Hawking Radiation of Grumiller Black Hole

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(Date textdate; Received textdate; Revised textdate; Accepted textdate;

Published text1 May 2014)

arXiv:1402.5514v2 [gr-qc] 26 Apr 2014

Abstract

In this paper, we consider the relativistic Hamilton-Jacobi (HJ) equation and study the Hawking radiation (HR) of scalar particles from uncharged Grumiller black hole (GBH) which is affordable for testing in astrophysics. GBH is also known as Rindler modified Schwarzschild BH. Our aim is not only to investigate the effect of the Rindler parameter a on the Hawking temperature (T_H), but to examine whether there is any discrepancy between the computed horizon temperature and the standard T_H as well. For this purpose, in addition to its naive coordinate system, we study on the three regular coordinate systems which are Painlevé-Gullstrand (PG), ingoing Eddington-Finkelstein (IEF) and Kruskal-Szekeres (KS) coordinates. In all coordinate systems, we calculate the tunneling probabilities of incoming and outgoing scalar particles from the event horizon by using the HJ equation. It has been shown in detail that the considered HJ method is concluded with the conventional T_H in all these coordinate systems without giving rise to the famous factor-2 problem. Furthermore, in the PG coordinates Parikh-Wilczek's tunneling (PWT) method is employed in order to show how one can integrate the quantum gravity (QG) corrections to the semiclassical tunneling rate by including the effects of self-gravitation and back reaction. We then show how these corrections yield a modification in the T_H.

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I. INTRODUCTION

Rindler acceleration [1], which acts on an observer accelerated in a flat spacetime has recently become rage anew. This is due to its similarity with the mysterious acceleration that revealed after the long period observations on the Pioneer spacecraft – Pioneer 10 and Pioneer 11 – after they covered a distance about $3 \times 10^9 km$ on their paths out of the Solar System [2]. Contrary to the expectations, that mysterious acceleration is attractive i.e., directed toward the Sun and this phenomenon is known as the Pioneer anomaly. Firstly, Grumiller [3] (and later together with his collaborators [4, 5]) showed the correlation between the a and the Pioneer anomaly. On the other hand, Turyshev et al. [6] have recently made an alternative study to the Grumiller's ones in which the Pioneer anomaly is explained by thermal heat loss of the satellites.

Another intriguing feature of the a is that it may play the role of dark matter in galaxies [3, 4]. Namely, the incorporation of the Newton's theory with the a might serve to explain rotation curves of spiral galaxies without the presence of a dark matter halo (a reader may refer to the study of Lin et al. [7]). For the galaxy-Sun pair, the a with the order $\sim 10^{-11} m/s^2$ in physical units is a very close value to MOND's acceleration which successfully describes rotation curves without a dark matter halo (see [8] and references therein). However, very recently the studies [7, 8] have been retested and criticized by Cervantes-Cota and Gómez-

López [9].

As stated in [3, 4], the main function of the a is to constitute a crude model which casts doubts on the description of rotation curves with a linear growing of the velocity with the radius. By virtue of this, in the novel study of [3] it was suggested that the effective potential of a point mass M should include $r-$dependent acceleration term. Moreover, in the studies of [3, 4] it is explicitly shown that dilatonic field source in general relativity (GR) is required for deriving a spacetime with the a. However, in performing this process one should be cautious about the physical energy conditions. It has been recently revived by [10] that the GBH spacetime [4] does not satisfy the all essential energy conditions of the GR. Very recently, Mazharimousavi and Halilsoy (MH) [11] have shown that the GBH metric becomes physically acceptable in the $f(\Re)$ gravity. In other words, in the $f(\Re)$ gravity the problematic energy conditions are all fixed. The physical source that has been used in [11] possesses a perfect fluid-type energy momentum tensor, and the pressure of the fluid becomes negative with a particular choice. So, one can infer that the a plays the role of the dark matter. From now on, we designate the metric of [11] as Grumiller-Mazharimousavi-Halilsoy BH and abbreviate it as GMHBH. Meanwhile, very recently detailed analysis of the geodesics of this BH has been made by [12].

