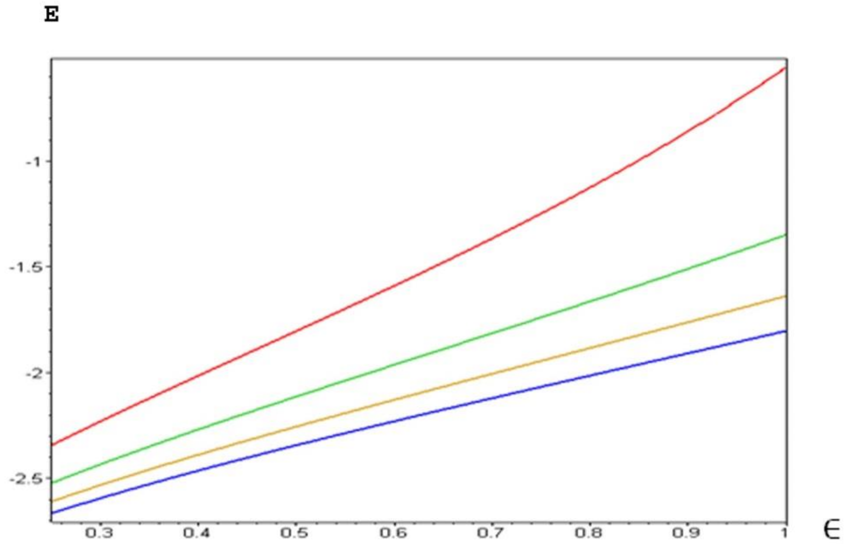

Figure 4.6: The plot of E versus $\in$, using Eq. (4.15) and taking $\left(m = 0,\ n_\rho = 2,\ \tilde{n}_z = 1,\ \alpha = \gamma = -\frac{1}{4} \text{ and } \beta = -\frac{1}{2}, \text{for } D = 5,10,15,20\right)$.

From Fig. 4.6, for the radial coulomb, it is obvious that $E \propto \in$. It shows that as $\in$ decreases, the energy of the system also decrease and the spacing between the line spectra narrows down. However, as $\in$ increases, the Energy of the system also increases and the space between the line spectra widen-up.

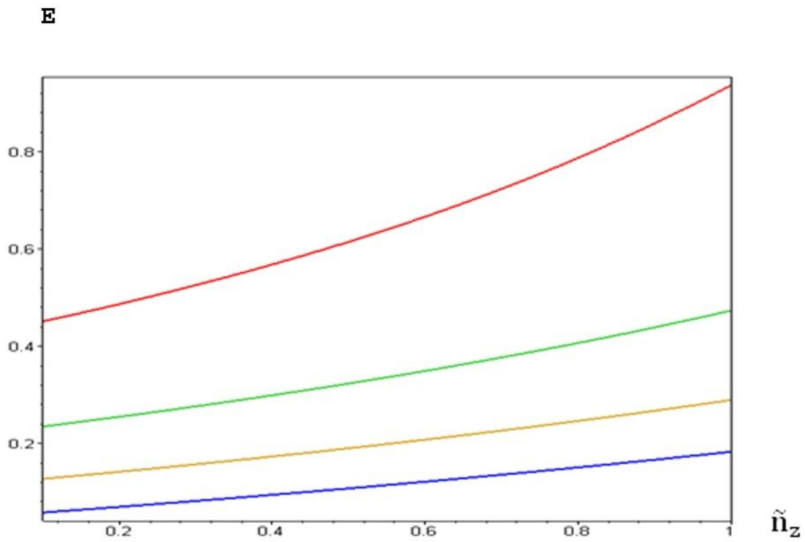

Figure 4.7: The plot of E versus $\tilde{n}_z$, using Eq. (4.15) and taking $\left(m = 0,\ n_\rho = 2\,, \in = 0.7, \text{for } D = 2,4,6,8\ \ \alpha = \gamma = -\frac{1}{4} \text{ and } \beta = -\frac{1}{2} \right)$.

