Dilatonic interpolation between Reissner-Nordström and Bertotti-Robinson spacetimes with physical consequences

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Abstract

We give a general class of static, spherically symmetric, non-asymptotically flat and asymptotically non-(anti) de Sitter black hole solutions in Einstein-Maxwell-Dilaton (EMD) theory of gravity in 4-dimensions. In this general study we couple a magnetic Maxwell field with a general dilaton potential, while double Liouville-type potentials are coupled with the gravity. We show that the dilatonic parameters play the key role in switching between the Bertotti-Robinson and Reissner-Nordström spacetimes. We study the stability of such black holes under a linear radial perturbation, and in this sense we find exceptional cases that the EMD black holes are unstable. In continuation we give a detailed study of the spin-weighted harmonics in dilatonic Hawking radiation spectrum and compare our results with the previously known ones. Finally, we investigate the status of resulting naked singularities of our general solution when probed with quantum test particles.

I. INTRODUCTION

We revisit the 4−dimensional Einstein-Maxwell-Dilaton (EMD) theory and show that there are still plenty of rooms available to contribute the subject. Double Liouville potential and general dilaton coupling is considered to obtain more general solutions with extra parameters and diagonal metric in the theory. From the outset we remind that, depending on the relative parameters, the double Liouville potential has the advantage of admitting local extrema and critical points. The Higgs potential also shares such features, whereas single Liouville potential lacks these properties. Double Liouville-type potentials arise also when higher-dimensional theories are compactified to 4−dimensional spacetimes and expectedly bring in further richness. All known solutions to date can be obtained [1–3] as particular limits of our general solution, and it contains new solutions as well. In the most general form our solution covers Reissner-Nordstrom (RN) type black holes and Bertotti-Robinson (BR) spacetimes interpolated within the same metric. Interpolation of two different solutions in general relativity is not a new idea [4]. Particular limits of the dilatonic parameter yield the RN and BR spacetimes. In between the two, the linear dilaton black hole (LDBH) lies for the specific choice of the parameters. It is well-known that the near horizon geometry of the extremal RN black hole yields the BR electromagnetic universe. The latter [5] is important for various reasons: It is a singularity free non-black hole solution which admits maximal symmetry and finds application in conformal field theory correspondence (i.e. AdS/CFT). Particles in the BR universe move with uniform acceleration in a conformally flat background. These features are mostly valid not only in $N = 4$ but in higher dimensions ($N > 4$) as well. The topological structure of the BR spacetime is still $AdS_2 \times S^{N-2}$ in $N-$dimensions with the radius of S^{N-2} depending on the dimension of the space. Recently we have extended the Maxwell part of the BR spacetime to cover the Yang-Mills (YM) field and obtained common features that share with the Maxwell field [6]. The dilatonic black hole solution involved in the general solution obtained in this paper is non-asymptotically flat, therefore we expressed it in terms of the quasi local mass (M_{QL}) [7]. The metric is regular at horizons with only available singularity at $r = 0$. Another feature is the asymptotic ($r \to \infty$) absence of (anti) de-Sitter property which was discovered also within the context of different models [3]. Our general solution has been tested for stability against the radial, linear perturbations. We found that presence of dilaton can trigger instability in the RN black hole which is stable otherwise. Our analysis proves that the BR sector remains manifestly stable against such perturbations. Thermodynamic stability has also been discussed briefly by considering the specific heat of the metric. Divergence in the specific heat for specific values of the parameters signals phase transition in our thermodynamic system, i.e., topology change in the spacetime.

Next, we concentrate ourselves on the LDBH case and analyze the Hawking temperature both from semi-classical and standard surface gravity methods [8, 9]. We point out the contrasts between the two methods when there are single and double horizons. The high frequency limit of the semi-classical radiation spectrum method (*SCRSM*) does not agree with the Hawking's result. It is observed, as an interesting contribution in this work that the coupling between scalar field charge and the magnetic charge of the spacetime gives rise to spin-weighted spheroidal harmonics which plays a dominant role in the difference. In the absence of such coupling, when the scalar field is assumed chargeless for instance, similar analysis was carried out previously and we had recovered the same results easily. It turns out that the very existence of a spin-weighted spheroidal harmonics in the theory transforms a divergent temperature spacetime to a finite one. We argue that such a behavior may play a leading role in the detection of such LDBHs.

In the final section of the paper we appeal once more to the test scalar field equation, but this time with the purpose to investigate the quantum nature of the naked singularities. We identify first the particular solution that yields horizonless naked singularity at $r = 0$. By invoking the Horowitz-Marolf [10] criterion on quantum nature of classical singularities we explore under which set of parameters classically singular but quantum mechanically regular metrics can occur in our general solution.

The organization of our paper is as follows. In Sec. II we introduce our action, field equations and obtain the general solution. Sec. III singles out the linear dilaton case and investigates the stability of our general solution. Application of the *SCRSM* and its connection with the Hawking temperature is employed in Sec. IV. Sec. V discusses the status of naked singularities from quantum picture. We summarize our results in conclusion which appears in Sec. VI.

II. FIELD EQUATIONS AND THE METRIC ANSATZ FOR EMD GRAVITY

The 4−dimensional action in the EMD theory is given by $(8\pi G = 1)$

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - \frac{1}{2}W(\phi)\left(F_{\lambda\sigma}F^{\lambda\sigma}\right)\right), \tag{1}$$

where

$$V(\phi) = V_1 e^{\beta_1\phi} + V_2 e^{\beta_2\phi}, \quad W(\phi) = \lambda_1 e^{-2\gamma_1\phi} + \lambda_2 e^{-2\gamma_2\phi}. \tag{2}$$

ϕ refers to the dilaton scalar potential and γ_i denotes the dilaton parameter, λ_i is a constant and $V(\phi)$ is a double Liouville-type potential. We note that we exclude the simultaneous values $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$ in general, since these particular values lead to the already known cases.

Let us remark that although double Liouville potential in $V(\phi)$, which renders local minima, necessary for construction of vacuum states possible, the similar choice for $W(\phi)$ seems less appealing. It will be justified from the exact solutions below, however, that there are asymptotics which remains inaccessible by the choice of a single Liouville term in $W(\phi)$. Stated otherwise, at both asymptotes of $r = 0$ and $r = \infty$ (or $\tilde{r} = 0$ and $\tilde{r} = \infty$ for LDBH) dilatonic coupling to the magnetic field becomes much stronger. Choosing a single Liouville potential simply looses the strength at one end of the range. Besides, it is all a matter of choice to set $\lambda_1(\lambda_2) = 0$, which makes the dilatonic coupling asymptotically free. In the LDBH case as it will be proved, if we set $\lambda_1 = 0$, we shall remove the possibility of an inner (Cauchy) horizon which justifies the advantages and motivation for choosing the double Liouville-type potential in $W(\phi)$. In (1) R is the usual Ricci scalar and $\mathbf{F} = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ is the Maxwell 2−form (with $\wedge$ indicating the wedge product) given by

$$\mathbf{F} = \mathbf{dA}, \tag{3}$$

for $\mathbf{A} = A_\mu dx^\mu$, the potential 1−form. Our pure magnetic potential with charge Q, which is given by

$$\mathbf{A} = -Q\cos\theta \; d\varphi, \tag{4}$$

leads to

$$\mathbf{F} = Q\sin\theta \; d\theta \wedge d\varphi. \tag{5}$$

Let us note that with the present choice of $W(\phi)$ the electric-magnetic symmetry that exists in the standard dilatonic coupling, i.e., $\lambda_1(\lambda_2) = 0$, is no more valid. Our choice in this paper relies entirely on the magnetic choice. Variations of the action with respect to the gravitational field $g_{\mu\nu}$ and the scalar field ϕ lead, respectively to the EMD field equations

$$R_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + V(\phi) g_{\mu\nu} + W(\phi) \left(2F_{\mu\lambda} F_\nu{}^\lambda - \frac{1}{2} F_{\lambda\sigma} F^{\lambda\sigma} g_{\mu\nu} \right), \tag{6}$$

$$\nabla^2 \phi - V'(\phi) - \frac{1}{2} W'(\phi) (F_{\lambda\sigma} F^{\lambda\sigma}) = 0, \tag{7}$$

$$\left(\prime \equiv \frac{d}{d\phi} \right),$$

where $R_{\mu\nu}$ is the Ricci tensor. Variation with respect to the gauge potential $\mathbf{A}$ yields the Maxwell equation

$$\mathbf{d} \left(W(\phi)^\star \mathbf{F} \right) = 0, \tag{8}$$

in which the hodge star * means duality.