More than forty years ago, Bekenstein unraveled that the entropy of a BH is proportional to its surface area [13–15]. Afterwards Hawking studied the particle creation around the event horizon of a BH to ascertain that the BH possesses a black body type thermal radiation with the temperature subject to its surface gravity [16, 17]. After these novel studies of Hawking, up to date there is a rapidly growing literature on the thermodynamics of various kinds of BHs. Moreover, deriving alternative methods to the HR which divulges the underlying BH spacetime has always remained on the agenda. For the topical review of the HR together with its available methods, a reader may consult to [18]. Among those alternative methods for the HR, utilization of the relativistic HJ equation is one of the runproof methods. This method is developed by [19] that basically employs the complex path analysis of Padmanabhan et al. [20–22]. The associated method involves the WKB approximation and calculates the imaginary part of the action of the tunneling particles. In performing this process one should ignore the self-gravitational effects of the tunneling particle and the energy conservation. In general, the relativistic HJ equation can be solved by substituting a suitable ansatz. For the separability of the equation the chosen ansatz should take account of the Killing vectors of the spacetime. Thus we obtain an integral equation which yields the classically forbidden trajectory that starts from inside of the BH and finishes at the outside observer. On the other hand, the integral under question has always a pole located at the event horizon of the BH. We recall that such integrals are evaluated by applying the method of complex path analysis in order to circumvent the pole. Result of the integral leads us to get the tunneling rate for the GMHBH which renders possible to read the T_H. On the other hand, PWT method [23–25] uses the null geodesics to derive the T_H as a quantum tunneling process. In this method, self-gravitational interaction of the radiation and energy conservation are taken into account. As a result, the HR spectrum can not be strictly thermal for many well-known BHs, like Schwarzschild, Reissner-Nordström etc. [25, 26].

Here we plan to investigate the HR of a static and spherically symmetric GMHBH via the well-known HJ and PWT methods. We restate that, in this paper, we shall make only an application of the associated methods to the GMHBH. By doing this, we aim not only to make an analysis about the influences of the a on the HR, but to test whether the associated methods employing for the GMHBH with different coordinates yield the true T_H without admitting the factor-2 problem or not. For the review of the factor-2 problem arising in the HR, a reader may refer to [27–33].

First of all, we shall review the GMHBH which has a fluid source in the context of $f(\Re)$ gravity [11]. Then we use the HJ method in order to calculate the imaginary part of the classical action for outgoing trajectories crossing the horizon. In addition to the naive coordinates, three more coordinate systems (all regular) which are PG, IEF and KS, respectively, are considered. Slightly different from the other coordinate systems, during the application of the HJ method in the KS coordinates, we will first reduce the GMHBH spacetime to a Minkowski type space with a conformal factor, and then show in detail how one recovers the T_H. Furthermore, in the PG coordinate system we shall study the PWT method in order to give a QG correction to the tunneling probability by considering the back reaction effect. To this end, the log-area correction to the Bekenstein-Hawking entropy will be taken into account. Finally, the modified T_H due to the back reaction effect will be computed.

The paper uses the signature $(-,+,+,+)$ and the geometrical units $c = G = \hbar = k_B = 1$. The paper is organized as follows. In Sec. II, we review some of the geometrical and thermodynamical features of the GMHBH. We also show how the HJ equation is separated by a suitable ansatz within the naive coordinates. The calculations of the tunneling rate and henceforth the T_H via the HJ method are also represented. In Sec. III the HR of the GMHBH in the PG coordinates is analyzed in the frameworks of the HJ and PWT methods. The back reaction effect on the T_H is also examined. Sec. IV and V are devoted to the application of the HJ method in the IEF and KS coordinate systems, respectively. Finally, the conclusion and future directions are given in Sec. VI.

II. GMHBH AND HJ METHOD

In this section we will first present the geometry and some thermodynamical properties of the GMHBH. Then, with aid of a suitable ansatz we will get the radial equation for the relativistic HJ equation in the background of the GMHBH. Finally, we represent how the HJ method culminates in the T_H.

