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‘Square root’ of the Maxwell Lagrangian versus confinement in general relativity

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ABSTRACT

We employ the ‘square root’ of the Maxwell Lagrangian (i.e. $\sqrt{F_{\mu\nu}F^{\mu\nu}}$), coupled with gravity to search for the possible linear potentials which are believed to play role in confinement. It is found that in the presence of magnetic charge no confining potential exists in such a model. Confining field solutions are found for radial geodesics in pure electrically charged Nariai–Bertotti–Robinson (NBR)-type spacetime with constant scalar curvature. Recently, Guendelman, Kaganovich, Nissimov and Pacheva (2011) [7] have shown that superposed square root with standard Maxwell Lagrangian yields confining potentials in spherically symmetric spacetimes with new generalized Reissner–Nordström–de Sitter/anti-de Sitter black hole solutions. In NBR spacetimes we show that confining potentials exist even when the standard Maxwell Lagrangian is relaxed.

A power-law extension of the Maxwell action coupled with gravity was considered by [1–3]

$$I=\frac{1}{2}\int dx^4\sqrt{-g}\big(R-2\Lambda-\alpha\mathcal{F}^s\big), \tag{1}$$

in which s and α are real constants, $\mathcal{F}=F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant with $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and Λ stands for the cosmological constant. The first study with this form of nonlinear electrodynamic (NED) was made in spherical symmetry and ever since many authors have considered different aspects/applications of this action [2]. Although the original paper [3] considered a conformally invariant action (i.e. $s=d/4$) this requirement was subsequently relaxed. It was shown that $s=1/2$ raised problems in connection with the energy conditions [4] and for this reason it was abandoned. Nielsen and Olesen [5] proposed such a magnetic ‘square root’ Lagrangian (i.e. $\sqrt{F_{\mu\nu}F^{\mu\nu}}$) in string theory while ’t Hooft [6] highlighted a linear potential term to be effective toward confinement. More recently Guendelman et al. [7] investigated confining electric potentials in black hole spacetimes in the presence of the standard Maxwell Lagrangian.

In this Letter we suppress the standard Maxwell Lagrangian, keeping only the ‘square root’ of the Maxwell Lagrangian, to search for confining potentials. It is known that under the scale transformation, i.e. $x_\mu\to\lambda x_\mu$, $A_\mu\to\frac{1}{\lambda}A_\mu$ ($\lambda=\text{const.}$) in $d=4$ the latter doesn’t remain invariant. Even in this reduced form we prove the existence of such potentials in some spacetimes identified as the Nariai–Bertotti–Robinson (NBR)-type spacetime. Due to the absence of Maxwell Lagrangian $\sim F_{\mu\nu}F^{\mu\nu}$, however, the Coulomb potential will be missing in our formalism. We choose the case $s=1/2$ in $d=4$ with a general line element

$$ds^2=-f(r)\,dt^2+\frac{dr^2}{f(r)N(r)^2}+R(r)^2\big(d\theta^2+\sin^2\theta\,d\varphi^2\big), \tag{2}$$

where $f(r)$, $N(r)$ and $R(r)$ are three unknown functions of r. Our choice of Maxwell 2-form is

$$F=E(r)\,dt\wedge dr+P\sin(\theta)\,d\theta\wedge d\varphi \tag{3}$$

in which P stands for the magnetic charge constant and $E(r)$ is to be determined. From the variational principle the nonlinear Maxwell equation reads

$$d\left(\frac{{}^\star\mathbf{F}}{\sqrt{\mathcal{F}}}\right)=0, \tag{4}$$

in which ${}^\star\mathbf{F}$ is dual of $\mathbf{F}$. Using the line element one finds

$${}^\star\mathbf{F}=ENR^2\sin\theta\,d\theta\wedge d\varphi-\frac{P}{NR^2}\,dt\wedge dr, \tag{5}$$

and

$$\mathcal{F}=-2E^2N^2+\frac{2P^2}{R^4}. \tag{6}$$

The nonlinear Maxwell equation yields

$$\frac{ENR^2}{\sqrt{-2E^2N^2+\frac{2P^2}{R^4}}}=\frac{\beta}{\sqrt{2}} \tag{7}$$