A. Ansatz and the Solutions:

Our ansatz line element for EMD gravity is chosen to be

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + R(r)^2 \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right), \tag{9}$$

with $f(r)$ and $R(r)$ only function of r while the Maxwell invariant takes the form

$$F_{\lambda\sigma} F^{\lambda\sigma} = \frac{2Q^2}{R^4}. \tag{10}$$

The Maxwell equation (8) is satisfied automatically and the field equations become

$$\nabla^2 \phi := \frac{1}{R^2} \left(R^2 f \phi' \right)' = V'(\phi) + \frac{1}{2} W'(\phi) (F_{\lambda\sigma} F^{\lambda\sigma}), \tag{11}$$

$$R_t^t := -\frac{\left(f' R^2 \right)'}{2R^2} = V(\phi) - \frac{W(\phi)}{2} (F_{\lambda\sigma} F^{\lambda\sigma}), \tag{12}$$

$$R_r^r := -\frac{2fR''}{R} - \frac{\left(f' R^2 \right)'}{2R^2} = f\phi'^2 + V(\phi) - \frac{W(\phi)}{2} (F_{\lambda\sigma} F^{\lambda\sigma}), \tag{13}$$

$$R_\theta^\theta = R_\varphi^\varphi := \frac{1 - \left(fRR' \right)'}{R^2} = V(\phi) + \frac{2Q^2 W(\phi)}{R(r)^4} - \frac{W(\phi)}{2} (F_{\lambda\sigma} F^{\lambda\sigma}), \tag{14}$$

in which a prime stands for derivative with respect to the argument of the function. We start with an ansatz for $R(r)$ as

$$R(r) = Ae^{\eta\phi} \tag{15}$$

in which A and η are constants to be found. Substitution in (12) and (13), implies

$$\phi(r) = \frac{2\eta}{2\eta^2+1}\ln r. \tag{16}$$

Finally by putting these results into Eq.s (11) and (14) one finds that by setting

$$\eta = -\frac{1}{\alpha\sqrt{2}}, \tag{17}$$

and

$$\begin{aligned} \gamma_1 &= -\frac{\alpha}{\sqrt{2}}, \quad \gamma_2 = \frac{1}{\alpha\sqrt{2}}, \\ \beta_1 &= \sqrt{2}\alpha, \quad \beta_2 = \frac{\sqrt{2}}{\alpha}, \end{aligned} \tag{18}$$

a general solution for $f(r)$ reads

$$\begin{aligned} f(r) = \left(1+\alpha^2\right)^2 \Bigg[& \frac{Q^2\lambda_1 r^{\frac{-2}{1+\alpha^2}}}{(1+\alpha^2)A^4} + \\ & \left(\frac{Q^2\lambda_2}{A^4} - V_2\right)\frac{r^{\frac{2\alpha^2}{1+\alpha^2}}}{(1+\alpha^2)\alpha^2} - \frac{V_1 r^{\frac{2}{1+\alpha^2}}}{3-\alpha^2} - Mr^{-\frac{1-\alpha^2}{1+\alpha^2}}\Bigg], \end{aligned} \tag{19}$$

with the constraint condition

$$-V_2\left(1-\alpha^2\right)A^4 - \alpha^2 A^2 + \lambda_2 Q^2\left(1+\alpha^2\right) = 0. \tag{20}$$

Herein M is a mass-related integration constant and α and A are constants that will serve to parametrize the solution. We note that in case that we are interested in the Newtonian limit, when $\alpha = 0 = \lambda_2 = V_2$, and $r \to \infty$, we must choose $M \to 2M$, so that M represents the Newtonian mass. Therefore the dilatonic function ϕ, Liouville potential V and W in terms of α become

$$\begin{aligned} \phi(r) &= -\frac{\alpha\sqrt{2}}{1+\alpha^2}\ln r, \quad R(r) = Ar^{\frac{1}{1+\alpha^2}}, \\ V &= V_1 r^{\frac{-2\alpha^2}{1+\alpha^2}} + V_2 r^{\frac{-2}{1+\alpha^2}}, \; W = \lambda_1 r^{\frac{-2\alpha^2}{1+\alpha^2}} + \lambda_2 r^{\frac{2}{1+\alpha^2}}. \end{aligned}$$

We remark that the solution (16-20) is the general diagonal solution that covers all particular solutions of this kind known so far. For arbitrary value of α, other then 0, 1 and ∞, it yields a new solution in accordance with our ansatz. It is observed also that our metric and potentials are invariant under $\alpha \to -\alpha$, whereas $\phi \to -\phi$. The asymptotic behavior of the metric function, $f(r)$ and other limiting cases can be summarized as follows

$$\lim_{r\to\infty} f(r) \to \begin{cases} \left(1+\alpha^2\right)^2 \left(-\frac{V_1 r^{\frac{2}{1+\alpha^2}}}{3-\alpha^2}\right) & 0 \leq \alpha^2 < 1 \\ 2\left(\frac{Q^2\lambda_2}{A^4} - V_1 - V_2\right) r & \alpha^2 = 1 \\ \left(\frac{1+\alpha^2}{\alpha^2}\right)\left(\frac{Q^2\lambda_2}{A^4} - V_2\right) r^{\frac{2\alpha^2}{1+\alpha^2}} & 1 < \alpha^2 \end{cases} \tag{21}$$

$$\lim_{r\to 0^+} f(r) \to \left(1+\alpha^2\right)\left(\frac{Q^2\lambda_1}{A^4 r^{\frac{2}{1+\alpha^2}}}\right). \tag{22}$$

The case $\alpha^2 = 1$ will be studied separately, while the case $\alpha^2 = 0$, with the choice of $\lambda_2 = 0$, $V_2 = 0$, and $A = 1 = \lambda_1$ leads to

$$\begin{aligned} f(r) &= 1 - \frac{V_1}{3}r^2 - \frac{M}{r} + \frac{Q^2}{r^2}, \\ R(r) &= r, \phi = 0, \end{aligned} \tag{23}$$

which corresponds to the action

$$S_{\alpha^2=0} = \int d^4x\sqrt{-g}\left(\frac{1}{2}R - V_1 - \frac{1}{2}(F_{\lambda\sigma}F^{\lambda\sigma})\right). \tag{24}$$

This is recognized as the 4−dimensional action in the EM theory with the solution representing a RN black hole with a cosmological constant. Another limiting case of interest consists of the case with $\alpha^2 \to \infty$, $\lambda_2 = 1$, with the action

$$S_{\alpha^2=\infty} = \int d^4x\sqrt{-g}\left(\frac{1}{2}R - V_2 - \frac{1}{2}(F_{\lambda\sigma}F^{\lambda\sigma})\right), \tag{25}$$

leading to the solution

$$\begin{aligned} f(r) &= \left(\frac{Q^2}{A^4} - V_2\right) r^2 - \tilde{M} r, \\ A^2\left(V_2 A^2 + 1\right) &= Q^2, \\ R(r) &= A, \quad \phi(r) = 0, \end{aligned} \tag{26}$$

in which $\tilde{M}$ is the mass related integration constant. Here also we have a 4−dimensional action in the EM theory with cosmological constant but the metric function represents a BR space time.

By looking at the asymptotic behaviors of the general solution one finds that $0 \leq \alpha^2 < 1$ and $1 < \alpha^2$ correspond to the cases of RN and BR solutions, respectively. Here $\alpha^2 = 1$ acts much like a phase transition which changes the structure of space time from RN into BR. The thermodynamic instability from the expression of specific heat capacity C_Q, (Eq. (58) given below) justifies this fact. It is quite interesting to see what will be the answer if one chooses $\alpha^2 = 1$. In the next section we concentrate on this critical value for α^2.