The $4D$ action obtained from $f(\Re)$ gravity is given by

$$S = \frac{1}{2\lambda} \int \sqrt{-g} f(\Re) d^4x + S_M, \tag{1}$$

where $\lambda = 8\pi G = 1$, $\Re$ is the curvature scalar and $f(\Re) = \Re - 12a\xi \ln |\Re|$ in which a and ξ are positive constants. S_M denotes the physical source for a perfect fluid-type energy momentum tensor

$$T_\mu^\upsilon = diag.[-\rho, p, q, q], \tag{2}$$

with the thermodynamic pressure p being a function of the rest mass density of the matter (for short: matter density) ρ only, so that $p = -\rho$. Meanwhile, q is also a state function which is to be determined. Recently, MH has obtained the GMHBH solution to the above action in their landmark paper [11]. Their solution is described by the following $4D$ static and spherically symmetric line element

$$ds^2 = -Hdt^2 + \frac{dr^2}{H} + r^2 d\Omega^2, \tag{3}$$

where $d\Omega^2$ is the standard metric on $2-$sphere and the metric function $H(r)$ is computed as

$$H = 1 - \frac{2M}{r} + 2ar = \frac{2a}{r}(r - r_h)(r - r_0), \tag{4}$$

which is nothing but the metric function of the GBH without the cosmological constant [3]. Here, M represents the constant mass and

$$r_0 = -\frac{\sqrt{1+16aM}+1}{4a}, \tag{5}$$

which cannot be horizon due to its negative signature. Therefore, the GMHBH possesses only one horizon (event horizon, r_h) which is given by

$$r_h = \frac{\sqrt{1+16aM}-1}{4a}, \tag{6}$$

Further, it is found that the energy-momentum components are

$$p = -\rho = \frac{\left[6a\xi - f(\Re)\right] r^2 + 4(\xi - a)r - 6M\xi}{2r^2}, \tag{7}$$

$$q = -\frac{f(\Re)r - 2\xi + 8a}{2r}, \tag{8}$$

where

$$f(\Re) = -\left[\frac{12a}{r} + 12a\xi \ln\left(\frac{12a}{r}\right)\right], \tag{9}$$

One can easily observe from the last three equations that the a is decisive for the fluid source. This can be best seen by simply taking the limit of $a \to 0$ which corresponds to the vanishing fluid and Ricci scalar, and so forth $\xi \to 0$. In short, $f(\Re)$ gravity reduces to the usual $\Re$-gravity. In short, while $a \to 0$ the GMHBH reduces to the well-known Schwarzschild BH.

Surface gravity [34] of the GMHBH can simply be calculated through the following expression

$$\kappa(M) = \left.\frac{H'}{2}\right|_{r=rh} = \frac{a\left(r_h - r_0\right)}{r_h}, \tag{10}$$

where a prime "$\prime$" denotes differentiation with respect to r. From here on in, one obtains the Hawking temperature of the GMHBH as

$$\begin{aligned} T_H &= \frac{\kappa(M)}{2\pi} = \frac{a\left(r_h - r_0\right)}{2\pi r_h}, \\ &= \frac{a\sqrt{1+16aM}}{\pi\left(\sqrt{1+16aM}-1\right)}, \end{aligned} \tag{11}$$

From the above expression, it is seen that while the GMHBH losing its M by virtue of the HR, T_H increases (i.e., $T_H \to \infty$) with $M \to 0$ in such a way that its divergence speed is tuned by a. Meanwhile, one can check that $\lim_{a\to 0} T_H = \frac{1}{8\pi M}$ which is well-known Hawking temperature computed for the Schwarzschild BH. The Bekenstein-Hawking entropy is given by

$$S_{BH} = \frac{A_h}{4} = \pi r_h^2, \tag{12}$$

Its differential form is written as

$$dS_{BH} = \frac{4\pi}{\sqrt{1+16aM}} r_h dM, \tag{13}$$

By using the above equation, the validity of the first law of thermodynamics for the GMHBH can be approved via

$$T_H dS_{BH} = dM. \tag{14}$$

Here, we consider the problem of a scalar particle (spin-0) which crosses the event horizon from inside to outside while there is no back-reaction effect and self-gravitational interaction. Within the semi-classical framework, the classical action I of the particle satisfies the relativistic HJ equation [19] is given by

$$g^{\mu\nu}\partial_\mu I \partial_\nu I + m^2 = 0, \tag{15}$$

in which m is the mass of the scalar particle, and $g^{\mu\nu}$ represents the invert metric tensors derived from the metric (3). By considering Eqs. (3), (4) and (15), we get

$$\frac{-1}{H}(\partial_t I)^2 + H(\partial_r I)^2 + \frac{1}{r^2}(\partial_\theta I)^2 + \frac{1}{r^2 \sin^2\theta}(\partial_\varphi I)^2 + m^2 = 0, \tag{16}$$

