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Exact solutions to a massive charged scalar field equation in the magnetically charged stringy black hole geometry and Hawking radiation

I. Sakalli[1], **A. Al-Badawi**[2]

[1] Department of Physics, Eastern Mediterranean University, Gazimagosa, North Cyprus, Mersin 10, Turkeye-mail: `izzet.sakalli@emu.edu.tr`
[2] Department of Physics, AL-Hussein Bin Talal University, Ma'an, Jordan e-mail: `dr.ahmadbadawi@ahu.edu.jo`.

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Abstract Exact solutions of a massive complex scalar field equation in the geometry of a Garfinkle-Horowitz-Strominger (stringy) black hole with magnetic charge is explored. The separated radial and angular parts of the wave equation are solved exactly in the non-extreme case. The angular part is shown to be an ordinary spin-weighted spheroidal harmonics with a spin-weight depending on the magnetic charge. The radial part is achieved to reduce a confluent Heun equation with a multiplier. Finally, based on the solutions, it is shown that Hawking temperature of the magnetically charged stringy black hole has the same value as that of the Schwarzschild black hole.

Key words Klein-Gordon Equation, Charged String Black Hole, Confluent Heun Equation, Hawking Radiation.

1 INTRODUCTION

Exact solution to a sourceless charged massive scalar field equation in the Kerr-Newman black hole geometry was first found by Wu and Cai [1]. Later on, they have extended their previous study [1] to exact solution of a scalar wave equation in a Kerr-Sen black hole geometry [2], which reduces to the electrically charged Garfinkle-Horowitz-Strominger (GHS) geometry in the limit of vanishing rotation parameter. However, no one has considered to solve the problem of the massive complex scalar field equation in the magnetically charged GHS geometry, yet. So we think that it might be useful

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to fill this absence in the literature. The main goal of this paper is to show that exact solutions to a massive charged scalar field equation on magnetically charged GHS background is somewhat different than the electrically charged GHS geometry [2] and investigate Hawking evaporation of scalar particles.

Recall that according to the laws of black hole mechanics [3], Hawking [4,5] proved that a stationary black hole can emit particles from its event horizon with a constant temperature proportional to the surface gravity. Since Hawking's this pioneering study, many methods have been proposed in order to calculate the Hawking radiation during the last three decades, see for instance [6,7,8,9,10]. One of the commonly used methods is the method of Damour-Ruffini-Sannan (DRS) [11,12]. This method is applicable to any Hawking's problems in which asymptotic behaviors of the wave equation are known near the event horizon.

According to the DRS method, it is plausible to investigate the exact solutions of the scalar wave equation in order to get more accurate calculations about Hawking radiation. In general, obtaining an exact solution to the wave equation in a given geometry is so difficult. On the other hand, the main reason of a researcher's motivation is indeed such difficulties. Today, a wider perspective of the properties and physics of black holes can be acquired by studying other types of black hole solutions appearing in string theory. Of particular interest is considering GHS [13] black hole, which is a member of a family of solutions to low-energy limit of string theory. It is discovered when the field content of Einstein-Maxwell theory is enlarged to include a dilaton field ϕ, which couples to the metric and the gauge field, non-trivially. This causes the charged stringy black holes to differ significantly from the Reissner-Nordström (RN) black hole. Very recently, the Hawking radiation of the GHS black hole (in the string frame) has been studied by using the method of cancellation of anomalies at the horizon [14].

In this paper, we shall not discuss the extreme case since it does not have the characteristics of a black hole solution. In generally, the crucial equation, which plays an essential role in the calculation of the Hawking radiation is the radial equation. In the non-extreme case, it is shown that the radial equation reduces to a confluent Heun equation. Although the Heun differential equations are less known than the hypergeometric family in the literature, due to the necessities of their using in various physical problems, they have been intensively attracting much interest. One may refer to [15, 16] in order to see the applications of the Heun equations to many modern physical problems. Heun equations also appear in the quantum mechanical problems of general relativity. For instance, it can be seen that more recently Al-Badawi and Sakalli [17] have shown that the angular part of the Dirac equation in the rotating Bertotti-Robinson geometry is solved in terms of the confluent Heun functions.

The paper is organized as follows: In Sec. II, a brief overview of the GHS black hole solution is given. Next, we separate a massive magnetically charged scalar field equation on the GHS spacetime into the angular and radial parts. The solutions to radial equation in non-extreme case is devoted to Sec. III. Next, we shall employ the DRS method to discuss Hawking radiation in Sec. IV. Finally, we draw our conclusions.

