*Research Article*

# Quantum Probes of Timelike Naked Singularities in $2 + 1$-Dimensional Power-Law Spacetimes

## O. Gurtug, M. Halilsoy, and S. Habib Mazharimousavi

*Department of Physics, Eastern Mediterranean University, Gazimağusa, Northern Cyprus, Mersin 10, Turkey*

Correspondence should be addressed to S. Habib Mazharimousavi; habib.mazhari@emu.edu.tr

The formation of naked singularities in 2 + 1-dimensional power-law spacetimes in linear Einstein-Maxwell and Einstein-scalar theories sourced by azimuthally symmetric electric field and a self-interacting real scalar field, respectively, are considered in view of quantum mechanics. Quantum test fields obeying the Klein-Gordon and Dirac equations are used to probe the classical timelike naked singularities developed at $r = 0$. We show that when the classically singular spacetimes probed with scalar waves, the considered spacetimes remain singular. However, the spinorial wave probe of the singularity in the metric of a self-interacting real scalar field remains quantum regular. The notable outcome in this study is that the quantum regularity/singularity cannot be associated with the energy conditions.

## 1. Introduction

In recent years, general relativity in 2+1-dimensions has been one of the attractive arenas for understanding the general aspects of black holes physics. The main motivation of this attraction is the existing tractable mathematical structure when compared with the higher dimensional counterparts. The preliminary works in this field were popularized by the black hole solution of Banados-Teitelboim-Zanelli (BTZ), a spacetime sourced by a negative cosmological constant [1–3]. Extension of 2+1-dimensional solutions to Einstein-Maxwell (EM) cases followed in [4] and its massive gravity version is given in [5]. The static and rotating charged black hole in 2 + 1-dimensional Brans-Dicke theory was studied in [6] and rotating black holes with torsion were considered in [7]. The 2 + 1-dimensional charged black hole with nonlinear electrodynamic coupled to gravity has been studied in [8] and with a scalar hair has been given in [9]. The peculiar feature in the aforementioned studies in the EM theory is that the electric field is considered in the radial direction.

In recent decades, the physical properties of the solutions presented in both linear and nonlinear electromagnetism have been investigated by researchers. The solutions admitting black holes are analyzed in terms of thermodynamical aspects such as temperature and entropy [3, 10, 11]. Furthermore, the AdS/CFT correspondence which relates thermal properties of black holes in the AdS space to a dual CFT is another important achievement of 2 + 1-dimensional gravity [12].

On the other hand, the solutions admitting naked singularities are not analyzed in detail and, hence, it requires further care as far as the cosmic censorship hypothesis is concerned. Therefore, the resolution of singularities becomes important not in 3 + 1-dimensional gravity, but also in both lower and higher dimensional gravity. Because of the scales where these singularities are forming, their resolution requires a consistent theory at these small scales. The theory of quantum gravity seems to be the most promising theory; however, it is still "under construction." An alternative method for resolving the singularities is proposed by Horowitz and Marolf (HM) [13] by developing the work of Wald [14]. According to this method, the classical notion of a curvature singularity that is regarded as *geodesics incompleteness* with respect to point particle probe is replaced by *quantum singularity* with respect to wave probes.

In this paper, our focus will be on the 2 + 1-dimensional power-law spacetimes. Spherically symmetric power-law

metrics for dimensions $n \geq 4$ has been investigated in view of quantum mechanics by Blau, Frand, and Weiss (BFW) in [15], by employing the method of HM. The formation of naked singularities in 2-parameter family of

$$ds^2 = \eta x^p \left(-dx^2 + dy^2\right) + x^q d\Omega_d^2, \tag{1}$$

Szekeres-Iyer [16–18], metrics is probed with scalar field. It has been shown that the timelike naked singularity at $x = 0$ for $\eta = -1$ satisfying the dominant energy condition (DEC) is quantum mechanically singular in the sense of the HM criterion.

Another study in line with power-law spacetimes was considered by Helliwell and Konkowski (HK) in [19]. HK considered cylindrically symmetric four-parameter power-law metrics in 3 + 1-dimension in the form of

$$ds^2 = -r^\alpha dt^2 + r^\beta dr^2 + \frac{1}{C^2} r^\gamma d\theta^2 + r^\delta dz^2, \tag{2}$$

in the limit of small $r$, where $\alpha$, $\beta$, $\gamma$, $\delta$, and $C$ are constant parameters. HK classified the metric (2) as Type I, if $\alpha = \beta$, and given by

$$ds^2 = r^\beta \left(-dt^2 + dr^2\right) + \frac{1}{C^2} r^\gamma d\theta^2 + r^\delta dz^2. \tag{3}$$

According to the analysis of HK, a large set of classically singular spacetimes emerges quantum mechanically non-singular, if it is probed with scalar waves having nonzero azimuthal quantum number $m$ and axial quantum number $k$ in the sense of HM criterion. HK have also argued the possible relation with the energy conditions that can be used to eliminate the quantum singular spacetimes.

In 2 + 1-dimensional gravity, the method of HM has been used in the following works to probe the timelike naked curvature singularities: the BTZ black hole is considered in [20]. The EM extension of BTZ black hole both in linear and nonlinear theory and in Einstein-Maxwell dilaton theory is considered in [21]. The formation of naked singularities for a magnetically charged solution in Einstein-Power-Maxwell theory is considered in [22]. Occurrence of naked singularities in Einstein-nonlinear electrodynamics with circularly symmetric electric field is considered in [23]. In these studies, the timelike naked singularity is probed with waves that differs in spin structure. Namely, the bosonic and the fermionic waves are used that obey the Klein-Gordon and the Dirac equation, respectively. The common outcome in these studies is that the naked singularity remains quantum singular when it is probed with bosonic waves. However, probing the singularity with fermionic waves has revealed that only the magnetically charged solution in Einstein-Power-Maxwell theory is singular. The other spacetimes considered so far behave as quantum regular against fermionic waves.