From Fig. 4.7, for the radial coulomb type, it is clear that, $E \propto -\tilde{n}_z$. It shows that as $\tilde{n}_z$ decreases, the energy of the system also decreases. However, as $\tilde{n}_z$ increases the energy of the system also increases and the spacing between the line spectra widens-up.

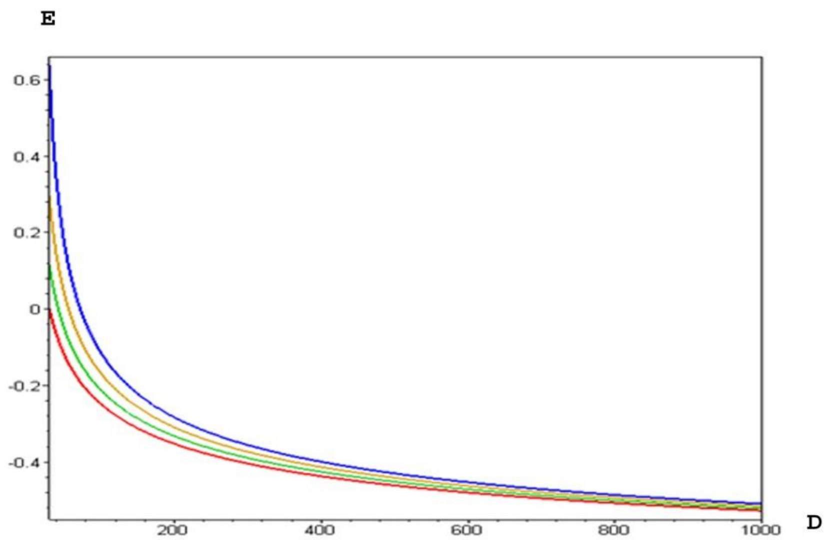

Figure 4.8: The plot of E versus D, using Eq. (4.15) and taking $\left(m = 0,\ n_\rho = 2\,, \in = 1, \text{for } \tilde{n}_z = 1,2,3,4\ \ \alpha = \gamma = -\frac{1}{4} \text{ and } \beta = -\frac{1}{2} \right)$.

From Fig. 4.8, for the radial coulomb type, it is clear that $E \propto -1/D$. It shows that, as $D \rightarrow 0$, the energy of the system increases, and the spacing between the line spectra also increase. However, as D increases, the Energy of the system decreases and the spacing between the line spectra narrows down to a straight line.

Next, we consider the radial harmonic oscillator $V(\rho) = \frac{\alpha^2\rho^2}{4}$. Along with the Morse potential $\tilde{V}(z)$ Eq. (4.12). In this case

$$\lambda_Z^2 = \alpha\left(\frac{\sqrt{D}}{\epsilon} - \tilde{n}_z - \frac{1}{2}\right). \quad \tilde{n}_z = 0,1,2,\ldots \tag{4.16}$$

and

$$V(\rho,\varphi,z) = \frac{\alpha^2\rho^2}{4} + D\rho^2\left(e^{-2\epsilon z} - 2e^{-\epsilon z}\right)\ , D > 0. \tag{4.17}$$

Substituting Eq. (4.16) into Eq. (3.15) we obtain

$$E = \left(\frac{m^2+3}{2}\right) - (\xi - \beta) - \frac{1}{2}\left(\frac{1}{\alpha}\left[\frac{\sqrt{D}}{\epsilon} - \tilde{n}_z - \frac{1}{2}\right] + 2n_\rho + 1\right)^2. \tag{4.18}$$

This is energy for the radial harmonic oscillator along with the z-dependent Morse potential.