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where β is an integration constant. This equation admits a solution for the electric field as

$$E = \frac{P\beta}{NR^2\sqrt{R^4+\beta^2}}, \tag{8}$$

and therefore

$$\mathcal{F} = \frac{2P^2}{R^4+\beta^2}. \tag{9}$$

We note here that $\mathcal{F}$ is positive which is needed for our choice of square root expression. Variation of the action with respect to $g_{\mu\nu}$ gives Einstein–Maxwell equations

$$G_\mu^\nu + \Lambda g_\mu^\nu = T_\mu^\nu \tag{10}$$

in which

$$T_\nu^\mu = -\frac{\alpha}{2}\left(\delta_\nu^\mu\sqrt{\mathcal{F}} - \frac{2(F_{\nu\lambda}F^{\mu\lambda})}{\sqrt{\mathcal{F}}}\right). \tag{11}$$

Explicitly we find

$$T_t^t = T_r^r = -\frac{\alpha}{\sqrt{2}}\left(\frac{P\sqrt{R^4+\beta^2}}{R^4}\right), \tag{12}$$

and

$$T_\theta^\theta = T_\phi^\phi = \frac{\alpha}{\sqrt{2}}\frac{P\beta^2}{R^4\sqrt{R^4+\beta^2}}. \tag{13}$$

Having $T_t^t = T_r^r$ means that $G_t^t = G_r^r$ which leads to $N(r) = C$ and $R(r) = r$. Note that C is an integration constant which is set for convenience to $C = 1$. The Einstein equations admit a black hole solution for the metric function given by

$$f(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2 - \frac{P\alpha}{\sqrt{2}r}\int\sqrt{1+\frac{\beta^2}{r^4}}\,dr. \tag{14}$$

Here by using the expansion $\sqrt{1+t} = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + \mathcal{O}(t^3)$ for $|t| < 1$ one finds for large r (i.e. $\frac{r^4}{\beta^2} > 1$)

$$f(r_{large}) = 1 - \frac{P\alpha}{\sqrt{2}} - \frac{2m}{r} - \frac{\Lambda}{3}r^2 + \frac{P\alpha\beta^2\sqrt{2}}{12r^4} + \mathcal{O}\left(\frac{1}{r^8}\right), \tag{15}$$

and for small r (i.e. $\frac{r^4}{\beta^2} < 1$) we rewrite $\int\sqrt{1+\frac{\beta^2}{r^4}}\,dr = \int\frac{\beta}{r^2}\times\sqrt{1+\frac{r^4}{\beta^2}}\,dr$ which implies

$$f(r_{small}) = 1 - \frac{2m}{r} + \frac{P\alpha\beta}{\sqrt{2}r^2} - \left(\frac{\Lambda}{3} + \frac{P\alpha\sqrt{2}}{12\beta}\right)r^2 + \frac{P\alpha\sqrt{2}}{112\beta^3}r^6 + \mathcal{O}\left(r^{10}\right), \tag{16}$$

where m is an integration constant related to mass. The Ricci scalar of the spacetime is given by

$$R = 2 + \frac{4m}{r^3} - \frac{2\sqrt{2}\alpha P\sqrt{r^4+\beta^2}}{r^4} + \frac{\sqrt{2}P\alpha}{\sqrt{r^4+\beta^2}} - \frac{\sqrt{2}P\alpha}{r^3}\int\sqrt{1+\frac{\beta^2}{r^4}}\,dr, \tag{17}$$

which at infinity is convergent while at $r = 0$ is singular. For a moment in order to see the structure of the electromagnetic field (3) we resort to the flat spacetime given by the line element

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right). \tag{18}$$

The electric field reads as

$$E = \frac{P}{r^2\sqrt{1+\frac{r^4}{\beta^2}}} \tag{19}$$

which results in the potential

$$V = -P\int\frac{dr}{r^2\sqrt{1+\frac{r^4}{\beta^2}}}. \tag{20}$$

Here also we use the expansion $\frac{1}{\sqrt{1+t}} = 1 - \frac{1}{2}t + \frac{3}{8}t^2 + \mathcal{O}(t^3)$ for $|t| < 1$ to obtain

$$V(r_{small}) = -P\int\frac{dr}{r^2}\left(1+\frac{r^4}{\beta^2}\right)^{-\frac{1}{2}} = \frac{P}{r} + \frac{Pr^3}{6\beta^2} - \frac{3Pr^7}{56\beta^4} + \mathcal{O}\left(r^{11}\right), \tag{21}$$

for small r and

$$V(r_{large}) = -P\beta\int\frac{dr}{r^4}\left(1+\frac{\beta^2}{r^4}\right)^{-\frac{1}{2}} = \frac{P\beta}{3r^3} - \frac{P\beta^3}{14r^7} + \frac{3P\beta^5}{88r^{11}} + \mathcal{O}\left(\frac{1}{r^{15}}\right), \tag{22}$$

for large r. It is readily seen that the magnetic charge P is indispensable for an electric solution to exist in the flat spacetime.