To our knowledge, the analysis of power-law metrics in 2 + 1-dimension has not been considered so far. This fact will be the main motivation for the present study. The solutions admitting timelike naked singularities in the linear Einstein-Maxwell (EM) [24] and Einstein-scalar (ES) [25] theories sourced by azimuthally symmetric electric field and a self-interacting real scalar field, respectively, will be investigated within the framework of quantum mechanics. The peculiar feature of both solutions is that they admit metrics in power-law form in 2 + 1-dimensional gravity.

The solution in linear EM theory with azimuthally symmetric electric field in 2 + 1-dimension was given in [24]. To our knowledge, the singularity structure of the solution presented in [24] has not been studied so far. We are aiming in this study to investigate the solution admitting naked singularity in [24]. In our analysis, the classical naked singularity will be probed with quantum fields obeying the massless Klein-Gordon and Dirac equations. We showed that against both probes the spacetime remains quantum mechanically singular. This happens in spite of the fact that the weak, strong, and dominant energy conditions are manifestly satisfied.

The electric field component in EM extensions was considered to be radial so far while the possibility of a circular electric field went unnoticed. Recall from the Maxwell equations, $\nabla \times \mathbf{E} \sim \partial \mathbf{B}/\partial t$ and $\nabla \times \mathbf{B} \sim \partial \mathbf{E}/\partial t$, that the possibility of a constant $\mathbf{E}$ ($= E_0 = F_{t\theta} =$ constant) and gradient form of $\mathbf{B}$ (i.e., $\mathbf{B} = \nabla b$, for $b$ a scalar function, independent of time) may occur. When confined to 2 + 1-dimension such $\mathbf{E}$ and $\mathbf{B}$ satisfy Maxwell's equations trivially with singularity due to a physical source at $r = 0$. It is this latter case that we wish to point out and investigate in this paper.

The paper is organized as follows. In Section 2, the solutions obtained in [24] and in [25] are reviewed and the structure of the resulting spacetimes is briefly introduced. In Section 3, the definition of quantum singularity is summarized and the timelike naked singularity in the considered spacetimes is analyzed with quantum fields obeying the Klein-Gordon and Dirac equations. The paper is concluded with a conclusion in Section 4.

## 2. Review of the Solutions Admitting Power-Law Metrics in $(2+1)$-Dimension

*2.1. Rederivation of the Linear Einstein-Maxwell Solution with Azimuthally Symmetric Electric Field.* We start with the Einstein-Maxwell action given by

$$I = \frac{1}{2} \int d^3 x \sqrt{-g} \left(R - \mathscr{F}\right) \tag{4}$$

in which $R$ is the Ricci scalar and $\mathscr{F} = F_{\mu\nu} F^{\mu\nu}$ is the Maxwell invariant with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The circularly symmetric line element is given by

$$ds^2 = -A(r) dt^2 + \frac{1}{B(r)} dr^2 + r^2 d\theta^2, \tag{5}$$

where $A(r)$ and $B(r)$ are unknown functions of $r$ and $0 \leq \theta \leq 2\pi$. The electric field ansatz is chosen to be normal to radial direction and uniform; that is,

$$\mathbf{F} = E_0 dt \wedge d\theta \tag{6}$$

in which $E_0 =$ constant [23]. The dual field is found as $^\star\mathbf{F} = (E_0/r)\sqrt{B/A} dr$. It is known that the integral of $^\star\mathbf{F}$ gives the total charge. Let us note that even in a flat space with

$A = B = 1$ we obtain a logarithmic expression for the charge; that is, $Q(r) \sim \ln r$. This electric field is derived from an electric potential one-form given by

$$\mathbf{A} = E_0 \left( a_0 t d\theta - b_0 \theta dt \right), \tag{7}$$

in which $a_0$ and $b_0$ are constants satisfying $a_0 + b_0 = 1$. Maxwell's equation,

$$d \left( {}^{*}\mathbf{F} \right) = 0, \tag{8}$$

is trivially satisfied. Note that the invariant of electromagnetic field is given by

$$\mathcal{F} = 2F_{t\theta}F^{t\theta} = \frac{-2E_0^2}{A(r) r^2}. \tag{9}$$

Next, the Einstein-Maxwell equations are given by

$$G_\mu^\nu = T_\mu^\nu \tag{10}$$

in which

$$T_\nu^\mu = -\frac{1}{2} \left( \delta_\nu^\mu \mathcal{F} - 4F_{\nu\lambda}F^{\mu\lambda} \right). \tag{11}$$

Having $\mathcal{F}$ known one finds

$$T_t^t = T_\theta^\theta = \frac{1}{2}\mathcal{F},$$
$$T_r^r = -\frac{1}{2}\mathcal{F} \tag{12}$$

as the only nonvanishing energy-momentum components. To proceed further, we must have the exact form of the Einstein tensor components given by

$$G_t^t = \frac{B'}{2r},$$
$$G_r^r = \frac{BA'}{2rA}, \tag{13}$$
$$G_\theta^\theta = \frac{2A''AB - A'^2B + A'B'A}{4A^2},$$

in which a "prime" means $d/dr$. The field equations then read as follows:

$$\frac{B'}{2r} = \frac{1}{2}\mathcal{F},$$
$$\frac{BA'}{2rA} = -\frac{1}{2}\mathcal{F}, \tag{14}$$
$$\frac{2A''AB - A'^2B + A'B'A}{4A^2} = \frac{1}{2}\mathcal{F}.$$

