q-Polynomials and Location of Their Zeros

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ABSTRACT

In this thesis, we define the q-Bernoulli numbers and polynomials, q-Euler numbers and polynomials, q-Frobenius-Euler numbers and polynomials and q-Genocchi numbers and polynomials of higher order in two variables x and y, by using two q-exponential functions. We also prove some properties and relationships of these polynomials and q-analogue of the Srivastava and Pinter addition theorem. Furthermore, we represent the figures of the q-Bernoulli, q-Euler and q-Genocchi numbers and polynomials. We find the solutions of these q-polynomials, for $n \in \mathbb{N}$, x and $q \in \mathbb{C}$ by using a computer package Mathematica®software. Finally, we discuss the reflection symmetries of these q-polynomials.

Keywords: q-analogues of Bernoulli - Euler - Genocchi - Frobenius-Euler numbers and Polynomials, Srivastava Pinter addition Theorems, shapes and roots of q-polynomials

ÖZ

Bu tezde, iki q-üstel fonksiyonlarını kullanarak q-Bernoulli, q-Euler, q-Frobenius-Euler ve q-Genocchisayıları ve polinomlari iki değişken x ve y yüksek düzenin polinomları tanımlanır ve bu polinomların bazı özellikleri, ilişkileri ve Srivastava-Pinter ilave teoremin q-analogu kanıtlanır. Ayrıca bilgisayar kullanarak q-Bernoulli, q-Euler ve q-Genocchi numaralarının şekilleri keşfedilir ve indeks n değerleri için q-Bernoulli, q-Euler ve q-Genocchi polinomların köklerinin yapısı tarif edilir.

Anahtar Kelimeler: Genelleştirilmiş Bernoulli-Euler- Genocchi -Frobenius-Euler sayıları ve Polinomları ve Srivastava - Pinter ilave teoremi, q-polinomlarının kökleri ve grafikleri

DEDICATION

TO MY OLD SUPERVISOR Prof. Dr. PETER KAS

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Chapter 1

INTRODUCTION

In mathematics, the Bernoulli numbers B_n and polynomials $B_n(x)$, Euler numbers E_n and polynomials $E_n(x)$ and Genocchi numbers G_n and polynomials $G_n(x)$ are important topics in number theory, analysis and differential topology and have applications in statistics, combinatorics, numerical analysis and so on.

In the 17th century, mathematician studied to find out a formula for the sum of the first n natural numbers with k-th powers, where k is a positive integer:

$$S_k(n) = 1^k + 2^k + 3^k + \ldots + n^k$$

The Swiss mathematician Jakob Bernoulli (1654-1705) would solve this problem with the following equality:

$$1^k + 2^k + 3^k + \ldots + (n-1)^k = k! \int_0^n B_k(x)dx$$

where $B_k(x)$ is the Bernoulli polynomials. Jakob Bernoulli discovered the Bernoulli numbers, B_n, famous work *"Ars conjectandi"* published in 1713 in related with sums of powers of consecutive integers. Independently, Japanese mathamatician Seki Kőwa studied with Bernoulli numbers, B_n, in his posthumous work *"Kutsoyo Sampo"* published in 1712. Lady Ada Lovelace (1815-1852) wrote the first computer program in the world to investigate the Bernoulli numbers. Then, G.-S. Cheon in [6] and H.M. Srivastava, Á. Pintér in [11] studied on the Bernoulli and Euler polynomials their properties and relationships.

Over 7 decades ago, Carlitz studied on q-analogues of the ordinary Bernoulli numbers B_n and polynomials $B_n(x)$ and introduced the q-Bernoulli numbers and polynomials (see [3], [4] and [5]). Then, many other mathematicians studied with q-analogues of Bernoulli numbers and polynomials and introduced new definitions of B_n and $B_n(x)$ such as Simsek ([33], [34] and [35]), Cenki et al. ([13], [14], [15]), Choi et al. ([16] and [17]), Srivastava et al. [36], Ryoo et al. [32], Luo and Srivastava [12], Ozden and Simsek [29]. N.I.Mahmudov in [28],[44],[45],[56],[57] introduced new generating functions to define q analogues of $B_n(x)$ and B_n, Euler polynomials $E_n(x)$, numbers E_n and Genocchi polynomials $G_n(x)$ and numbers G_n and Frobenious-Euler numbers and polynomials. In [20]-[27] Kim et al introduced a new notion for the q-Genocchi numbers and polynomials, studied on basic properties and gaves relationships of the q-analogues of Euler and Genocchi polynomials. Many other authors studied on this subject such as : Cenkci et al [14], Luo and Srivastava [8], [9], [12], Simsek et al [37], Cheon [6], Srivastava et al. [11], Nalci and Pashaev [49], Gabaury and Kurt B. [41], Kurt V. [46], Araci et al. [51]. D.S. Kim, T. Kim, and J Seo [60] studied on the new q-extension of Frobenious-Euler numbers and polynomials, also D.S. Kim and T. Kim[62], studied on higher order Frobenious-Euler numbers and polynomials-Bernoulli mixed type polynomials.

In recent years, Woon [58], Veselov and Ward [59] investigated the real and complex roots of Bernoulli polynomials There are a lot of studies on investigated roots of q-numbers and polynomials, see [63], [64], [65], [67], [56], [57]. By using these numerical results we can understand the structure of these q-numbers and q-polynomials, examine the properties, give relationships and make some comparisons between them.

In this thesis, we give new definitions for the higher order q-numbers and polynomials in two variables x and y, by using two q-exponential functions (see [2])

$$e_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q!} = \prod_{k=0}^{\infty} \frac{1}{(1-(1-q)q^k z)}, \; 0 < |q| < 1, \quad |z| < \frac{1}{|1-q|}, \tag{1.1}$$

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n-1)} z^n}{[n]_q!} = \prod_{k=0}^{\infty} (1+(1-q)q^k z), \quad 0 < |q| < 1, \quad z \in \mathbb{C}, \tag{1.2}$$

From this form we get $e_q(z) E_q(-z) = 1$. Then, we have

$$\begin{aligned} D_q e_q(z) &= e_q(z), \\ D_q E_q(z) &= E_q(qz), \end{aligned}$$

where D_q is the q-derivative and defined by

$$D_q f(z) := \frac{f(qz) - f(z)}{qz - z}.$$

Two q-exponential functions (1.1) and (1.2) help us easily to prove some properties of these q-polynomials and q-analogue of the Srivastava and Pinter addition theorem. Moreover we investigate the shapes of the q- numbers and polynomials. We define the structure of the real and complex roots of the q- polynomials for values of the n, q and $x \in \mathbb{C}$ where n is the degree of these polynomials by using a computer package Mathematica .

This thesis consist of four chapters and is organized as follows:

In chapter 2, we give some fundamental definitions and some properties of Bernoulli numbers B_n and polynomials $B_n(x)$, Euler numbers E_n and polynomials $E_n(x)$ and Genocchi numbers G_n and polynomials $G_n(x)$. We discuss relationships of $B_n(x)$ and $E_n(x)$. The classical Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ are defined in [45] by the generating functions given as follows:

$$\begin{aligned} \frac{t}{e^t-1} &= \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad |t| < 2\pi \\ \left(\frac{t}{e^t-1}\right)e^{tx} &= \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad |t| < 2\pi \end{aligned}$$

the rational numbers B_n, are Bernoulli numbers for $n \in \mathbb{N}_0$. The classical Euler numbers E_n and polynomials $E_n(x)$ are defined in [45] by means of generating function as follows:

$$\begin{aligned} \frac{2}{e^t+1} &= \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad |t| < \pi \\ \left(\frac{2}{e^t+1}\right)e^{tx} &= \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad |t| < \pi \end{aligned}$$

The classical Genocchi numbers G_n and polynomials $G_n(x)$ are defined in [28] by means of generating functions:

$$\begin{aligned} \frac{2t}{e^t+1} &= \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad |t| < \pi \\ \left(\frac{2t}{e^t+1}\right)e^{tx} &= \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi \end{aligned}$$

In this chapter, we also give few values of Bernoulli numbers B_n and polynomials $B_n(x)$, Euler numbers E_n and polynomials $E_n(x)$ and Genocchi numbers G_n and polynomials $G_n(x)$.

In chapter 3, we give some basic definitions and elementary properties related to q-integers. We always make use of quantum concepts as follows: for more details see [1]. The q-shifted factorial is defined by

$$
\begin{aligned}
(a;q)_0 &= 1, \quad (a;q)_n = \prod_{j=0}^{n-1}\left(1-q^j a\right), \quad n \in \mathbb{N}, \\
(a;q)_\infty &= \prod_{j=0}^{\infty}\left(1-q^j a\right), \quad |q|<1, \ a \in \mathbb{C}.
\end{aligned}
$$

The q-numbers and q-numbers factorial is defined by

$$
\begin{aligned}
[a]_q &= \frac{1-q^a}{1-q} \quad (q \neq 1); \\
[0]_q! &= 1; \quad [n]_q! = [1]_q [2]_q \dots [n]_q \quad n \in \mathbb{N}, \ a \in \mathbb{C}
\end{aligned}
$$

respectively. The q-polynomial coefficient is defined by

$$
\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}, \quad k \leq n, \ n \in \mathbb{N}.
$$

N. I. Mahmudov define the $(w,q)-$Bernoulli numbers and polynomials in [56] as follows

$$
\begin{aligned}
\frac{t}{we_q(t)-1} &= \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(w)} \frac{t^n}{[n]_q!}, \quad |t|<2\pi, \\
\frac{t}{we_q(t)-1} e_q(tx) e_q(ty) &= \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(w)}(x,y) \frac{t^n}{[n]_q!} \quad |t|<2\pi.
\end{aligned}
$$

where $q \in \mathbb{C}$,and $0 < |q| < 1$. The $(w,q)-$Euler numbers and polynomials is defined in [56] as follows:

$$\frac{2}{we_q(t)+1} = \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(w)} \frac{t^n}{[n]_q!}, \quad |t| < \pi,$$
$$\frac{2}{we_q(t)+1} e_q(tx) e_q(ty) = \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(w)}(x,y) \frac{t^n}{[n]_q!}, \quad |t| < \pi.$$

We also study on relationships between the $(w,q)-$Bernoulli polynomials and $(w,q)-$Euler polynomials. Then we discuss some elementary properties and get new formulas which are extensions of the formulas studied by other authors like Cheon, Srivastava and Pinter, and so on. (see [6], [11]). Furthermore, we explore the shapes of the q-Bernoulli and q-Euler numbers and polynomials. We describe the structure of the roots of the q-Bernoulli and q-Euler polynomials for values of the n, q and $x \in \mathbb{C}$ where n is the degree of polynomials by using a computer. In this chapter, we also give the definition of higher order Frobenius-Euler numbers and polynomials $H_{n,q}^{\alpha,\lambda}(x)$ and we investigate some elementary properties of these polynomials (see [57]).

In chapter 4, we define the q-Bernoulli numbers $\mathfrak{B}_{n,q}$ and q-Bernoulli polynomials $\mathfrak{B}_{n,q}(x,y)$ in x, y by means of the generating functions in [57] as follows:

$$\left(\frac{t}{e_q(t)-1}\right) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q} \frac{t^n}{[n]_q!}, \quad |t| < 2\pi,$$
$$\left(\frac{t}{e_q(t)-1}\right) e_q(tx) e_q(ty) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(x,y) \frac{t^n}{[n]_q!}, \quad |t| < 2\pi.$$

We also define the q-Genocchi numbers $\mathfrak{G}_{n,q}$ and q-Genocchi polynomials $\mathfrak{G}_{n,q}(x,y)$ in x,y by means of the generating functionsin [57] as follows:

$$\left(\frac{2t}{e_q(t)+1}\right) = \sum_{n=0}^{\infty} \mathfrak{G}_{n,q} \frac{t^n}{[n]_q!}, \quad |t| < \pi,$$
$$\left(\frac{2t}{e_q(t)+1}\right) e_q(tx)\, e_q(ty) = \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}(x,y) \frac{t^n}{[n]_q!}, \quad |t| < \pi.$$

where $q \in \mathbb{C}$, and $0 < |q| < 1$.We give some elementary properties of the $q-$genocchi polynomials $\mathfrak{G}_{n,q}(x,y)$. Also, we prove an interesting relationship between the q-Genocchi and the q-Bernoulli polynomials. Then, we obtain q-analogous of some properties. Moreover, we display the shapes of the q-Genocchi numbers and polynomials. Then we investigate the roots of the q-Genocchi polynomials for values of the n, q and $x \in \mathbb{C}$ where n is the degree of these polynomials by using a computer package Matematica.

Throughout this thesis, we always make use the following notations and symbols:

- - Bernoulli numbers: B_n
  - Bernoulli polynomials: $B_n(x)$
  - Euler numbers : E_n
  - Euler polynomials: $E_n(x)$
  - Genocchi numbers: G_n
  - Genocchi polynomials: $G_n(x)$
  - q-Bernoulli numbers : $\mathfrak{B}_{n,q}$
  - q-Bernoulli polynomials in x,y : $\mathfrak{B}_{n,q}(x,y)$

- $(w,q)-$Bernoulli numbers of order w : $\mathfrak{B}_{n,q}^{(w)}$
- $(w,q)-$Bernoulli polynomials of order w in x,y : $\mathfrak{B}_{n,q}^{(w)}(x,y)$
- $(w,q)-$Euler numbers of order w : $\mathfrak{E}_{n,q}^{(w)}$
- $(w,q)-$Euler polynomials of order w in x,y : $\mathfrak{E}_{n,q}^{(w)}(x,y)$
- q-Genocchi numbers: $\mathfrak{G}_{n,q}$
- q-Genocchi polynomials in x,y : $\mathfrak{G}_{n,q}(x,y)$
- Higher order, Frobenius-Euler numbers: $H_{n,q}^{\alpha,\lambda}$
- Higher order Frobenius-Euler polynomials $H_{n,q}^{\alpha,\lambda}(x)$

Chapter 2

BERNOULLI, EULER AND GENOCCHI NUMBERS AND POLYNOMIALS

In this chapter, we mention about fundamental definitions and some elementary properties of B_n, $B_n(x)$, E_n, $E_n(x)$, G_n and $G_n(x)$. For more details of this topics see [1], [6] and [11].