For the HJ equation it is general to use the separation of variables method for the action $I = I(t, r, \theta, \varphi)$ as follows

$$I = -Et + W(r) + J(x^i), \tag{17}$$

where

$$\partial_t I = -E, \quad \partial_r I = \partial_r W(r), \quad \partial_i I = J_i, \tag{18}$$

and J_i's are constants in which $i = 1, 2$ identifies angular coordinates θ and φ, respectively. The norm of the timelike Killing vector ∂_t becomes (negative) unity at a particular location:

$$r \equiv R_d = \frac{r_h + r_0}{2} + \frac{1 + \sqrt{4(r_h - r_0)^2 + 4(r_h + r_0)a + 1}}{4a}, \tag{19}$$

It means that when a detector of an observer is located at R_d which is outside the horizon, the energy of the particle measured by the observer is E. Solving Eq. (16) for $W(r)$ yields

$$W(r) = \pm \int \frac{\sqrt{E^2 - \frac{H}{r^2}\left(J_\theta^2 + \frac{J_\varphi^2}{\sin^2\theta} + m^2 r^2\right)}}{H} dr, \tag{20}$$

The quadratic form of Eq. (16) is the reason of $\pm$ signatures that popped up in the above equation. Solution of Eq. (20) with "$+$" signature corresponds to the outgoing scalar particles and the other solution i.e., the solution with "$-$" signature refers to the ingoing particles. Evaluating the above integral around the pole at the horizon (following to the prescription given by [35]), one reaches to

$$W_{(\pm)} = \pm \frac{i\pi E r_h}{2a(r_h - r_0)} + \delta, \tag{21}$$

where δ is a complex integration constant. Thus, we can deduce that imaginary parts of the action arises due to the pole at the horizon and from the complex constant δ. Thence, we can determine the probabilities of ingoing and outgoing particles while crossing r_h as

$$P_{out} = e^{-2\,\mathrm{Im}\,I} = \exp\left[-2\mathrm{Im}W_{(+)}\right], \tag{22}$$

$$P_{in} = e^{-2\,\mathrm{Im}\,I} = \exp\left[-2\mathrm{Im}W_{(-)}\right], \tag{23}$$

In the classical point of view, a BH absorbs any ingoing particles passing its horizon. In other words, there is no reflection for the ingoing waves which corresponds to $P_{in} = 1$. This is enabled by setting $\mathrm{Im}\delta = \frac{\pi E r_h}{2a(r_h - r_0)}$. This choice also implies that the imaginary part of the action I for a tunneling particle can only come out $W_{(+)}$. Namely, we get

$$\mathrm{Im}\,I = \mathrm{Im}W_{(+)} = \frac{\pi r_h E}{a(r_h - r_0)}. \tag{24}$$

Therefore, the tunneling rate for the GMHBH can be obtained as

$$\Gamma = P_{out} = e^{\frac{-2\pi E r_h}{a(r_h - r_0)}}, \tag{25}$$

and according to [25]

$$\Gamma = e^{-\beta E}, \tag{26}$$

in which β denotes the Boltzmann factor and $T = \frac{1}{\beta}$, one can easily read the horizon temperature of the GMHBH as

$$\check{T}_H = \frac{a(r_h - r_0)}{2\pi r_h}. \tag{27}$$

This nothing but the T_H obtained in Eq. (11).

III. HJ AND PWT METHODS WITHIN PG COORDINATES

In the literature, PG coordinates are known as the first coordinate system which is non-singular at the event horizon and allow us to describe timelike or null worldlines inward crossing the horizon. In other words, we use the PG coordinates [36, 37] in order to describe the spacetime on either side of the event horizon of a static BH. In this coordinate system, the generic spherically metric (3) loses its diagonal or static form. Instead it allows a cross term which makes the metric stationary and no longer symmetric, but oriented. Thus, an observer does not consider the surface of the horizon to be in any way special. In this section, we consider the PG coordinates of the GMHBH not only in the HJ method, but in the PWT method as well. Then we show how both methods yield the T_H. Besides, the back reaction effect on the T_H is thoroughly discussed.