The above field equations admit the following solutions for $A$ and $B$:

$$A = \frac{1}{B} = \chi r^{2E_0^2} \tag{15}$$

in which $\chi > 0$ is an integration constant and without loss of generality we set it to $\chi = 1$. Hence the line element becomes

$$ds^2 = r^{2E_0^2} \left( -dt^2 + dr^2 \right) + r^2 d\theta^2. \tag{16}$$

This is a black point solution with the horizon at the origin which is the singular point of the spacetime with Kretschmann scalar:

$$\mathcal{K} = \frac{12E_0^4}{r^{4(1+E_0^2)}}. \tag{17}$$

The solution has a single parameter which is the electric field $E_0$. Setting $E_0 = 0$ makes the solution the $(2+1)$-dimensional flat spacetime. Note also that for the choice $E_0 = 1$, from (16), we obtain a conformally flat metric with conformal factor $r^2$. It is observed that the strength of $E_0$ serves to increase the degree of divergence of the scalar curvature. Based on our energy momentum tensor components one finds that the energy density and the radial and tangential pressures are given by

$$\rho = -T_t^t = \frac{E_0^2}{r^{2(1+E_0^2)}},$$
$$p = T_r^r = \rho = \frac{E_0^2}{r^{2(1+E_0^2)}}, \tag{18}$$
$$q = T_\theta^\theta = -\rho = -\frac{E_0^2}{r^{2(1+E_0^2)}}.$$

Therefore the weak energy conditions (WEC), that is, (i) $\rho \geq 0$, (ii) $\rho + p \geq 0$, and (iii) $\rho + q \geq 0$, all are satisfied. The strong energy conditions are also satisfied, that is, the WECs together with (iv) $\rho + p + q \geq 0$. Dominant energy condition (DEC), that is, $p_{\mathrm{eff}} \geq 0$, and causality condition (CC), that is, $0 \leq p_{\mathrm{eff}} \leq 1$, are also easily satisfied knowing that $p_{\mathrm{eff}} = (p+q)/2 = 0$.

*2.2. Exact Radial Solution to $(2+1)$-Dimensional Gravity Coupled to a Self-Interacting Real Scalar Field.* Exact radial solution with a self-interacting, real, scalar field coupled to the $(2+1)$-dimensional gravity is given by Schmidt and Singleton in [25]. The action representing $(2 + 1)$-dimensional gravity with a self-interacting scalar field is given by

$$I = \frac{1}{2\kappa} \int d^3x \sqrt{-g} \left( R + \mathcal{L}_S \right) \tag{19}$$

in which $\kappa = 8\pi G$ is the coupling constant, $G$ denotes the Newton's constant, $R$ Ricci scalar, and $\mathcal{L}_S$ represents the Lagrangian of the self-interacting scalar field $\phi$ given by

$$\mathcal{L}_S = -2\partial_\mu \phi \partial^\mu \phi - V(\phi). \tag{20}$$

The potential $V(\phi)$ is a Liouville potential described by

$$V(\phi) = e^{-\sqrt{\kappa}\phi}. \tag{21}$$

With reference to [25], the metric and the scalar field are found as

$$ds^2 = -Kr^2dt^2 + dr^2 + r^2d\theta^2, \tag{22}$$

$$\phi(r) = \frac{1}{\sqrt{\kappa}}\ln(r), \tag{23}$$

where $K$ is a constant parameter. The only nonvanishing energy-momentum tensor component is the radial pressure component, $T_{rr} = 1/\kappa r^2$. The Kretschmann scalar is given by $\mathscr{K} = 4/r^4$, which indicates a scalar curvature singularity at $r = 0$. This solution has no horizon and, hence, the singularity at $r = 0$ is a timelike naked singularity.

According to the energy-momentum tensor components one finds that the energy density and the radial and tangential pressures are given by

$$\rho = 0,$$

$$p = \frac{1}{r^2}, \tag{24}$$

$$q = 0.$$

As a consequence, the weak energy conditions (WEC), that is, (i) $\rho \geq 0$, (ii) $\rho + p \geq 0$, and (iii) $\rho + q \geq 0$, the strong energy conditions (SEC), that is, the WECs together with (iv) $\rho + p + q \geq 0$, are all satisfied. The DEC, $p_{\text{eff}} = (p + q)/2 \geq 0$ is also satisfied.

## 3. Singularity Analysis

It has been known that the spacetime singularities inevitably arise in Einstein's theory of relativity. It describes the "end point" or incomplete geodesics for timelike or null trajectories followed by classical particles. Among the others, naked singularities which is visible from outside needs further care as far as the weak cosmic censorship hypothesis is concerned. It is believed that naked singularity forms a threat to this hypothesis. As a result of this, understanding and the resolution of naked singularities seems to be extremely important for the deterministic nature of general relativity. The general belief in the resolution of the singularities is to employ the methods imposed by the quantum theory of gravity. However, the lack of a consistent quantum gravity leads the researchers to alternative theories in this regard. String theory [26, 27] and loop quantum gravity [28] constitute two major study fields in resolving singularities. Another alternative method, following the work of Wald [14], was proposed by Horowitz and Marolf (HM) [13], which incorporates the "self-adjointness" of the spatial part of the wave operator. Hence, the classical notion of *geodesics incompleteness* with respect to point-particle probe will be replaced by the notion of *quantum singularity* with respect to wave probes.