III. THE LINEAR DILATON

From the asymptotic behavior of the metric function one may see that $\alpha^2 = 1$ is a critical value and the behavior of spacetime changes. In this chapter we only concentrate on this specific value for α^2, and will be referred to as linear dilaton. The general solution after this setting reads

$$\begin{aligned}
-\gamma_1 = \gamma_2 &= \frac{1}{\sqrt{2}}, \\
\beta_1 = \beta_2 &= \sqrt{2}, \\
\phi(r) = -\frac{1}{\sqrt{2}} \ln r, &\quad R(r) = A\sqrt{r}, \\
V = \frac{\tilde{V}}{r}, &\quad W = \frac{\lambda_1}{r} + \lambda_2 r, \\
A^2 = 2\lambda_2 Q^2, &\quad (\lambda_2 > 0)
\end{aligned} \tag{27}$$

$$f(r) = \left[\frac{\lambda_1}{\lambda_2 A^2 r} + \left(\frac{1}{A^2} - 2\tilde{V}\right) r - \tilde{M}\right], \tag{28}$$

where $\tilde{V} = V_1 + V_2$.

In order to explore the physical properties of the linear dilaton case we perform the transformation $R(r) = A\sqrt{r} \to \widetilde{r}$. This transforms the metric into,

$$ds^2 = -f(\widetilde{r})dt^2 + \frac{4\widetilde{r}^2}{A^4 f(\widetilde{r})} d\widetilde{r}^2 + \widetilde{r}^2 d\Omega^2, \tag{29}$$

in which

$$f(\widetilde{r}) = \frac{1}{\widetilde{r}^2}\left(\left(\frac{1}{A^2} - 2\tilde{V}\right)\frac{\widetilde{r}^4}{A^2} - M_{QL}\widetilde{r}^2 + \frac{\lambda_1}{\lambda_2}\right), \tag{30}$$

where the mass M_{QL} denotes the quasilocal mass whose general definition is given below in Eq. (46). Other related parameters transform into the following forms,

$$\phi(\widetilde{r}) = \sqrt{2}\ln\left(\frac{A}{\widetilde{r}}\right), \tag{31}$$

$$V(\widetilde{r}) = \frac{A^2}{\widetilde{r}^2}(V_1 + V_2), \qquad W(\widetilde{r}) = \frac{\lambda_1 A^2}{\widetilde{r}^2} + \frac{\lambda_2 \widetilde{r}^2}{A^2}.$$

The location of horizons can be found if we set the metric function $g_{tt} = 0$. The solution is

$$\widetilde{r}_h = \frac{1}{\sqrt{2a}}\sqrt{M_{QL} \pm \sqrt{M_{QL}^2 - 4ac}}, \tag{32}$$

where

$$a = \left(\frac{1}{A^2} - 2\tilde{V}\right)\frac{1}{A^2}, \qquad c = \frac{\lambda_1}{\lambda_2}. \tag{33}$$

The linear dilaton solution admits single or double-horizons if the parameters are chosen appropriately. Another possible case is the extremal limit that occurs if $M_{QL}^2 = 4ac$. The horizon in this particular case is given by $\widetilde{r}_h = \sqrt{\frac{M_{QL}}{2a}}$. The double horizon case occurs if the parameters simultaneously satisfy $M_{QL} > \sqrt{M_{QL}^2 - 4ac}$ and $M_{QL}^2 > 4ac$. This choice leads to the horizons

$$\widetilde{r}_+ = \sqrt{\frac{M_{QL} + \sqrt{M_{QL}^2 - 4ac}}{2a}}, \tag{34}$$

$$\widetilde{r}_- = \sqrt{\frac{M_{QL} - \sqrt{M_{QL}^2 - 4ac}}{2a}}.$$

The relations between the parameters and double-Liouville-type potentials in the formation of black holes becomes evident if one looks for the critical case. This is the case when $M_{QL} = \sqrt{M_{QL}^2 - 4ac}$, which follows that $2\tilde{V} = \frac{1}{A^2}$. Hence if $2\tilde{V} < \frac{1}{A^2}$, no horizon forms and the central singularity $\widetilde{r} = 0$ becomes a *naked* singularity. It can easily be seen that for $\lambda_1 = 0$, or for the single Liouville-type potential in $W(\phi)$, we have automatically single, outer event horizon alone. Another interesting property is in the behavior of the curvature scalar R. The curvature scalar for the metric function (29) is,

$$R = -\frac{4\widetilde{r}^4(aA^4 - 1) - A^4(a\widetilde{r}^4 - M_{QL}\widetilde{r}^2 + c)}{2\widetilde{r}^6}. \tag{35}$$

Note that the curvature scalar is finite at the location of horizons. Furthermore, when $\widetilde{r} \to \infty$, the Kretschmann and curvature scalars, the Liouville-type potentials and the coupling term of dilaton with Maxwell field all vanish. The mass and charge are finite and the dominant field is gravity with finite curvature. Consequently, the solution given in Eq. (29) is well-behaved. However, the $Q = 0$ limit does not exist.

A. Linear stability analysis of the general solution

By employing a similar method used by Yazadjiev [11] we investigate the stability of the possible EMD solution, in terms of a linear, radial perturbation. To do so we assume that our dilatonic scalar field $\phi(r)$ changes into $\phi_\circ(r) + \psi(t,r)$, in which $\psi(t,r)$ is very weak compared to the original dilaton field $\phi_\circ(r)$ and we call it the perturbed term. As a result we choose our perturbed metric as

$$ds^2 = -f(r)\, e^{\Gamma(t,r)} dt^2 + e^{\chi(t,r)} \frac{dr^2}{f(r)} + R(r)^2\, d\Omega_2^2. \tag{36}$$

One should notice that, since our gauge potentials are magnetic, the Maxwell equations (Eq.(8)) are satisfied. The linearized version of the field equations (11-14) plus one extra term for R_{tr} are given by

$$R_{tr} : \frac{\chi_t(t,r)\, R'(r)}{R(r)} = \partial_r \phi_\circ(r)\, \partial_t \psi(t,r) \tag{37}$$

$$\nabla_\circ^2 \psi - \chi \nabla_\circ^2 \phi_\circ + \frac{1}{2}(\Gamma - \chi)_r\, \phi_\circ' f - \partial_{\phi_\circ}^2 V(\phi_\circ)\, \psi = \frac{Q^2}{R(r)^4} \partial_{\phi_\circ}^2 W(\phi_\circ)\, \psi \tag{38}$$

$$R_{\theta\theta} : (1 - R_{\circ\theta\theta})\, \chi - \frac{1}{2} R R' f (\Gamma - \chi)_r = \left(R^2 \partial_{\phi_\circ} V(\phi_\circ) + \frac{Q^2}{R^2} \partial_{\phi_\circ} W(\phi_\circ) \right) \psi \tag{39}$$

in which a lower index $_\circ$ represents the quantity in the unperturbed metric. First equation in this set implies

$$\chi(t,r) = \frac{1}{\eta} \psi(t,r) \tag{40}$$

which after making substitutions in the two latter equations and eliminating the $(\Gamma - \chi)_r$ one finds

$$\nabla_\circ^2 \psi(t,r) - U(r)\, \psi(t,r) = 0 \tag{41}$$

where

$$U(r) = \frac{2}{r^{\frac{2}{1+\alpha^2}}} \left\{ \frac{\alpha^2}{A^2} + \left(\frac{Q^2 \lambda_2}{A^4} + V_2 \right) \left(\frac{1 - \alpha^4}{\alpha^2} \right) \right\}. \tag{42}$$

To get these results we have implicitly used the constraint (20) on A. Again by imposing the same constraint , one can show that $U(r)$ is positive. It is not difficult to apply the separation method on (41) to get

$$\psi(t,r) = e^{\pm\epsilon t}\zeta(r), \quad \nabla_{\circ}^{2}\zeta(r) - U_{eff}(r)\zeta(r) = 0, \quad U_{eff}(r) = \left(\frac{\epsilon^2}{f} + U(r)\right), \tag{43}$$

where ϵ is a constant. If one shows that the effective potential $U_{eff}(r)$ is positive for any real value for ϵ it means that there exists a solution for $\zeta(r)$ which is not bounded. In other words by the linear perturbation our black hole solution is stable for any value of ϵ.