We can pass to the PG coordinates by applying the following transformation [38] to the metric (3)

$$dt_{PG} = dt + \frac{\sqrt{1-H}}{H} dr, \tag{28}$$

where t_{PG} is our new time coordinate (let us call it as PG time). One of the main properties of these coordinates is that t_{PG} concurrently corresponds to the proper time. After substituting Eq. (28) into the metric (3), one obtains the PG line-element as follows

$$ds^2 = -H dt_{PG}^2 + 2\sqrt{1-H}\, dt_{PG} dr + dr^2 + r^2 d\Omega^2, \tag{29}$$

For the metric (29), the HJ equation (15) becomes

$$-(\partial_{t_{PG}}I)^2+2\sqrt{1-H}(\partial_{t_{PG}}I)(\partial_r I)+H(\partial_r I)^2+\frac{1}{r^2}(\partial_\theta I)^2+\frac{1}{r^2\sin^2\theta}(\partial_\varphi I)^2=0 \tag{30}$$

Letting

$$I=-Et_{PG}+W_{PG}(r)+J(x^i), \tag{31}$$

and now by substituting for the above ansatz in Eq. (30), we obtain

$$W_{PG}(r)=\int\frac{E\sqrt{1-H}}{H}\left(1\pm\sqrt{1-\frac{HF}{(1-H)\,E^2}}\right)dr, \tag{32}$$

where

$$F=m^2-E^2+\frac{J_\theta^2}{r^2}+\frac{J_\varphi^2}{r^2\sin^2\theta}, \tag{33}$$

Thus one can see that near the horizon Eq. (32) reduces to

$$W_{PG(\pm)}=E\int\frac{1}{H}(1\pm 1)dr, \tag{34}$$

Since $W_{PG(-)}=0$ which is a warranty condition for non-reflection of the ingoing particles, we thus have

$$W_{PG(+)}=\frac{ir_h\pi E}{a(r_h-r_0)}. \tag{35}$$

So, we get the imaginary part of the I as

$$\mathrm{Im}\,I=\mathrm{Im}W_{PG(+)}=\frac{\pi r_h E}{a(r_h-r_0)}, \tag{36}$$

After recalling Eqs. (25) and (26), we can readily read the horizon temperature of the GMHBH which is expressed in the PG coordinates as

$$\check{T}_H=\frac{a(r_h-r_0)}{2\pi r_h}. \tag{37}$$

This result is full measure of the standard value of the T_H (11).

Now, employing the tunneling method prescribed by [25] we recalculate the imaginary part of the I for an outgoing positive energy particle which crosses the horizon outwards in the PG coordinates. In the metric (29), the radial null geodesics of a test particle has a rather simple form

$$\dot{r} = \frac{dr}{d_{t_{PG}}} = -\sqrt{1-H} \pm 1, \tag{38}$$

where upper (lower) sign corresponds to outgoing (ingoing) geodesics. After expanding the metric function H around the horizon r_h, we get

$$H = H'(r_h)(r - r_h) + O(r - r_h)^2, \tag{39}$$

and hence by using Eq. (10), the radial outgoing null geodesics, $\dot{r}$, can be approximately expressed as

$$\dot{r} \cong \kappa(M)(r - r_h), \tag{40}$$

The imaginary part of the I for an outgoing positive energy particle which crosses the horizon from inside (r_{in}) to outside (r_{out}) is given by

$$\mathrm{Im}\, I = \mathrm{Im} \int_{r_{in}}^{r_{out}} p_r dr = \mathrm{Im} \int_{r_{in}}^{r_{out}} \int_0^{p_r} d\tilde{p}_r dr, \tag{41}$$

Hamilton's equation for the classical trajectory is given by

$$dp_r = \frac{d\Pi}{\dot{r}}, \tag{42}$$

where p_r and Π denote radial canonical momentum and Hamiltonian, respectively. So, one obtains

$$\mathrm{Im}\, I = \mathrm{Im} \int_{r_{in}}^{r_{out}} \int_0^{\Pi} \frac{d\widetilde{\Pi}}{\dot{r}} dr, \tag{43}$$

Now, if we consider the whole system as a spherically symmetric system of total mass M, which is kept fixed, then this system consists of a GMHBH with varying mass $M - \omega$, emitting a spherical shell of mass ω such that $\omega \ll M$. This phenomenon is known as self-gravitational effect [23]. After taking this effect into account, the above integration is expressed as

$$\begin{aligned}\operatorname{Im} I &= \operatorname{Im}\int_{r_{in}}^{r_{out}}\int_{M}^{M-\omega}\frac{d\widetilde{\Pi}}{\dot{r}}dr,\\ &= -\operatorname{Im}\int_{r_{in}}^{r_{out}}\int_{0}^{\omega}\frac{d\widetilde{\omega}}{\dot{r}}dr,\end{aligned} \tag{44}$$