In this paper, the method proposed by HM will be used in analyzing the naked singularity. This method in fact has been used successfully in $(3 + 1)$ and higher dimensional spacetimes. Ishibashi and Hosoya [29] have studied several spacetimes by employing Sobolev space instead of the

usual Hilbert space. HK have considered quasiregular [30], Gal'tsov-Letelier-Tod spacetimes [31], Levi-Civita spacetimes [32, 33], and conformally static spacetimes [34, 35]. Pitelli and Letelier have considered spherical and cylindrical topological defects [36], the global monopole spacetime [37], cosmological spacetime [38], and Horava-Lifshitz cosmology [39]. Mazharimousavi and his collaborators have considered Lovelock theory [40], linear dilaton black hole spacetimes [41], model of $f(R)$ gravity [42], and weak field regime of $f(R)$ global monopole spacetimes [43]. The main purpose in these studies is to understand whether these classically singular spacetimes turn out to be quantum mechanically regular if they are probed with quantum fields rather than classical particles. The idea is in analogy with the fate of a classical atom in which the electron should plunge into the nucleus but rescued with quantum mechanics. The main concepts of this method which can be applied only to static spacetimes having timelike singularities are summarized briefly as follows.

Let us consider the Klein-Gordon equation for a free particle that satisfies $i(d\psi/dt) = \sqrt{\mathscr{A}_E}\psi$, whose solution is $\psi(t) = \exp[-it\sqrt{\mathscr{A}_E}]\psi(0)$ in which $\mathscr{A}_E$ denotes the extension of the spatial part of the wave operator. If the wave operator $\mathscr{A}$ is not essentially self-adjoint, in other words if $\mathscr{A}$ has an extension, the future time evolution of the wave function $\psi(t)$ is ambiguous. Then, the HM method defines the spacetime as quantum mechanically singular. However, if there is only a single self-adjoint extension, the wave operator $\mathscr{A}$ is said to be essentially self-adjoint and the quantum evolution described by $\psi(t)$ is uniquely determined by the initial conditions. According to the HM method, this spacetime is said to be quantum mechanically nonsingular. The essential self-adjointness of the operator $\mathscr{A}$ can be verified by using the deficiency indices and the Von Neumann's theorem that considers the solutions of the equation

$$\mathscr{A}^*\psi \pm i\psi = 0, \tag{25}$$

and showing that the solutions of (25) does not belong to Hilbert space $\mathscr{H}$. (We refer to; [29, 44–46] for detailed mathematical analysis.)

### 3.1. Quantum Probes of the Linear Einstein-Maxwell Solution with Azimuthally Symmetric Electric Field

*3.1.1. Klein-Gordon Fields.* The massless Klein-Gordon equation for a scalar wave can be written as

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\right]\psi = 0. \tag{26}$$

The Klein-Gordon equation can be written for the metric (16), by splitting the temporal and spatial part as

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + r^{2(E_0^2-1)}\frac{\partial^2\psi}{\partial\theta^2}. \tag{27}$$

This equation can also be written as

$$\frac{\partial^2\psi}{\partial t^2} = -\mathscr{A}\psi, \tag{28}$$

where $\mathscr{A}$ is the spatial operator given by

$$\mathscr{A} = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - r^{2(E_0^2-1)}\frac{\partial^2}{\partial \theta^2}. \tag{29}$$

And, according to the HM method, it is subjected to be investigated whether its self-adjoint extensions exist or not. This is achieved by assuming a separable solution to (25) in the form of $\psi(r,\theta) = R(r)Y(\theta)$, which yields the radial equation as

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r} + \left(cr^{2(E_0^2-1)} \pm i\right)R(r) = 0, \tag{30}$$

with $c$ the separation constant. The essential self-adjointness of the spatial operator $\mathscr{A}$ requires that neither of the two solutions of the above equation is square integrable over all space $L^2(0,\infty)$. The square integrability of the solution of the above equation for each sign $\pm$ is checked by calculating the squared norm of the obtained solution in which the function space on each $t = $ constant hypersurface $\Sigma$ is defined as $\mathscr{H} = \{R \mid \|R\| < \infty\}$. The squared norm for the metric (2) is given by

$$\|R\|^2 = \int_0^\infty \frac{|R_\pm(r)|^2\, r}{\sqrt{A(r)B(r)}}dr. \tag{31}$$

The squared norm is investigated for three different cases of the value of electric field $E_0$.

*Case 1* ($E_0^2 = 1$). For this particular case, (30) transforms to

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r} + (c \pm i)R(r) = 0, \tag{32}$$

whose solution is given by

$$R(r) = a_1 J_0\left(\sqrt{\hat{c}}\,r\right) + a_2 N_0\left(\sqrt{\hat{c}}\,r\right), \tag{33}$$

such that $\hat{c} = c \pm i$.

*Case 2* ($0 < E_0^2 < 1$). In this case, (30) becomes

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r} + \frac{c}{r^{2(1-E_0^2)}}R(r) = 0, \tag{34}$$

and the solution is given by

$$R(r) = a_3 J_0\left(\frac{\sqrt{c}}{E_0^2}r^{E_0^2}\right) + a_4 N_0\left(\frac{\sqrt{c}}{E_0^2}r^{E_0^2}\right). \tag{35}$$

*Case 3* ($E_0^2 > 1$). For this choice of $E_0$, (30) remains the same and the solution is given by

$$R(r) = a_5 J_0\left(\frac{\sqrt{c}}{E_0^2}r^{E_0^2}\right) + a_6 N_0\left(\frac{\sqrt{c}}{E_0^2}r^{E_0^2}\right). \tag{36}$$

Note that $J_0(x)$ and $N_0(x)$ in (33), (35), and (36) are the first kind of Bessel and Neumann functions, respectively, with integration constants $a_i$ in which $i = 1 \cdots 6$.

Our calculations have revealed that, in general, in each case for appropriate $a_i$, the squared norm $\|R\|^2 < \infty$, which is always square integrable. Hence, the spatial part of the operator is not essentially self-adjoint for all space $L^2(0,\infty)$. Therefore, the classical singularity at $r = 0$ remains quantum singular as well when probed with massless scalar, bosonic waves.