But in our case one must be careful. For instance let's go back to the general solution (19) and set $V_1 = 0$,

$$f(r) = \frac{(1+\alpha^2)}{r^{\frac{2}{1+\alpha^2}}}\left\{\left(\frac{Q^2\lambda_2}{A^4} - V_2\right)\frac{r^2}{\alpha^2} - M\left(1+\alpha^2\right)r + \frac{Q^2\lambda_1}{A^4}\right\}, \tag{44}$$

this solution may have double horizons, single horizon (extremal) or no horizon. These depend on the values of the parameters. One may notice that this solution is a non-asymptotically flat metric and therefore the ADM mass is not defined in general. Following the quasilocal mass formalism introduced by Brown and York [7] it is known that, a spherically symmetric $N-$dimensional metric solution as

$$ds^2 = -F(R)^2 dt^2 + \frac{dR^2}{G(R)^2} + R^2 d\Omega_{N-2}^2, \tag{45}$$

admits a quasilocal mass M_{QL} defined by [6, 7]

$$M_{QL} = \frac{N-2}{2} R_B^{N-3} F(R_B)\left(G_{ref}(R_B) - G(R_B)\right). \tag{46}$$

Here $G_{ref}(R)$ is an arbitrary non-negative reference function, which yields the zero of the energy for the background spacetime, and R_B is the radius of the spacelike hypersurface boundary. Applying this formalism to the solution (44), one obtains the horizon M in terms of M_{QL} as

$$M = \frac{2}{(1+\alpha^2)A^2} M_{QL}, \tag{47}$$

after which the metric function becomes

$$f(r) = \frac{(1+\alpha^2)}{r^{\frac{2}{1+\alpha^2}}}\left\{\left(\frac{Q^2\lambda_2}{A^4} - V_2\right)\frac{r^2}{\alpha^2} - \frac{2M_{QL}}{A^2}r + \frac{Q^2\lambda_1}{A^4}\right\}. \tag{48}$$

Indeed, since we wish to cover all known solutions in the literature of this kind, we consider λ_i, $M_{QL} \geq 0$, and

$$\left(\frac{Q^2\lambda_2}{A^4} - V_2\right) \geq 0. \tag{49}$$

This condition together with Eq. (20) give a transparent view of $U_{eff}(r)$. In other words, after simplification, one can rewrite $U(r)$ as

$$U(r)=\frac{2}{r^{\frac{2}{1+\alpha^2}}}\left\{\left(\frac{Q^2\lambda_2}{A^4}-V_2\right)+\left(\frac{Q^2\lambda_2}{A^4}+V_2\right)\frac{1}{\alpha^2}\right\}, \tag{50}$$

which reveals for $-\frac{Q^2\lambda_2}{A^4}\leq V_2\leq\frac{Q^2\lambda_2}{A^4}$, $U(r)$ and then $U_{eff}(r)$ are positive, which means that the corresponding metric is stable. But for $V_2<-\frac{Q^2\lambda_2}{A^4}$, if $\alpha^2<\alpha^2_{critical}$ where

$$\alpha^2_{critical}=\frac{|V_2|-\frac{Q^2\lambda_2}{A^4}}{|V_2|+\frac{Q^2\lambda_2}{A^4}}, \tag{51}$$

then $U(r)$ gets negative value and therefore our solution faces an instability condition. Here it is interesting to note that $\alpha^2_{critical}<1$ belongs to the RN type black hole solutions, i.e. BR type solution is automatically stable for any value of α^2.

The general solution reveals another interesting case after we set $V_1=0$, and $\lambda_2=0$ i.e.

$$f(r)=\frac{(1+\alpha^2)}{r^{\frac{2}{1+\alpha^2}}}\left\{-\frac{V_2}{\alpha^2}r^2-\frac{2M_{QL}}{A^2}r+\frac{Q^2\lambda_1}{A^4}\right\}. \tag{52}$$

Upon choosing $V_2<0$ ($V_2>0$) this admits the effective potential

$$U(r)=\frac{2|V_2|}{r^{\frac{2}{1+\alpha^2}}}\left(\frac{\alpha^2-1}{\alpha^2}\right) \tag{53}$$

which clearly from (20), for $\alpha^2<1$ ($\alpha^2>1$) manifests an unstable black hole solution. As a result we observe that a stable RN black hole becomes unstable under certain conditions in the presence of a dilaton and a Liouville potential.

B. Thermodynamic stability

Concerning the solution (19), we set the parameters $\lambda_1=\lambda_2=1$ and $V_1=0$ to get

$$f(r)=\frac{(1+\alpha^2)}{r^{\frac{2}{1+\alpha^2}}}\left\{\left(\frac{Q^2}{A^4}-V_2\right)\frac{r^2}{\alpha^2}-\frac{2M_{QL}}{A^2}r+\frac{Q^2}{A^4}\right\} \tag{54}$$

which in terms of the radius of horizon r_h one finds the quasilocal mass as

$$M_{QL}=\frac{r_h^2\left(Q^2-V_2A^4\right)+Q^2\alpha^2}{2A^2\alpha^2r_h}. \tag{55}$$

The Hawking temperature

$$T_H=\frac{f'(r_h)}{4\pi}=\frac{(1+\alpha^2)\left[r_h^2\left(A^2-2Q^2\right)-Q^2\left(1-\alpha^2\right)\right]}{4\left(1-\alpha^2\right)A^4\pi r_h^{\frac{3+\alpha^2}{1+\alpha^2}}}. \tag{56}$$

and the Bekenstein-Hawking entropy

$$S = \frac{\mathfrak{a}}{4} = \pi r_h^2, \tag{57}$$

where $\mathfrak{a}$ is the area of the black hole, together lead to the heat capacity C_Q for constant Q as

$$C_Q = T_H \left(\frac{\partial S}{\partial T_H}\right)_Q = \frac{(\alpha^2+1)\left[r_h^2\left(2-\left(\frac{A}{Q}\right)^2\right)+1-\alpha^2\right]}{(\alpha^2-1)\left[r_h^2\left(2-\left(\frac{A}{Q}\right)^2\right)+3+\alpha^2\right]} 2\pi r_h^2. \tag{58}$$

Our black hole solution becomes thermodynamically stable /unstable depending on $C_Q > 0$ /$C_Q < 0$ which is not difficult to test from this expression. For $\left(\frac{A}{Q}\right)^2 < 2$ and $\alpha^2 < 1$, as an example, our black hole becomes thermodynamically unstable. Also for $\left(\frac{A}{Q}\right)^2 = 2$ one gets

$$C_Q = -\frac{\alpha^2+1}{3+\alpha^2} 2\pi r_h^2,$$

which shows an instability independent of the values of α. Tab. 1 illustrates the stable and unstable regions in terms of α^2 and $x = r_h^2\left(2-\left(\frac{A}{Q}\right)^2\right)$.

	$\alpha^2 - 1 < x$	$-1 < x < \alpha^2 - 1$	$-(3+\alpha^2) < x < -1$	$x < -(3+\alpha^2)$
$\alpha^2 < 1$	Unstable	Stable	Stable	Unstable
$\alpha^2 > 1$	Stable	Unstable	Unstable	Stable

(Table: 1)

Eq. (58) reveals also that $\alpha^2 = 1$ (i.e., the linear dilaton) is a phase transition point, however, there may be other possible transition points following a solution for α in the quadratic equation

$$r_h^2\left(2-\left(\frac{A}{Q}\right)^2\right)+3+\alpha^2 = 0. \tag{59}$$

IV. APPLICATION OF THE SCRSM AND HAWKING TEMPERATURE

In this section, we shall attempt to make a more precise temperature calculation for the non-extreme LDBHs, $\alpha^2 = 1$ given in Eq. (28), by using a method of semi-classical radiation spectrum, which has been recently designated as *SCRSM* [9]. The main difference between our present work with others [8, 9] (and references therein) is that the considered non-extreme LDBHs possess two horizons, due to having magnetic charge, instead of one.

