

Spectroscopy of Rindler Modified Schwarzschild Black Hole

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Abstract

We study quasinormal modes (QNMs) of uncharged Grumiller black hole (GBH). This massive BH has a Rindler acceleration a , and hence it is also called Rindler modified Schwarzschild BH. After reducing the radial equation of the massless Klein-Gordon equation to the Zerilli equation, we compute the complex frequencies of the QNMs of the GBH. To this end, an approximation method which considers small perturbation around its horizon is used. Considering the highly damped QNMs in the process proposed by Maggiore, the quantum entropy and area spectra of these BHs are found. Although the QNM frequencies are tuned by the Rindler term, we show in detail that the spacing does not depend on it. Here, dimensionless constant ϵ of the area spectrum is found to be double of its Schwarzschild value. The latter result is also discussed.

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I. INTRODUCTION

One of the trend subjects in the thermodynamics of BHs is the quantization of the BH horizon area and entropy. The pioneer works in this regard dates back to 1970s, in which Bekenstein showed that BH entropy is proportional to the area of the BH horizon [1, 2]. Furthermore, Bekenstein [3–5] conjectured that if the BH horizon area is an adiabatic invariant, according to Ehrenfest’s principle it has a discrete and equally spaced spectrum as the following

$$A_n = \epsilon n l_p^2, \quad (n = 0, 1, 2, \dots), \quad (1)$$

where ϵ is a dimensionless constant and l_p is the Planck length. A_n denotes the area spectrum of the BH horizon and n is the quantum number. One can easily see that when the BH absorbs a test particle the minimum increase of the horizon area is $\Delta A_{\min} = \epsilon l_p^2$. In units with $c = G = 1$ and $l_p^2 = \hbar$, the undetermined dimensionless constant ϵ is considered as the order of unity. Bekenstein purported that the BH horizon is formed by patches of equal area $\epsilon \hbar$ with $\epsilon = 8\pi$. After that, many studies have been done in order to obtain such equally spaced area spectrum, whereas the spacing could be different than $\epsilon = 8\pi$ (a reader may refer to [6] and references therein).

Since a BH is characterized by mass, charge and angular momentum, it can be treated as an elementary particle. Since each object made by the elementary particles has its own characteristic vibrations known as the QNM frequencies, QNMs should reveal some information about the BH. Specially they are important for observational aspect of gravitational

waves phenomena. In the same conceptual framework, Hod [7, 8] suggested that ϵ can be obtained by using the QNM of a BH. Based on Bohr's correspondence principle [9], Hod theorized that the real part of the asymptotic QNM frequency (ω_R) of a highly damped BH is associated with the quantum transition energy between two quantum levels of the BH. This transition frequency allows a change in the BH mass as $\Delta M = \hbar\omega_R$. For the Schwarzschild BH, Hod calculated the value of the dimensionless constant as $\epsilon = 4 \ln 3$. Thereafter, Kunstatter [10] considered the natural adiabatic invariant I_{adb} for system with energy E and vibrational frequency $\Delta\omega$ (for a BH, E is identified with the mass M) which is given by

$$I_{adb} = \int \frac{dE}{\Delta\omega}. \quad (2)$$

At large quantum numbers, the adiabatic invariant is quantized via the Bohr-Sommerfeld quantization; $I_{adb} \simeq n\hbar$. Thus, Hod's result ($\epsilon = 4 \ln 3$) is also derived by Kunstatter. Later on, Maggiore [11] set up another method in which the QNM of a perturbed BH is considered as a damped harmonic oscillator. This approach was plausible since the QNM has an imaginary part. In other words, Maggiore considered the proper physical frequency of the harmonic oscillator with a damping term in the form of $\omega = (\omega_R^2 + \omega_I^2)^{\frac{1}{2}}$, where ω_R and ω_I are the real and imaginary parts of the frequency of the QNM, respectively. In the $n \gg 1$ limit which is equal to the case of highly excited mode, $\omega_I \gg \omega_R$. Therefore, one infers that ω_I should be used rather than ω_R in the adiabatic quantity. As a result, it was found that $\epsilon = 8\pi$, which corresponds to the same area spectrum of Bekenstein's original result of the Schwarzschild BH [12, 13]. By this time, we can see numerous studies in the literature in which Maggiore's method (MM) was employed (see for instance [14–20]).