It is remarkable to note that the considered spacetime in this paper is in the form of a power-law metrics that can be expressed as

$$ds^2 = r^{2E_0^2}\left(-dt^2 + dr^2\right) + r^2 d\theta^2, \tag{37}$$

which satisfies all energy conditions.

Quantum singularity structure of the four-parameter, power-law metrics in $3 + 1$-dimensional cylindrically symmetric spacetime has already been considered by Helliwell and Konkowski (HK) in [19]. According to the classification presented in [19], the metric (28) is Type I with the only parameters $\beta = 2E_0^2$ and $\gamma = 2$ such that $C = 1$. In our metric the parameter $\beta$ is related to the intensity of the constant electric field $E_0$ so that $\beta = 2E_0^2 > 0$. Hence, $\beta < 0$ is not allowed in our study. Furthermore, the absence of an extra dimension in our study does not provide a wave component with a mode $k$ that has a crucial effect in the study performed in [19]. HK have shown that a large set of classically singular spacetimes emerges quantum mechanically nonsingular, if it is probed with waves having non-zero azimuthal quantum number $m$ and axial quantum number $k$. The presence of an extra dimension in [19], with a parameter $\delta$, is closely related to the geometry of the spacetime and, hence, the considered spacetime in this paper is different. It is also important to note that the presence of the electric field in our study increases the rate of divergence in scalar curvature. This feature amounts to increase the strength of the naked singularity. As a result, unlike the case considered in [19], the classical naked singularity remains quantum singular with respect to bosonic wave (spin 0) probe.

*3.1.2. Dirac Fields.* In $2 + 1$-dimensional curved spacetimes, the formalism leading to a solution of the Dirac equation was given in [47]. This formalism has been used in [20] and in our earlier studies [21–23]. The Dirac equation in $2 + 1$-dimensional curved background for a free particle with mass $m$ is given by

$$i\sigma^\mu(x)\left[\partial_\mu - \Gamma_\mu(x)\right]\Psi(x) = m\Psi(x), \tag{38}$$

where $\Gamma_\mu(x)$ is the spinorial affine connection given by

$$\Gamma_\mu(x) = \frac{1}{4}g_{\lambda\alpha}\left[e_{\nu,\mu}^{(i)}(x)e_{(i)}^\alpha(x) - \Gamma_{\nu\mu}^\alpha(x)\right]s^{\lambda\nu}(x),$$
$$s^{\lambda\nu}(x) = \frac{1}{2}\left[\sigma^\lambda(x), \sigma^\nu(x)\right]. \tag{39}$$

Since the fermions have only one spin polarization in $2 + 1$-dimension, the Dirac matrices $\gamma^{(j)}$ can be given in terms of Pauli spin matrices $\sigma^{(i)}$ so that

$$\gamma^{(j)} = \left(\sigma^{(3)}, i\sigma^{(1)}, i\sigma^{(2)}\right), \tag{40}$$

where the Latin indices represent internal (local) frame. In this way,

$$\{\gamma^{(i)}, \gamma^{(j)}\} = 2\eta^{(ij)} I_{2\times 2}, \tag{41}$$

where $\eta^{(ij)}$ is the Minkowski metric in $2+1$-dimension and $I_{2\times 2}$ is the identity matrix. The coordinate dependent metric tensor $g_{\mu\nu}(x)$ and matrices $\sigma^{\mu}(x)$ are related to the triads $e_{\mu}^{(i)}(x)$ by

$$g_{\mu\nu}(x) = e_{\mu}^{(i)}(x)\, e_{\nu}^{(j)}(x)\, \eta_{(ij)},$$
$$\sigma^{\mu}(x) = e_{(i)}^{\mu} \gamma^{(i)}, \tag{42}$$

where $\mu$ and $\nu$ stand for the external (global) indices. The suitable triads for the metric (16) are given by

$$e_{\mu}^{(i)}(t, r, \theta) = \mathrm{diag}\left(r^{E_0^2}, r^{E_0^2}, r\right). \tag{43}$$

With reference to our earlier studies in [21–23], the following ansatz is used for the positive frequency solutions:

$$\Psi_{n,E}(t, x) = \begin{pmatrix} R_{1n}(r) \\ R_{2n}(r)\, e^{i\theta} \end{pmatrix} e^{in\theta} e^{-iEt}. \tag{44}$$

The radial part of the Dirac equations governing the propagation of the fermionic waves should be examined for a unique self-adjoint extensions for all space $L^2(0, \infty)$. In doing this, the possible values of the electric field intensity $E_0$ should also be taken into consideration. To consider all these, the behavior of the solution of the radial part of the Dirac equation $R_{in}(r)$ ($i = 1, 2$) will be investigated near $r \to 0$, and $r \to \infty$.

*(a) The Case of $r \to 0$.* The behavior of the radial part of the Dirac equation for $E_0^2 > 1$ is given by

$$R_{in}''(r) - \frac{\alpha_1}{r} R_{in}'(r) + \frac{\alpha_1(\alpha_1 + 2)}{4r^2} R_{in}(r) = 0,$$
$$i = 1, 2, \tag{45}$$

in which $\alpha_1 = E_0^2 - 1$, and the solution is

$$R_{in}(r) = b_1 r^{\alpha_1/2 + 1} + b_2 r^{\alpha_1/2}, \quad i = 1, 2. \tag{46}$$

For $E_0^2 = 1$ case, the Dirac equation simplifies to

$$R_{in}''(r) + R_{in}'(r) - \left[E^2 + n(n+1)\right] R_{in}(r) = 0,$$
$$i = 1, 2 \tag{47}$$

and the solution is given by

$$R_{in}(r) = b_3 e^{r(-1+\sqrt{1+4\xi})/2} + b_4 e^{-r(1+\sqrt{1+4\xi})/2}, \quad i = 1, 2 \tag{48}$$

in which $\xi = E^2 + n(n+1)$ and $E$ is the energy of the Dirac particle.