2.1 Bernoulli Numbers

A sequence of rational numbers called B_n plays an important role in mathematıcs for instance in number theory, analysis and differential topology. The Bernoulli numbers, B_n have relationships with the Euler numbers E_n, Genocchi numbers G_n , Stirling numbers and the tangent numbers. The first few values of B_n 's are given below:

$$\begin{aligned} B_0 &= 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = -\frac{1}{30},\ B_5 = 0, \\ B_6 &= \frac{1}{42},\ B_7 = 0,\ B_8 = -\frac{1}{30}, \cdots \end{aligned}$$

Some authors used $B_1 = +\frac{1}{2}$ and this sequence is called the *second Bernoulli numbers* where $B_1 = -\frac{1}{2}$ is called the *first Bernoulli numbers*.

And some called *even-index Bernoulli numbers* since $B_n = 0$ for all odd index n where $n > 1$ and denoted by B_n instead B_{2n}.

Definition 1 [1] *To find a formula for the sum of the first n natural numbers with r-th* *powers, where r is a positive integer is called the power sum problem*

$$S_r(n) = \sum_{k=1}^{n} k^r = 1^r + 2^r + \ldots + n^r \tag{2.1}$$

For small values of r one can easily derive the formulas for example

for $r = 1$ we get

$$S_1(n) = \sum_{k=1}^{n} k^1 = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

for $r = 2$ we have

$$S_2(n) = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for $r = 3$ we have

$$S_3(n) = \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

The coefficients of the sum formula (2.1) are related to the B_n by Bernoulli's formula:

$$S_r(n) = \frac{1}{r+1} \sum_{k=0}^{r} \binom{r+1}{k} B_k n^{r+1-k} \tag{2.2}$$

where B_k is Bernoulli numbers and $B_1 = +\frac{1}{2}$ is used.

For $B_1 = -\frac{1}{2}$, Bernoulli's formula is stated as

$$S_r(n) = \frac{1}{r+1} \sum_{k=0}^{r} (-1)^k \binom{r+1}{k} B_k n^{r+1-k} \tag{2.3}$$

According to above sum formula (2.2) we can get some known number sets for example, for $r = 0$ and $B_0 = 1$ we get $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

$$0 + 1 + 1 + ... + 1 = \frac{1}{1}(B_0.n) = n$$

for $r = 1$ and $B_1 = \frac{1}{2}$ we get $\{0, 1, 3, 6, \ldots$ which are called the triangular numbers .

$$0 + 1 + 2 + ... + n = \frac{1}{2}(B_0.n^2 + 2B_1.n) = \frac{1}{2}(n^2 + n)$$

for $r = 2$ and $B_2 = \frac{1}{6}$ we get $\{0, 1, 5, 14, \ldots\}$ which are called the pyramidal numbers .

$$\begin{aligned} 0 + 1^2 + 2^2 + ... + n^2 &= \frac{1}{3}(B_0.n^3 + 3B_1.n^2 + 3B_2.n) \\ &= \frac{1}{3}(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n) \end{aligned}$$

Bernoulli numbers, B_n,can be introduced by using different characterizations Three of them are given as follows:

1. a generating function
2. a recursive equation
3. an explicit formula

Definition 2 [1] *(Generating function) The Bernoulli numbers are defined by the generating functions:*

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad |t| < 2\pi.$$

Here the rational numbers B_n, are called Bernoulli numbers for $n \in \mathbb{N}$. In this equation we replace B^n by B_n $(n \geq 0)$ symbolically.

Since $\frac{t}{e^t-1}$ has simple poles at $t = \pm 2\pi ni$, $n = 1, 2, \ldots$, the expansion converges for $|t| < 2\pi$.

In definition (2) let t approaches to 0 then we get $B_0 = 1$.

Next we have

$$\frac{t}{2} + \frac{t}{e^t - 1} = \frac{t}{2}\frac{e^t + 1}{e^t - 1} = \frac{t}{2}\coth\frac{t}{2}$$

is an even function of t.So in its power series expansion about $t = 0$ the odd order coefficients after $n = 1$ are zero.

$$\begin{aligned} B_1 &= -\frac{1}{2} \\ B_{2n+1} &= 0, \quad n \in \mathbb{N}_0. \end{aligned}$$

Now, we obtain a reccurrence formula to compute of the Bernoulli numbers

$$\frac{t}{e^t - 1}e^t = t + \frac{t}{e^t - 1}$$

then, we have

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \sum_{n=0}^{\infty} \frac{t^n}{n!} = t + \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

now, by applying Cauchy product we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} B_k \frac{t^k}{k!}\frac{t^{n-k}}{(n-k)!} = t + \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

on both sides compare coefficients of t^n for $n > 1$ we obtain

$$\frac{B_n}{n!} = \sum_{k=0}^{n} B_k \frac{1}{k!(n-k)!},$$

therefore we get

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k, \quad n > 1.$$

This relation can be written symbolicaly as

$$B_n = (1+B)_n, \quad n > 1.$$

Definition 3 [1] *(Recursive equation) The binomial recursion formula for B_n is given for all $n \in \mathbb{N}$*

$$\sum_{k=0}^{n} \binom{n}{k} B_k - B_n = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}$$

For $n = 1$, we obtain the value of B_0

$$\begin{aligned} \sum_{k=0}^{1} \binom{1}{k} B_k - B_1 &= 1, \\ B_0 + B_1 - B_1 &= 1, \\ B_0 &= 1. \end{aligned}$$

For $n = 2$, we obtain the value of B_1

$$\begin{aligned}
\sum_{k=0}^{2}\binom{2}{k}B_k - B_2 &= 0,\\
B_0 + 2B_1 + B_2 - B_2 &= 0,\\
B_1 &= -\frac{1}{2}.
\end{aligned}$$

Definition 4 [1] *(Explicit formula) An explicit formula for Bernoulli numbers is given by*

$$B_n(x) = \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{(x+j)^n}{k+1}.$$

For $x = 0$,we get the following form,

$$B_n = \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{j^n}{k+1},$$

and for, $x = 1$, we get the following form

$$B_n = \sum_{k=1}^{n+1}\sum_{j=1}^{k}(-1)^{j+1}\binom{k-1}{j-1}\frac{j^n}{k}.$$

2.2 Bernoulli Polynomials

In mathematics, the Bernoulli polynomials $B_n(x)$ arrise in many special functions like Riemann zeta function,

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots = \sum_{k=1}^{\infty}\frac{1}{k^s}$$

and the Hurwitz zeta function

$$\zeta(s,q) = \sum_{n=1}^{\infty} \frac{1}{(q+n)^s}.$$

For more details see ([1]).

Definition 5 [1] *The Bernoulli polynomials $B_n(x)$ are defined by the generating function*

$$\frac{t}{e^t - 1} e^{tx} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

for each nonnegative integer n.

Definition 6 [1] *The explicit formula for $B_n(x)$ is given*

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}$$

for $n \geq 0$ where B_k are the Bernoulli numbers.

Symbollically one can use

$$B_n(x) = (B + x)^n.$$

The first few $B_n(x)$ for $n \in \mathbb{N}$ are listed below:

$$\begin{aligned}
B_0(x) &= 1, \\
B_1(x) &= x - \frac{1}{2}, \\
B_2(x) &= x^2 - x + \frac{1}{6}, \\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30} \\
B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x.
\end{aligned}$$

2.2.1 Properties of Bernoulli Polynomials

For more details see [1]:

1. The Bernoulli polynomials at $x = 0$ are equal to Bernoulli numbers

$$B_n(0) = B_n.$$

2. If we differentiate the generating function with respect to x, we get the following relation

$$B_n^{'}(x) = nB_{n-1}(x).$$

3. The difference relation property is given below

$$\triangle B_n(x) = B_n(x+1) - B_n(x) = nx^{n-1}$$

where $\triangle$ is the difference operator.

4. $B_n(1-x) = (-1)^n B_n(x)$ for $n \geq 0$.

2.3 Euler Numbers

A sequence of integers called Euler numbers, E_n, are defined by Taylor series expansion given as follow:

$$\sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = \frac{1}{\cosh t} = \frac{2}{e^t + e^{-t}} \tag{2.4}$$

where $\cosh t$ is the hyperbolic cosine. Here we replace E^n by E_n ($n \geq 0$) symbolically.

The equation (2.4) is equivalent to following identitiy

$$(E+1)_n + (E-1)_n = \begin{cases} 2 & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n > 0 \end{cases}.$$

The values of the first few E_n are given below

$$\begin{aligned} E_0 &= 1,\ E_1 = 0,\ E_2 = -1,\ E_3 = 0,\ E_4 = 5,\ E_5 = 0, \\ E_6 &= -61,\ E_7 = 0,\ E_8 = 1385, \cdots \end{aligned}$$

For all $n > 0$, the E_n with odd indexed are all zero

$$E_{2n+1} = 0$$

and the even indexed ones have alternating signs.

2.4 Euler Polynomials

Definition 7 [1] *The Euler polynomials $E_n(x)$, is defined by the generating function:*

$$\frac{2}{e^t+1}e^{tx} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!},$$

where we replace E^n by E_n $(n \geq 0)$ symbolically.

Definition 8 [1] *An explicit formula for the $E_n(x)$ is given by*

$$E_m(x) = \sum_{n=0}^{m} \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+k)^m.$$

Now, from the above equation we get $E_n(x)$ in terms of the E_k as

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} (x - \frac{1}{2})^{n-k}.$$

The first few $E_n(x)$ are listed below:

$$\begin{aligned}
E_0(x) &= 1, \\
E_1(x) &= x - \frac{1}{2}, \\
E_2(x) &= x^2 - x, \\
E_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \\
E_4(x) &= x^4 - 2x^3 + x \\
E_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^3 - \frac{1}{2}.
\end{aligned}$$

2.4.1 Properties of Euler Polynomials

1. The $E_n(x)$ at $x = 0$ are equal to Euler numbers

$$E_n(0) = E_n.$$

2. If we differentiate the generating function with respect to x, we get the following relation

$$E_n'(x) = nE_{n-1}(x).$$

3.

$$\triangle E_n(x) = E_n(x+1) + E_n(x) = 2x^n$$

where $\triangle$ is the difference operator.

4.

$$E_n(1-x) = (-1)^n E_n(x)$$

for $n \geq 0$.

2.5 Properties of Bernoulli and Euler polynomials

In recent years, Cheon [6] obtained the following results:

1.

$$B_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) \text{ where } n \in \mathbb{N}_0. \tag{2.5}$$

2.

$$E_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) \tag{2.6}$$

where $n = 1, 2, 3, \ldots$

3.

$$B_n(x) = \sum_{\substack{k=0 \\ k \neq 1}}^{n} \binom{n}{k} B_k E_{n-k}(x) \tag{2.7}$$

where $n = 1, 2, 3, \ldots$

Here equations 2.5 and 2.6 are special cases of addition theorems given below:

$$E_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k(x) y^{n-k}.$$

and equation 2.7 is equivalent to the following idetities

$$B_n(x) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} B_{n-k} E_k(2x),$$

$$2^n B_n(\frac{x}{2}) = \sum_{k=0}^{n} \binom{n}{k} B_k E_{n-k}(x).$$

2.6 Genocchi Numbers and Polynomials

Definition 9 *Genocchi numbers G_n are defined in* [68] *by means of the generating function:*

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad |t| < \pi,$$

where we replace G^n by G_n $(n \geq 0)$ symbolically.

The values of the first few G_n are listed below:

$$\begin{aligned} G_1 &= 1, G_2 = -1, G_3 = 0, G_4 = -1, G_5 = 0, G_6 = -3, \\ G_7 &= 0, G_8 = 17, G_9 = 0, G_{10} = -155, \\ G_{11} &= 0, G_{12} = 2073, \cdots \end{aligned}$$

The odd indexed of G_n for $n > 1$ is zero

$$G_{2n+1} = 0$$

and even ones have the following relationships with B_n and E_n

$$G_n = 2(1-2^{2n})B_{2n} = 2nE_{2n-1.}$$

Definition 10 *In* [68] *Ryoo, C.S. define the Genocchi polynomials for $x \in \mathbb{R}$, as follows:*

$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi.$$

Also, the following recurrence relation is given

$$G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}.$$

where G_k is Genocchi numbers. For $x = 0$, we obtain $G_n(0) = G_n$.

The first few Genocchi polynomials are listed below:

$$\begin{aligned}
G_1(x) &= 1, \\
G_2(x) &= 2x - 1, \\
G_3(x) &= 3x^2 - 3x, \\
G_4(x) &= 4x^3 - 6x^2 + 1, \\
G_5(x) &= 5x^4 - 10x^3 + 5x, \\
G_6(x) &= 6x^5 - 15x^4 + 15x^2 - 3, \\
G_7(x) &= 7x^6 - 21x^5 + 35x^3 - 21x, \cdots
\end{aligned}$$

Chapter 3

THE q-ANALOGUES OF BERNOULLI AND EULER POLYNOMIALS

The main aim of this chapter is to give new definitions of two dimensional (w,q)-Bernoulli and (w,q)-Euler numbers and polynomials by using generating functions and study on relationships between the $(w,q)-$Bernoulli polynomials and (w,q)-Euler polynomials. We also discuss some elementary properties and get new formulas which are extensions of the formulas studied by other authors like Cheon, Srivastava and Pinter, and so on. (see [6], [11]). Furthermore, we explore the shapes of the q-Bernoulli and q-Euler numbers and polynomials. We describe the structure of the roots of the q-Bernoulli and q-Euler polynomials for values of the n, q and $x \in \mathbb{C}$ where n is the degree of polynomials by using a computer. In this chapter, we also give the definition of higher order Frobenius-Euler numbers and polynomials $H_{n,q}^{\alpha,\lambda}(x)$ and we investigate some elementary properties of these polynomials (see [57]).

3.1 The q-integers

In this section we will give basic definitions related to q-integers. For more details see [1].

Definition 11 [1]*Let $f(x)$ be an arbitrary function and $q \in \mathbb{R}^{+} \setminus \{1\}$. Then the following* *statement is called q-derivative of the function $f(x)$.*

$$D_q f(x) = \frac{D_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{qx - x} \tag{3.1}$$

Let c and d are any two constants then D_q is a linear operator on the space of polynomials since it satisfies the following property:

$$D_q(cf(x) + dg(x)) = cD_q f(x) + dD_q(x).$$

Definition 12 [1] *q-shifted factorial: The following expression is defined as q-shifted factorial*

$$\begin{aligned}(a;q)_0 &= 1, \quad (a;q)_n = \prod_{j=0}^{n-1}(1-q^j a), \quad n \in \mathbb{N}, \\ (a;q)_\infty &= \prod_{j=0}^{\infty}(1-q^j a), \quad |q| < 1, \quad a \in \mathbb{C}\end{aligned}$$

Definition 13 [1] *q-integer: The q-analogue of n is defined by*

$$[n]_q \;:\; = \begin{cases} n & \text{if } q = 1 \\ \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1} & \text{if } q \neq 1 \end{cases} \tag{3.2}$$

$$\text{and} \quad [0]_q \;:\; = 0. \tag{3.3}$$

where $n \in \mathbb{N}$ and $q \in \mathbb{R}^+$.

