Chaos in electrovac and non-Abelian plane wave spacetimes

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Superposed electrovac pp-waves causes chaos. To show this, we project the particle geodesics onto the (x, y) plane and simulate the phase space's Poincaré section numerically. Similar considerations apply, with minor modifications, to the geodesics in a non-Abelian plane wave spacetime.

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PP-waves form the best known class of exact solutions to Einstein's field equations [1]. Interest in this class has been revived due to the fact that string theory admits exact solutions on such backgrounds. Beside the pure gravitational pp-waves, it admits pure electromagnetic (em) waves and their natural mixture which we refer to as "superposed electrovac pp-waves". Extension of such plane waves to non-Abelian gauge theory is also well-known [2,3]. We investigate the particle geodesics in such backgrounds and verify the emergence of chaotic behavior under certain conditions.

The line-element describing electrovac pp-waves is given by [4–6]

$$ds^2 = 2dudv - 2dzd\bar{z} + |f(u, z)|^2 du^2, \tag{1}$$

where $f(u, z)$ is an arbitrary holomorphic function expressed by a Laurent series expansion

$$f(u, z) = \sum_{i=-\infty}^{\infty} h_i(u) z^i, \tag{2}$$

in which $h_i(u)$ stands for an arbitrary function of u and $z = x + iy$. The nonvanishing Weyl and Ricci components (in the Newman-Penrose formalism) are

$$\Psi_4 = \bar{f} f_{zz}, \qquad \Phi_{22} = |f_z|^2. \tag{3}$$

in which a bar denotes complex conjugation and a subscript implies partial derivative. It is trivially seen that pure em pp-waves correspond to the special case in which f is a linear function of z.

Our primary interest here is to investigate whether the superposed electrovac pp-wave spacetime exposes a chaotic behavior or not. A previous analysis proved that the space of pure impulsive gravitational waves exhibits chaos [7], and from physics standpoint this result does concern the particle behavior in string theory.

The geodesics equation for (1) amounts to

$$\dot{u} = \alpha = \text{const} \neq 0,$$

or

$$u(\tau) = \alpha\tau, \tag{4}$$

(i.e. we discard an additive constant)

$$\ddot{v} + \frac{\alpha^2}{2}|f|^2_{,u} + \alpha(\dot{x}|f|^2_{,x} + \dot{y}|f|^2_{,y}) = 0, \tag{5}$$

$$\ddot{x} + \frac{\alpha^2}{4}|f|^2_{,x} = 0, \tag{6}$$

$$\ddot{y} + \frac{\alpha^2}{4}|f|^2_{,y} = 0, \tag{7}$$

in which a "dot" denotes $\frac{d}{d\tau}$, with τ being an affine parameter. The metric condition requires also that

$$\alpha\dot{v} = \left(\dot{x}^2 + \dot{y}^2 - \frac{1}{2}|f|^2\alpha^2 + \frac{\epsilon}{2}\right), \tag{8}$$

to replace (5), where $\epsilon = 1, 0, -1$ for timelike, null or spacelike geodesics, respectively. In this report, we shall concentrate mainly on Eqs. (6) and (7) with the choice that f is independent of u, which represents a 2D dynamical system, described by the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y). \tag{9}$$

The potential $V(x, y)$ is expressed in terms of the metric function f (with the specific parameter $\alpha = \sqrt{2}$) by

$$V(x, y) = \frac{1}{2}|f|^2. \tag{10}$$

It is readily seen that the pure em pp-waves, which correspond to a linear holomorphic function, makes trivially an integrable system. More generally, any $f(z) = z^k$, k being an arbitrary parameter, not necessarily an integer (and suppressing a multiplicative constant) implies a potential $V = \frac{1}{2}(x^2 + y^2)^k$, which is integrable in the electrovac theory as well. These forms are all axially symmetric and integrable in the polar coordinates. Thus the electrovac pp-waves admit a large class of regular motions for the geodesics particles.