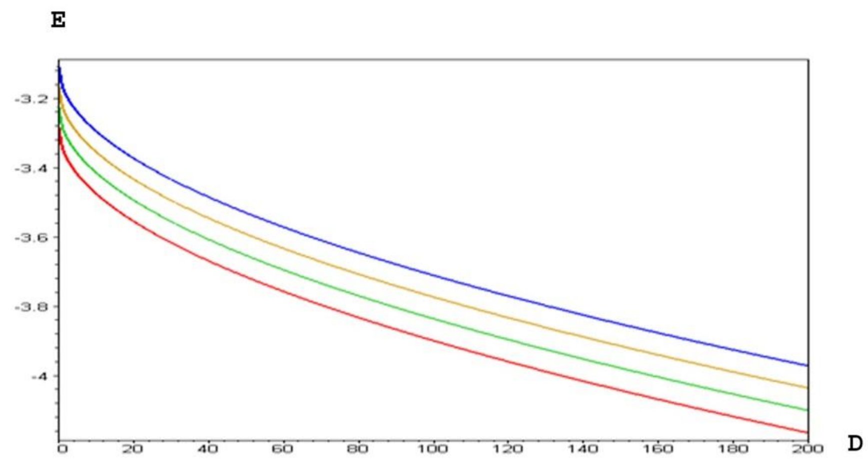

Figure 4.9: The plot of E versus D, using Eq. (4.18) and taking $\left(m = 0,\ n_\rho = 2\ , \in= 1, \mathrm{a} = 50, \text{for } \tilde{n}_z = 1{,}2{,}3{,}4\ ,,\ \alpha = \gamma = -\frac{1}{4} \text{ and } \beta = -\frac{1}{2} \right)$.

From Fig. 4.9, for the radial harmonic oscillator, it is clear that $E \propto -D$. It shows that, as $D \rightarrow 0$, the energy of the system increases. However, as D increases, the energy of the system decreases and the spacing between the line spectra increases.

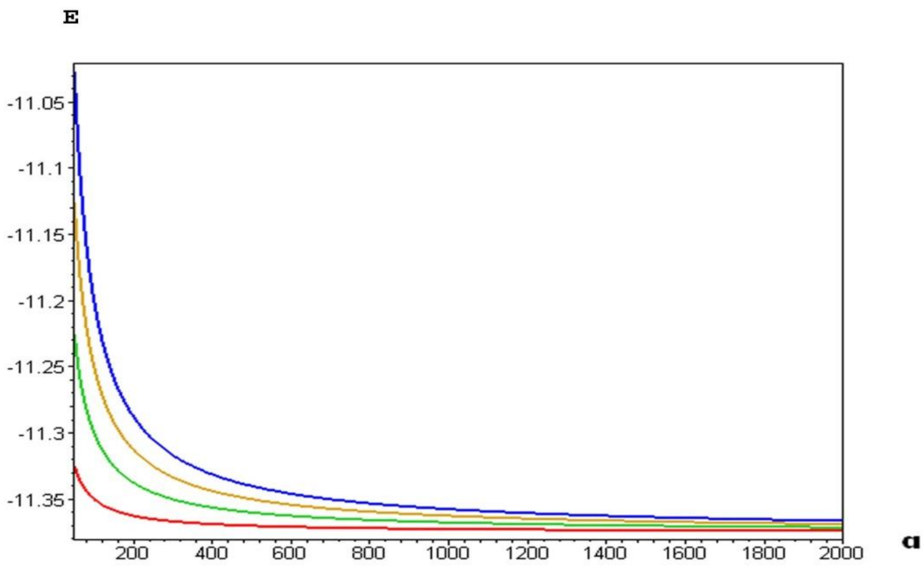

Figure 4.10: The plot of E versus a, using Eq. (4.18) and taking $\left(m = 0,\ n_\rho = 2\ , \in= 1, D = 1, \text{for } \tilde{n}_z = 1{,}2{,}3{,}4,\ \alpha = \gamma = -\frac{1}{4} \text{ and } \beta = -\frac{1}{2} \right)$.