Definition 14 *The set of q-integers $\mathbb{N}_q$ is defined by*

$$\mathbb{N}_q = \{[n], \;\; \text{with} \;\; n \in \mathbb{N}\}. \tag{3.4}$$

By putting $q = 1$ we get the set of nonnegative integers $\mathbb{N}$.

Definition 15 [1] *q-factorial: The following expression is defined as q-analogue of $n!$*

$$[n]! = [n]!_q := \begin{cases} 1 & \text{if} \quad n = 0 \\ [1][2]\cdots[n] & \text{if} \;\; n = 1, 2, 3, \cdots \end{cases} . \tag{3.5}$$

where $n \in \mathbb{N}$ and $q \in \mathbb{R}^+$.

Definition 16 *The following expression is defined as q-analogue of the function $(x-a)^n$*

$$(x-a)_q^n = \begin{cases} 1 & \text{if} \quad n = 0 \\ (x-a)(x-qa)\ldots(x-q^{n-1}a) & \text{if} \;\; n \geq 1 \end{cases} . \tag{3.6}$$

Definition 17 [1] *q-binomial coefficient: The q-analogue of binomial coefficient is defined by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \; : \; = \frac{[n][n-1]\ldots[n-k+1]}{[k]!} \tag{3.7}$$

$$= \frac{[n]_q!}{[k]_q![n-k]_q!} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q \tag{3.8}$$

where $n, k \in \mathbb{N}$ and $0 \leq k \leq n$.

Lemma 18 *Gauss's Binomial formula: The following expression is defined as q-analogue of the function $(x+y)^n$*

$$(x+y)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} x^k y^{n-k}, \;\; n \in \mathbb{N}_0. \tag{3.9}$$

Lemma 19 [1] *Heine's Binomial formula: The following formula*

$$\frac{1}{(1-x)_q^n} = 1 + \sum_{k=0}^{n} \frac{[n][n+1][n+k-1]}{[k]!} x^k, \quad n \in \mathbb{N}_0. \tag{3.10}$$

is called Heine's Binomial formula.

From lemma 18, let replace x by 1 and y by x.then we have the following Gauss's Binomial formula[1]

$$(1+x)_q^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\frac{1}{2}k(k-1)} x^k, \quad n \in \mathbb{N}_0.$$

Now, let $n \to \infty$ then we obtain infinite product given as follow:

$$(1+x)_q^\infty = (1+x)(1+qx)(1+q^2x)\cdots$$

Also, when $|q| < 1$, we have

$$\lim_{n\to\infty} [n]_q = \lim_{n\to\infty} \frac{1-q^n}{1-q} = \frac{1}{1-q} \tag{3.11}$$

which converges some finite limit and

$$\begin{aligned} \lim_{n\to\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q &= \lim_{n\to\infty} \frac{[n][n-1]\cdots[n-k+1]}{[k]!} \\ &= \lim_{n\to\infty} \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\ldots(1-q^k)} \\ &= \lim_{n\to\infty} \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}. \end{aligned} \tag{3.12}$$

Now, assume that $|q| < 1$, if we apply 3.11 and 3.12 to Gauss's and Heine's binomial formulas then we get the following two expressions:

$$(1+x)_q^\infty = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} \frac{x^k}{(1-q)(1-q^2)\cdots(1-q^k)} \tag{3.13}$$

and

$$\frac{1}{(1-x)_q^\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(1-q)(1-q^2)\cdots(1-q^k)} \tag{3.14}$$

The two identities above that are found by Euler relate infinite products to infinite sums. When $q = 1$, the formulas 3.13 and 3.14 are undefined so they have not got ordinary analogues.The formula 3.13 is called Euler's first identity, $E1$, and the formula 3.14 is called Euler's second identity, $E2$.

Moreover, the formula $E2$ 3.14 becomes

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{x^k}{(1-q)(1-q^2)\dots(1-q^k)} &= \sum_{k=0}^{\infty} \frac{\left(\frac{x}{1-q}\right)^k}{(\frac{1-q}{1-q})(\frac{1-q^2}{1-q})\dots(\frac{1-q^k}{1-q})} \\ &= \sum_{k=0}^{\infty} \frac{\left(\frac{x}{1-q}\right)^k}{[k]!},\end{aligned} \tag{3.15}$$

which corresponds Taylor's expansion of the ordinary exponential function:

$$e^x = \sum_{k=0}^{\infty} \frac{x^n}{n!}.$$

Definition 20 [1] *A q-analogue of the ordinary exponential function e^x is defined by*

$$e_q^x := e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!}.$$

Then from 3.14 and 3.15 we have

$$e_q^{\frac{x}{1-q}} = \frac{1}{(1-x)_q^{\infty}}$$

or

$$e_q^x = \frac{1}{(1-(1-q)x)_q^{\infty}}$$

where $|x| < \frac{1}{1-q}$ and $|q| < 1$.

One can define another q-exponential function by using $E1$ 3.13.

Definition 21 [1] *Another q-analogue of the classical exponential function e^x is*

$$E_q^x := E_q(x) = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} \frac{x^k}{[k]!} = (1+(1-q)x)_q^{\infty}, \quad |q| < 1.$$

From definitions 20 and 21 we can easily see that

$$e_q(x)E_q(-x) = 1.$$

Moreover, we have

$$D_q e_q(x) = \sum_{k=0}^{\infty} \frac{D_q x^k}{[k]!} = \sum_{k=1}^{\infty} \frac{[k]x^{k-1}}{[k]!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{[k-1]!} = \sum_{k=0}^{\infty} \frac{x^k}{[k]!},$$

and

$$\begin{aligned} D_q E_q(x) &= \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} \frac{D_q x^k}{[k]!} = \sum_{k=1}^{\infty} q^{\frac{1}{2}k(k-1)} \frac{[k]x^{k-1}}{[k]!} \\ &= \sum_{k=1}^{\infty} q^{\frac{1}{2}(k-1)(k-2)} q^{k-1} \frac{x^{k-1}}{[k-1]!} = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} q^k \frac{x^k}{[k]!}, \end{aligned}$$

so, we have

$$D_q e_q(x) = e_q(x) \text{ and } D_q E_q(x) = E_q(qx).$$

In addition, by using $E1$(3.13) and $E2$ (3.14) we have

$$\begin{aligned} e_{1/q}(x) &= \sum_{k=0}^{\infty} \frac{(1-1/q)^k x^k}{(1-1/q)(1-1/q^2)...(1-1/q^k)} \\ &= \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} \frac{(1-q)^k x^k}{(1-q)(1-q^2)...(1-q^k)} \end{aligned}$$

and so we obtain

$$e_{1/q}(x) = E_q(x).$$

Definition 22 [1] *The following identity is called q-Jackson integral of* $f(x)$

$$\int f(x) d_q x = (1-q)x \sum_{k=0}^{\infty} q^k f(q^k x).$$

3.2 (w,q)-Bernoulli polynomials and the (w,q)-Euler polynomials

In this section, we define the (w,q)-Bernoulli numbers $\mathfrak{B}_{n,q}^{(w)}$ and polynomials $\mathfrak{B}_{n,q}^{(w)}(x,y)$ and the (w,q)-Euler numbers $\mathfrak{E}_{n,q}^{(w)}$ and polynomials $\mathfrak{E}_{n,q}^{(w)}(x,y)$ as follows. see [57]

Definition 23 [57] *The (w,q)-Bernoulli numbers $\mathfrak{B}_{n,q}^{(w)}$ and (w,q)-Bernoulli polynomials $\mathfrak{B}_{n,q}^{(w)}(x,y)$ in two dimensions x,y are defined by the generating functions respectively:*

$$\frac{t}{we_q(t)-1} = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(w)} \frac{t^n}{[n]_q!}, \quad |t| < 2\pi,$$
$$\frac{t}{we_q(t)-1} e_q(tx) e_q(ty) = \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(w)}(x,y) \frac{t^n}{[n]_q!} \quad |t| < 2\pi$$

in a suitable neighborhood of $t=0$, where $q \in \mathbb{C}$, and $0 < |q| < 1$.

Definition 24 [57] *The (w,q)-Euler numbers $\mathfrak{E}_{n,q}^{(w)}$ and (w,q)-Euler polynomials $\mathfrak{E}_{n,q}^{(w)}(x,y)$ in two dimensions x,y are defined by the generating functions respectively:*

$$\frac{2}{we_q(t)+1} = \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(w)} \frac{t^n}{[n]_q!}, \quad |t| < \pi,$$
$$\frac{2}{we_q(t)+1} e_q(tx) e_q(ty) = \sum_{n=0}^{\infty} \mathfrak{E}_{n,q}^{(w)}(x,y) \frac{t^n}{[n]_q!}, \quad |t| < \pi$$

in a suitable neighborhood of $t=0$, where $q \in \mathbb{C}$, and $0 < |q| < 1$.

From the previous definitions, one can easilly observe the following

$$\mathfrak{B}_{n,q}^{(w)} = \mathfrak{B}_{n,q}^{(w)}(0), \quad \lim_{q\to 1^-} \mathfrak{B}_{n,q}^{(w)}(x,y) = B_n^{(w)}(x+y), \quad \lim_{q\to 1^-} \mathfrak{B}_{n,q}^{(w)} = B_n^{(w)}.$$
$$\mathfrak{E}_{n,q}^{(w)} = \mathfrak{E}_{n,q}^{(w)}(0), \quad \lim_{q\to 1^-} \mathfrak{E}_{n,q}^{(w)}(x,y) = E_n^{(w)}(x+y), \quad \lim_{q\to 1^-} \mathfrak{E}_{n,q}^{(w)} = E_n^{(w)}.$$

Here $B_n^{(w)}(x)$ and $E_n^{(w)}(x)$ denote the w-Bernoulli and w-Euler polynomials which are defined by

$$\frac{t}{we^t-1} e^{tx} = \sum_{n=0}^{\infty} B_n^{(w)}(x) \frac{t^n}{[n]_q!}$$

and

$$\frac{2}{we^t+1}e^{tx} = \sum_{n=0}^{\infty} E_n^{(w)}(x)\frac{t^n}{[n]_q!} \quad .$$

3.3 Properties of (w,q)-Bernoulli and the (w,q)-Euler polynomials

In this section, we discuss some fundamental properties and their proofs for the $\mathfrak{B}_{n,q}^{(w)}(x,y)$ and $\mathfrak{E}_{n,q}^{(w)}(x,y)$ which are q-extensions of properties of $B_n^{(w)}(x)$ and $E_n^{(w)}(x)$.

Lemma 25 [57] *For all $x \in \mathbb{C}$, let $y = 0$ then $\mathfrak{B}_{n,q}^{(w)}(x,y)$ and $\mathfrak{E}_{n,q}^{(w)}(x,y)$ satisfies following properties respectively*

$$\mathfrak{B}_{n,q}^{(w)}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{(w)} x^{n-k}, \tag{3.16}$$

and

$$\mathfrak{E}_{n,q}^{(w)}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{E}_{k,q}^{(w)} x^{n-k}. \tag{3.17}$$

Proof. [57] The proof of (3.16) is based on the following identity

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(w)}(x)\frac{t^n}{[n]_q!} &= \frac{t}{we_q(t)-1}e_q(tx) \\
&= \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}^{(w)}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n x^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathfrak{B}_{k,q}^{(w)}\frac{t^k}{[k]_q!}\frac{t^{n-k}x^{n-k}}{[n-k]_q!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{(w)} x^{n-k}\frac{t^n}{[n]_q!}.
\end{aligned}$$

■ Similarly, (3.17) can be proved.

Lemma 26 [57] (q-analogue of Differential relations) *If we take q-derivative of* $\mathfrak{B}_{n,q}^{(w)}(x)$ *and* $\mathfrak{E}_{n,q}^{(w)}(x)$ *then we have following identities respectively:*

$$D_{q,x}\mathfrak{B}_{n,q}^{(w)}(x) = [n]_q\,\mathfrak{B}_{n-1,q}^{(w)}(x), \tag{3.18}$$

and

$$D_{q,x}\mathfrak{E}_{n,q}^{(w)}(x) = [n]_q\,\mathfrak{E}_{n-1,q}^{(w)}(x). \tag{3.19}$$

for all $x, y \in \mathbb{C}$.

Proof. [57] To prove (3.18) lets take the first q-derivative of the following expression with respect to x

$$\frac{t}{we_q(t)-1}e_q(tx)$$

Then, we have

$$\begin{aligned}
D_{q,x}\left(\frac{t}{we_q(t)-1}e_q(tx)\right) &= \frac{t}{we_q(t)-1}te_q(tx) \\
&= t\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x)\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}\mathfrak{B}_{n-1,q}^{(w)}(x)\frac{t^n}{[n-1]_q!} \\
&= \sum_{n=0}^{\infty}[n]_q\,\mathfrak{B}_{n-1,q}^{(w)}(x)\frac{t^n}{[n]_q!}
\end{aligned}$$

Therefore,

$$D_{q,x}\mathfrak{B}_{n,q}^{(w)}(x) = [n]_q \mathfrak{B}_{n-1,q}^{(w)}(x).$$

■ Similarly, one can be proved for (w,q)-Euler polynomials (3.19).