in which the Hamiltonian $\Pi = M - \omega$ *i.e.* $d\Pi = -d\omega$ is used. Hence, $\dot{r}$ (40) can be reexperienced as follows

$$\dot{r} \cong \kappa_{QG}(r - r_h), \tag{45}$$

where $\kappa_{QG} = \kappa(M-\omega)$ is the modified horizon gravity, which is the so-called quantum gravity corrected surface gravity [39, 40]. Thus, after r integration (the integration over r is done by deforming the contour), Eq. (44) becomes

$$\operatorname{Im} I = -\pi\int_0^{\omega}\frac{d\tilde{\omega}}{\kappa_{QG}}, \tag{46}$$

So, let us express the "modified Hawking temperature" in the form of $T_{QG} = \frac{\kappa_{QG}}{2\pi}$. From here on ,we get

$$\begin{aligned}\operatorname{Im} I &= -\frac{1}{2}\int_0^{\omega}\frac{d\tilde{\omega}}{T_{QG}},\\ &= -\frac{1}{2}\int_{S_{QG}(M)}^{S_{QG}(M-\omega)} dS,\\ &= -\frac{1}{2}\Delta S_{QG},\end{aligned} \tag{47}$$

then the modified tunneling rate is computed via

$$\Gamma_{QG} \sim e^{-2\operatorname{Im} I} = e^{\Delta S_{QG}}. \tag{48}$$

In string theory and loop quantum gravity, it is introduced with a logarithmic correction (see for instance [41, 42] and references therein)

$$S_{QG} = \frac{A_h}{4} + \alpha\ln A_h + O(\frac{1}{A_h}), \tag{49}$$

where α is a dimensionless constant, and it arises due to the back reaction effects. It takes different values according to which theory is considered [41]. Thus, with the aid of Eqs. (12) and (49) one can compute ΔS_{QG} as follows

$$\Delta S_{QG} = -\frac{\pi\left(8a\omega + \sqrt{1+16a(M-\omega)} - \sqrt{1+16aM}\right)}{8a^2} + \alpha ln\left(\frac{1+8a(M-\omega)-\sqrt{1+16a(M-\omega)}}{1+8aM-\sqrt{1+16aM}}\right), \tag{50}$$

Now, using the second law of thermodynamics

$$T_{QG} dS_{QG} = dM, \tag{51}$$

one can find the QG corrected form of the Hawking temperature T_{QG} due to the back reaction. After a straightforward calculation, we can derive T_{QG} from Eq. (51) in terms of the Hawking temperature as follows

$$T_{QG} = \left(1 + \frac{\alpha}{\pi r_h^2}\right)^{-1} T_H \tag{52}$$

Thus, one can easily see that once we ignore the back reaction effect (i.e., $\alpha = 0$) we just produce the semiclassical Hawking temperature, T_H. Meanwhile, it is also possible to obtain T_{QG} from Eq. (48). For this purpose, we expand ΔS_{QG} (50) and recast terms up to leading order in ω. So, one finds

$$\begin{aligned}\Delta S_{QG} &\cong -\left[\frac{\pi}{a}\left(\frac{\sqrt{1+16aM})-1}{\sqrt{1+16aM}}\right) + \frac{16a\alpha}{(1+16aM-\sqrt{1+16aM})}\right]\omega + O(\omega^2), \\ &= -\left(\frac{1}{T_H} + \alpha\frac{16\pi T_H}{1+16aM}\right)\omega + O(\omega^2),\end{aligned} \tag{53}$$

Based on Eqs. (26) and (48), we obtain

$$\Gamma_{QG} \sim e^{\Delta S_{QG}} = e^{-\frac{\omega}{T}}, \tag{54}$$

The inverse temperature, identified with the coefficient of ω is equal to

$$T = \left(\frac{1}{T_H} + \alpha\frac{16\pi T_H}{1+16aM}\right)^{-1}. \tag{55}$$

After manipulating the above equation, one can find that T is nothing but the QG corrected Hawking temperature (52). Namely, $T = T_{QG}$.