2 GHS SPACETIME AND SEPARATION OF KLEIN-GORDON EQUATION ON IT

In the low-energy limit of string field theory, the four-dimensional action (in Einstein frame) describing the dilaton field ϕ coupled to a $U(1)$ gauge field is

$$S = \int d^4x\sqrt{-g}(R - 2(\nabla\phi)^2 - e^{-2\phi}F^2) \tag{1}$$

where $F_{\mu\upsilon}$ is the Maxwell field associated with a $U(1)$ subgroup of $E_8 \times E_8$ or Spin$(32)/Z_2$. In the presence of a magnetic charge the dilaton cannot be constant and the static, spherically symmetric solutions designated with GHS black holes [13] are given by

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r(r - \frac{Q^2e^{-2\phi_0}}{M})(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2}$$

with

$$e^{-2\phi} = e^{-2\phi_0}(1 - \frac{Q^2e^{-2\phi_0}}{Mr}),$$

$$F_{\theta\varphi} = Q\sin\theta, \tag{3}$$

where ϕ_0 is the asymptotic constant value of the dilaton and Q is the magnetic charge. For electric charge case one can generate the solutions by applying the duality transformations

$$\widetilde{F}_{\mu\upsilon} \to \frac{1}{2}e^{-2\phi}\epsilon^{\rho\sigma}_{\mu\upsilon}F_{\rho\sigma} \quad \text{and} \quad \phi \to -\phi. \tag{4}$$

Note that this transformation does not modify the geometry (2). In this case, the solution for the dilaton and electromagnetic field are given by

$$e^{2\phi} = e^{-2\phi_0}(1 - \frac{Q^2e^{-2\phi_0}}{Mr}),$$

$$F_{rt} = \frac{Q}{r^2}, \tag{5}$$

where Q refers now to the electric charge. Although the Einstein metric (2) is the same for both electrically and magnetically charged black holes, one of the black hole solutions should be considered while separating the Klein-Gordon equation in this geometry. This is because the electromagnetic four-vector potentials are different for the both black holes. Here, we are interseted in metric (2) together with fields (3), and for simplicity, we set $\phi_0 = 0$. It is easy to derive the electromagnetic four-vector potential of the magnetically charged stringy black holes as follows

$$A_\mu = -Q\cos\theta\delta^\mu_\varphi, \tag{6}$$

Let us compare solution (2) to RN which represents the solution of a charged black hole given by the following metric

$$ds^2 = -(1 - \frac{2M}{r} + \frac{Q}{r^2})dt^2 + \frac{dr^2}{1 - \frac{2M}{r} + \frac{Q}{r^2}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{7}$$

One can see some important differences immediately. First of all, contrast to the pure Einstein-Maxwell theory there is only one horizon at $r = 2M$. In fact the R^2 part of metric (2) is identical to the Schwarzschild black hole. This implies that also the surface gravity [18] coincides with Schwarzschild

$$\kappa = \left[\lim_{r\to 2M}(-\frac{1}{4}g^{tt}g^{ij}g_{tt,i}g_{tt,j})\right]^{\frac{1}{2}} = \frac{1}{4M}. \tag{8}$$

Important differences appear in the angular part. There is a curvature singularity (spacelike), hidden inside the horizon, when the radius of two-sphere vanishes at $r = \frac{Q^2}{M}$. Since the DRS method concerns the asymptotic behaviors of the scalar waves near the event horizon, this spacelike singularity does not make any trouble on using of this method. Another difference with respect to the RN solution concerns the extremal configuration. Here it is given by $|Q| = \sqrt{2}M$ instead of the condition $|Q| = M$ for the RN black holes. In the extreme limit, the area of the event horizon $A = 8\pi M(2M - \frac{Q^2}{M})$ shrinks to zero and turns out to be singular. Namely, a naked singularity appears, so the solution is no longer a black hole.

In curved spacetime, a massive charged test scalar field Φ with mass μ and charge q obeys the covariant Klein-Gordon equation [1]. The massive scalar wave function Φ in metric (2) can be separated as $\Phi(r, t, \theta, \varphi) = R(r)G(\theta)e^{i(m\varphi - \omega t)}$, in which the angular part $G(\theta)$ satisfies the following equation

$$G'' + \cot\theta G' + \left[\frac{\lambda}{M} - \frac{(m + qQ\cos\theta)^2}{\sin^2\theta}\right]G = 0. \tag{9}$$

where λ is a separation constant. (Throughout the paper, a prime denotes the derivative with respect to its argument.)