Here, we first consider a massless scalar field Ψ with charge q obeying the covariant Klein-Gordon equation in the LDBH geometry. Namely, we look for the exact solution of the following equation,

$$\Box\Psi = 0, \tag{60}$$

where the d' Alembertian operator $\Box$ is given by

$$\Box = \frac{1}{\sqrt{-g}} D_\mu(\sqrt{-g} g^{\mu\nu} D_\nu), \tag{61}$$

in which D_μ symbolizes the covariant gauge differential operator as being

$$D_\mu = \partial_\mu - iqA_\mu. \tag{62}$$

The scalar wave function Ψ of Eq. (60) can be separated to the angular and radial equations by letting

$$\Psi = Z(r)\pounds(\theta)e^{i(m\varphi-\omega t)}, \tag{63}$$

the separated angular equation can be found as

$$\pounds'' + \cot\theta\pounds' + \left[\lambda\!\!\!^{-} - \frac{(m+p\cos\theta)^2}{\sin^2\theta}\right]\pounds = 0, \tag{64}$$

where $p = qQ$ and $\lambda\!\!\!^{-}$ is a separation constant. (From now on, a prime denotes the derivative with respect to its argument.) After setting the eigenvalue $\lambda\!\!\!^{-} = l(l+1) - p^2$ in Eq. (64), one can see that solutions to the angular part, $\pounds(\theta)e^{im\varphi}$, are the spin-weighted spheroidal harmonics ${}_pY_{lm}(\theta,\varphi)$ with spin-weight p [13].

On the other hand, before proceeding to the radial equation, one may rewrite the metric function $f(r)$ in Eq. (28) as

$$f(r) = \frac{b}{r}(r-r_2)(r-r_1), \tag{65}$$

where r_2 and r_1 denote the outer and inner horizons of the LDBHs, respectively. In the new form of the metric function Eq. (65), the physical parameters are

$$b=\frac{1}{A^2}-2\tilde{V}, \tag{66}$$
$$r_2=\frac{1}{2b}\left(c+\sqrt{c^2-4ab}\right),$$
$$r_1=\frac{1}{2b}\left(c-\sqrt{c^2-4ab}\right),$$

in which

$$c=\tilde{M}=4M \quad \text{and} \quad a=\frac{\lambda_1}{\lambda_2 A^2}. \tag{67}$$

Since the algorithm in the calculations of the *SCRSM* cover only the outer region of the black hole ($r>r_2$), we must impose a condition in order to keep $f(r)$ positive i.e. $b>0$. Henceforth, one can derive the following radial equation as

$$b(r-r_2)(r-r_1)Z''+b(2r-r_2-r_1)Z'+(\left(\frac{r^2\omega^2}{b(r-r_2)(r-r_1)}-\frac{\lambda\!\!\!^{-}}{A^2}\right)Z=0. \tag{68}$$

The above equation can be solved in terms of hypergeometric functions. Here, we give the final result as

$$Z(r)=C_1(r-r_2)^{i\tilde{\omega}r_2}(r-r_1)^{-i\tilde{\omega}r_1}F\left[\hat{a},\hat{b};\hat{c};\frac{r_2-r}{r_2-r_1}\right]+$$
$$C_2(r-r_2)^{-i\tilde{\omega}r_2}(r-r_1)^{-i\tilde{\omega}r_1}F\left[\hat{a}-\hat{c}+1,\hat{b}-\hat{c}+1;2-\hat{c};\frac{r_2-r}{r_2-r_1}\right]. \tag{69}$$

The parameters of the hypergeometric functions are

$$\hat{a}=\frac{1}{2}+i(\frac{\omega}{b}+\sigma), \quad \hat{b}=\frac{1}{2}+i(\frac{\omega}{b}-\sigma), \text{ and } \hat{c}=1+2i\tilde{\omega}r_2, \tag{70}$$

where

$$\sigma=\frac{1}{b}\sqrt{\omega^2-\frac{\lambda\!\!\!^{-} b}{A^2}-\left(\frac{b}{2}\right)^2}, \quad \tilde{\omega}=\omega\tilde{\eta}, \quad \text{and} \quad \tilde{\eta}=\frac{1}{b(r_2-r_1)}. \tag{71}$$

Here, σ is assumed to have real values. Furthermore, setting

$$r-r_2=\exp(\frac{x}{\tilde{\eta}r_2}), \tag{72}$$

one gets the behavior of the partial wave near the outer horizon ($r \to r_2$) as

$$\Psi \simeq C_1 e^{i\omega(x-t)} + C_2 e^{-i\omega(x-t)}. \tag{73}$$

One may infer the constants C_1 and C_2 as being the amplitudes of the near-horizon outgoing and ingoing waves, respectively.

In the literature, there exists a useful feature of the hypergeometric functions, which is a transformation of the hypergeometric functions of any argument (say z) to the hypergeometric functions of its inverse argument ($1/z$). The relevant transformation is given by [14]

$$\begin{aligned} F(\bar{a}, \bar{b}; \bar{c}; z) = & \frac{\Gamma(\bar{c})\Gamma(\bar{b}-\bar{a})}{\Gamma(\bar{b})\Gamma(\bar{c}-\bar{a})}(-z)^{-\bar{a}} F(\bar{a}, \bar{a}+1-\bar{c}; \bar{a}+1-\bar{b}; 1/z) \\ & + \frac{\Gamma(\bar{c})\Gamma(\bar{a}-\bar{b})}{\Gamma(\bar{a})\Gamma(\bar{c}-\bar{b})}(-z)^{-\bar{b}} F(\bar{b}, \bar{b}+1-\bar{c}; \bar{b}+1-\bar{a}; 1/z). \end{aligned} \tag{74}$$

The above transformation leads us to obtain the asymptotic behavior of the partial wave, easily. After applying the transformation to the general solution (69), we obtain the partial wave near-infinity as follows

$$\Psi \simeq \frac{(r-r_1)^{-i\tilde{\omega}r_1}}{\sqrt{r-r_2}} \left\{ B_1 \exp i \left[\frac{x}{\tilde{\eta}r_2}(\sigma + \omega\tilde{\eta}r_1) - \omega t \right] + B_2 \exp i \left[\frac{x}{\tilde{\eta}r_2}(-\sigma + \omega\tilde{\eta}r_1) - \omega t \right] \right\}. \tag{75}$$

On the other hand, since we consider the case of $r \to \infty$, the overall-factor term

$$(r-r_1)^{-i\tilde{\omega}r_1} \cong \exp i(-\frac{x\omega r_1}{r_2}), \tag{76}$$

whence the partial wave (75) reduces to

$$\Psi \simeq \frac{1}{\sqrt{r-r_2}} \left\{ B_1 \exp i \left[\frac{x}{\tilde{\eta}r_2}\sigma - \omega t \right] + B_2 \exp i \left[-\frac{x}{\tilde{\eta}r_2}\sigma - \omega t \right] \right\}, \tag{77}$$

where B_1 and B_2 correspond to the amplitudes of the asymptotic outgoing and ingoing waves, respectively. One can derive the relations between B_1,B_2 and C_1, C_2 as follows

$$B_1 = C_1 \frac{\Gamma(\widehat{c})\Gamma(\widehat{a}-\widehat{b})}{\Gamma(\widehat{a})\Gamma(\widehat{c}-\widehat{b})} + C_2 \frac{\Gamma(2-\widehat{c})\Gamma(\widehat{a}-\widehat{b})}{\Gamma(\widehat{a}-\widehat{c}+1)\Gamma(1-\widehat{b})}, \tag{78}$$

$$B_2 = C_1 \frac{\Gamma(\widehat{c})\Gamma(\widehat{b}-\widehat{a})}{\Gamma(\widehat{b})\Gamma(\widehat{c}-\widehat{a})} + C_2 \frac{\Gamma(2-\widehat{c})\Gamma(\widehat{b}-\widehat{a})}{\Gamma(\widehat{b}-\widehat{c}+1)\Gamma(1-\widehat{a})}.$$

Hawking radiation can be considered as the inverse process of scattering by the black hole such that the outgoing mode at the spatial infinity should be absent [8]. Briefly $B_1 = 0$, and it naturally yields the coefficient for reflection by the black hole as

$$R = \frac{|C_1|^2}{|C_2|^2} = \frac{\left|\Gamma(\widehat{c}-\widehat{b})\right|^2 |\Gamma(\widehat{a})|^2}{\left|\Gamma(1-\widehat{b})\right|^2 |\Gamma(\widehat{a}-\widehat{c}+1)|^2}, \tag{79}$$

which is equivalent to

$$R = \frac{\cosh \pi \left[\sigma - \frac{\omega}{b}(\frac{r_2+r_1}{r_2-r_1})\right] \cosh \pi \left(\sigma - \frac{\omega}{b}\right)}{\cosh \pi \left[\sigma + \frac{\omega}{b}(\frac{r_2+r_1}{r_2-r_1})\right] \cosh \pi (\sigma + \frac{\omega}{b})}. \tag{80}$$