Now, going back to the curved space metric ansatz

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) \tag{23}$$

one obtains, for $\beta = 0$, the exact solution from (14) as

$$f(r) = 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2 - \frac{P\alpha}{\sqrt{2}}, \tag{24}$$

with $E(r) = 0$. Such a metric represents a global monopole [8] with a deficit angle which is valid only for $P \neq 0$. We note that this represents a non-asymptotically flat black hole with mass, cosmological constant and global monopole charge.

For the case of pure electric field let us consider now in (3) $P = 0$ and due to the sign problem we revise our square root term as $\sqrt{-F_{\mu\nu}F^{\mu\nu}}$ in the action. Further, to remove the ambiguity in arbitrariness of $E(r)$ from the Maxwell equation (4) we require that the spacetime has constant scalar curvature. This restricts our $E(r)$ only to be a constant. This yields with reference to the metric ansatz (2), as a result of the Maxwell equation, for the choice $N(r) = 1$ that one obtains $R(r) = r_0 =$ constant and $E(r) = E_0 =$ constant.

The tt and rr components of the Einstein equations yield

$$\Lambda = \frac{1}{r_0^2}, \tag{25}$$

so that the solution for $f(r)$ takes the form

$$f = -\left(\Lambda + \frac{\alpha E_0}{\sqrt{2}}\right)r^2 + C_1 r + C_2 \tag{26}$$

where C_1 and C_2 are constants of integration. With this $f(r)$ the line element reads

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r_0^2\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) \tag{27}$$

in which the electric field (E_0) and cosmological constant ($\Lambda = \frac{1}{r_0^2}$) are both essential parameters. By setting $E_0 = 0 = C_1$, it reduces to the Nariai [9] line element. For this reason (27) is known for $E_0 \neq 0$ to be the Nariai–Bertotti–Robinson (NBR)-type [9] line element. Let us add that since our case is an NED, rather than the linear Maxwell theory our solution shows minor digression from the standard NBR spacetime [9]. Due to this fact we prefer to label it simply as NBR-type. In the sequel we consider radial geodesics for both $P \neq 0$ and $P = 0$.

1. Absence of linear potential for $P \neq 0$, $\beta \neq 0$

We study the radial geodesics of a charged particle with electric charge q_0 and unit mass ($m = 1$) for simplicity in the spacetime (2) (for $N(r) = 1$ and $R(r) = r$). For the radial geodesics we set $\theta = \theta_0 =$ constant and $\varphi = \varphi_0 =$ constant, so that the particle Lagrangian is given by (a 'dot' in the sequel stands for derivative with respect to the proper distance s)

$$L = \frac{1}{2}\left(-f\dot{t}^2 + \frac{1}{f}\dot{r}^2\right) + q_0 P\dot{t}\int \frac{P}{r^2\sqrt{1+\frac{r^4}{\beta^2}}}\,dy. \tag{28}$$

Herein a constant of motion is given by

$$\frac{\partial L}{\partial \dot{t}} = -\mathcal{E} \tag{29}$$

where $\mathcal{E}$ represents the energy of the particle. The geodesic equation reads

$$f\dot{t} = q_0 P\int \frac{P}{r^2\sqrt{1+\frac{r^4}{\beta^2}}} - \mathcal{E} \tag{30}$$

with geodesic condition

$$\dot{r}^2 + f = (f\dot{t})^2. \tag{31}$$

A substitution yields the equation of motion for the particle

$$\dot{r}^2 + V_{eff} = \mathcal{E}^2 \tag{32}$$

where

$$\begin{aligned} V_{eff} = {} & 1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2 - \frac{P\alpha}{\sqrt{2}r}\int\sqrt{1+\frac{\beta^2}{r^4}}\,dr \\ & - (q_0 P)^2\left(\int\frac{dr}{r^2\sqrt{1+\frac{r^4}{\beta^2}}}\right)^2 \\ & + 2\mathcal{E}q_0 P\int\frac{dr}{r^2\sqrt{1+\frac{r^4}{\beta^2}}}. \end{aligned} \tag{33}$$