Letting $\pounds = \frac{\lambda}{M} = l(l+1) - p^2$, one can see that $G(\theta)e^{im\varphi}$ is nothing but a spin-weighted spherical harmonics ${}_pY_{lm}(\theta, \varphi)$. Here, it should be highlighted that the spin-weight appears as $p = qQ$ [19], which is seen as a differentness when we compare it with the electrically charged case [2]. Basically, this result is the consequence of the four-vector potential (6) involving an angular term. One should notice that by the modified separation constant $\pounds$, the magnetic charge is carried into the radial equation, which is going to be discussed in the next section.

3 REDUCTION OF THE RADIAL TO A CONFLUENT HEUN EQUATION

In this section, we shall show that the radial part of the massive complex scalar field equation is in fact a confluent Heun equation. Using metric (2) as being a background for the covariant Klein-Gordon equation, we see that the separated radial part of the massive charged test scalar field Φ with mass μ is governed by the following equation

$$(r-2M)(Q^2-rM)R''+\left[Q^2-rM-M(r-2M)\right]R'+\left[\lambda-\mu^2 r(Q^2-rM)+\frac{r^2\omega^2(Q^2-rM)}{r-2M}\right]R=0, \tag{10}$$

Making the following coordinate transformation

$$r=2M-Dx, \tag{11}$$

where $D=\frac{2M^2-Q^2}{M}$, letting $k=\sqrt{\omega^2-\mu^2}$ (assuming that $\omega>\mu$) and substituting

$$R(r)=x^{2i\omega x}e^{-ikDx}H(x), \tag{12}$$

into differential equation (10), then we can reduce it to a confluent form of Heun equation [20]

$$H''+(\alpha+\frac{\beta+1}{x}+\frac{\gamma+1}{x-1})H'+\frac{1}{x(x-1)}\left\{\eta+\frac{\beta}{2}+\frac{1}{2}(\beta+1)(\gamma-\alpha)+\left[\frac{\alpha}{2}(\beta+\gamma+2)+\delta\right]x\right\}H=0, \tag{13}$$

which shows that it has two singular points at $x=0,1$ and one irregular singular point at the infinity, $x=\infty$ [15,16]. It yields the following specific parameters:

$$\begin{aligned}&\gamma=0, \qquad \alpha=-2ikD, \qquad \beta=4i\omega M,\\ &\delta=-2DM(\omega^2+k^2) \qquad \eta=-(\pounds+\delta),\end{aligned} \tag{14}$$

In the literature, see for instance [21], there are special transformations, which make possible to express the Heun functions in terms of ordinary special functions. Today, all well-known transformations from confluent Heun functions to other special functions are listed in the famous computer package, *MAPLE* 10 and its higher versions. Adapting Maple's notation for the confluent Heun functions, we obtain the following canonical solutions of equation (13)

$$H(x) = C_1 HeunC(\alpha, \beta, \gamma, \delta, \eta; x) + C_2 x^{-\beta} HeunC(\alpha, -\beta, \gamma, \delta, \eta; x). \quad (15)$$

The convergent Taylor series expansion of the confluent Heun functions with respect to the independent variable x around regular singular point $x = 0$ (i.e. around event horizon $r = 2M$) is obtained using the known three-terms recurrence relation [15,16] and initial conditions:

$$HeunC(\alpha, \beta, \gamma, \delta, \eta; 0) = 1, \quad (16)$$

and

$$HeunC'(\alpha, \beta, \gamma, \delta, \eta; x)|_{x=0} = \frac{(1+\beta)(\gamma-\alpha)+\beta+2\eta}{2(1+\beta)}. \quad (17)$$

4 HAWKING RADIATION OF SCALAR PARTICLES

It is easy to see from equations (11) and (15) that the radial solutions (12) near to the horizon behave asymptotically as

$$R(r) \sim C_1 (r-2M)^{2i\omega M} + C_2 (r-2M)^{-2i\omega M}, \quad (18)$$

Therefore, just outside the horizon ($r > 2M$) two linearly independent solutions exist:

1- The outgoing wave solution

$$\Phi^{out} \to C_1 (r-2M)^{2i\omega M} e^{-i\omega t} \ {}_{qQ}Y_{lm}(\theta, \varphi). \quad (19)$$