Rindler acceleration "a" [21] has recently become popular anew. This is because of Grumiller's BH (GBH) having a static and spherically symmetric metric [22, 23] which attempts to describe the gravity of a central mass at large distances. In [22], it was suggested that the effective potential of a central gravitating mass M should include r -dependent acceleration term. Therefore the problem effectively degrades to a $2D$ system where the Newton's gravitational force modifies into $F_G = -m(\frac{M}{r^2} + a)$ in which m is the mass of a test particle. For $a < 0$, the two forces represent repulsive property however while $a > 0$ gives an inward attractive force. Here, unless stated otherwise, throughout the paper we shall use $a > 0$. Firstly, this attractive acceleration was hypothesized to decipher the

inexplicable acceleration that revealed after the long period observations on the Pioneer spacecrafts – Pioneer 10 and Pioneer 11 – after they covered a distance about $3 \times 10^9 km$ on their trajectories out of the Solar System [24]. The associated acceleration is unlikely attractive i.e., directed toward the Sun, and this phenomenon is known as the Pioneer anomaly. However, the newest and widely acclaimed study [25] on this subject has shown that the Pioneer anomaly could be explained by thermal heat loss of the satellite. Besides these, it is also speculated that the Rindler acceleration may play the role of dark matter in galaxies [22, 23]. So, the integration of the Newton’s theory with the Rindler acceleration could explain rotation curves of spiral galaxies without the presence of a dark matter halo. In brief, the main role of a is to constitute a rough model involving rotation curves with a linear growing of the velocity with the radius. Up to the present, the main studies on the GBH are [26, 27] in which light bending, gravitational redshift and perihelion shifts were computed for the planets in our solar system. For the most recent work on the GBH, which is about its geodesics a reader may consult [28]. As a last remark, when the Rindler term in the GBH metric is terminated ($a = 0$), all results reduce to those of Schwarzschild BH as it must.

In this paper, our main motivation is to examine how the influence of the Rindler acceleration effects the GBH spectroscopy. We shall first compute the QNMs of the GBH and subsequently use them in the MM. For this purpose, we organize the the paper as follows. In the next section, we describe the GBH metric and its basic thermodynamical features. We also represent that how the massless Klein Gordon equation reduces to the one dimensional Schrödinger-type wave equation which is the so-called the Zerilli equation [29] in the GBH geometry. Sect. 3 is devoted to the calculation of the QNMs of the GBH by considering the small perturbations around the horizon. After that, we employ the MM for the GBH in order to compute its entropy and area spectra. Finally the conclusion is given in Sect. 4.

II. GBH AND ITS ZERILLI EQUATION

In this section, we represent the geometry and some thermodynamical properties of the GBH. We explicitly show how one gets the radial equation for a massless scalar field in the background of the GBH. Then, we give the one-dimensional Schrödinger wave equation form (the so-called Zerilli equation [29]) of the GBH.

GBH's line element [22] without cosmological constant is given by

$$ds^2 = -H(r)dt^2 + \frac{dr^2}{H(r)} + r^2 d\Omega^2, \quad (3)$$

where $d\Omega^2$ is the standard metric on 2-sphere and the metric function $H(r)$ is computed as

$$\begin{aligned} H(r) &= 1 - \frac{2M}{r} + 2ar, \\ &= \frac{2a}{r}(r - r_h)(r - r_0), \end{aligned} \quad (4)$$

where M is the mass of the BH and a is a positive real constant, which corresponds to the Rindler acceleration parameter. One can easily see that when $a = 0$, spacetime (1) reduces to the well-known Schwarzschild BH. On the other hand, herein r_0 is found to be

$$r_0 = -\frac{\sqrt{1 + 16aM} + 1}{4a}, \quad (5)$$

which cannot be horizon due to its negative signature. Therefore, the GBH possesses only one horizon (event horizon, r_h) which is given by

$$r_h = \frac{\sqrt{1 + 16aM} - 1}{4a}, \quad (6)$$

such that while $a \rightarrow 0$ we get $r_h = 2M$, as it is expected. After computing the scalars of the metric, we obtain

$$\begin{aligned} K &= R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = 32\frac{a^2}{r^2} + 48\frac{M^2}{r^6}, \\ R &= -12\frac{a}{r}, \\ R_{\alpha\beta}R^{\alpha\beta} &= 40\frac{a^2}{r^2}. \end{aligned} \quad (7)$$

which obviously show that central curvature singularity is at $r = 0$.

Surface gravity [30] of the GBH can simply be calculated through the following expression

$$\kappa = \frac{H'(r)}{2} \Big|_{r=r_h} = \frac{a(r_h - r_0)}{r_h}, \quad (8)$$

where a prime "′" denotes differentiation with respect to r . From here on in, one obtains the Hawking temperature T_H of the GBH as

$$\begin{aligned} T_H &= \frac{\hbar\kappa}{2\pi}, \\ &= \frac{a(r_h - r_0)\hbar}{2\pi r_h}, \end{aligned} \quad (9)$$