Once more we expand this potential to get for small r

$$\begin{aligned} V_{eff}(r_{small}) = {} & 1 - \frac{2m + 2\mathcal{E}q_0 P}{r} + \frac{P\alpha\beta\sqrt{2} - 2q_0^2P^2}{2r^2} \\ & - \left(\frac{P\alpha\sqrt{2}}{12\beta} + \frac{\Lambda}{3} + \frac{q_0^2P^2}{3\beta^2}\right)r^2 - \frac{\mathcal{E}q_0 P}{3\beta^2}r^3 \\ & + \mathcal{O}(r^6), \end{aligned} \tag{34}$$

and for large r

$$\begin{aligned} V_{eff}(r_{large}) = {} & 1 - \frac{1}{2}P\alpha\sqrt{2} - \frac{2m}{r} - \frac{\Lambda}{3}r^2 - \frac{2\mathcal{E}q_0 P\beta}{3r^3} \\ & + \frac{P\alpha\sqrt{2}\beta^2}{12r^4} + \mathcal{O}\left(\frac{1}{r^6}\right), \end{aligned} \tag{35}$$

where both manifestly show the absence of a linear ($\sim r$) term in the effective potential V_{eff}. Our expansions, however, cover only the asymptotic regions for small/large r values. For a general proof arbitrary r should be accounted which can be expressed in terms of elliptic functions.

2. Linear potential for $P = 0$ and $E = E_0 =$ constant

With the constant electric field E_0 now we have $V(r) = -E_0 r$, up to a disposable constant. The Lagrangian of a charged particle in the spacetime (27) (with charge q_0 and unit mass) is given by

$$L = \frac{1}{2}\left(-f\dot{t}^2 + \frac{1}{f}\dot{r}^2\right) + q_0 E_0 r\dot{t}. \tag{36}$$

For simplicity we set $r_0 = q_0 = 1$ and $\alpha = 2\sqrt{2}$ so that the geodesic motion takes the form

$$\left(\frac{dr}{ds}\right)^2 = \mathcal{E}^2 + Ar^2 + Br - C_2. \tag{37}$$

Here $\mathcal{E} > 0$ is the conserved energy and the constants A and B are abbreviated as

$$A = (1 + E_0)^2, \qquad B = -C_1 + 2E_0\mathcal{E}. \tag{38}$$

We set now $A = 0 = C_1$, $C_2 = -1$, so that (37) integrates with the effective linear potential $V_{eff} = 2\mathcal{E}r$ to yield

$$\sqrt{\mathcal{E}^2 - 2\mathcal{E}r + 1} = \pm\mathcal{E}(s_0 - s) \tag{39}$$

in which s_0 is an integration constant. Clearly the potential is confined by $0 < r < \frac{\mathcal{E}^2+1}{2\mathcal{E}}$ and the underlying geometry is an NBR spacetime transformable (for $r = \cosh\chi$) to the line element

$$ds^2 = -\sinh^2\chi\,dt^2 + d\chi^2 + d\theta^2 + \sin^2\theta\,d\varphi^2 \tag{40}$$

with electric field $E_0 = -1$ and cosmological constant $\Lambda = 1$.

In conclusion, the square root Lagrangian $\sqrt{F_{\mu\nu}F^{\mu\nu}}$ (or $\sqrt{-F_{\mu\nu}F^{\mu\nu}}$ for the pure electric case) with a cosmological constant in the absence of the standard Lagrangian $F_{\mu\nu}F^{\mu\nu}$ admits solution with a uniform electric field (and zero magnetic charge) which provides a linear potential believed to be effective in confinement [6]. The Coulomb part of the 'Cornell potential' [10] will naturally be absent in our formalism. We have shown also that magnetic charge ($P \neq 0$) acts against confinement. The spacetime in which the square root Maxwell Lagrangian yields confinement happens to be the constant curvature NBR spacetime even in this form of the square root NED. In such a spacetime we have both electric field, cosmological constant and the freedom of choice of one in terms of the other renders a linear potential in the effective potential V_{eff} possible.

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