For $0 < E_0^2 < 1$ case, the Dirac equation becomes

$$R_{in}''(r) + \frac{\beta_0}{r} R_{in}'(r) + \frac{\beta_0(\beta_0 + 2)}{4r^2} R_{in}(r) = 0,$$
$$i = 1, 2 \tag{49}$$

in which $\beta_0 = 1 - E_0^2$, and the solution is

$$R_{in}(r) = b_5 r^{(E_0^2 + \sqrt{1-4\beta_0})/2} + b_6 r^{(E_0^2 - \sqrt{1-4\beta_0})/2}, \quad i = 1, 2 \tag{50}$$

such that for a real solution the electric field intensity is bounded to $3/4 \leq E_0^2 < 1$. Note that a prime in (45), (47), and (49) denotes a derivative with respect to $r$.

The square integrability of the above solutions corresponding to different values of electric field intensity $E_0$ is checked by calculating the squared norm given in (31). Calculations have indicated that irrespective of the integration constants $b_k$ ($k = 1 \cdots 6$) and the electric field intensity $E_0$, all the solutions obtained near $r \to 0$ are square integrable; that is to say $\|R_{in}\|^2 < \infty$.

*(b) The Case of $r \to \infty$.* For the asymptotic case, the radial part of the Dirac equation for the electric field intensity $E_0^2 > 1$ is given by

$$R_{in}''(r) + r^{\alpha_1} R_{in}'(r) - n(n+1) r^{2\alpha_1} R_{in}(r) = 0,$$
$$i = 1, 2, \tag{51}$$

whose solution is given by

$$R_{in}(r) = b_7 e^{-r^{E_0^2(\sqrt{1+4\eta}+1)}/2E_0^2}$$
$$\times \mathrm{KM}\left(\frac{(\alpha_1 + 2)\sqrt{1+4\eta} + \alpha_1}{2E_0^2\sqrt{1+4\eta}}, \frac{\alpha_1 + 2}{E_0^2}, \right.$$
$$\left.\frac{\sqrt{1+4\eta}\, r^{E_0^2}}{E_0^2}\right) r + b_8 e^{-r^{E_0^2(\sqrt{1+4\eta}+1)}/2E_0^2}$$
$$\times \mathrm{KU}\left(\frac{(\alpha_1 + 2)\sqrt{1+4\eta} + \alpha_1}{2E_0^2\sqrt{1+4\eta}}, \frac{\alpha_1 + 2}{E_0^2}, \right.$$
$$\left.\frac{\sqrt{1+4\eta}\, r^{E_0^2}}{E_0^2}\right) r \tag{52}$$

in which $\eta = n(n+1)$, KM and KU stand for KummerM and KummerU.

For $E_0^2 = 1$, the behavior of the Dirac equation is

$$R_{in}''(r) + R_{in}'(r) - mr^2 R_{in}(r) = 0, \quad i = 1, 2, \tag{53}$$

in which $m$ is the mass of the Dirac particle which is taken to be unity for practical reasons and the solution is given in terms of Kummer function as

$$R_{in}(r) = b_9 \mathrm{KummerM}\left(\frac{13}{16}, \frac{3}{2}, r^2\right) r e^{-r(1+r)/2}$$
$$+ b_{10} \mathrm{KummerU}\left(\frac{13}{16}, \frac{3}{2}, r^2\right) r e^{-r(1+r)/2}. \tag{54}$$

For $0 < E_0^2 < 1$ case, the Dirac equation becomes

$$R''_{in}(r) - mr^{2(1-\beta_0)}R_{in}(r) = 0, \quad i = 1, 2, \tag{55}$$

and the solution for $m = 1$ and $\beta_0 = 1/2$ is given by

$$R_{in}(r) = b_{11}\sqrt{r}I_{1/3}(x) + b_{12}\sqrt{r}K_{1/3}(x) \tag{56}$$

in which $I_{1/3}(x)$ and $K_{1/3}(x)$ are the first and second kind modified Bessel functions and $x = -2r^{3/2}/3$.

The obtained solutions in asymptotic case $r \to \infty$, for three different electric field intensities $E_0$, are checked for a square integrability. Our calculations have revealed that the solutions for $b_8 = b_{10} = 0$ and $b_7 \neq 0$, $b_9 \neq 0$ together with $b_{11} \neq 0$ and $b_{12} \neq 0$, the squared norm $\|R_{in}\|^2 \to \infty$, indicating that the solutions do not belong the Hilbert space.

From this analysis, we conclude that the radial part of the Dirac operator is not essentially self-adjoint for all space $L^2(0, \infty)$, and, therefore, the formation of the classical timelike naked singularity in the presence of the azimuthally symmetric electric field in $2 + 1$-dimensional geometry remains quantum mechanically singular even if it is probed with spinorial fields.

### 3.2. Quantum Probes of the Radial Solution to $(2 + 1)$-Dimensional Gravity Coupled to a Self-Interacting Real Scalar Field

#### 3.2.1. Klein-Gordon Fields.
The massless Klein-Gordon equation for the metric (22) after separating the temporal and spatial parts can be written as

$$\frac{\partial^2 \psi}{\partial t^2} = -\mathscr{A}\psi, \tag{57}$$

in which the spatial operator $\mathscr{A}$ is given by

$$\mathscr{A} = -K\left(r^2\frac{\partial^2}{\partial r^2} + 2r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2}\right). \tag{58}$$

As a requirement of the HM criterion, the spatial operator $\mathscr{A}$ should be investigated for unique self-adjoint extensions. Hence, it is required to look for a separable solution to (57), in the form of $\psi(r, \theta) = R(r)Y(\theta)$ which gives the radial solution as