Lemma 27 [57] (q-analogue of Difference Equation) *For all $x \in \mathbb{C}$ we have*

$$w\mathfrak{B}_{n,q}^{(w)}(x,1) - \mathfrak{B}_{n,q}^{(w)}(x,0) = [n]_q x^{n-1},$$

Proof. *[57]* Let us use the following identities to prove the lemma

$$\begin{aligned} \frac{wt}{we_q(t)-1}e_q(tx)e_q(t) - \frac{t}{we_q(t)-1}e_q(tx) &= \frac{t}{we_q(t)-1}e_q(tx)\Big(we_q(t)-1\Big) \\ &= te_q(tx) \end{aligned}$$

Indeed,

$$\begin{aligned} \sum_{n=0}^{\infty}\Big(w\mathfrak{B}_{n,q}^{(w)}(x,1) - \mathfrak{B}_{n,q}^{(w)}(x,0)\Big)\frac{t^n}{[n]_q!} &= t\sum_{n=0}^{\infty}\frac{t^n x^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty}\frac{t^n x^{n-1}}{[n-1]_q!} \\ &= \sum_{n=0}^{\infty}[n]_q x^{n-1}\frac{t^n}{[n]_q!}. \end{aligned}$$

It remains to compare the coefficients of $\frac{t^n}{[n]_q!}$. ■

Lemma 28 [57] (q-analogue of Difference Equation) *For all $x, y \in \mathbb{C}$ we have*

$$w\mathfrak{E}_{n,q}^{(w)}(x,1) + \mathfrak{E}_{n,q}^{(w)}(x,0) = 2x^n$$

Proof. [57] Let us use the following identities to prove the lemma

$$\begin{aligned} \frac{2w}{we_q(t)+1}e_q(tx)e_q(t) + \frac{t}{we_q(t)+1}e_q(tx) &= \frac{2}{we_q(t)+1}e_q(tx)\Big(we_q(t)+1\Big) \\ &= 2e_q(tx) \end{aligned}$$

Indeed,

$$\sum_{n=0}^{\infty}\Big(w\mathfrak{E}_{n,q}^{(w)}(x,1) + \mathfrak{E}_{n,q}^{(w)}(x)\Big)\frac{t^n}{[n]_q!} = 2\sum_{n=0}^{\infty}\frac{t^n x^n}{[n]_q!} = \sum_{n=0}^{\infty}2x^n\frac{t^n}{[n]_q!}.$$

■

Lemma 29 [57] (q-analogue of Theorem of complement) *For all $x \in \mathbb{C}$ we have*

$$\mathfrak{B}_{n,q}^{(w)}(x) = \frac{1}{w}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-1)}\mathfrak{B}_{k,1/q}^{(1/w)}(1)\, x^{n-k}$$

Proof. [57] Let us use the following identities to prove the lemma

$$\begin{aligned} \sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x)\frac{t^n}{[n]_q!} &= \frac{tE_q(-t)}{w-E_q(-t)}e_q(tx) = \frac{-t}{e_{1/q}(-t)-w}e_{1/q}(-t)e_q(tx) \\ &= \frac{1}{w}\sum_{n=0}^{\infty}q^{\frac{1}{2}n(n-1)}\mathfrak{B}_{n,1/q}^{(1/w)}(1)\frac{(-t)^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{x^n t^n}{[n]_q!} \\ &= \sum_{n=0}^{\infty}\frac{1}{w}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k-1)}\mathfrak{B}_{k,1/q}^{(1/w)}(1)\, x^{n-k}\frac{t^n}{[n]_q!}. \end{aligned}$$

It remains to compare the coeffcients of $\frac{t^n}{[n]_q!}$. ■

3.4 q-analoques of the addition theorems

In this section, we study on relationships between the $\mathfrak{B}_{n,q}^{(w)}(x,y)$ and $\mathfrak{E}_{n,q}^{(w)}(x,y)$. We also discuss some elementary properties and get new formulas which are extensions of the formulas studied by other authors like Cheon, Srivastava and Pinter, and so on. (see [6], [11], [9]).

Theorem 30 [57] *For $n \in \mathbb{N}_0$, the following relationship*

$$\begin{aligned}\mathfrak{B}_{n,q}^{(w)}(x,y) &= \frac{1}{2m^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \left[m^k \mathfrak{B}_{k,q}^{(w)}(x) + w\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q m^j \mathfrak{B}_{j,q}^{(w)}(x)\right] \\ &\quad \times \mathfrak{E}_{n-k,q}^{(w)}(my).\end{aligned}$$

holds true between the $\mathfrak{B}_{n,q}^{(w)}(x,y)$ and $\mathfrak{E}_{n,q}^{(w)}(x,y)$.

Proof. [57] Using the following identity

$$\begin{aligned}\frac{t}{we_q(t)-1}e_q(tx)e_q(ty) &= \frac{2}{we_q\left(\frac{t}{m}\right)+1}\cdot e_q\left(\frac{t}{m}my\right) \\ &\quad \times \frac{we_q\left(\frac{t}{m}\right)+1}{2}\cdot\frac{t}{we_q(t)-1}e_q(tx)\end{aligned}$$

we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x,y)\frac{t^n}{[n]_q!} &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(w)}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\frac{wt^n}{m^n[n]_q!} \\
&\quad\times\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x)\frac{t^n}{[n]_q!} \\
&\quad+\frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(w)}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x)\frac{t^n}{[n]_q!} \\
&=: I_1+I_2.
\end{aligned}
$$

It is clear that

$$
\begin{aligned}
I_2 &= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(w)}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x)\frac{t^n}{[n]_q!} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q m^{k-n}\mathfrak{B}_{k,q}^{(w)}(x)\,\mathfrak{E}_{n-k,q}^{(w)}(my)\frac{t^n}{[n]_q!}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
I_1 &= \frac{1}{2}w\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q m^{-n}\mathfrak{E}_{j,q}^{(w)}(my)\frac{t^n}{[n]_q!} \\
&= \frac{1}{2}w\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{B}_{k,q}^{(w)}(x)\sum_{j=0}^{n-k}\begin{bmatrix} n-k \\ j \end{bmatrix}_q m^{k-n}\mathfrak{E}_{j,q}^{(w)}(my)\frac{t^n}{[n]_q!} \\
&= \frac{1}{2}w\sum_{n=0}^{\infty}m^{-n}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q \mathfrak{E}_{j,q}^{(w)}(my)\sum_{k=0}^{j}\begin{bmatrix} j \\ k \end{bmatrix}_q m^k\mathfrak{B}_{k,q}^{(w)}(x)\frac{t^n}{[n]_q!}.
\end{aligned}
$$

Therefore, we obtain the following relationship between $\mathfrak{B}_{n,q}^{(w)}(x,y)$ and $\mathfrak{E}_{n,q}^{(w)}(x,y)$

$$\begin{aligned}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(x,y)\frac{t^n}{[n]_q!} &= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q m^{k-n}\\ &\times\left[\mathfrak{B}_{k,q}^{(w)}(x)+m^{-k}w\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}_q m^j\mathfrak{B}_{j,q}^{(w)}(x)\right]\\ &\times\mathfrak{E}_{n-k,q}^{(w)}(my)\frac{t^n}{[n]_q!}.\end{aligned}$$

■ Next, the following corollary gives us some special cases of Theorem 30.

Corollary 31 *The following relationships given in [9] holds true .*

$$B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}\left(B_k(y)+\frac{k}{2}y^{k-1}\right)E_{n-k}(x),$$

$$B_n(x+y)=\frac{1}{2m^n}\sum_{k=0}^{n}\binom{n}{k}\begin{bmatrix}m^kB_k(x)+m^kB_k\left(x-1+\frac{1}{m}\right)\\ +km(1+m(x-1))^{k-1}\end{bmatrix}$$

$$\times E_{n-k}(my).$$

for $n\in\mathbb{N}_0$ and $m\in\mathbb{N}$

Corollary 32 *[45] The following relationship holds true*

$$\mathfrak{B}_{n,q}(x,y) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\left(\mathfrak{B}_{k,q}(0,y)+q^{\frac{1}{2}(k-1)(k-2)}\frac{[k]_q}{2}y^{k-1}\right) \tag{3.20}$$

$$\times\mathfrak{E}_{n-k,q}(x,0). \tag{3.21}$$

for $n\in\mathbb{N}_0$

Corollary 33 *[45] For all $x,y\in\mathbb{C}$ and $n\in\mathbb{N}_0$ the following relationships between $\mathfrak{B}_{n,q}(x,y)$*

and $\mathfrak{E}_{n,q}(x,y)$ holds true.

$$\mathfrak{B}_{n,q}(x,0)=\sum_{\substack{k=0\\(k\neq 1)}}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{B}_{k,q}\mathfrak{E}_{n-k,q}(x,0)+\left(\mathfrak{B}_{1,q}+\frac{1}{2}\right)\times\mathfrak{E}_{n-1,q}(x,0), \tag{3.22}$$

$$\mathfrak{B}_{n,q}(0,y)=\sum_{\substack{k=0\\(k\neq 1)}}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{B}_{k,q}\mathfrak{E}_{n-k,q}(0,y)+\left(\mathfrak{B}_{1,q}+\frac{1}{2}\right)\times\mathfrak{E}_{n-1,q}(0,y). \tag{3.23}$$

The formulas (3.20)-(3.23) are q-analogues of the Cheon's main result [6]. Notice that $\mathfrak{B}_{1,q}=-\frac{1}{[2]_q}$, see [30], and for $q\to 1^{-}$the extra term will be zero .

Theorem 34 [57] *For $n\in\mathbb{N}_0$, the following relationship*

$$\begin{aligned}\mathfrak{E}^{(w)}_{n,q}(x,y) &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{1}{[k+1]_q}m^{k+1-n}\left(w\mathfrak{E}^{(w)}_{k+1,q}\left(x,\frac{1}{m}\right)-\mathfrak{E}^{(w)}_{k+1,q}(x)\right)\\ &\quad\times\mathfrak{B}^{(w)}_{n-k,q}(my).\end{aligned}$$

holds true between the $\mathfrak{E}^{(w)}_{n,q}(x,y)$ and $\mathfrak{B}^{(w)}_{n,q}(x,y)$.

Proof. [57] Using the following identity

$$\begin{aligned}\frac{2}{we_q(t)+1}e_q(tx)e_q(ty) &= \frac{2}{we_q(t)+1}e_q(tx)\cdot\frac{we_q\left(\frac{t}{m}\right)-1}{t/m}\\ &\quad\times\frac{t/m}{we_q\left(\frac{t}{m}\right)-1}e_q\left(\frac{t}{m}my\right).\end{aligned}$$

Indeed, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(w)}(x,y)\frac{t^n}{[n]_q!} &= \frac{m}{t}\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{(w)}(x)\frac{t^n}{[n]_q!}\left(w\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}-1\right)\\
&\times\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(my)\frac{t^n}{m^n[n]_q!}\\
&= m\sum_{n=1}^{\infty}\left(w\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q m^{k-n}\mathfrak{E}_{k,q}^{(w)}(x)-\mathfrak{E}_{n,q}^{(w)}(x)\right)\frac{t^{n-1}}{[n]_q!}\\
&\times\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(my)\frac{t^n}{m^n[n]_q!}\\
&= m\sum_{n=0}^{\infty}\left(w\sum_{k=0}^{n+1}\begin{bmatrix} n+1\\ k\end{bmatrix}_q m^{k-n-1}\mathfrak{E}_{k,q}^{(w)}(x)-\mathfrak{E}_{n+1,q}^{(w)}(x)\right)\frac{t^{n}}{[n+1]_q!}\\
&\times\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(my)\frac{t^n}{m^n[n]_q!}\\
&= m\sum_{n=0}^{\infty}\left(w\mathfrak{E}_{n+1,q}^{(w)}\left(x,\frac{1}{m}\right)-\mathfrak{E}_{n+1,q}^{(w)}(x)\right)\frac{t^{n}}{[n+1]_q!}\\
&\times\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{(w)}(my)\frac{t^n}{m^n[n]_q!}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{[k+1]_q}\begin{bmatrix} n\\ k\end{bmatrix}_q m^{k+1-n}\left(w\mathfrak{E}_{k+1,q}^{(w)}\left(x,\frac{1}{m}\right)-\mathfrak{E}_{k+1,q}^{(w)}(x)\right)\\
&\times\mathfrak{B}_{n-k,q}^{(w)}(my)\frac{t^n}{[n]_q!}.
\end{aligned}$$

■

Corollary 35 *The following relationships given in* [9] *holds true*

$$E_n(x+y) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k}\left(y^{k+1} - E_{k+1}(y)\right)B_{n-k}(x),$$

$$\begin{aligned} E_n(x+y) = &\sum_{k=0}^{n}\binom{n}{k}\frac{m^{k-n+1}}{k+1} \\ &\times\left[2\left(x+\frac{1-m}{m}\right)^{k+1} - E_{k+1}\left(x+\frac{1-m}{m}\right) - E_{k+1}(x)\right] \\ &\times B_{n-k}(my). \end{aligned}$$

for $n \in \mathbb{N}_0$ *and* $m \in \mathbb{N}$.

Next, the following corollaries gives us some special cases of Theorem 34. The relations are q-extensions of previous corollarry studied by Luo in [9].

Corollary 36 [44] *For* $n \in \mathbb{N}_0$ *the following relationship holds true.*

$$\mathfrak{E}_{n,q}(x,y) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{2}{[k+1]_q}\left(y^{k+1} - \mathfrak{E}_{k+1,q}(0,y)\right)\mathfrak{B}_{n-k,q}(x,0).$$

Corollary 37 [44] *For* $n \in \mathbb{N}_0$ *the following relationship holds true.*

$$\mathfrak{E}_{n,q}(x,0) = -\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{2}{[k+1]_q}\mathfrak{E}_{k+1,q}\mathfrak{B}_{n-k,q}(x,0),$$

$$\mathfrak{E}_{n,q}(0,y) = -\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{2}{[k+1]_q}\mathfrak{E}_{k+1,q}\mathfrak{B}_{n-k,q}(0,y).$$

3.5 Location of zeros of the q-Bernoulli polynomials

We can understand the structure of $\mathfrak{B}_{n,q}(x)$ and $\mathfrak{B}_{n,q}$ by using computer experiments. By this way. we can easily study with $\mathfrak{B}_{n,q}(x)$ and $\mathfrak{B}_{n,q}$. Also, we can examine the structure and properties and make some comparisons. These results are used in many areas, for instance pure mathematics, applied mathematics, mathematical physics and so on .

In this section, we represent the figures of the q-Bernoulli polynomials. Then we find the solutions of the $\mathfrak{B}_{n,q}(x) = 0$ by using a computer package Mathematica®software. Finally, we discuss the reflection symmetries of the $\mathfrak{B}_{n,q}(x)$see [57].

In figures 3.1−3.3 the graphs of q-Bernoulli polynomials $\mathfrak{B}_{n,q}(x)$ for $q = 0.5, 0.9$ and 0.9999, $n = 1$, 5, 10, 15 and 20 where $-1 \leq x \leq 2$ is given.