From the above expression, it is seen that while the GBH losing its M by virtue of the Hawking radiation, T_H increases (i.e., $T_H \rightarrow \infty$) with $M \rightarrow 0$ in such a way that its divergence speed is tuned by a . Meanwhile, one can check that $\lim_{a \rightarrow 0} T_H = \frac{1}{8\pi M}$ which is well-known Hawking temperature computed for the Schwarzschild BH. The Bekenstein-Hawking entropy is given by

$$\begin{aligned} S_{BH} &= \frac{A_h}{4\hbar}, \\ &= \frac{\pi r_h^2}{\hbar}, \end{aligned} \quad (10)$$

Its differential form is written as

$$dS_{BH} = \frac{4\pi}{\sqrt{1 + 16aM\hbar}} r_h dM, \quad (11)$$

By using the above equation, the validity of the first law of thermodynamics for the GBH can be approved via

$$T_H dS_{BH} = dM. \quad (12)$$

In order to find the entropy spectrum by using the MM, here we shall firstly consider the massless scalar wave equation on the geometry of the GBH. The general equation of massless scalar field in a curved spacetime is written as

$$\square\Psi = 0, \quad (13)$$

where \square denotes the Laplace-Beltrami operator. Thus, the above equation is equal to

$$\frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} \partial^i \Psi), \quad i = 0 \dots 3, \quad (14)$$

Using the following ansatz for the scalar field Ψ in the above equation

$$\Psi = \frac{R(r)}{r} e^{i\omega t} Y_L^m(\theta, \varphi), \quad \text{Re}(\omega) > 0, \quad (15)$$

in which $Y_L^m(\theta, \varphi)$ is the well-known spheroidal harmonics which admits the eigenvalue $-L(L+1)$ [31], one obtains the following Zerilli equation [29] as

$$\left[-\frac{d^2}{dr^{*2}} + V(r) \right] R(r) = \omega^2 R(r), \quad (16)$$

where the effective potential is computed as

$$V(r) = H(r) \left[\frac{L(L+1)}{r^2} + \frac{2M}{r^3} + \frac{a}{r} \right], \quad (17)$$

The tortoise coordinate r^* is defined as,

$$r^* = \int \frac{dr}{H(r)}, \quad (18)$$

which yields

$$r^* = \frac{1}{2a(r_h - r_0)} \ln \left[\frac{(\frac{r}{r_h} - 1)^{r_h}}{(r - r_0)^{r_0}} \right], \quad (19)$$

Finally, one can easily check the asymptotic limits of r^* as follows

$$\lim_{r \rightarrow r_h} r^* = -\infty \quad \text{and} \quad \lim_{r \rightarrow \infty} r^* = \infty. \quad (20)$$

III. QNMS AND ENTROPY/AREA SPECTRA OF GBH

In this section, we intend to derive the entropy and area spectra of the GBH by using the MM. Gaining inspiration from the studies [32–34], here we use an approximation method in order to define the QNMs. According to this method, to obtain QNM it is sufficient to use one of the two definitions of the QNMs; only ingoing waves should exist near the horizon. Namely,

$$R(r)|_{QNM} \sim e^{i\omega r^*} \text{ at } r^* \rightarrow -\infty, \quad (21)$$

Now we can proceed to solve Eq. (16) in the near horizon limit and then impose the above boundary condition to find the frequency of QNM i.e., ω . Expansion of the metric function $H(r)$ around the event horizon is given by

$$\begin{aligned} H(r) &= H'(r_h)(r - r_h) + \mathcal{O}(r - r_h)^2, \\ &\simeq 2\kappa(r - r_h), \end{aligned} \quad (22)$$

where κ is the surface gravity, which is nothing but $\frac{1}{2}H'(r_h)$. From Eq. (18) we now obtain

$$r^* \simeq \frac{1}{2\kappa} \ln(r - r_h), \quad (23)$$

Furthermore, after setting $y = r - r_h$ and inserting Eq. (22) into Eq. (17) together with performing Taylor expansion around $r = r_h$, we find the near horizon form of the effective potential as

$$V(y) \simeq 2\kappa y \left[\frac{L(L+1)}{r_h^2} \left(1 - \frac{2y}{r_h}\right) + \frac{2\kappa}{r_h} \left(1 - \frac{y}{r_h}\right) \right]. \quad (24)$$

After substituting Eq. (24) into the Zerilli equation (16), one gets

$$-4\kappa^2 y^2 \frac{d^2 R(y)}{dy^2} - 4\kappa^2 y \frac{dR(y)}{dy} + V(y)R(y) = \omega^2 R(y), \quad (25)$$