IV. HJ METHOD WITHIN IEF COORDINATES

IEF coordinates are another regular coordinate system at the event horizon which was originally constructed by [43, 44]. These coordinates are aligned with radially moving photons. The generic metric (3) takes the following form in the IEF coordinates (e.g. [45])

$$ds^2 = -H d\upsilon^2 + 2\sqrt{1-H}\, d\upsilon dr + dr^2 + r^2 d\Omega^2, \tag{56}$$

in which υ is a null coordinate which is the so-called advanced time. It is given by

$$\upsilon = t + r_*, \tag{57}$$

where r_* is known as the tortoise coordinate. For the outer region of the GMHBH, it is found to be

$$r_* = \int \frac{dr}{H} = \frac{1}{2a(r_h - r_0)} \ln\left[\frac{(\frac{r}{r_h}-1)^{rh}}{(r-r_0)^{r_0}}\right], \tag{58}$$

Since the metric (56) has a Killing vector field of $\xi^\mu = \partial_\upsilon$, in this coordinate system an observer measures the scalar particle's energy by $E = -\partial_\upsilon I$. In this regard, the action is assumed to be of the form

$$I = -E\upsilon + W_{EF}(r) + J(x^i). \tag{59}$$

By using the above ansatz in the Eq. (15) for the metric (56), the final expression for $W_{EF}(r)$ is found as

$$W_{EF}(r) = \int \frac{E}{H}\left(1 \pm \sqrt{1 - \frac{\tau H}{E^2}}\right) dr, \tag{60}$$

in which

$$\tau = m^2 + \frac{J_\theta^2}{r^2} + \frac{J_\varphi^2}{r^2 \sin^2\theta}, \tag{61}$$

Around the event horizon, we see that $W_{EF}(r)$ reduces to the following expression

$$W_{EF(\pm)} = E \int \frac{1}{H}(1 \pm 1) dr, \tag{62}$$

which is nothing but the same expression obtained in Eq. (34). Hereupon, applying our standard procedure we get

$$W_{EF(-)} = 0, \;\; W_{EF(+)} = \frac{i r_h \pi E}{a(r_h - r_0)} \rightarrow \mathrm{Im}I = \mathrm{Im}W_{EF(+)} = \frac{\pi E r_h}{a(r_h - r_0)}, \tag{63}$$

and likewise to Sec. IV, the horizon temperature computed for the GMHBH in the IEF coordinates is of course that of the Hawking temperature:

$$\check{T}_H = \frac{a(r_h - r_0)}{2\pi r_h} = T_H. \tag{64}$$

V. HJ METHOD WITHIN KS COORDINATES

Another non-singular coordinate system which covers the whole spacetime manifold of the maximally extended BH solution is known as the KS coordinates [46, 47]. These coordinates are generally used to properly chart the spacetimes with the form of metric (3). Namely, the KS coordinates are able to squeeze infinity into a finite distance, and thus the entire spacetime can be visualized on a stamp-like diagram. In this section, we shall employ the HJ equation for the KS form of the GMHBH in order to represent how one gets the T_H via the HJ method.

We can rewrite the metric (3) in the following form, as made in [48],

$$ds^2 = -H du dv + r^2 d\Omega^2, \tag{65}$$

where

$$du = dt - dr_* \;\; dv = dt + dr_*, \tag{66}$$

After defining new coordinates (U, V) in terms of the surface gravity (10) which are given by

$$U = -e^{-\kappa u}, \;\; V = e^{\kappa v}, \tag{67}$$

we transform metric (65) to the KS form

$$ds^2 = \frac{H}{\kappa^2}\frac{dU dV}{UV} + r^2 d\Omega^2. \tag{68}$$

More explicitly, Eq. (68) becomes

$$ds^2 = -\pounds dUdV + r^2 d\Omega^2, \tag{69}$$

where

$$\pounds = \frac{2r_h^3}{ar(r_h - r_0)^2}(r - r_0)^{1+\frac{r_0}{r_h}}, \tag{70}$$

This metric is regular everywhere except at the physical singularity $r = 0$. Alternatively, the metric (69) can be transformed into

$$ds^2 = -\pounds(d\Im^2 - dR^2) + r^2 d\Omega^2, \tag{71}$$

which can be made by the following transformations

$$\Im = \frac{1}{2}(V + U) = \frac{(\frac{r}{r_h} - 1)^{\frac{1}{2}}}{(r - r_0)^{\frac{r_0}{2r_h}}} \sinh(\kappa t), \tag{72}$$

$$R = \frac{1}{2}(V - U) = \frac{(\frac{r}{r_h} - 1)^{\frac{1}{2}}}{(r - r_0)^{\frac{r_0}{2r_h}}} \cosh(\kappa t), \tag{73}$$