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{2}{r}\frac{\partial R(r)}{\partial r} + \frac{1}{r^2}\left(c \pm \frac{i}{K}\right)R(r) = 0, \tag{59}$$

in which $c$ stands for the separation constant. The essential self-adjointness of the spatial operator $\mathscr{A}$ requires that neither of the two solutions of the above equation is square integrable over all space $L^2(0, \infty)$. The square integrability of the solution of the above equation for each sign $\pm$ is checked by calculating the squared norm of the obtained solution in which the function space on each $t = $ constant hypersurface $\Sigma$ is defined as $\mathscr{H} = \{R \mid \|R\| < \infty\}$. The squared norm for the metric (22) is given by

$$\|R\|^2 = \int_0^\infty \sqrt{\frac{g_{rr}g_{\theta\theta}}{g_{tt}}}|R_\pm(r)|^2 \, dr \simeq \int_0^\infty |R_\pm(r)|^2 \, dr. \tag{60}$$

We first consider the case when $r \to 0$. In this limiting case (59) simplifies to

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r^2}\left(c \pm \frac{i}{K}\right)R(r) = 0, \tag{61}$$

whose solution is

$$R(r) = C_1 r^{\delta_1} + C_2 r^{\delta_2}, \tag{62}$$

in which

$$\delta_1 = \frac{1}{2}\left(1 + \sqrt{1 - 4v}\right),$$

$$\delta_1 = \frac{1}{2}\left(1 - \sqrt{1 - 4v}\right), \tag{63}$$

$$v = c \pm \frac{i}{K}.$$

The above solution is checked for square integrability with the norm given in (60). Our analysis have shown that for specific mode of solution (depending on the value of $v$) the squared norm diverges; that is, $\|R\|^2 \to \infty$, indicating that the solution does not belong to Hilbert space.

We consider also the case when $r \to \infty$, so that (59) approximates to

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{2}{r}\frac{\partial R(r)}{\partial r} = 0, \tag{64}$$

and its solution is given by

$$R(r) = C_3 + \frac{C_4}{r}, \tag{65}$$

in which $C_k$ ($k = 1\cdots4$) are the integration constants. The above solution is square integrable if and only if the integration constant $C_4 = 0$. As a consequence, although there exist some modes of solution that do not belong to the Hilbert space near $r \to 0$ and $r \to \infty$, the generic result is that the considered spacetime remains quantum singular against bosonic wave probe in view of quantum mechanics.

#### 3.2.2. Dirac Fields.
The steps demonstrated previously for the solution of Dirac equation are applied for the metric given in (37). The exact radial part of the Dirac equation is given by

$$R''_{1n}(r) + \frac{1}{r}\left(1 + \frac{E}{M\sqrt{K}r + E}\right)R'_{1n}(r)$$

$$+ \left[\frac{1}{r^2}\left(\frac{E^2}{K} - n^2\right) - \frac{En}{r^2\left(M\sqrt{K}r + E\right)} - M^2\right]$$

$$\cdot R_{1n}(r) = 0,$$

$$R''_{2n}(r) + \frac{1}{r}\left(1 - \frac{E}{M\sqrt{K}r - E}\right)R'_{2n}(r) \tag{66}$$

$$+ \left[\frac{1}{r^2}\left(\frac{E^2}{K} - (n+1)^2\right) - \frac{E(n+1)}{r^2\left(M\sqrt{K}r + E\right)}\right.$$

$$\left. - M^2\right]R_{2n}(r) = 0.$$

The solution to these equations should be investigated for a unique self-adjoint extensions for all space $L^2(0, \infty)$. This will be done by considering the behavior of the solution near $r \to 0$ and $r \to \infty$.

*(a) The Case When $r \to 0$.* The behavior of the radial part of the Dirac equation given in (66) when $r \to 0$ is

$$R_{1n}''(r) + \frac{2}{r}R_{1n}'(r) - \frac{En}{M\sqrt{K}r^3}R_{1n}(r) = 0,$$

$$R_{2n}''(r) + \frac{2}{r}R_{2n}'(r) - \frac{E(n+1)}{M\sqrt{K}r^3}R_{2n}(r) = 0, \tag{67}$$

whose solutions are given, respectively, by

$$R_{1n}(r) = \frac{d_1}{\sqrt{r}}J_1(x_1) + \frac{d_2}{\sqrt{r}}N_1(x_1),$$

$$R_{2n}(r) = \frac{d_3}{\sqrt{r}}J_1(x_2) + \frac{d_4}{\sqrt{r}}N_1(x_2), \tag{68}$$

where $d_k$ ($k = 1 \cdots 4$) are the integration constants, $J_1(x_i)$ and $N_1(x_i)$ are the Bessel and Neumann functions with order the 1 such that $x_1 = 2\sqrt{-\lambda_1/r}$ and $x_2 = 2\sqrt{-\lambda_2/r}$, $\lambda_1 = En/M\sqrt{K}$, $\lambda_2 = E(n+1)/M\sqrt{K}$. The square integrability of these solutions is checked with the norm given in (60). The outcome of our analysis is that none of the obtained solutions belong to the Hilbert space. In other words the squared norm $\|R\|^2 \to \infty$.

*(b) The Case When $r \to \infty$.* The behavior of the radial part of the Dirac equation given in (66) when $r \to \infty$ is

$$R_{1n}''(r) + \frac{1}{r}R_{1n}'(r) - \frac{En}{M\sqrt{K}r^3}R_{1n}(r) = 0,$$

$$R_{2n}''(r) + \frac{2}{r}R_{2n}'(r) - \frac{E(n+1)}{M\sqrt{K}r^3}R_{2n}(r) = 0, \tag{69}$$

whose solutions are given, respectively, by

$$R_{in}(r) = l_i J_0(x_3) + l_i N_0(x_3), \quad i = 1, 2 \tag{70}$$

in which $l_i$ ($i = 1, 2$) are the integration constants and $J_0(x_3)$ and $N_0(x_3)$ are the Bessel and Neumann functions with order 0 such that $x_3 = \sqrt{-M}r$. The analysis for square integrability has shown that this solution is not square integrable. Alternatively, $\|R\|^2 \to \infty$.

