We consider a \( n \)-dimensional domain wall (DW) \( \Sigma \) in a \( n + 1 \)-dimensional bulk \( \mathcal{M} \). This DW splits the background bulk into two \( n + 1 \)-dimensional spacetimes which will be referred to as \( \mathcal{M}_\pm \). Here \( \pm \) is assumed with respect to the DW. Our action of Gauss-Bonnet (GB) extended gravity is chosen as

\[
S = \frac{1}{2\kappa^2} \int_\mathcal{M} \sqrt{-g} \left(R + \alpha L_{\text{GB}}\right) + \frac{1}{\kappa^2} \int_\Sigma \sqrt{-h} L_{\text{DW}},
\]

in which \( L_{\text{DW}} = -\sigma = \text{constant} \) is the Nambu-Goto form of the DW Lagrangian, and \( K \) is the extrinsic curvature of DW with \( h = \left| g_{ij} \right| \). (Latin indices run over the DW coordinates while Greek indices refer to the bulk’s coordinates). The GB Lagrangian \( L_{\text{GB}} \) is given by

\[
L_{\text{GB}} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2,
\]

with the GB parameter \( \alpha \). A variation of the action with respect to the space-time metric \( g_{\mu\nu} \) yields the field equations

\[
G_{\mu\nu}^E + \alpha G_{\mu\nu}^{GB} = 0,
\]

where

\[
G_{\mu\nu}^{GB} = 2\left(-R_{\mu\rho\sigma\nu}R^{\rho\sigma} - 2R_{\mu\rho\nu\sigma}R^{\rho\sigma} - 2R_{\rho\sigma\nu}R^{\rho\sigma}\right) + RR_{\mu\nu} - \frac{1}{2} L_{\text{GB}} g_{\mu\nu}.
\]

Our bulk metric is a \( n + 1 \)-dimensional static, spherically symmetric space-time,

\[
ds^2_b = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega_{n-1}^2,
\]

in which \( f(r) \) is the only metric function to be determined and \( d\Omega_{n-1}^2 \) is the line element of \( S^{n-1} \). Upon imposing the constraint

\[
-f(a)\left(\frac{dt}{d\tau} \right)^2 + \frac{1}{f(a)} \left(\frac{da}{d\tau} \right)^2 = -1,
\]

with the DW position at \( r = a(\tau) \), the DW’s line element takes the form

\[
ds^2_{\text{DW}} = -dr^2 + a(\rho)^2 d\Omega_{n-1}^2.
\]

This is the standard Friedmann-Robertson-Walker metric and its only degree of freedom is \( a(\tau) \) in which \( \tau \) is the proper time measured by the observer on the DW. Now, we wish to consider the rules satisfied by the DW as the boundary of \( \mathcal{M}_\pm \). These boundary conditions are the generalized Israel conditions which correspond to the Einstein equations on the wall. [1]

The generalized Darmois-Israel junction conditions on \( \Sigma \) apt for the GB extension is [2]

\[
-\frac{1}{\kappa^2} ((K^i_j - \alpha \delta^i_j) - \frac{\alpha}{2\kappa^2} (3J^i_j - J \delta^i_j + 2P^i_mn K^{mn}) = S^i_j,
\]

where the surface energy-momentum tensor \( S_{ij} \) is given by [3]

\[
S_{ij} = \frac{1}{\sqrt{-h}} \sqrt{g_{ij}} \int d^n x \sqrt{-h} (-\sigma).
\]

The form of the stress-energy tensor can be written as

\[
S^i_j = -\sigma \delta^i_j
\]

in which \( \sigma = \text{constant} \), stands for the wall tension (or energy density of the wall \( \Sigma \)). Considering the energy-momentum tensor in the form \( S_i^j = \text{diag}(-\rho, p, p, \ldots) \), we observe that \( \sigma = \rho = -p \), and satisfies the weak energy condition. Here in (8) a bracket implies a jump across \( \Sigma \). The divergence-free part of the Riemann tensor \( P_{abcd} \) and the tensor \( J_{ab} \) (with trace \( J = J^i_i \)) are given by [2]