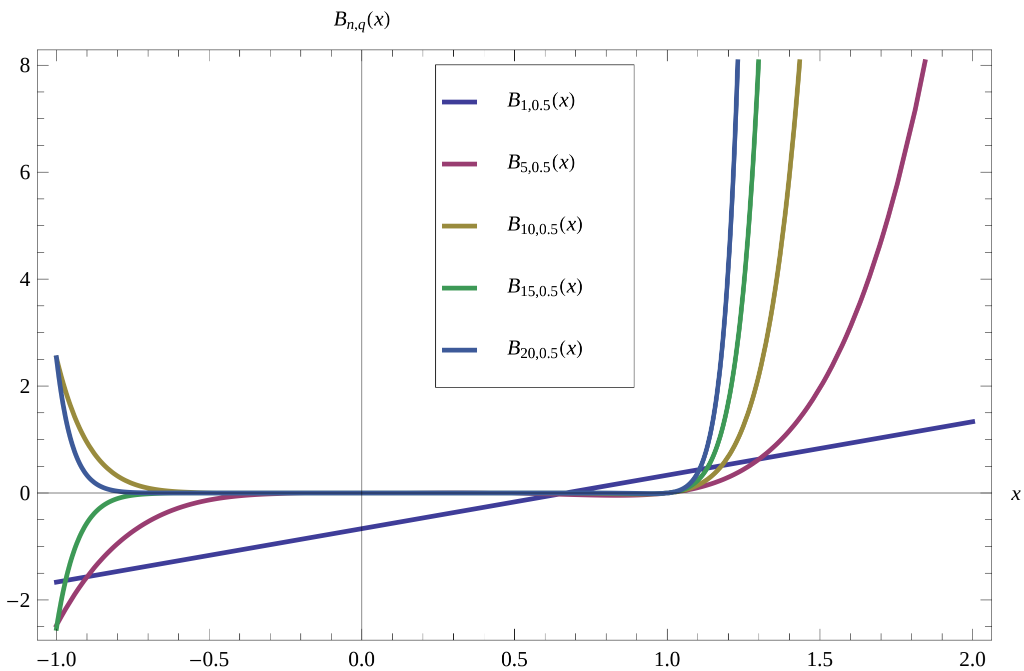

Figure 3.1. Shape of $\mathfrak{B}n, 0.5(x)$

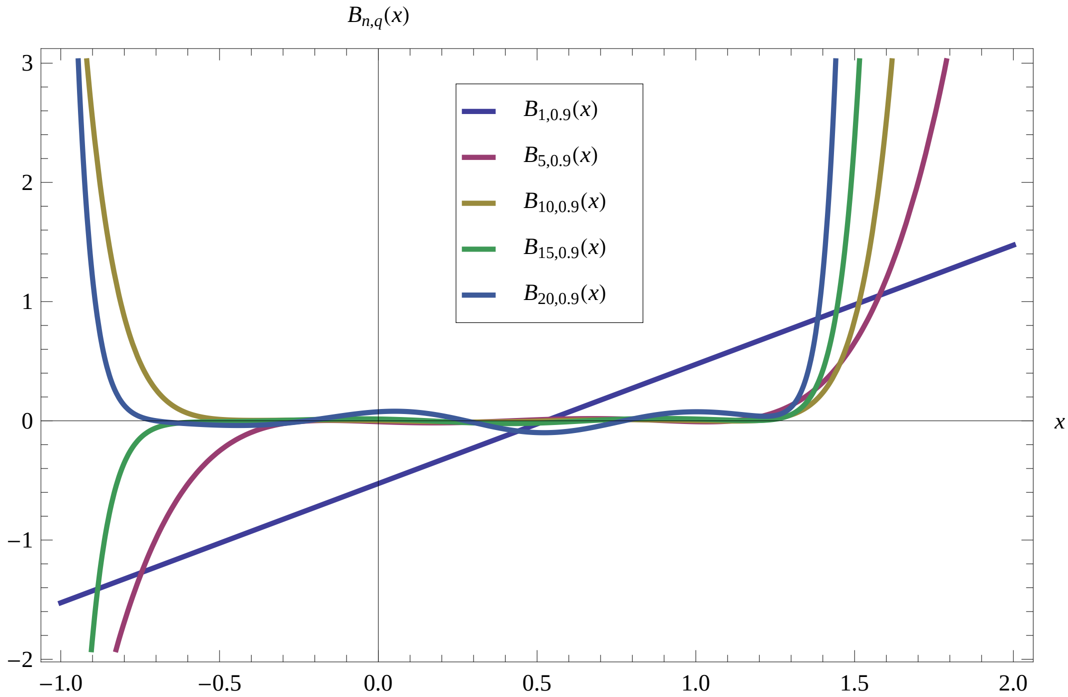

Figure 3.2. Shape of $\mathfrak{B}_{n,0.9}(x)$

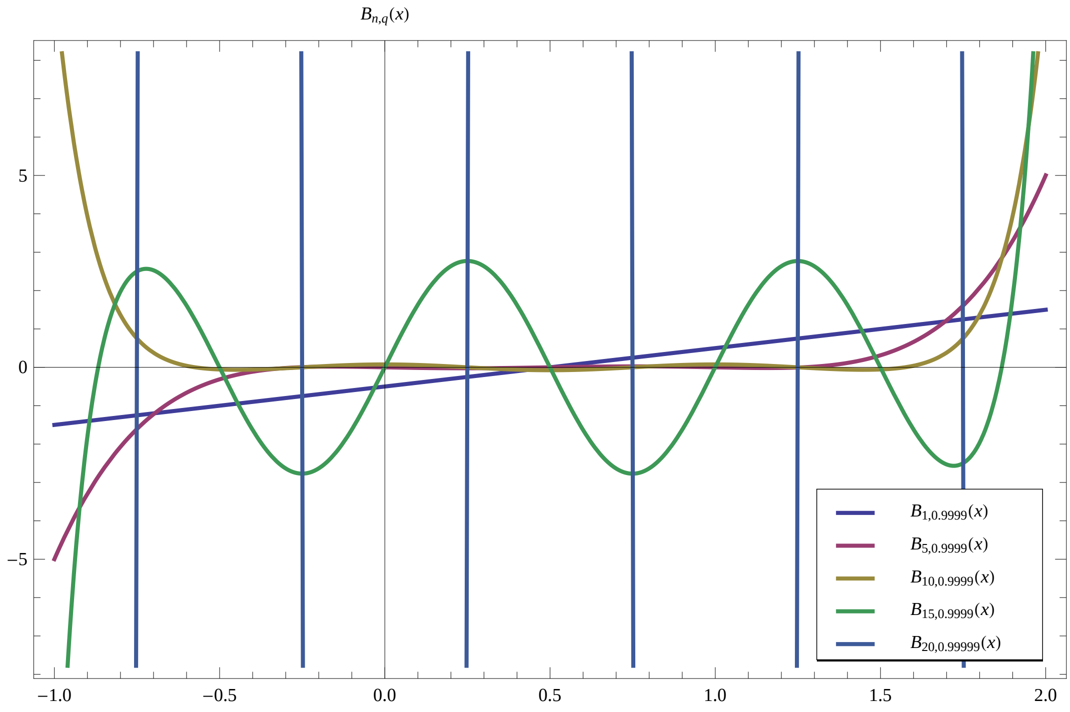

Figure 3.3. Shape of $\mathfrak{B}_{n,0.9999}$

The roots of the $\mathfrak{B}_{n,q}(x)$, are plotted in figures 3.4-3.12 for $n = 10, 15, 20$ and $q = \frac{1}{2}, \frac{9}{10}, 0.\overline{9}$, where $x \in C$ and $-1 \le x \le 2$.

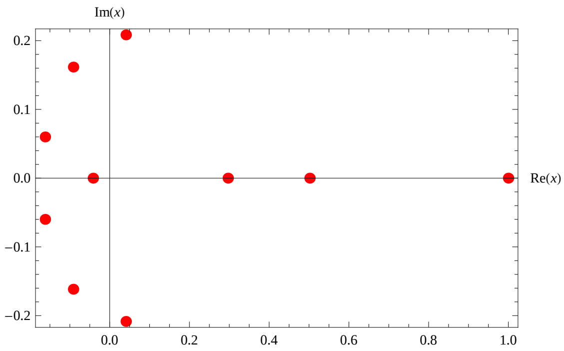

Figure 3.4. Zeros of $\mathfrak{B}_{10,0.5}(x)$

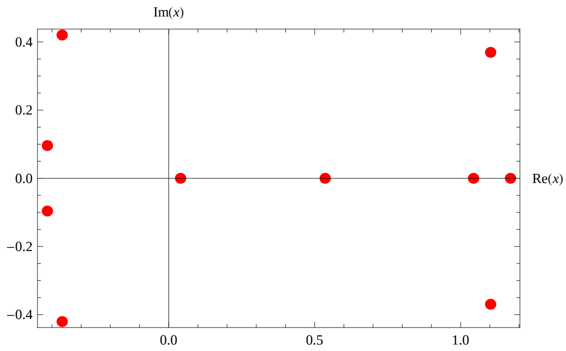

Figure 3.5. Zeros of $\mathfrak{B}_{10,0.9}(x)$

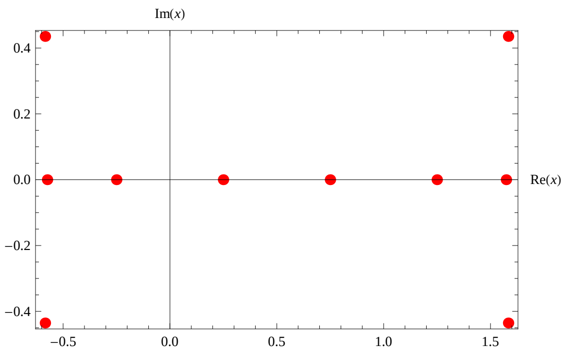

Figure 3.6. Zeros of $\mathfrak{B}_{10,0.9999}(x)$

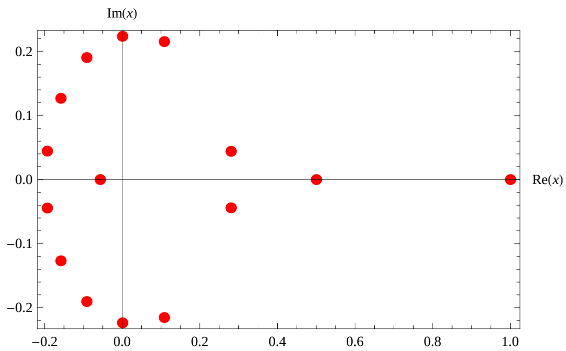

Figure 3.7. Zeros of $\mathfrak{B}_{15,0.5}(x)$

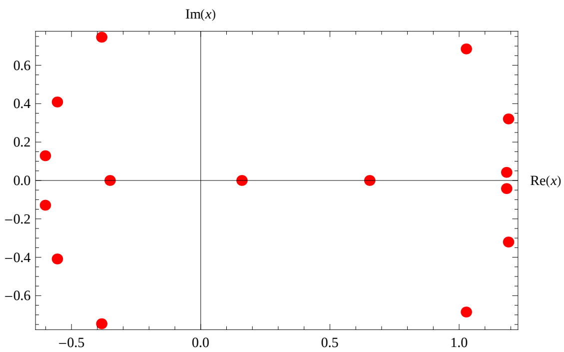

Figure 3.8. Zeros of $\mathfrak{B}_{15,0.9}(x)$

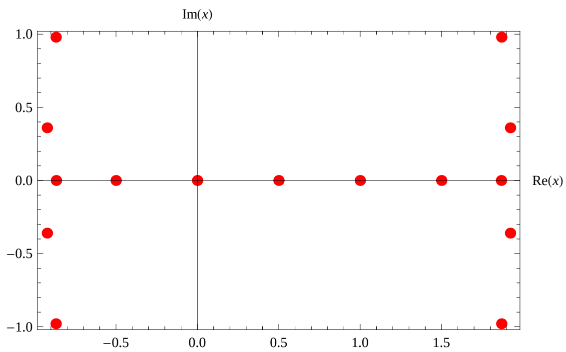

Figure 3.9. Zeros of $\mathfrak{B}_{15,0.9999}(x)$

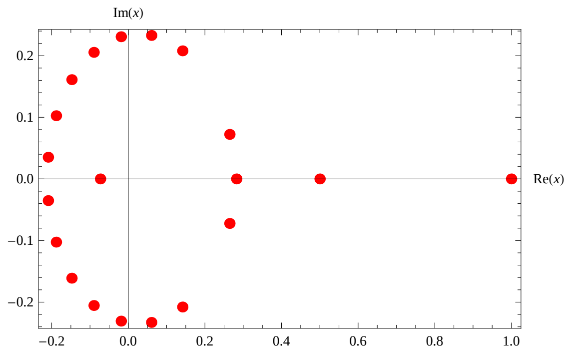

Figure 3.10. Zeros of $\mathfrak{B}_{20,0.5}(x)$

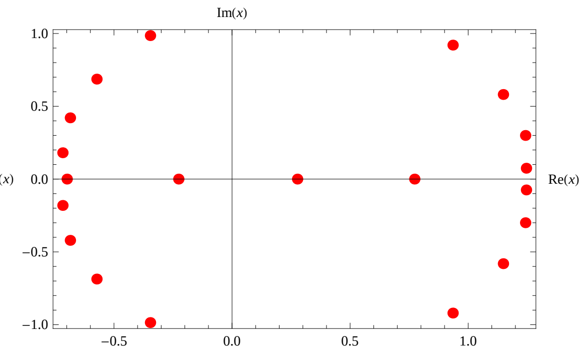

Figure 3.11. Zeros of $\mathfrak{B}_{20,0.9}(x)$

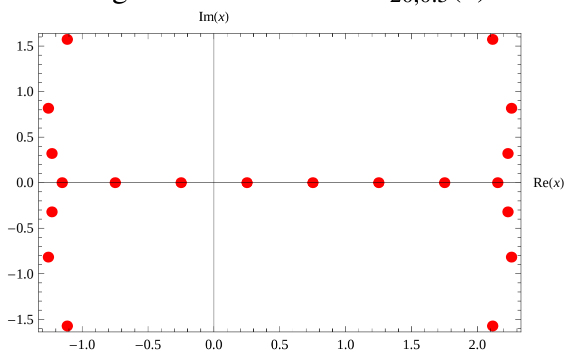

Figure 3.12. Zeros of $\mathfrak{B}_{20,0.9999}(x)$

In figures 3.4−3.12, $\mathfrak{B}_{n,q}(x)$, $x \in C$ have $Im(x) = 0$ reflection symmetry. In table 3.1, the real roots of $\mathfrak{B}_{n,q}(x)$, $q = 0.5$, are shown. In Table 3.2, the real roots of $\mathfrak{B}_{n,q}(x)$,