Solution of the above equation yields

$$R(y) \sim y_1^{\frac{i\omega}{2\kappa}} {}_1F_1(\hat{a}, \hat{b}; \hat{c}), \quad (26)$$

where ${}_1F_1(\hat{a}, \hat{b}; \hat{c})$ is the confluent hypergeometric function [35]. The parameters of the confluent hypergeometric functions are found to be

$$\begin{aligned} \hat{a} &= \frac{1}{2} + i\left(\frac{\omega}{2\kappa} - \frac{\hat{\alpha}}{\hat{\beta}\sqrt{\kappa}}\right), \\ \hat{b} &= 1 + i\frac{\omega}{\kappa}, \\ \hat{c} &= i\frac{\hat{\beta}x}{2r_h\sqrt{\kappa}}, \end{aligned} \quad (27)$$

where

$$\begin{aligned}\hat{\beta} &= 4\sqrt{r_h}\sqrt{L(L+1) + \kappa r_h}, \\ \hat{\alpha} &= L(L+1) + 2\kappa r_h,\end{aligned}\tag{28}$$

In the limit of $y \ll 1$, the solution (26) becomes

$$R(y) \sim C_1 y^{-\frac{i\omega}{2\kappa}} \frac{\Gamma(i\frac{\omega}{\kappa})}{\Gamma(\hat{a})} + C_2 y^{\frac{i\omega}{2\kappa}} \frac{\Gamma(-i\frac{\omega}{\kappa})}{\Gamma(1 + \hat{a} - \hat{b})},\tag{29}$$

where constants C_1 and C_2 represent the amplitudes of the near-horizon outgoing and ingoing waves, respectively. Now, since there is no outgoing wave in the QNM at the horizon, the first term of Eq. (29) should be terminated. This is possible with the poles of the Gamma function of the denominator. Therefore, the poles of the Gamma function are the policy makers of the frequencies of the QNMs. Thus, the frequencies of the QNM of the GBH are read as

$$\omega_s = \frac{2\sqrt{\kappa}\hat{\alpha}}{\hat{\beta}} + i\frac{2\pi}{\hbar}(2s+1)T_H, \quad (s = 1, 2, 3, \dots)\tag{30}$$

where m is the overtone quantum number of the QNM. Thus, the imaginary part of the frequency of the QNM is

$$\omega_I = \frac{2\pi}{\hbar}(2s+1)T_H,\tag{31}$$

As it can be seen from above, the Rindler acceleration plays a crucial role on ω_I . While $a \rightarrow 0$, $\omega_I = \frac{(2s+1)}{4M}$ which is consistent with the Schwarzschild BH result [36, 37]. Hence the transition frequency between two highly damped neighboring states becomes $\Delta\omega \equiv \Delta\omega_I = \omega_{s+1} - \omega_s = 4\pi T_H/\hbar$. Hence, the adiabatic invariant quantity (2) turns out to be

$$I_{adb} = \frac{\hbar}{4\pi} \int \frac{dM}{T_H},\tag{32}$$

According to the first law of thermodynamics (12), it reads

$$I_{adb} = \frac{S_{BH}}{4\pi}\hbar,\tag{33}$$

Finally, recalling the Bohr-Sommerfeld quantization rule $I_{adb} = \hbar n$, one gets the spacing of the entropy spectrum as

$$S_n = 4\pi n, \quad (34)$$

Since $S = \frac{A}{4\hbar}$, the area spectrum is obtained as

$$A_n = 16\pi n\hbar, \quad (35)$$

From the above, we can simply measure the area spacing as

$$\Delta A = 16\pi\hbar. \quad (36)$$

It is easily seen that unlike to ω_I the spectroscopy of the GBH is completely independent of the Rindler term a . The obtained spacings between the levels are double of the Bekenstein's original result which means that $\epsilon = 16\pi$. The discussion on this discrepancy is made in the conclusion part.

IV. CONCLUSION

In this paper, the BH spectroscopy of the GBH is investigated through the MM. We applied an approximation method given in [32–34] to the Zerilli equation (16) in order to compute the QNM of the GBH. After a straightforward calculation, the QNM frequency of the GBH are analytically found. The obtained result shows that imaginary part of the frequency, ω_I depends on the Rindler term a . Then, with the aid of Eq. (2), we obtained the entropy/area spectra of the GBH. Both spectra are found to be independent of the Rindler term a , and they are equally spaced. Furthermore, we read the dimensionless constant as $\epsilon = 16\pi$ which means that the equi-spacing is the double of its Schwarzschild value: $\epsilon = 8\pi$ [11]. This differentness may arise due to the Schwinger mechanism [38]. Because, in the Bekenstein's original work [2], one gets the entropy spectrum by combining both the Schwinger mechanism and the Heisenberg quantum uncertainty principle. However, the method that we applied here considers only the uncertainty principle via the Bohr-Sommerfeld quantization. Therefore, as stated in [8], the spacings between two neighboring levels may become different depending on the which method is preferred. Thus, finding $\epsilon = 16\pi$ rather than its known value $\epsilon = 8\pi$ is not an unexpected result. As a last remark, evenly spaced structure of the entropy and area spectra of the GBH is also in agreement

with the Wei et al.'s conjecture [16], which proposes that static BHs of Einstein's gravity theory has equidistant quantum spectra of both entropy and area .

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