2- The ingoing wave solution

$$\Phi^{in} \to C_2 (r-2M)^{-2i\omega M} e^{-i\omega t} \ {}_{qQ}Y_{lm}(\theta, \varphi), \quad (20)$$

in which ω is assumed to be a positive. As $r \to \infty$ the outgoing wave has an infinite number of oscillations and therefore cannot be straightforwardly extended to the interior region of the black hole in contrast with the ingoing wave. On the other hand, the outgoing wave can be analytically extended from outside into the interior of the black hole by the lower half complex r-plane

$$(r-2M) \to (2M-r)e^{-i\pi}. \quad (21)$$

According to the DRS method [11,12], a correct wave describing a particle flying off of the black hole is

$$\Phi = N_\omega \left[\Theta(r-2M)\Phi^{out}_{r>2M} + e^{4\pi M\omega}\Theta(2M-r)\Phi^{out}_{r<2M}\right], \qquad (22)$$

where Θ is the conventional Heaviside function and N_ω is a normalization factor. Since Φ^{out} differs from Φ^{in} as a factor $(r-2M)^{-4i\omega M}$, the above complexified analytical treatment (21) requires to put a difference factor of $e^{4\pi M\omega}$ into equation (22). Thus we can derive the relative scattering probability of the scalar wave at the event horizon

$$\mathcal{R} = \left|\frac{\Phi^{out}_{r>2M}}{\Phi^{out}_{r<2M}}\right|^2 = e^{-8\pi M\omega}, \qquad (23)$$

and the resulting radiation spectrum of scalar particles are obtained as follows

$$|N_\omega|^2 = \frac{\mathcal{R}}{1-\mathcal{R}} = (e^{8\pi M\omega}-1)^{-1}. \qquad (24)$$

Using the formal definition of the radiation spectrum [18], one can easily read the Hawking temperature as

$$T_H = \frac{1}{8\pi M} = \frac{\kappa}{2\pi}. \qquad (25)$$

This result shows that the statistical Hawking temperatures of the Schwarzschild black hole and the GHS black hole (independent of its charge type) are the same.

5 CONCLUSION

In this paper, our target was to investigate exact solutions of a massive complex scalar field equation in the magnetically charged string black hole background, which is referred to as the GHS black hole. After getting the exact solution, we have applied the method of DRS to derive the Hawking radiation of the magnetically charged GHS black holes. On the other hand, we should state that the present calculation of the Hawking temperature reproduces the result expected from more general analyses, which has been recently made in [22] due to the inspection of the form of the metric in Eq.(2).

The separated angular part is obtained in terms of the spin-weighted spheroidal harmonics with a spin-weight, which peculiarly depends on the product of the charges, qQ. On the other hand, the separated radial part is successfully reduced to a confluent form of the Heun equation. After using the initial conditions of the confluent Heun functions, which are obtained by the virtue of the Taylor series expansion around the event horizon, the asymptotic behaviors of the ingoing and outgoing scalar waves are defined near to the horizon. Here, after using the DRS method, we have shown that the thermal property of the magnetically charged stringy black holes closely resemble to the electrically charged stringy black holes. Namely, a charged stringy black hole shares similar quantum thermal effect as the Schwarzschild spacetime exhibits. Nevertheless, there might be a way to reveal differences between these black hole radiations. To the end this, one may consider the classical approximation [9] to compute the Hawking Radiation. This is an equivalent procedure to calculating the Bogoliubov coefficients relating two vacua: The vacuum for a quantum field near the horizon is not same to the observer's vacuum at infinity. Briefly, it is a procedure to compute the reflection and absorption coefficients of a wave by the black hole. However, the coefficient for reflection by the black hole can be best calculated whether one may find a relevant transformation between the confluent Heun functions $HeunC(\alpha, \beta, \gamma, \delta, \eta; x)$ and $HeunC(\alpha, \beta, \gamma, \delta, \eta; 1/x)$ such as in the hypergeometric functions [9]. But, a transformation from x to $1/x$ of the argument of the confluent Heun functions does not exist in the literature. The main difficulty in making such a transformation arises due to the fact that the point $x = 0$ is a regular singular point in contrast to the point $x = \infty$, which is an irregular singular point. In summary, nowadays useful asymptotic expansions of the confluent Heun functions are still open questions.

Finally, it should be emphasized that to properly study Hawking radiation in a selected geometry, the backreaction of the quantum effects must be taken into account. However, such an attempt resorts to the complete theory of quantum.

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