Fig. 4.10, is for the radial harmonic oscillator. It is clear that $E \propto -1/\mathrm{a}$. It shows that, as $\mathrm{a} \rightarrow 0$, the energy of the system increases, and the spacing between the line spectra also increases. However, as ɑ an increase, the energy of the system decreases and the spacing between the line spectra narrows down to a straight line.

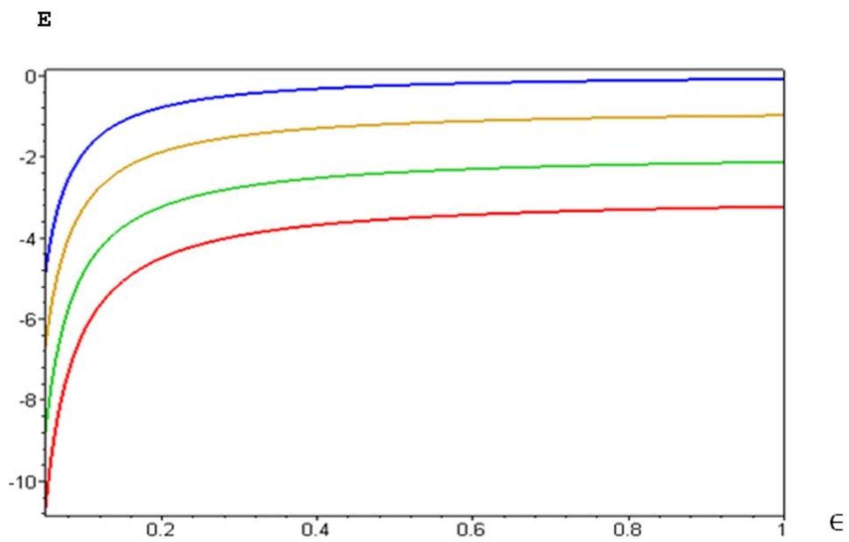

Figure 4.11: The plot of E versus $\in$, using Eq. (4.18) and taking $\left(m = 0,\ n_\rho = 1, D = 1, \mathrm{a} = 10 \right.$, for $\tilde{n}_z = 1,5,10,15,\ \alpha = \gamma = -\frac{1}{4}$ and $\left. \beta = -\frac{1}{2} \right)$.

Fig. 4.11, is for the radial harmonic oscillator. It shows that as $\in$ increases, the energy increases to some points, after which, the energy of the system becomes steady while the spacing between the line spectra is maintained.

Chapter 5

CONCLUSION

We started by using the Hamiltonian operator Eq. (1.1) with kinetic energy and potential energy to obtain the position dependent mass (PDM) equation of the Von Rose Hamiltonian [1] with ambiguity parameters where we looked at the position-dependent mass equation;

$m(\vec{r}) \equiv M(\rho,\varphi,z) = g(\rho)f(\varphi)k(z)$ under the azimuthally symmetric settings. By using the general power-law radial position dependent mass, Eq. (2.30), where $M(\rho,\varphi,z) = g(\rho) = \rho^{-2}$.

By using separation of variables method, we obtained Eqs. (2.47), (2.49) and (2.50). Using the radial columbic potential $\tilde{V}(\rho) = -2\rho^{-1}$, we obtained the eigenenergy Eq. (3.9), and for the radial harmonic oscillator potential $\tilde{V}(\rho) = \frac{\alpha^2\rho^2}{4}$, obtained the eigenenergy (3.15).

With combining the solution of this two energy Eqs. (2.47) and (2.49), we were able to determine the energies for the coulomb potential, Eq. (3.9), and for the Harmonic oscillator potential, Eq. (3.15). Applying the value of elements with the eiginenergies equation we can find the effects of the impenetrable walls potential, and the Morse potential, on the radial coulomb and harmonic oscillator were analyzed.

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