Thus the resulting radiation spectrum is

$$N = \left(e^{\frac{\omega}{T}} - 1\right)^{-1} = \frac{R}{1-R} \quad \rightarrow \quad T = \frac{\omega}{\ln(\frac{1}{R})}, \tag{81}$$

and finally, one can read the more precise value of the temperature as

$$T = \omega / \ln \left[\frac{\cosh \pi \left[\sigma + \frac{\omega}{b}(\frac{r_2+r_1}{r_2-r_1})\right] \cosh \pi (\sigma + \frac{\omega}{b})}{\cosh \pi \left[\sigma - \frac{\omega}{b}(\frac{r_2+r_1}{r_2-r_1})\right] \cosh \pi \left(\sigma - \frac{\omega}{b}\right)} \right]. \tag{82}$$

This must be considered as the equilibrium temperature of the quantum field at the vacuum state valid for all frequencies. In the limit of ultrahigh frequencies ($\sigma \simeq \frac{\omega}{b}$), Eq. (82) reduces to

$$T_{high} \simeq \lim_{\omega \gg 1} T \simeq \frac{\omega}{\ln\left[\exp(\frac{4\pi\omega}{b})\right]} \simeq \frac{b}{4\pi}, \tag{83}$$

which smears out the $\omega-$dependence and results in a pure thermal spectrum. One can immediately observe that Eq. (82) is independent from the horizons of the non-extreme LDBHs similar to the other 4-dimensional LDBH solutions [8, 9] possessing one horizon. But, contrary to the others [8, 9], the resulting high frequency temperature T_{high} Eq. (83) differs from the standard Hawking temperature T_H [15], which is computed as usual by dividing the surface gravity by 2π :

$$T_H = \frac{\kappa}{2\pi} = \left.\frac{f'}{4\pi}\right|_{r=r_2} = \frac{b}{4\pi}(1 - \frac{r_1}{r_2}). \tag{84}$$

Let us note that this same result for the T_H can be obtained from Hawking's period argument of the Euclideanized line element. For this purpose we complexify time in (29) by $t \to i\tau$ and rearrange the terms so that the line element reads in the form $R^2 \times S^2$, given by

$$ds^2 \sim \left(\frac{d\rho}{\Sigma_\circ}\right)^2 + (\rho d\tau)^2 + \tilde{r}_{\circ h}^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{85}$$

Here $\tilde{r}_{\circ h}^2$ stands for the value of the radial coordinate on the outer horizon (when it exists) and the constant $\Sigma_\circ$ reads

$$\Sigma_\circ = \frac{1}{2}\left(\frac{1}{A^2} - 2\tilde{V}\right)\left|1 - \frac{4\lambda_1}{\lambda_2 A^2}\frac{\left(\frac{1}{A^2} - 2\tilde{V}\right)}{\sqrt{\tilde{M} + \tilde{M}^2 - \frac{4\lambda_1}{\lambda_2 A^2}\left(\frac{1}{A^2} - 2\tilde{V}\right)}}\right| \tag{86}$$

which relates to the period of the angle τ, upon the overall multiplication by $\Sigma_\circ$. Since T_H is the inverse of the period we obtain

$$T_H = \frac{1}{2\pi}\Sigma_\circ \tag{87}$$

which is identical with (84), valid for double-horizon Hawking temperature. In order to find the vacuum 'in' and 'out' states for the scalar field we have to choose the metric such that the surface gravity and mass of the black hole both vanish. This can be done from (66), by choosing $c^2 = 4ab$ first, to make an extremal LDBH (with zero temperature), and next, to let $c \to 0$ to make the mass also zero. These conditions cast our LDBH metric into

$$ds^2 = -brdt^2 + \frac{dr^2}{br} + A^2 r d\Omega^2. \tag{88}$$

By simple arrangement this vacuum metric transforms into

$$ds^2 = \rho^2\left(-d\tau^2 + dx^2 + d\Omega^2\right) \tag{89}$$

where

$$r = e^{\beta x}, \quad t = \frac{\beta}{b}\tau, \quad \rho = Ae^{\frac{\beta}{2}x}, \quad \beta = A\sqrt{b}.$$

The massless Klein-Gordon equation $\nabla^2\Phi = 0$, with $\Phi = \frac{1}{\rho}\Psi$ takes the form

$$\frac{1}{\rho^3}\left(\partial_{\tau\tau} - \partial_{xx} + \frac{\beta^2}{4} + \ell(\ell+1)\right)\Psi = 0. \tag{90}$$

The vacuum 'in' and 'out' solutions for the scalar field are

$$\Phi_{in} \sim \frac{1}{\sqrt{r}}e^{-i(\beta\sigma x + \omega t)} \tag{91}$$

$$\Phi_{out} \sim \frac{1}{\sqrt{r}}e^{i(\beta\sigma x - \omega t)}$$

where σ has the meaning from (71). Once these states propagate from vacuum they turn into thermal states as described above.

So the question arises here as in which case does the temperature Eq. (82) matches with the value of T_H in Eq. (84)? The answer is absolutely related with the value of physical parameter, σ. Let us assume that the value of the parameter σ is so great that it predominates term $\frac{\omega}{b}(\frac{r_2+r_1}{r_2-r_1})$ (but $\frac{\omega}{b}(\frac{r_2+r_1}{r_2-r_1})$ is still comparable with σ) in the expression of the temperature (82). Unless this assumption is not violated, the corresponding limit of T will be T_H. In summary,

$$T_H \simeq \lim_{\left[\sigma > \frac{\omega}{b}(\frac{r_2+r_1}{r_2-r_1})\right] \gg 1} T \simeq \omega / \ln \left\{ \frac{\exp \pi \left[\sigma + \frac{\omega}{b}(\frac{r_2+r_1}{r_2-r_1}\right] \exp \pi(\sigma + \frac{\omega}{b})}{\exp \pi \left[\sigma - \frac{\omega}{b}(\frac{r_2+r_1}{r_2-r_1})\right] \exp \pi \left(\sigma - \frac{\omega}{b}\right)} \right\} \tag{92}$$

$$\simeq \frac{\omega}{\ln \left\{ \exp 2\pi \left[\frac{\omega}{b}\left(\frac{r_2+r_1}{r_2-r_1}\right)\right] \exp 2\pi \left(\frac{\omega}{b}\right) \right\}} \simeq \frac{b}{4\pi}(1 - \frac{r_1}{r_2}).$$

Another question may immediately come out: how does σ maintain its predomination against $\frac{\omega}{b}(\frac{r_2+r_1}{r_2-r_1})$? To clarify the question, one can check Eq. (71) in order to see that the predomination of σ strictly depends on negative values of $\lambda\!\!\!^{-}$. However, this is possible only with the case of $p^2 = q^2 Q^2 > l(l+1)$. Hence, a significant remark is revealed that obtaining T_H of the non-extreme LDBHs from the *SCRSM*, the only possibility is to consider charged scalar waves instead of chargeless ones.

Furthermore, we want to serve most intriguing figures about the spectrum temperature Eq. (82). To this end, first we plot T versus frequency ω of non-extreme LDBHs with $r_1, r_2 \neq 0$ for low and high $|p|$-values, and display all graphs in Fig. 1. As it can be seen from Fig. 1 in the high frequencies the thermal behaviors of the LDBHs with different $|p|$-values exhibits similar behaviors in which their temperatures approach to T_{high} while $\omega \to \infty$. The plot with low $|p|$-value in Fig. 1 does not behave like the Hawking temperature. On the other hand, the other plot in Fig. 1, which has high $|p|$-value represents the Hawking temperature T_H in the low frequencies ($\omega > 0$). Beside this, once the parameter σ is lost its predomination against $\frac{\omega}{b}(\frac{r_2+r_1}{r_2-r_1})$, the latter plot increases to reach the T_{high} with increasing frequency as well. In the case of the non-extreme LDBHs with $r_1 = 0$, there is no difference between T_{high} and T_H because of Eq. (84), and at the low $|p|$-values the temperature T exhibits similar behavior as in the case $r_1, r_2 \neq 0$, which is the well-known thermal character in the EMD theory [8]. By the way, one should exclude $\omega = 0$ during the plotting of the temperature. Because it causes uncertainty for the temperature Eq. (82) and physically this case is not acceptable since we consider the propagation of scalar waves. Fig. 2 is about the graph of T versus frequency ω of non-extreme LDBHs with $r_1 = 0$ in a high $|p|$-value. In this figure, it is illustrated that by increasing the frequency from 0^+, the temperature first starts from a constant value, which is T_H and then makes a peak (not much higher than T_H), and then decreases back to T_H while $\omega \to \infty$. Rousingly, one can observe that the behavior of the graph in Fig. 2 is very similar to the graph obtained from the well-known Planck radiation formula, see for instance [16]. Besides, both Fig. 1 and Fig. 2 show us that whenever high $|p|$-values are present, the frequency of the scalar wave needed to detect the temperature of the LDBHs as to be the Hawking temperature T_H can either be very high (only for $r_1 = 0$ case, which is already known before [8]) or low. The latter information about the relationship between T_H and low frequencies is completely new for us, and may play crucial role for the thermal detection of the LDBHs in the future.