From these foregoing equations, we immediately observe that

$$R^2 - \Im^2 = \frac{(\frac{r}{r_h} - 1)}{(r - r_0)^{\frac{r_0}{r_h}}}, \tag{74}$$

which means that $R = \pm\Im$ corresponds to the future and past horizons. In other respects, here $\partial_\Im$ is not a timelike Killing vector for the metric (71). So, it is profitable to consider the timelike Killing vector of the metric in the following form

$$\partial_{\widehat{T}} = N(R\partial_\Im + T\partial_R), \tag{75}$$

where N denotes the normalization constant. It admits a specific value that the norm of the Killing vector becomes negative unity at R_d (19) which is the outer region of the GMHBH. Therefore, at that specific location the normalization constant is found to be

$$N = \frac{r_h - r_0}{r_h}\sqrt{\frac{ar}{2(r - r_h)(r - r_0)}}\Bigg|_{r=R_d} = \frac{a(r_h - r_0)}{r_h}, \tag{76}$$

Without loss of generality, we may only consider the (1+1) dimensional form of the KS metric (71) which is

$$ds^2 = -\pounds(dT^2 - dR^2), \tag{77}$$

The calculation of the HJ method is more straightforward in this case. The HJ equation (15) for the above metric reads

$$-\pounds^{-1}\left[-\left(\partial_{\Im} I\right)^2 + \left(\partial_R I\right)^2\right] + m^2 = 0, \tag{78}$$

This equation implies that the ansatz for the I could be written as

$$I = \rho(R - \Im) + J(x^i), \tag{79}$$

For simplicity, we may further set $J(x^i) = 0$ and $m = 0$. Now, the energy can be defined as

$$E = -\partial_{\widehat{T}} I, \tag{80}$$

which is equivalent to

$$E = -\frac{a(r_h - r_0)}{r_h}(R\partial_{\Im} I + T\partial_R I), \tag{81}$$

Using the above equation with ansatz (79), one derives the following expression.

$$\rho(y) = \int \frac{Er_h}{a(r_h - r_0)y} dy, \tag{82}$$

where $y = R - \Im$. The above expression has a divergence at the horizon $y = 0$, namely $R = \Im$. Thus, it leads to a pole at the horizon which could be overcome by doing a semi-circular contour of integration in the complex plane. The result is found to be

$$\mathrm{Im} I = \frac{\pi r_h E}{a(r_h - r_0)}. \tag{83}$$

which means that the Hawking temperature, $T_H = \frac{a(r_h - r_0)}{2\pi r_h}$, is impeccably recovered in the background of the KS metric of the GMHBH.

VI. CONCLUSION

In this paper, by using the relativistic HJ equation we have studied the HR in the GMHBH background engendered by the theory of $f(\Re)$ gravity. Today, the GMHBH has become prominent since it is considered as one of the significant theoretical astrophysical models in which the dark matter halo and the flat galactic rotation curves are taken into account. In addition to its naive coordinates, three different regular coordinate systems which are PG, IEF and KS have been employed throughout the present study. It has been shown in detail that the computed horizon temperatures via the HJ method exactly matches with the standard Hawking temperature. Among the ansätze that we have used for the HJ equation (15) in the former sections, the one belonging to the KS coordinates is different than others. Because in the KS coordinates the time coordinate is not in a simplex form. To this end, we have first found a proper timelike Killing vector having a normalization constant N (76) such that the norm of this Killing vector becomes negative unity at R_d (19). Subsequently, with aid of this Killing vector we have managed to identify an ansatz I which results in T_H within the process of HJ method. During this computation, without loss of generality, we have discarded the mass of the scalar particle and neglected the angular dependence of the HJ equation.

In the PG coordinates, we have also considered the back reaction effects in the PWT method for the HR of the GMHBH. The modified tunneling rate (48) has been computed via the log-area correction to the Bekenstein-Hawking entropy (49). From this, QG corrected Hawking temperature (i.e., T_{QG}) have also been found.

Finally, it is of interest to extend our analysis to yet another particle other than spin-0, which could be photon and fermion. In other words, it will be interesting to examine whether Maxwell and Dirac equations [49] on the GMHBH geometry within the HJ and PWT methods yield the T_H or not. This is going to be our next work in the near future.

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