As a result of this analysis, the radial part of the Dirac operator on this spacetime is essentially self-adjoint and, therefore, the timelike naked singularity in the $(2 + 1)$-dimensional gravity sourced by a real scalar field is quantum mechanically wave regular. This result indicates that the spin of the wave is effective in healing the singularity.

## 4. Conclusion

In this paper, the formation of timelike naked singularities in $2 + 1$-dimensional power-law spacetimes in linear Einstein-Maxwell and Einstein-scalar theories powered by azimuthally

symmetric electric field and a self-interacting real scalar field, respectively, are investigated from quantum mechanical point of view. Two types of waves with different spin structures are used to probe the timelike naked singularities that develops at $r = 0$.

We showed that the scalar (bosonic, spin 0) wave probe is not effective in healing the classical timelike naked singularities formed both in $2 + 1$-dimensional power-law metrics sourced by azimuthally symmetric electric field and a real scalar field. From the Kretschmann scalar given in (17), it is easy to observe that the azimuthally symmetric electric field $E_0$ serves to increase the rate of divergence and, hence, in some sense yields a stronger naked singularity at $r = 0$.

It is important to compare our study with that of HK in [19] and of BFW in [15], where the occurrence of timelike naked singularities is analyzed in quantum mechanical point of view. First, we compare with the work of HK. The metric considered by HK is a four-parameter power-law metrics in $3 + 1$-dimension, whose metric coefficients behave as power laws in the radial coordinate $r$, in the small $r$ approximation. The common point of our study with that of HK is that both metrics are in the power-law form. However, there is a significant difference between our metrics and that of HK. Although, the duality of the Maxwell field 2-form in $3 + 1$-dimension is still a 2-form, but in $2 + 1$-dimension, duality of the Maxwell field maps a 2-form into 1-form or vice versa. Another distinction is that the presence of an extra dimension in $3 + 1$-dimension, allows the mode solution in the wave equation to depend on the axial quantum number $k$, which has a crucial effect on healing the singularity. We may add also that $2 + 1$-dimensional spacetime is a brane in $3 + 1$-dimension and therefore dilution of the gravitational singularity in higher dimensions is not unexpected. In view of all these, the spacetimes considered in this paper and that of HK is topologically different. Another interesting result of this study is that all the energy conditions WEC, DEC, and SEC and the causality conditions are satisfied, but the spacetime is quantum singular. There was an attempt to relate the quantum regular/singular spacetimes with the energy conditions in [19]. The result obtained in our case indicates that eliminating the quantum singular spacetimes by just invoking the energy conditions is not a reliable method. This result confirms the comment made by HK in [19] that invoking energy conditions is not guaranteed physically.

On the other hand, the metric considered by BFW in [15] is a spherically symmetric power-law metrics with dimensions $n \geq 4$. Here, the timelike naked singularity is probed with scalar field and shows that the resulting spacetime is quantum singular. The possible connection with the energy conditions on the quantum resolution of the timelike naked singularity is also addressed in [15]. The general conclusion drawn for $n \geq 4$-dimensional spherically symmetric power-law metrics is that "*metrics with timelike singularities of power-law type satisfying the strict dominant energy condition remain singular when probed with scalar waves.*" This statement is in conformity with our results obtained in this study. As it was shown, the considered spacetimes in $(2 + 1)$-dimensional gravity (Einstein-Maxwell solution with azimuthally symmetric electric field and Einstein-scalar

solution with a self-interacting real scalar field) satisfies the DEC, and the classical timelike naked singularities in both metrics remain quantum singular when probed with scalar wave obeying the Klein-Gordon equation. What we believe is that the statement of BFW cannot be generalized for any wave probe.

A contradicting result to above statement is obtained for the exact radial solution with a self-interacting, real, scalar field coupled to the (2 + 1)-dimensional gravity. The timelike naked singularity is probed with Dirac field (fermionic, spin 1/2) that obeys the Dirac equation. We showed that the spatial radial part of the Dirac operator has a unique extension so that it is essentially self-adjoint. And as a result, the classical timelike naked singularity in the considered spacetime remains quantum regular when probed with fermions.

The notable result obtained in this paper and also in earlier studies along this direction has indicated that the quantum healing of the classically singular spacetime crucially depends on the wave that we probe the singularity. In order to understand the generic behavior of the 2 + 1-dimensional spacetimes, more spacetimes should be investigated. So far, vacuum Einstein, Einstein-Maxwell (both linear and nonlinear), and Einstein-Maxwell-dilaton solutions are investigated. As a future research, timelike naked singularities in 2 + 1-dimensional Einstein-scalar (minimally coupled) and Einstein-Maxwell-scalar solutions should also be investigated from quantum mechanical point of view. It will be interesting also to consider the spinorial wave generalization of the power-law metrics considered in [15, 19].
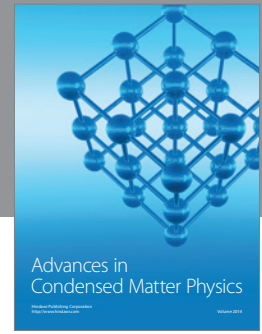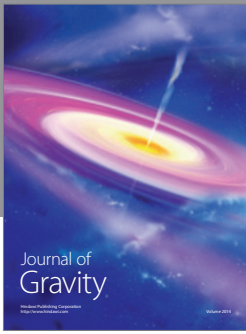
## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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