\[
P^m_{imnj} = R_{mijn} + (R_{mn}g_{ij} - R_{mj}g_{in}) - (R_{im}g_{mj} - R_{ij}g_{mn}) + \frac{1}{2} R (g_{im}g_{nj} - g_{ij}g_{mn}),
\]

\[
J_{ij} = \frac{1}{8} \left[ -2K_{ij}g_{mn}K_{nm} + K_{mn}K_{mn}K_{ij} - 2K_{im}K_{mn}K_{nj} - K^2 K_{ij} \right].
\]
By employing these expressions through (8) and (10) we find the energy density and surface pressures for a generic metric function \( f(r) \), with \( r = a(\tau) \). The results are given by [4]

\[
- \Delta(n - 1) \left[ \frac{2}{a} - \frac{4\bar{a}}{3a^2} (\Delta^2 - 3(1 + \dot{a}^2)) \right] = \kappa^2 \sigma, \tag{13}
\]

\[
\frac{2(n - 2)\Delta}{a} + \frac{2\ell}{\Delta} - \frac{4\bar{a}}{3a^2} \left[ 3\ell\Delta - \frac{3\ell}{\Delta} (1 + \dot{a}^2) + \frac{\Delta^3}{a} (n - 4) \right] - 6\Delta \left( \frac{n - 4}{2} (1 + \dot{a}^2) \right) = -\kappa^2 \sigma, \tag{14}
\]

where \( \ell = \bar{a} + f'(a)/2 \) and \( \Delta = \sqrt{f(a) + \dot{a}^2} \) in which

\[
f(a) = f_-(r)|_{r = a}. \tag{15}
\]

Note that a dot \( \cdot \cdot \cdot \) implies derivative with respect to the proper time.

We differentiate (13) to get (with \( \dot{a} = \ell - f'(a)/2 \))

\[
\ell = \frac{\Delta^2}{a} a^2 + \bar{a} \left[ 2(af' - \Delta^2) + 6(1 + \dot{a}^2) \right],
\]

which, after substitution into (14) we recover (13). In other words, Eqs. (13) and (14) are not independent, the solution of one satisfies also the other. Now, we analyze the first equation (13) of the junction conditions. By some manipulation, \( \Delta \) above can be expressed in the form

\[
\Delta = \sqrt{\xi} - \frac{\epsilon}{3\sqrt{s}}, \tag{17}
\]

where

\[
\xi = -\frac{s}{8} \pm \frac{1}{16} \sqrt{12e^3 + 81s^2}, \tag{18}
\]

\[
s = \frac{3}{8} \frac{\kappa^2 \sigma a^3}{\bar{a}(d - 1)}, \quad \epsilon = \frac{3}{2} \left( 1 - f + \frac{\dot{a}^2}{2\bar{a}} \right). \tag{19}
\]

From \( \Delta = \sqrt{f(a) + \dot{a}^2} \) and (17) it follows that

\[
\dot{a}^2 + V(a) = 0, \tag{20}
\]

where

\[
V(a) = f - \left( \sqrt{\xi} - \frac{\epsilon}{3\sqrt{s}} \right)^2. \tag{21}
\]

In the sequel we consider the wall to be a classical one-dimensional particle which moves with zero total energy under the effective potential \( V(a) \). It is clear from (20) that only \( V(a) < 0 \) has a physical meaning. By plotting \( V(a) \) in terms of \( a \) we investigate the possible types of motion for the wall.

The metric function \( f(r) \) is the solution of the Einstein equations in the \( n + 1 \) - dimensional bulk, i.e., from Eq. (5). In terms of the Arnowitt-Deser-Misner mass and GB parameter \( \bar{a} = (n - 2)(n - 3)\alpha \), the solution for \( f(r) \) is [5]

\[
f_{\pm}(r) = 1 + \frac{r^2}{2\bar{a}} \left( 1 \mp \sqrt{1 + \frac{16\bar{a}M}{(n - 1)r^2}} \right). \tag{22}
\]

Here, the negative branch gives the correct limit of general relativity, i.e.,

\[
\lim_{\bar{a} \to 0} f_{-(r)} = 1 - \frac{4M}{(n - 1)r^2}, \tag{23}
\]

\[
\lim_{\bar{a} \to \infty} f_{-(r)} = 1. \tag{24}
\]