Table 3.1. Approximate solutions of $\mathfrak{B}_{n,0.5}(x) = 0$

Degree n	Real Zeros
10	-0.0416672, 0.296755,0.501855, 1.0
15	-0.0569424, 0.49992, 1.0
20	-0.0730929, 0.282403, 0.500003, 1.0

$q = 0.\overline{9}$, are shown. In figure 3.13, the 3 dimensional graph of the roots of $\mathfrak{B}_{n,q}(x)$, $x \in C$

Table 3.2. Approximate solutions of $\mathfrak{B}_{n,0.9999}(x) = 0$, $x \in \mathbb{R}$

Degree n	Real Zeros
10	-0.369208,-0.210514, 0.290877, 0.789895, 1.29165, 1.61001
15	-0.562566, -0.394066, 0.105868, 0.60595, 1.10587, 1.6059,
20	-0.4876, -0.249796, 0.250325, 0.750206, 1.25032

for $q = 0.5$ and $n = 1, \ldots, 20$ is given Let $RE_{\mathfrak{B}_{n,q}(x)}$ denotes the number of real roots and

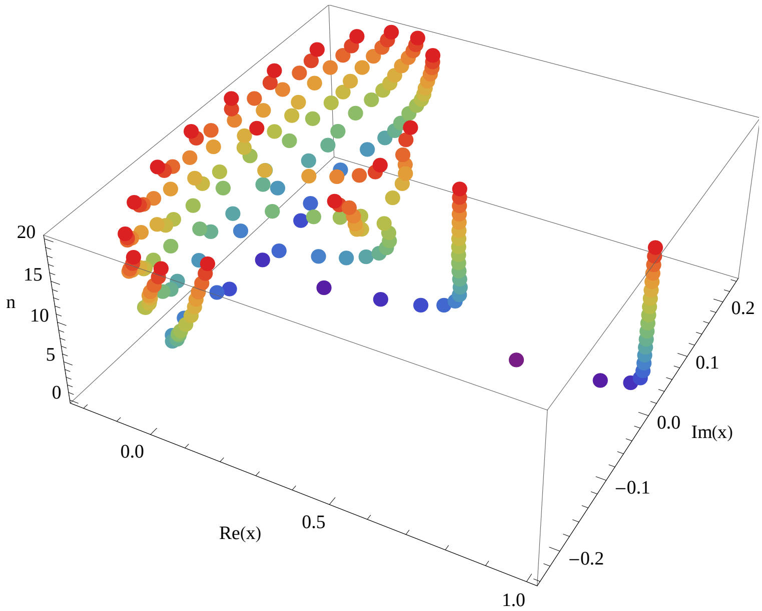

Figure 3.13. 3D shape of $B_{n=20,0.5}(x)$

$CM_{\mathfrak{B}_{n,q}(x)}$ denotes the number of complex roots then we obtain following identity

$$n = RE_{\mathfrak{B}_{n,q}(x)} + CM_{\mathfrak{B}_{n,q}(x)},$$

where n is the degree of $\mathfrak{B}_{n,q}(x)$. See Tables 3.1 and 3.2.

3.6 Location of zeros of the q-Euler polynomials

In this section, we demonstrate the figures and find the solutions of $\mathfrak{E}_{n,q}(x) = 0$ by using a computer package Mathematica®software. Then, according to shapes of the roots of $\mathfrak{E}_{n,q}(x)$ we analyze the reflection symmetries. [57]

In figures 3.14, 3.15 and 3.16, the shapes of the $\mathfrak{E}_{n,q}(x)$ for $n = 20$ and $\frac{1}{2} \leq q \leq 1$ are shown . In figures 3.17-3.19, the roots of the $\mathfrak{E}_{n,q}(x)$ are plotted for $n = 20$ and $q = \frac{1}{2}, \frac{9}{10}, 0.\overline{9}$ where $x \in C$ In table 3.3, the real roots of $\mathfrak{E}_{n,q}(x)$, ,$x \in C$, for $n = 10$ and 20 and $q = 0.\overline{9}$

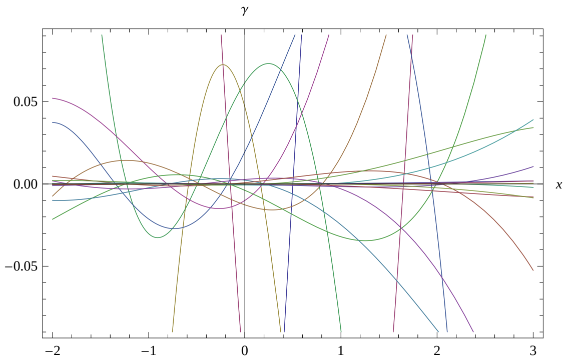

Figure 3.14. Shape of $\mathfrak{E}_{20,0.5}(x)$

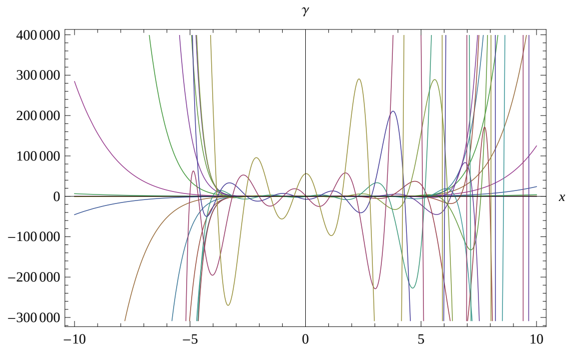

Figure 3.15. Shape of $\mathfrak{E}_{20,0.9}(x)$

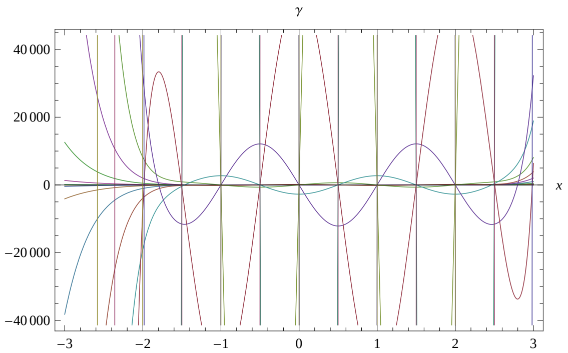

Figure 3.16. Shape of $\mathfrak{E}_{20,0.9999}(x)$

are shown. In figure 3.20, the 3 dimensional graph of the roots of q-Euler polynomials

Table 3.3. Approximate solutions of $\mathfrak{E}_{n,0.9999}(x) = 0,\ x \in \mathbb{R}$

Degree n	Real zeros of $\mathfrak{E}_{n,q}(x) = 0$ for $q = 0.\overline{9}$
10	-1.36555, -1.0152, -0.000225011, 0.999775, 2.01449, 2.36694
20	-2.58148, -2.00239, -1.00048, -0.000475024, 0.999525, 1.99953, 3.00139

$\mathfrak{E}_{n,q}(x)$, for $q = 0.\overline{9}$, $n = 1, ..., 20$ and $x \in C$ is given.

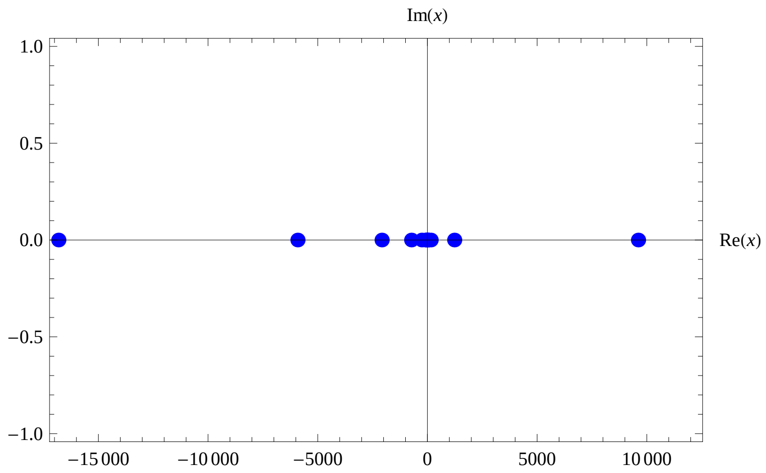

Figure 3.17. Zeros of $\mathfrak{E}_{20,0.5}(x)$

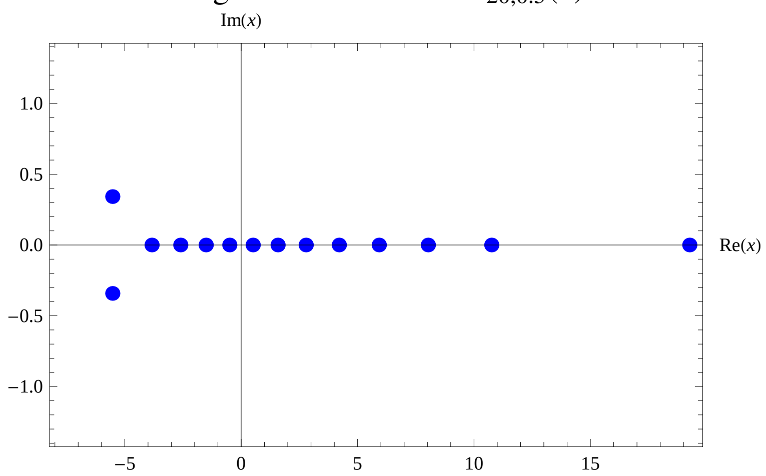

Figure 3.18. Zeros of $\mathfrak{E}_{20,0.9}(x)$

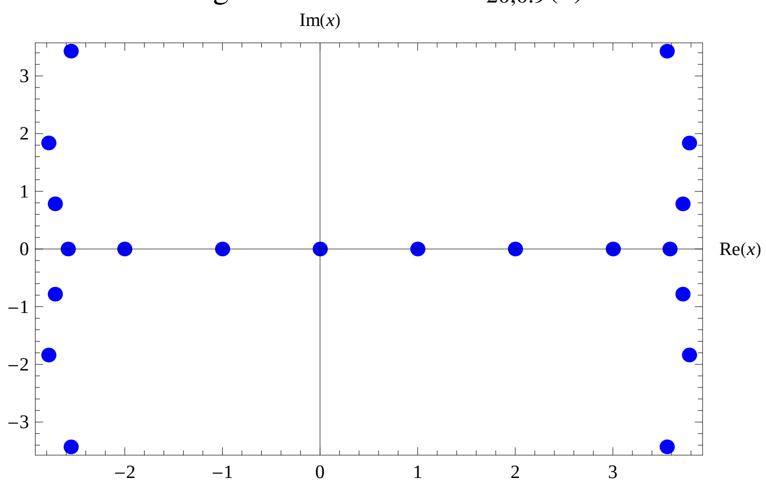

Figure 3.19. Zeros of $\mathfrak{E}_{20,0.9999}(x)$

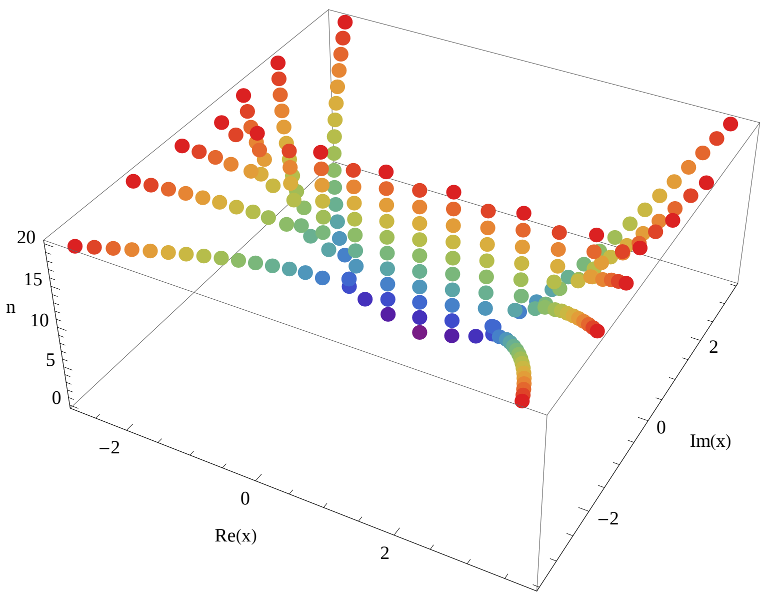

Figure 3.20. 3D shape of $\mathfrak{E}_{n=20,0.9999}(x)$

In figure 3.17, for $q = \frac{1}{2}$ and $n = 20$ the $\mathfrak{E}_{n,q}(x)$ has no complex roots.

In figures 3.17-3.19, $\mathfrak{E}_{n,q}(x)$, $x \in C$ have $\mathrm{Im}(x) = 0$ reflection symmetry.

3.7 Higher order q-Frobenius-Euler Numbers and Polynomials

In this section, Let $\lambda \in \mathbb{C}$ with $\lambda \neq 1$, $n \geq 0$ and $\alpha \in \mathbb{R}$. Now, we consider q-extension of higher order Frobenius-Euler numbers $H_{n,q}^{\alpha,\lambda}$ and polynomials, $H_{n,q}^{\alpha,\lambda}(x)$. We give some properties of higher order $q-$Frobenius-Euler numbers $H_{n,q}^{\alpha,\lambda}$ and polynomials $H_{n,q}^{\alpha,\lambda}(x)$.

Definition 38 *The higher order q-Bernoulli polynomials, $B_{n,q}^{\alpha}(x)$, are defined in [38] given as follows:*

$$\sum_{n=0}^{\infty} B_{n,q}^{\alpha}(x)\frac{t^n}{[n]_q!} = \left(\frac{t}{e_q(t)-1}\right)^{\alpha} e_q(tx).$$

for $q \in \mathbb{C}$ and $|q| < 1$

Definition 39 *The higher order q-Euler polynomials, $E_{n,q}^{\alpha}(x)$, are defined in [38] given as follows:*

$$\sum_{n=0}^{\infty} E_{n,q}^{\alpha}(x)\frac{t^n}{[n]_q!} = \left(\frac{2}{e_q(t)-1}\right)^{\alpha} e_q(tx).$$

for $q \in \mathbb{C}$ and $|q| < 1$.

Definition 40 *The higher order q-Frobenius-Euler numbers, $H_{n,q}^{\alpha,\lambda}$, and polynomials $H_{n,q}^{\alpha,\lambda}(x)$ are defined, by means of the generating functions as follows:*

$$\left(\frac{1-\lambda}{e_q(t)-\lambda}\right)^{\alpha} = \sum_{n=0}^{\infty} H_{n,q}^{\alpha,\lambda}\frac{t^n}{[n]_q!}, \quad |t| < 2\pi,$$

and

$$\left(\frac{1-\lambda}{e_q(t)-\lambda}\right)^{\alpha} e_q(tx) = \sum_{n=0}^{\infty} H_{n,q}^{\alpha,\lambda}(x)\frac{t^n}{[n]_q!}, \quad |t| < 2\pi.$$

for $q \in \mathbb{C}$ and $|q| < 1$.