V. SINGULARITY ANALYSIS

In section II, we present a solution in 4−dimensional static spherically symmetric EMD theory that incorporates two Liouville-type potential terms coupled with gravity together with magnetically charged dilatonic parameters. We have clarified that the solution possess a central singularity which is a characteristic feature for spherically symmetric systems. In the solutions that admit black holes this singularity is clothed by horizons. However, there are cases that this singularity is not hidden behind a horizons. In such cases the singularity is called a *naked* singularity.

In classical general relativity, singularities are described as incomplete geodesics. This simply means that the evolution of timelike or null geodesics is not defined after a finite proper time. There is a general consensus that a removal of classical singularities is not important only for quantum gravity but also for other fundamental theories. In view of this consensus, we are aiming to analyze whether these classical naked singularities that occur in the general solution described in Eqs. (16)-(19) and in its linear dilaton limit given in Eqs. (27) and (28), turn out to be ”strong” or ”smoothed out” when probed with quantum test particles. Our analysis will be based on the pioneering work of Wald [12] which was developed by Horowitz and Marolf (HM)[10]. HM, have proposed a criterion to test the classical singularities with quantum test particles that obey the Klein-Gordon equation for static spacetimes having timelike singularities. The criterion of HM has been applied successfully for several spacetimes [17–19] within the context of quantum mechanical concepts. Among the others, HM have already analyzed the quantum singularity for the extreme case of the charged dilatonic black hole in the absence of Liouville-type potentials. They confirmed that for a specific interval of dilaton parameter, the singularity is quantum mechanically regular. The brief review of the criterion is as follows.

A scalar quantum particle with mass m is described by the Klein-Gordon equation $(\nabla^{\mu}\nabla_{\mu} - m^2)\,\psi = 0$. This equation can be written by splitting the temporal and spatial portion as $\frac{\partial^2\psi}{\partial t^2} = -\mathcal{A}\psi$, such that the spatial operator $\mathcal{A}$ is defined by $\mathcal{A} = -\sqrt{f}D^i\left(\sqrt{f}D_i\right) + fm^2$, where $f = -\xi^{\mu}\xi_{\mu}$ with ξ^{μ} the timelike Killing field, while D_i is the spatial covariant derivative defined on the static slice Σ. Then, the Klein-Gordon equation for a free relativistic particle satisfies $i\frac{\partial\psi}{\partial t} = \sqrt{\mathcal{A}_E}\psi$, with the solution $\psi\,(t) = \exp\left(it\sqrt{\mathcal{A}_E}\right)\psi\,(0)$. If the extension of the operator $\mathcal{A}$ is not essentially self-adjoint, the future time evolution of the wave function is ambiguous. Then, HM criterion defines the spacetime quantum mechanically singular. However, if there is only one self-adjoint extension, the operator $\mathcal{A}$ is said to be essentially self-adjoint and the quantum evolution $\psi\,(t)$ is uniquely determined by the initial condition. According to the HM criterion, this spacetime is said to be quantum mechanically regular. Consequently, a sufficient condition for the operator $\mathcal{A}$ to be essentially self-adjoint is to investigate the solutions satisfying the following equation (see Ref. [20] for a detailed mathematical background),

$$\mathcal{A}\psi \pm i\psi = 0. \tag{93}$$

This equation admits separable solution and hence the radial part becomes,

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{fR^2}\frac{\partial\left(fR^2\right)}{\partial r}\frac{\partial\phi}{\partial r} - \frac{l\,(l+1)}{fr^2}\phi - \frac{m^2}{f}\phi \pm i\frac{\phi}{f^2} = 0, \tag{94}$$

in which $l\,(l+1) \geq 0$ is the eigenvalue of the Laplacian on the $2-$sphere. The necessary condition for the operator $\mathcal{A}$ to be essentially self adjoint is that at least one of the solutions to this equation fails to be of finite norm when $r \to 0$. In summery, the self adjointness of the operator $\mathcal{A}$, implies the well-posedness of the initial value problem. Therefore, the suitable norm $\|\phi\|$ for this case is the Sobolev norm which is used first time within this context by Ishibashi and Hosoya [20] defined by,

$$\|\phi\|^2 = \frac{q^2}{2}\int R^2 f^{-1}\left|\phi\right|^2 d\mu dr + \frac{1}{2}\int R^2 f\mid\left|\frac{\partial\phi}{\partial r}\right|^2 d\mu dr, \tag{95}$$

where q^2 is a positive constant and $d\mu$ is the volume element on the unit $2-$sphere. The regularity of the central singularity at $r = 0$ in quantum mechanical sense requires that the squared norm of the solutions of the Eq. (94) should be divergent for each $l\,(l+1)$ and each sign of imaginary term. The norm $\|\phi\|$ is divergent for $l\,(l+1) > 0$ if it is for $l = 0$, so essential self-adjointness will be examined for $l = 0$ ($S - wave$) case. This implies essential self adjointness for the operator $\mathcal{A}$. Furthermore, we assume, a massless case (i.e. $m = 0$), and ignoring the term $\pm i\frac{\phi}{f^2}$ (since it is negligible near the origin).

A. A more general case:

The general solution for any value of α^2 which is related to the dilaton parameters γ_1 and γ_2 is given in the Eq. (18). Since this solution is complicated enough for integrability, we consider the specific values of $\alpha^2 = 3$ and $V_1 = 0$. Hence, the general solution becomes;

$$f(r) = \frac{16a_2}{\sqrt{r}}\left(r - r_2\right)\left(r - r_1\right), \tag{96}$$

where

$$\begin{aligned} \tilde{r}_{1,2} &= \frac{M \pm \sqrt{M^2 - 4a_1a_2}}{2a_2}, \\ a_1 &= \frac{Q^2\lambda_1}{4A^4}, \qquad a_2 = \frac{1}{12}\left(\frac{Q^2\lambda_2}{A^4} - V_2\right). \end{aligned} \tag{97}$$

The extreme case occurs when $M^2 = 4a_1a_2$. In this case there is one horizon only and it is given by $r_h = \frac{M}{2a_2}$. If $M > \sqrt{M^2 - 4a_1a_2}$ and $M^2 > 4a_1a_2$, this particular case admits two horizons given by $\tilde{r}_{1,2}$. However, if $M^2 - 4a_1a_2 < 0$, no black hole forms and hence, the singularity at $r = 0$ becomes naked.

As a requirement of the HM criterion, the singularity at $r = 0$ must have a timelike character. This can be checked if one introduces tortoise coordinate defined by $r_* = \int \frac{dr}{f}$ and take its limit as $r \to 0$. We found that the limit is finite. Therefore, the singularity is timelike. The solution for Eq. (94) is

$$\phi(r) = \frac{A^{-2}}{16a_2\left(\tilde{r}_2 - \tilde{r}_1\right)} \ln\left|\frac{r-\tilde{r}_2}{r-\tilde{r}_1}\right|. \tag{98}$$

The first and the second terms of the squared norm (95) is finite, when $r \to 0$.

Consequently, the operator $\mathcal{A}$ is not essentially self-adjoint and therefore, the central singularity $r = 0$, remain quantum mechanically singular.