Next, by considering any finite sum of powers in the holomorphic function changes the picture completely and leads to chaotic motion. For example, the choice (let us choose all constants to be unity for convenience)

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$$f(z) = z^2 + z^3, \tag{11}$$

leads to the potential

$$V(x, y) = \frac{1}{2}(x^2 + y^2)^2(x^2 + y^2 + 2x + 1), \tag{12}$$

and the equations of motion are

$$\begin{aligned} \ddot{x} &= -(x^2 + y^2)(3x^3 + 5x^2 + 3xy^2 + 2x + y^2), \\ \ddot{y} &= -y(x^2 + y^2)(3x^2 + 4x + 3y^2 + 2). \end{aligned} \tag{13}$$

To study these geodesics initial points are chosen on the unit circle in the (x, y) plane. We parametrize the initial positions by an angle $\phi \epsilon [0, 2\pi]$ such that $x_k(0) = \cos(\frac{k\pi}{18})$ and $y_k(0) = \sin(\frac{k\pi}{18})$ (where $k = 0, 1, 2 \ldots 35$). Alternatively, our dynamical system can be investigated by the Poincaré section method which is a way of picturing the dynamics in the phase space. We follow the computational program, called Poincaré package [8]. The chaotic behavior is evident in Fig. 1, obtained by this method. Expectedly, more additional terms in the holomorphic function lead to much more tedious equations of motion, which we shall not discuss.

Non-Abelian plane waves, likewise are represented by the line-element

$$ds^2 = 2dudv - |dz|^2 + Y(u, x, y)du^2, \tag{14}$$

whose nonzero Weyl and Ricci scalars are

$$\Psi_4 = Y_{zz,} \qquad \Phi_{22} = Y_{z\bar{z}}. \tag{15}$$

The nonzero Ricci and energy-momentum tensors in the conventional notation are

$$R_{uu} = -T_{uu} = -2\Phi_{22} = -\frac{1}{2}\nabla^2 Y. \tag{16}$$

The Yang-Mills potential 1-form with the internal gauge index i is

$$A^i = A^i(u, x, y)du. \tag{17}$$

This leads to the field 2-form

$$F^i = A^i_{,a} dx^a \wedge du, \qquad (a = x, y) \tag{18}$$

and the Yang-Mills equations reduce to

$$A^i_{,aa} = 0. \tag{19}$$

A readily available class of solutions is given by

$$A^i(u, x, y) = \frac{1}{2}[\chi^i(u, z) + \bar{\chi}^i(u, \bar{z})], \tag{20}$$

where $\chi^i(u, z)$ are non-Abelian gauge valued functions, holomorphic in z and arbitrary in u. It should also be added that the solution will have a full non-Abelian character provided the gauge group is not restricted to its Abelian subgroup. The general form of the $Y(u, x, y)$, which incorporates gravitational waves added to the non-Abelian plane waves is given by

$$Y(u, x, y) = K(u, z) + \bar{K}(u, \bar{z}) + \frac{1}{4}\chi^i \bar{\chi}^i, \tag{21}$$

where $K(u, z)$ is another holomorphic function in z and arbitrary in u. The analogous geodesics equations to Eqs. (6) and (7) are now