Note that $\lim\limits_{q\to 1} H_{n,q}^{\alpha,\lambda}(x) = H_n^{\alpha,\lambda}(x)$, where $H_n^{\alpha,\lambda}(x)$ are the ordinary higher order Frobenius-Euler polynomials defined in [62] as follows:

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} H_n^{\alpha,\lambda}(x)\frac{t^n}{n!}, \quad |t| < 2\pi.$$

Definition 41 *The higher order q-Frobenius-Euler polynomials $H_{n,q}^{\alpha,\lambda}(x,y)$ in x, y are de-*

fined, by the generating functions as follows:

$$\left(\frac{1-\lambda}{e_q(t)-\lambda}\right)^{\alpha} e_q(tx)e_q(ty) = \sum_{n=0}^{\infty} H_{n,q}^{\alpha,\lambda}(x,y)\frac{t^n}{[n]_q!}, \quad |t| < 2\pi.$$

for $q \in \mathbb{C}$ and $|q| < 1$.

Lemma 42 *The following identity is true for $n \geq 0$,*

$$H_{n,q}^{\alpha,\lambda}(x) = \sum_{n=0}^{\infty} \binom{n}{k}_q H_{n-k,q}^{\alpha,\lambda} x^k.$$

Proof. From the previous definition , we easily see that

$$\sum_{n=0}^{\infty} H_{n,q}^{\alpha,\lambda}(x)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} H_{n,q}^{\alpha,\lambda}\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n x^n}{[n]_q!}$$

By Cauchy product, we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} H_{n-k,q}^{\alpha,\lambda}\frac{t^{n-k}}{[n-k]_q!}\frac{t^k x^k}{[k]_q!}$$
$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q H_{n-k,q}^{\alpha,\lambda} x^k \frac{t^n}{[n]_q!}$$

now, let us compare the coefficients of $\frac{t^n}{[n]_q!}$, we have

$$H_{n,q}^{\alpha,\lambda}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q H_{n-k,q}^{\alpha,\lambda} x^k.$$

■

Lemma 43 *(Differential relation)*

$$D_q H_{n,q}^{\alpha,\lambda}(x) = [n]_q H_{n-1,q}^{\alpha,\lambda} x^k.$$

Proof. Let's take q-derivative of q-Frobenius-Euler polynomials $H_{n,q}^{\alpha,\lambda}(x)$,with respect to x, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} D_{q,x} H_{n,q}^{\alpha,\lambda}(x) \frac{t^n}{[n]_q!} &= D_{q,x} \left(\frac{1-\lambda}{e_q(t)-\lambda} \right)^{\alpha} e_q(tx) \\
&= t \sum_{n=0}^{\infty} H_{n,q}^{\alpha,\lambda}(x) \frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty} H_{n,q}^{\alpha,\lambda}(x) \frac{t^{n+1}}{[n]_q!} \\
&= \sum_{n=1}^{\infty} H_{n-1,q}^{\alpha,\lambda}(x) \frac{t^n}{[n-1]_q!} \\
&= \sum_{n=1}^{\infty} [n]_q H_{n-1,q}^{\alpha,\lambda}(x) \frac{t^n}{[n]_q!}
\end{aligned}$$

then, by comparing coefficients on both sides we get

$$H_{n,q}^{\alpha,\lambda}(x) = [n]_q H_{n-1,q}^{\alpha,\lambda} x^k.$$

■

Lemma 44 *(Difference equation) For $n \geq 0$,we have*

$$H_{n,q}^{\alpha,\lambda}(x,1) - \lambda H_{n,q}^{\alpha,\lambda}(x,0) = (1-\lambda) H_{n,q}^{\alpha-1,\lambda}(x,0).$$

Proof. By using following identities we obtain

$$\begin{aligned}
\left(\frac{1-\lambda}{e_q(t)-\lambda}\right)^{\alpha} e_q(tx)e_q(t)-\lambda\left(\frac{1-\lambda}{e_q(t)-\lambda}\right)^{\alpha} e_q(tx) &= \left(e_q(tx)-\lambda\right)\left(\frac{1-\lambda}{e_q(t)-\lambda}\right)^{\alpha} e_q(tx) \\
&= \frac{(1-\lambda)^{\alpha}}{\left(e_q(t)-\lambda\right)^{\alpha-1}} e_q(tx) \\
&= (1-\lambda)\left(\frac{1-\lambda}{e_q(t)-\lambda}\right)^{\alpha-1} e_q(tx)
\end{aligned}$$

So, we get

$$H_{n,q}^{\alpha,\lambda}(x,1)-\lambda H_{n,q}^{\alpha,\lambda}(x,0)=(1-\lambda)H_{n,q}^{\alpha-1,\lambda}(x,0).$$

■

Chapter 4

ON TWO DIMENSIONAL q-BERNOULLI AND q-GENOCCHI NUMBERS AND POLYNOMIALS

The main aim of this chapter is to investigate two dimensional generalized q-Genocchi polynomials. We discuss q-extensions of some properties of $\mathfrak{G}_{n,q}(x,y)$ like Srivastava and Pintér's results given in [11]. It should be mentioned that probabilistics proof the Srivastava-Pintér addition theorems were given recently in [39]. Furthermore, we demonstrate the figures and find the solutions of $\mathfrak{G}_{n,q}(x)$ by using a computer package Mathematica®software. Then, according to shapes of the roots of $\mathfrak{G}_{n,q}(x)$ we analyze the reflection symmetries . (see [57])

Definition 45 *In* [56] *the q-Bernoulli numbers* $\mathfrak{B}_{n,q}$ *and polynomials* $\mathfrak{B}_{n,q}(x,y)$ *in two dimensions* x,y *are defined by the generating functions:*

$$\left(\frac{t}{e_q(t)-1}\right)=\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}\frac{t^n}{[n]_q!},\quad |t|<2\pi,$$
$$\left(\frac{t}{e_q(t)-1}\right)e_q(tx)\,e_q(ty)=\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x,y)\frac{t^n}{[n]_q!},\quad |t|<2\pi.$$

Definition 46 *In* [56] *the q-Genocchi numbers* $\mathfrak{G}_{n,q}$ *and polynomials* $\mathfrak{G}_{n,q}(x,y)$ *in two* *dimensions x, y are defined by the generating functions:*

$$\left(\frac{2t}{e_q(t)+1}\right)=\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}\frac{t^n}{[n]_q!},\quad |t|<\pi,$$
$$\left(\frac{2t}{e_q(t)+1}\right)e_q(tx)e_q(ty)=\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x,y)\frac{t^n}{[n]_q!},\quad |t|<\pi.$$

From the previous definitions, one can easilly observe the followings

$$\mathfrak{B}_{n,q}=\mathfrak{B}_{n,q}(0,0),\quad \lim_{q\to 1^-}\mathfrak{B}_{n,q}(x,y)=B_n(x+y),\quad \lim_{q\to 1^-}\mathfrak{B}_{n,q}=B_n,$$
$$\mathfrak{G}_{n,q}=\mathfrak{G}_{n,q}(0,0),\quad \lim_{q\to 1^-}\mathfrak{G}_{n,q}(x,y)=G_n(x+y),\quad \lim_{q\to 1^-}\mathfrak{G}_{n,q}=G_n.$$

Here $B_n(x)$ and $G_n(x)$ denote the classical Bernoulli and Genocchi polynomials are defined by

$$\left(\frac{t}{e^t-1}\right)e^{tx}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad \text{and}\quad \left(\frac{2t}{e^t+1}\right)e^{tx}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}.$$

4.1 Properties of q-Genocchi polynomials

In this section, we discuss some fundamental properties and their proofs for the q-Genocchi polynomials $\mathfrak{G}_{n,q}(x,y)$.

First of all we idefine a new q-extension of the following funtion $(x\oplus y)^n$.

Definition 47 [56] *The function $(x\oplus y)^n$ has the following q-extension given as follow*

$$(x\oplus y)_q^n:=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q x^k y^{n-k},\quad n\in\mathbb{N}_0.$$

One can easily derive the following fundamental properties of the q-Genocchi polynomials from Definition 46.

Property 1. *Summation formulas for the q-Genocchi polynomials:*

Lemma 48 [56] *The following formula is q-extension of summation formula*

$$\mathfrak{G}_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{G}_{k,q}(x\oplus y)_q^{n-k},$$

Proof. [56] Let us use the following identity to prove lemma

$$\begin{aligned}\left(\frac{2t}{e_q(t)+1}\right)e_q(tx)e_q(ty)&=\sum_{k=0}^{\infty}\mathfrak{G}_{k,q}\frac{t^k}{[k]_q!}\sum_{n=0}^{\infty}\frac{t^nx^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^ny^n}{[n]_q!}\\&=\sum_{k=0}^{\infty}\mathfrak{G}_{k,q}\frac{t^k}{[k]_q!}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{t^kx^k}{[k]_q!}.\frac{t^{n-k}y^{n-k}}{[n-k]_q!}\right)\\&=\sum_{k=0}^{\infty}\mathfrak{G}_{k,q}\frac{t^k}{[k]_q!}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q x^ky^{n-k}\right)\frac{t^n}{[n]_q!}\\&=\sum_{k=0}^{\infty}\mathfrak{G}_{k,q}\frac{t^k}{[k]_q!}\sum_{n=0}^{\infty}(x\oplus y)_q^{n-k}\frac{t^n}{[n]_q!}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{E}_{k,q}(x\oplus y)_q^{n-k}\frac{t^n}{[n]_q!}\end{aligned}$$

■

Lemma 49 [56] *We have following identity*

$$\mathfrak{G}_{n,q}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\mathfrak{G}_{k,q}(x)y^{n-k}.$$

for all $x,y\in\mathbb{C}$.

Proof. [56] By using following identity we get

$$\begin{aligned}\left(\frac{2t}{e_q(t)+1}\right)e_q(tx)\,e_q(ty) &= \sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\frac{t^n y^n}{[n]_q!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathfrak{G}_{k,q}(x)\frac{t^k}{[k]_q!}\frac{t^{n-k}y^{n-k}}{[n-k]_q!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\mathfrak{G}_{k,q}(x)y^{n-k}\frac{t^n}{[n]_q!}.\end{aligned}$$

■

Lemma 50 [56] *For all $x, y \in \mathbb{C}$ we have*

$$\mathfrak{G}_{n,q}(x) = \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q\mathfrak{G}_{k,q}x^{n-k}.$$

Proof. [56] The proof is readily derived from Definition 46. ■

,**Property 2.** *Difference equation:*

Lemma 51 [56] *We have following difference property*

$$\mathfrak{G}_{n,q}(x,1)+\mathfrak{G}_{n,q}(x,0) = 2\,[n]_q\,x^{n-1}.$$

for all $x, y \in \mathbb{C}$

Proof. [56] By using following identity we get

$$\begin{aligned}
\mathfrak{G}_{n,q}(x,1)+\mathfrak{G}_{n,q}(x,0) &= \left(\frac{2t}{e_q(t)+1}\right)e_q(tx)\,e_q(t)+\left(\frac{2t}{e_q(t)+1}\right)e_q(tx) \\
&= \left(\frac{2t}{e_q(t)+1}\right)e_q(tx)\left(e_q(t)+1\right) \\
&= 2te_q(tx) \\
&= 2t\sum_{n=0}^{\infty}\frac{t^n x^n}{[n]_q!} \\
&= 2\sum_{n=0}^{\infty}\frac{t^n x^{n-1}}{[n-1]_q!} \\
&= 2\sum_{n=0}^{\infty}\frac{t^n x^{n-1}}{[n]_q!}[n]_q \\
&= \sum_{n=1}^{\infty}2x^{n-1}[n]_q\frac{t^n}{[n]_q!}.
\end{aligned}$$

■

Property 3. *Differential relation:*

Lemma 52 [56] *We have*

$$D_{q,x}\mathfrak{G}_{n,q}(x) = [n]_q\,\mathfrak{G}_{n-1,q}(x).$$

for all $x, y \in \mathbb{C}$.

Proof. [56] It follows from the following relation

$$
\begin{aligned}
D_{q,x}\left(\frac{2t}{e_q(t)+1}\right)e_q(tx) &= \left(\frac{2t}{e_q(t)+1}\right)te_q(tx) \\
&= t\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x)\frac{t^n}{[n]_q!} \\
&= \sum_{n=0}^{\infty}[n]_q\,\mathfrak{G}_{n-1,q}(x)\frac{t^n}{[n]_q!}.
\end{aligned}
$$

■

4.2 Explicit relationship between the q-Genocchi and the q-Bernoulli polynomials

In this section we prove relationships between the q-Genocchi polynomials $\mathfrak{G}_{n,q}(x,y)$ and the q-Bernoulli polynomials $\mathfrak{B}_{n,q}(x,y)$.