For this reason we consider the negative branch solution, which means that \( f(a) = f_{-}(a) \). Upon substitution of \( f(a) \) in (21) we observe that

\[
\lim_{\bar{a} \to 0} V(a) = 1,
\]

which corresponds to a nonphysical case [i.e. Eq. (20)] and

\[
\lim_{\bar{a} \to \infty} V(a) = V_0 = 1 - \frac{4M}{(n - 1)\alpha^{n-2}} - \frac{\kappa^4 a^2 \sigma^2}{4 (n - 1)^2}. \tag{25}
\]

This shows that vanishing of the GB parameter yields a potential on the DW which contains a gravitational and antiharmonic oscillator potentials. The exact potential (with \( \alpha \neq 0 \), however, has a rather intricate structure which can be expanded in terms of the \( \alpha \) as

\[
V(a) = V_0 + V_1\alpha + V_2\alpha^2 + \ldots \tag{26}
\]

for \( V_0 \) was given in Eq. (26)

\[
V_1 = \frac{(n - 2)(n - 3)}{(n - 1)^2} \left( \frac{\sigma^4 k^8 a^2}{6(n - 1)^2} + \frac{4M\bar{a}^2k^4}{(n - 1)\alpha^{n-2}} + \frac{16(n - 3)M^2}{a^2(n - 1)} \right), \tag{27}
\]

and

\[
V_2 = \frac{(n - 2)^2(n - 3)^2}{(n - 1)^3} \left( -\frac{7}{36} \frac{\sigma^6 k^{12} a^2}{(n - 1)^3} - \frac{20}{3} \right)
\]

\[
\times \frac{M\bar{a}^4 k^8}{(n - 1)^2\alpha^{n-2}} - \frac{64M^2\sigma^2 k^4}{(n - 1)\alpha^{3(n - 1)}} - \frac{128M^3}{\alpha^{3(n - 2)}} \tag{28}
\]

In Figs. 1–3 we display \( V(a) \) and \( f(a) \) for \( \kappa^2 = 1, \sigma = 1, n = 4 \), with changing \( \alpha \) and \( M \). For different \( \alpha \) and \( M \) values we may obtain similar plots, such as for example \( 2a \) and \( 3c \). This implies that the effect of \( \alpha \) may be compensated with that of \( M \) and vice versa. Once inside the event horizon of the black hole the DW has no chance but crush to the central singularity as it should. This is the ultimate fate of our DW universe if it lies inside a large black hole. For favorable condition of the potential [i.e. \( V(a) < 0 \)] and in the vicinity (outside) of the horizon the DW collapses into the black hole much like shells [6]. The overall view, however, whether we have a black hole or not is that the potential provides a minimum bounce for the DW which is determined by the GB parameter \( \alpha \).
We should also add that in our analysis we were unable to see a maximum bounce. This implies that the GB extension of general relativity does not suffice to provide a closed universe on DW.

Figure 4 plots the same quantities in $n = 5$, for comparison with the previous ones in $n = 4$. What we observe is that going into higher dimensions does not change the general features except that some nonblack hole cases will turn into black holes. We should remark also that although the coupling constant $\sigma^2/C^2$ between the bulk and DW has been fixed as $\sigma^2/C^2 = 1$, its effect can be investigated by taking different values for $\sigma$. In general, larger $\sigma$ results smaller bouncing radii and vice versa.

In conclusion, if our 4-dimensional universe, assumed as a Friedmann-Robertson-Walker universe on a DW laying in a 5-dimensional Einstein-Gauss-Bonnet (EGB) bulk, the nature of a DW inside the black hole of course changes, since it turns into a dynamic and collapsing object toward the central singularity. This occurs in 2(a) and 2(b) more clearly. Fig. 2(d) is similar to 1(b), which means that the mass difference compensates with the difference in $\alpha$.
GB term protects us against the big crunch. Inclusion of physical fields such as Maxwell and Yang-Mills will definitely enrich our world on such a DW. Abiding by a bulk consisting of pure geometrical terms alone, however, the hierarchy of GB, known as the Lovelock gravity must be taken into account.