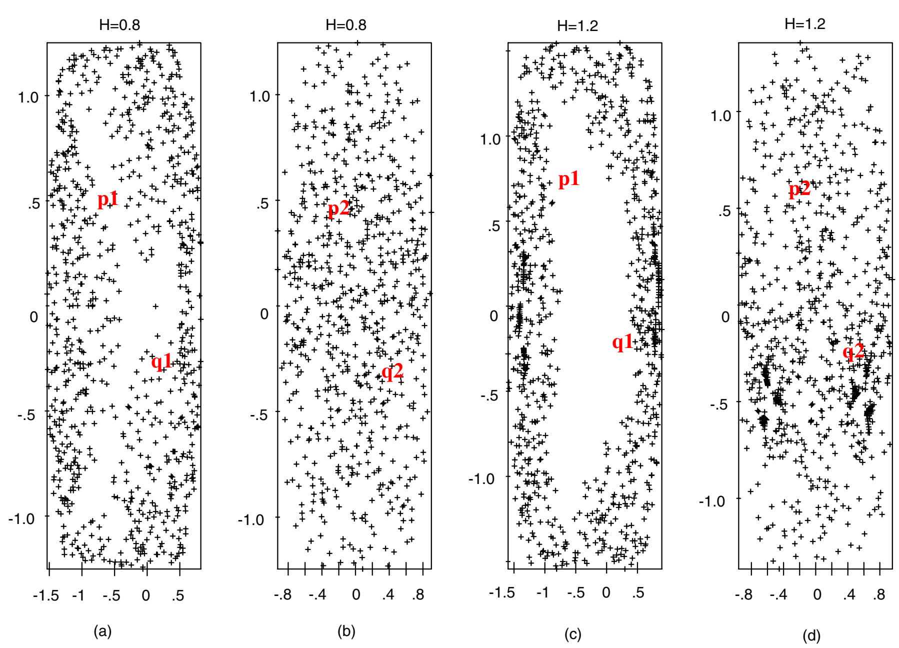

FIG. 1 (color online). Poincaré sections of $(\dot{x}, x)$ and $(\dot{y}, y)$ with $H = 0.8$ (i.e. 1a, 1b) and $H = 1.2$ (i.e. 1c, 1d) for the potential V (Eq. (12)). Each phase space with randomly distributed points represents a large chaotic sea. (Here, $x \to q1$, $y \to q2$, $\dot{x} \to p1$ and $\dot{y} \to p2$).

$$\ddot{x} + \alpha^2 Y_{,x} = 0, \tag{22}$$

$$\ddot{y} + \alpha^2 Y_{,y} = 0, \tag{23}$$

in which we have suppressed the u dependence of Y. This implies a u independent $K(z)$ and separable $\chi^i(u, z)$ in u and z such that the u dependence does not arise in the geodesics equation. This reduces the equations of geodesics to the familiar 2D Newtonian form. We recall that for the pure nonhomogeneous gravitational waves (i.e. $\chi^i = 0$), $K(z)$ can be chosen such that it leads to chaotic motion [7]. Specifically, for $K = (\text{const})z^3$, it leads to the Hénon-Heiles potential [9], which forms the prototype example of a chaotic system. When $K = 0$ (i.e. no independent gravity waves) and $\chi^i = g^i(u)z^k$, where $g^i(u)$ are gauge valued functions with the matrix constraint $g^i(u)\bar{g}^i(u) = \text{const}$, and the $k = \text{const}$, it leads to integrable geodesics. We note that the choice of u independent gauge matrices g^i also serves our purpose. To construct cases where the non-Abelian gauge field alone creates chaos it suffices to consider additive terms in the holomorphic function as we did in the electrovac case. For instance, the choice $K = 0$, $\chi^i(u, z) = g^i(u)(\sum_{k=2}^n z^k)$, for $n \geq 3$, and with the assumed constraint condition on the gauge function $g^i(u)$ yields chaotic geodesics. On the other hand, the special choice of $K = (\text{const})z^3$ with $\chi^i = g^i(u)z$ and with reference to Ref. [7] leads to a result in which the chaotic effect of gravity dominates over the gauge field in the asymptotic expansion.

Beside the non-Abelian gauge field other sources such as dilaton and axion can be superimposed to modify the energy-momenum T_{uu} in accordance with [10] as

$$2T_{uu} = \dot{\phi}^2 + \dot{\lambda}^2 + e^{-\phi}\chi^i_{,z}\bar{\chi}^i_{,\bar{z}} \tag{24}$$

Here $\phi(u)$ stands for the dilaton, the axion σ is defined through $e^{\phi}d\sigma = d\lambda$, and the gauge group is SU(2). Since all these physical fields can be considered local, vanishing asymptotically the chaos inherited from gravity renders the whole system to be chaotic. In the absence of gravity it is the choice of holomorphic function in the gauge function that plays the role of chaotic agent.

In conclusion, plane wave spacetimes give rise, under certain conditions to chaotic geodesics. This was already known for the pure nonhomogeneous gravitational pp-wave spacetimes. Similar properties hold true also for electrovac, non-Abelian plane wave backgrounds which may constitute sources such as dilaton and axion. As expected, this result may have far reaching implications in connection with Penrose limit [11] spacetimes and particle motion in string theory of higher dimensions.

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