Theorem 53 [56] *For $n\in\mathbb{N}_0$, the q-Genocchi and the q-Bernoulli polynomials has the following relationship*

$$
\begin{aligned}
\mathfrak{G}_{n,q}(x,y) \;=\; & \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{[k+1]_q}m^{k-n+1}\left(\mathfrak{G}_{k+1,q}\left(x,\frac{1}{m}\right)-\mathfrak{G}_{k+1,q}(x)\right) \\
& \times\mathfrak{B}_{n-k,q}(my).
\end{aligned}
$$

Proof. [56] Using the following identity

$$
\begin{aligned}
\frac{2t}{e_q(t)+1}e_q(tx)e_q(ty) \;=\; & \frac{2t}{e_q(t)+1}e_q(tx)\cdot\frac{e_q\left(\frac{t}{m}\right)-1}{t} \\
& \times\frac{t}{e_q\left(\frac{t}{m}\right)-1}\cdot e_q\left(\frac{t}{m}my\right)
\end{aligned}
$$

we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x,y)\frac{t^n}{[n]_q!} &= \frac{m}{t}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(x)\frac{t^n}{[n]_q!}\left(\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!}-1\right) \\
&\times\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(my)\frac{t^n}{m^n[n]_q!} \\
&= m\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q m^{k-n}\mathfrak{G}_{k,q}(x)-\mathfrak{G}_{n,q}(x)\right)\frac{t^{n-1}}{[n]_q!} \\
&\times\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(my)\frac{t^n}{m^n[n]_q!} \\
&= m\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n+1 \\ k\end{bmatrix}_q m^{k-n-1}\mathfrak{G}_{k,q}(x)-\mathfrak{G}_{n+1,q}(x)\right) \\
&\times\frac{t^n}{[n+1]_q!}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(my)\frac{t^n}{m^n[n]_q!} \\
&= m\sum_{n=0}^{\infty}\left(\mathfrak{G}_{n+1,q}\left(x,\frac{1}{m}\right)-\mathfrak{G}_{n+1,q}(x)\right)\frac{t^n}{[n+1]_q!} \\
&\times\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(my)\frac{t^n}{m^n[n]_q!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q\frac{1}{[k+1]_q}m^{k-n+1}\left(\mathfrak{G}_{k+1,q}\left(x,\frac{1}{m}\right)-\mathfrak{G}_{k+1,q}(x)\right) \\
&\times\mathfrak{B}_{n-k,q}(my)\frac{t^n}{[n]_q!}.
\end{aligned}$$

■

Corollary 54 [28] *For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ the following relationship holds true.*

$$G_n(x+y) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \frac{2}{k+1} \Big((k+1)y^k - G_{k+1,q}(y)\Big) B_{n-k}(x), \tag{4.1}$$

$$G_n(x+y) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \frac{1}{m^{n-k-1}(k+1)} \tag{4.2}$$

$$\times \left[2(k+1)G_k\left(y+\frac{1}{m}-1\right) - G_{k+1}\left(y+\frac{1}{m}-1\right) - G_{k+1}(y)\right] \tag{4.3}$$

$$\times B_{n-k,q}(mx)$$

between the ordinary Genocchi polynomials and the ordinary Bernoulli polynomials.

Corollary 55 [28] *For $n \in \mathbb{N}_0$ the $\mathfrak{G}_{n,q}(x,y)$ and $\mathfrak{B}_{n,q}(x,y)$ has the following relationship:*

$$\begin{aligned} \mathfrak{G}_{n,q}(x,y) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{2}{[k+1]_q} \left[[k+1]_q q^{\frac{1}{2}k(k-1)} y^k - \mathfrak{G}_{k+1,q}(y)\right] \\ &\quad \times \mathfrak{B}_{n-k,q}(x). \end{aligned}$$

In Corollary 55, by setting $y = 0$ we obtain the following explicit relationships:

Corollary 56 [28] *For $n \in \mathbb{N}_0$ the $\mathfrak{G}_{n,q}(x)$ and $\mathfrak{B}_{n,q}(x)$ has the following relationship:.*

$$\mathfrak{G}_{n,q}(x) = -\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{2}{[k+1]_q} \mathfrak{G}_{k+1,q} \mathfrak{B}_{n-k,q}(x),$$

Corollary 57 [28] *For $n \in \mathbb{N}_0$ the $\mathfrak{G}_{n,q}$ and $\mathfrak{B}_{n,q}$ has the following relationship:*

$$\mathfrak{G}_{n,q} = -\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{2}{[k+1]_q} \mathfrak{G}_{k+1,q} \mathfrak{B}_{n-k,q}.$$

Theorem 58 [56] *For $n \in \mathbb{N}_0$, the $\mathfrak{B}_{n,q}(x,y)$ and the $\mathfrak{G}_{n,q}(x,y)$ has the following relationship:*

$$\begin{aligned} \mathfrak{B}_{n,q}(x,y) &= \frac{1}{2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q m^{k-n} \\ &\quad \times \begin{bmatrix} \frac{1}{[k+1]_q} \mathfrak{B}_{k+1,q}(x) + m^{-k} \\ \times \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{1}{[j+1]_q} m^j \mathfrak{B}_{j+1,q}(x) \end{bmatrix} \\ &\quad \times \mathfrak{G}_{n-k,q}(my). \end{aligned}$$

Proof. [56] Using the following identity

$$\begin{aligned} \frac{t}{e_q(t)-1} e_q(tx) e_q(ty) &= \frac{t}{e_q(t)-1} e_q(tx) \frac{2t}{e_q\left(\frac{t}{m}\right)+1} \\ &\quad \times e_q\left(\frac{t}{m} my\right) \frac{e_q\left(\frac{t}{m}\right)+1}{2t} \end{aligned}$$

we have

$$\begin{aligned} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(x,y) \frac{t^n}{[n]_q!} &= \frac{1}{2t} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(x) \frac{t^n}{[n]_q!} \\ &\quad \times \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}(my) \frac{t^n}{m^n [n]_q!} \sum_{n=0}^{\infty} \frac{t^n}{m^n [n]_q!} \\ &\quad + \frac{1}{2t} \sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(x) \frac{t^n}{[n]_q!} \sum_{n=0}^{\infty} \mathfrak{G}_{n,q}(my) \frac{t^n}{m^n [n]_q!} \\ &= I_1 + I_2. \end{aligned}$$

It is clear that

$$\begin{aligned}
I_2 &= \frac{1}{2t}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(my)\frac{t^n}{m^n[n]_q!} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n+1,q}(x)\frac{t^n}{[n+1]_q!}\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(my)\frac{t^n}{m^n[n]_q!} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q m^{k-n}\frac{1}{[k+1]_q}\mathfrak{B}_{k+1,q}(x)\mathfrak{G}_{n-k,q}(my)\frac{t^n}{[n]_q!}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
I_1 &= \frac{1}{2t}\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}(x)\frac{t^n}{[n]_q!} \\
&\times\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}(my)\frac{t^n}{m^n[n]_q!}\sum_{n=0}^{\infty}\frac{t^n}{m^n[n]_q!} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\mathfrak{B}_{n+1,q}(x)\frac{t^n}{[n+1]_q!} \\
&\times\sum_{n=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q m^{-n}\mathfrak{G}_{j,q}(my)\frac{t^n}{[n]_q!} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{[k+1]_q}m^{k-n}\mathfrak{B}_{k+1,q}(x) \\
&\times\sum_{j=0}^{n-k}\begin{bmatrix} n-k \\ j \end{bmatrix}_q \mathfrak{G}_{j,q}(my)\frac{t^n}{[n]_q!}
\end{aligned}$$

Let use the following combinatorial property

$$\binom{n}{k}\binom{n-k}{j} = \binom{n}{j}\binom{n-j}{k}$$

and replace k by j we have

$$
\begin{aligned}
I_1 &= \frac{1}{2}\sum_{n=0}^{\infty} m^{-n} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \mathfrak{G}_{j,q}(my) \\
&\quad \times \sum_{k=0}^{n-j} \begin{bmatrix} n-j \\ k \end{bmatrix}_q \frac{1}{[k+1]_q} m^k \mathfrak{B}_{k+1,q}(x) \frac{t^n}{[n]_q!}.
\end{aligned}
$$

then replace j by $n-k$ and k by j we obtain

$$
\begin{aligned}
I_1 &= \frac{1}{2}\sum_{n=0}^{\infty} m^{-n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \mathfrak{G}_{n-k,q}(my) \\
&\quad \times \sum_{k=0}^{n-j} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{1}{[j+1]_q} m^j \mathfrak{B}_{j+1,q}(x) \frac{t^n}{[n]_q!}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sum_{n=0}^{\infty} \mathfrak{B}_{n,q}(x,y) \frac{t^n}{[n]_q!} = I_1 + I_2 = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q m^{k-n} \\
\times \left[\frac{1}{[k+1]_q} \mathfrak{B}_{k+1,q}(x) + m^{-k} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q \frac{1}{[j+1]_q} m^j \mathfrak{B}_{j+1,q}(x) \right] \\
\times \mathfrak{G}_{n-k,q}(my) \frac{t^n}{[n]_q!}.
\end{aligned}
$$

■

4.3 Location of zeros of the q-Genocchi polynomials

In this section, we demonstrate the figures of the q-Genocchi polynomials $\mathfrak{G}_{n,q}(x)$.and find the solutions of the $\mathfrak{G}_{n,q}(x) = 0$ by using a computer package Mathematica®software. Then, according to shapes of the roots of $\mathfrak{G}_{n,q}(x)$ we analyze the reflection symmetries of the $\mathfrak{G}_{n,q}(x)$ (see[56]).

In figures 4.1-4.3, the shapes of the $\mathfrak{G}_{n,q}(x)$ for $n = 20$ and $\frac{1}{2} \leq q \leq 1$ are shown.

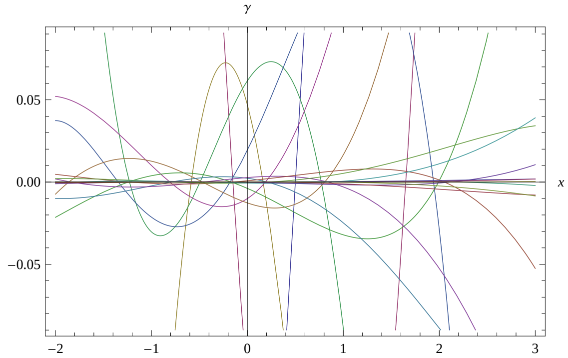

Figure 4.1. Shape of $\mathfrak{G}_{20,0.5}(x)$

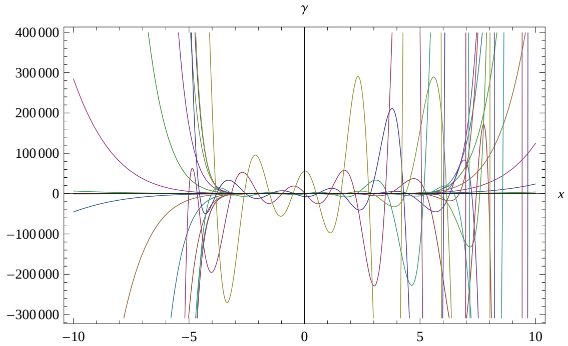

Figure 4.2. Shape of $\mathfrak{G}_{20,0.9}(x)$

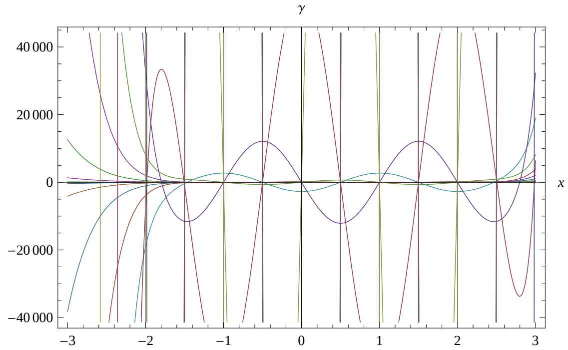

Figure 4.3. Shape of $\mathfrak{G}_{20,0.9999}(x)$

The roots of the $\mathfrak{G}_{n,q}(x)$, are plotted in figures 4.4, 4.5 and 4.6 for $n = 20$ and $q = \frac{1}{2}, \frac{9}{10}, 0.\overline{9}$, where $x \in C$.

In figures 4.4, 4.5 and 4.6, for $n = 20$, $q = 1/2, 0.9$ and 0.9999 $\mathfrak{G}_{n,q}(x)$, $x \in C$ have $\mathrm{Im}(x) = 0$ reflection symmetry.

In table 4.1, the real roots of $\mathfrak{G}_{n,q}(x)$, for $n = 20$ and $q = 0.5, 0.9$ and $0.\overline{9}$ are given.

Table 4.1. Approximate solutions of $\mathfrak{G}_{n,q}(x) = 0,\ x \in \mathbb{R}$

q	Real zeros of $\mathfrak{G}_{n,q}(x) = 0$ for $n = 20$
0.5	0.504495, 0.630159, 0.99995
0.9	-0.99901, 0.02681, 1.027078
$0.\overline{9}$	-2.357442, -1.5004759, -0.50045, 0.4995, 1.4995, 2.4995759, 3.3565

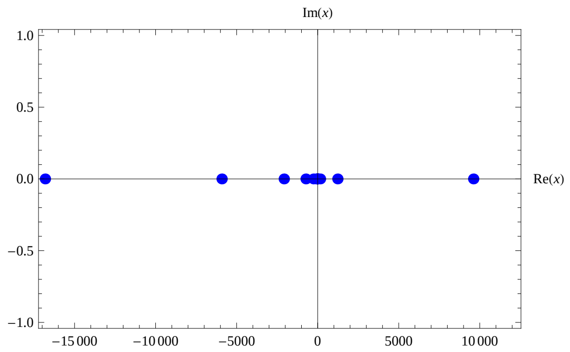

Figure 4.4. Zeros of $\mathfrak{G}_{20,0.5}(x)$

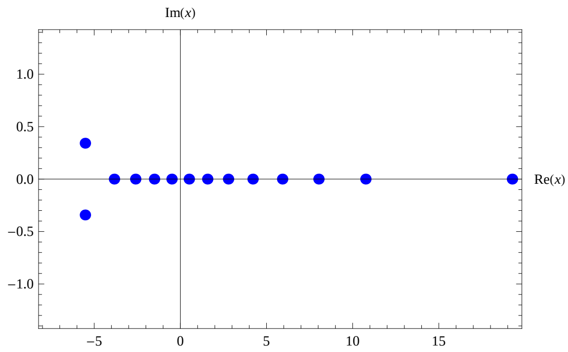

Figure 4.5. Zeros of $\mathfrak{G}_{20,0.9}(x)$

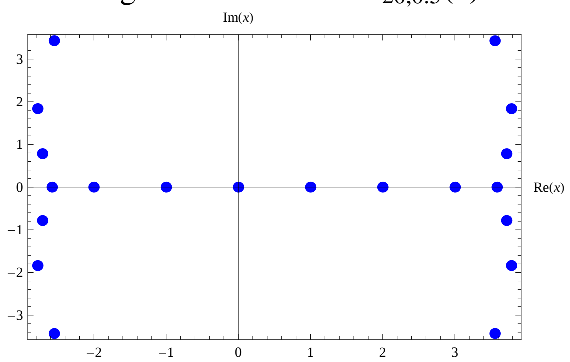

Figure 4.6. Zeros of $\mathfrak{G}_{20,0.9999}(x)$

Let n is the degree of $\mathfrak{G}_{n,q}(x)$, $RE_{\mathfrak{G}_{n,q}(x)}$ denotes the number of real roots and $CM_{\mathfrak{G}_{n,q}(x)}$ denotes the number of complex roots then we obtain following relationship:

$$n = RE_{\mathfrak{G}_{n,q}(x)} + CM_{\mathfrak{G}_{n,q}(x)},$$

See Table 4.1.

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