B. The linear dilaton case:

The metric function for the linear dilaton case can be written as, (from Eq. (65))

$$f(r) = \frac{b}{r}\left(r - r_2\right)\left(r - r_1\right),$$

where

$$r_{1,2} = \frac{\tilde{M} \pm \sqrt{\tilde{M}^2 - 4ab}}{2b},$$
$$a = \frac{\lambda_1}{A^2\lambda_2}, \qquad b = \left(\frac{1}{A^2} - 2\tilde{V}\right).$$

The naked singularity occurs when $\tilde{M}^2 - 4ab < 0$. The tortoise coordinate $r_* = \int \frac{dr}{f}$ is finite and indicating a timelike character at $r = 0$. The Penrose diagram of this particular case is shown in Fig. 3-a. The radial part of the separable Eq. (94) has solution for the linear dilaton case as,

$$\phi(r) = \frac{1}{bA^2\left(r_2 - r_1\right)} \ln\left|\frac{r-r_2}{r-r_1}\right|. \tag{99}$$

The first and the second term of the squared norm defined in Eq. (95) is finite. Therefore the spacetime is quantum mechanically singular. For the double-horizon case, $\tilde{M}^2 - 4ab > 0$, which implies $r_1 \neq r_2 \neq 0 \neq r_1$, the timelike singularity at $r = 0$ is not naked, and its Penrose diagram is depicted in Fig. 3-b. However, for a special case $\lambda_1 = 0$, the solution to Eq. (94) is

$$\phi(r) = \frac{A^2}{b\tilde{M}} \ln\left|\frac{r-r_h}{r}\right|, \tag{100}$$

in which $r_h = \frac{\tilde{M}}{b}$. The first term of the squared norm (95) is finite, whereas the second term behaves as,

$$\sim \left.(\ln|r|)\right|_{r=0} \to \infty. \tag{101}$$

Hence, under the condition $\lambda_1 = 0$, the central classical singularity becomes quantum mechanically non-singular. When we have a single-horizon, with the choice $\lambda_1 = 0$, for example, the singularity $r = 0$, is shown in the Penrose diagram (Fig. 3-c).

C. Near horizon behaviors

In order to study the global behavior of our solution, at least for specific choices of parameters, and to be able to sketch the Penrose diagrams, we cast the metric into the form apt for near horizons. With the choice $V_1 = V_2 = 0$ our metric function $f(r)$ takes the form

$$f(r) = \frac{1}{A^2 r^{1+\delta}} \left(r^2 - \frac{4M_{QL}}{1+\delta} r + \frac{\lambda_1}{\lambda_2} \frac{1-\delta}{1+\delta} \right), \tag{102}$$

in which

$$\delta = \frac{1-\alpha^2}{1+\alpha^2}, \quad -1 < \delta < 1 \tag{103}$$

and

$$A^2 = \frac{2}{1-\delta} \lambda_2 Q^2. \tag{104}$$

Upon the choice of parameters involved, we can have, double, single or no-horizon cases. By a redefinition for time, our line element reads, in brief,

$$ds^2 = A^2 d\tilde{s}^2 \tag{105}$$

in which

$$d\tilde{s}^2 = -\frac{(r-r_-)(r-r_+)}{r^{1+\delta}} dt^2 + \frac{r^{1+\delta}}{(r-r_-)(r-r_+)} dr^2 + r^{1+\delta} d\Omega^2 \tag{106}$$

and

$$r_{\pm} = \frac{2M_{QL}}{1+\delta} \left(1 \pm \sqrt{1 - \frac{\lambda_1}{\lambda_2} \frac{1-\delta^2}{4M_{QL}^2}} \right). \tag{107}$$

We note that the global structure of $d\tilde{s}^2$ is same with ds^2, and therefore we analyze $d\tilde{s}^2$. The singularity structure of (106) can be seen from the Kretchmann scalar-K, which reads

$$\lim_{r\to 0} K \sim \begin{cases} r^{-2(\delta+3)}, & \delta \neq \pm 1 \\ r^{-8}, & \delta = +1 \end{cases} \tag{108}$$

$$\lim_{r\to\infty} K \sim \begin{cases} r^{-2(\delta+1)}, & \delta \neq \pm 1 \\ \text{constant}, & \delta = -1 \end{cases} \tag{109}$$

We concentrate ourselves now to the near horizon geometry by the following reparametrization, with new coordinates $(\tilde{r}, \tilde{t})$

$$r_- = r_\circ, \quad r_+ = r_\circ + \epsilon b_0, \quad r = r_\circ + \epsilon \tilde{r}, \quad t = \frac{1}{\epsilon}\tilde{t} \tag{110}$$

in which $r_\circ$ and b_0 are constants and $\epsilon \to 0$, is a small parameter. We obtain, upon relabeling $\tilde{r} = r$ and $\tilde{t} = t$

$$d\tilde{s}^2 = -\frac{r\,(r - b_0)}{r_\circ^{1+\delta}} dt^2 + \frac{r_\circ^{1+\delta}}{r\,(r - b_0)} dr^2 + r_\circ^{1+\delta} d\Omega^2. \tag{111}$$

In Fig. 4 we plot the Penrose diagrams for the specific cases $b_0 = 0$, and $b_0 > 0$. The case $b_0 < 0$ doesn't differ from the case of $b_0 = 0$, and as a matter of fact this particular case corresponds to the BR limit, which is known to correspond to the near extreme geometry of the RN black hole. The more standard BR is obtained from the present one by the inversion $r \to \frac{1}{r}$.

VI. CONCLUSION

We have shown that dilaton field with Liouville's potential interpolates between RN black hole and non-black hole BR solution. The general solution for the metric function suggests that dilatonic presence induces significant changes in the solutions; for example, asymptotically flat black holes become non-asymptotically flat . It is shown, through radial linear perturbation, that dilaton can add instabilities to the otherwise stable RN black hole whereas BR remains stable. From the thermodynamic point of view also, by invoking specific heat the system can be tested against stability and phase transition. In the non-extreme LDBH case, which is a particular solution of our general solution the statistical and the standard Hawking temperatures are compared and plotted. It has been pointed out that with charged scalar waves and spin-weighted coupling the two results match for the case of double horizons. We recall that in the single horizon case, in spite of the existence of a linear dilaton such a discrepancy does not arise. It is remarkable that the spin-weighted spheroidal harmonics serve to convert the diverging temperature spectrum into a finite one. The presence of dilaton makes the spacetime highly singular at $r = 0$. Whether these singularities are quantum mechanically singular also, or not, we send a quantum test particle and apply the criterion due to Horowitz and Marolf. We find that under certain choice of our parameters the naked singularities create an infinite repulsive quantum potential so that the particle feels a regular space time.

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FIGURE CAPTIONS

Figure 1: Temperature T as a function of ω for the non-extreme LDBHs in the case of $r_1, r_2 \neq 0$. The plots are governed by Eq. (82). Different line styles belong to different $|p|$-values: Dotted line corresponds to $|p| = 0.5$ (as an example of low $|p|$-values) and solid line is for $|p| = 10$ (as an example for high $|p|$-values). The physical parameters in Eq. (82) are chosen as follows: $l = 1$, $b = 1$, $A = 1$, $r_1 = 0.5$ and $r_2 = 1$.

Figure 2: Temperature T as a function of ω for the non-extreme LDBHs in the case of $r_2 \neq 0$ and $r_1 = 0$, and when p has a high value. The plot is governed by Eq. (82). The physical parameters in Eq. (82) are chosen as follows: $|p| = 10$, $l = 1$, $b = 1$, $A = 1$ and $r_2 = 1$.

Figure 3-a: Penrose diagram for no-horizon case, $\tilde{M}^2 - 4ab < 0$ in which $r = 0$ is a naked singularity.

Figure 3-b: Penrose diagram of the LDBH with two distinct horizons $r_1 \neq r_2$, where $r = 0$ is a timelike singularity.

Figure 3-c: Penrose diagram of the LDBH with a single horizon at $r = r_h$. singular nature of $r = 0$ is not affected.

Figure 4-a: Penrose diagram for the line element (111) with $b_0 = 0$. There is no singularity and no horizon. By inverting coordinate $r \to \frac{1}{r}$ we obtain the standard BR diagram. We note that the choice $b_0 < 0$ is also similar to this case.

Figure 4-b: For $b_0 > 0$ there is a horizon and the Penrose diagram is as shown with singularities at the null boundaries. By inversion as in (3-a) we interchange $r = 0$ $(r \to \infty)$ with $r \to \infty$ $(r = 0)$.