Theorem to generate Einstein-Non Linear Maxwell Fields

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Abstract

We present a theorem in d-dimensional static, spherically symmetric spacetime in generic Lovelock gravity coupled with a non-linear electrodynamic source to generate solutions. The theorem states that irrespective of the order of the Lovelock gravity and non-linear Maxwell (NLM) Lagrangian, for the pure electric field case the NLM equations are satisfied by virtue of the Einstein-Lovelock equations. Applications of the theorem, specifically to the study of black hole solutions in Chern-Simons (CS) theory is given. Radiating version of the theorem has been considered, which generalizes the Bonnor-Vaidya (BV) metric to the Lovelock gravity with a NLM field as a radiating source. We consider also the radiating power - Maxwell source (i.e. $(F_{\mu\nu}F^{\mu\nu})^q$, $q =$ finely - tuned constant) within the context of Lovelock gravity.

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I. INTRODUCTION

The string theory motivated higher dimensional gravity, known as the Lovelock gravity [1] attracted much interest in recent years [2]. This theory is known to admit the most general higher order invariants in such combinations that the field equations preserve their second order form. These are important features concerning divergences at smaller scales and ghost free structure toward a quantum theory of gravity. The first few orders of the Lovelock Lagrangian are well known: zeroth /first order is just the cosmological constant (Λ) /Einstein-Hilbert (EH) term. The second order Lovelock term is known also as the Gauss-Bonnet (GB) term, which consists of quadratic invariants and is a highly non-trivial contribution, especially in higher dimensions[3]. Going to higher order corrections serves only to add contributions at higher levels of intricacy which are crucial in a possible quantum gravity. In spite of all complications it is remarkable that in a spherically symmetric line element exact black hole solutions are found in higher order Lovelock gravity [4]. These metrics are sourced by Maxwell and Yang-Mills (YM) fields. More recently, we obtained black holes in a power-YM source in Lovelock theory[5]. By the power-YM source it is meant that the source consists of a power of the YM invariant, i.e., $\left(F^{a}_{\mu\nu}F^{a\mu\nu}\right)^{q}$. Here $F^{a}_{\mu\nu}$ refers to the YM field with gauge group index a and $q > 0$ is an arbitrary constant parameter. Although q may stand arbitrary, energy and causality conditions restrict it to a certain set of admissible parameters [5].

By the same token, in this paper we consider Lovelock gravity of all higher orders coupled with a non-linear Maxwell (NLM) source[6]. We prove a Theorem, covering Lovelock terms to all orders - albeit with proper constant coefficients - which generates new metrics once the NLM energy-momentum is known. The energy-momentum is given by $\mathcal{L}\left(\mathcal{F}\right)$, in which $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ denotes the Maxwell invariant. The field equation satisfied by $F_{\mu\nu}$ will be referred to as the NLM equation. The Theorem involves spherically symmetric metric ansatz together with the NLM Lagrangian which leads to a general class of metrics. Static electric fields fall within the range of our Theorem which yields Born-Infeld (BI) electrodynamics as a particular example. A static, pure magnetic YM field constitutes another example of BI type.

For the first/second order Lovelock terms the Theorem addresses to EH/GB gravities, resulting in metrics expressed in solutions of first/second order algebraic equations. With equal ease, we extend this result to any higher order Lovelock gravity in terms of a higher order algebraic equations [7]. As an interesting example we obtain Chern-Simons-Born-Infeld (CSBI) black hole solutions in odd dimensions. We elaborate on $d = 5$, to show in particular that the thermodynamically well-behaving CS black hole preserves, its good features when coupled with a BI source in the general relativity limit. We note that combination of CS and BI types is in an artificial manner since CS/BI black holes arise naturally in odd /even dimensions. Such a combination works only when the ${}^{\star}F_{\mu\nu}$ (i.e., dual of $F_{\mu\nu}$) vanishes, which occurs in a restricted type of sources. BI black holes are known to arise naturally from Pfaffians only in even dimensional spacetimes [5]. Depending on the topological parameter $\chi = 0, \pm 1$ and $\Lambda \gtreqless 0$ we investigate the Hawking temperature of the 5-dimensional CSBI black holes. Finally, we investigate the implications of our Theorem in the Eddington-Bondi form of radiating metrics [8]. This is the time dependent version of both the mass and charge so that the metrics become time dependent. That is, we extend the Bonnor-Vaidya [8] form of radiating metric to the d-dimensional Lovelock gravity coupled with a NLM field. We consider various types of non-linearities in electromagnetic field as particular examples.

Organization of the paper is as follows. The Theorem, its proof and applications are given in Sec. II. Generalization of the Theorem to the radiating metrics in higher dimensions with NLM as source is studied in Sec. III. In section IV, we investigate the case of power-Maxwell non-linearity. The paper is completed with the conclusion in Sec. V.

II. A THEOREM FOR SOLVING EINSTEIN-LOVELOCK-NLM EQUATIONS

Theorem : *Let the d-dimensional static spherically symmetric spacetime sourced by non-linear electromagnetic field be described by the action* $(8\pi G = 1)$

$$S = \frac{1}{2}\int dx^d \sqrt{-g}\left\{-\frac{(d-2)(d-1)}{3}\Lambda + \mathcal{L}_1 + \alpha_2\mathcal{L}_2 + ... + \alpha_{\left[\frac{d-1}{2}\right]}\mathcal{L}_{\left[\frac{d-1}{2}\right]} + \mathcal{L}(\mathcal{F})\right\} \quad , \tag{1}$$

in which

$$\mathcal{L}_n = 2^{-n}\delta^{a_1...a_n b_1...b_n}_{c_1...c_n d_1...d_n} R^{c_1 d_1}{}_{a_1 b_1} ... R^{c_n d_n}{}_{a_n b_n}, \quad n \geq 1, \tag{2}$$

is the nth order Lovelock Lagrangian, α_n *is a real constant and the bracket* $[.]$ *refers to integer part. Herein* $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$, *is the Maxwell invariant for a static field 2-form*

$$F = E(r)\, dt \wedge dr. \tag{3}$$

If the $\mathcal{L}(\mathcal{F})$ satisfies the non-linear Maxwell (NLM) equation

$$d\left({}^{\star}F\mathcal{L}_{\mathcal{F}}\right)=0 \tag{4}$$

with $\mathcal{L}_{\mathcal{F}}=\frac{\partial\mathcal{L}}{\partial\mathcal{F}}$ and ${}^{\star}F$ the dual of F, then the Einstein equations admit the solution

$$ds^2=-(\chi-r^2H(r))dt^2+\frac{1}{(\chi-r^2H(r))}dr^2+r^2d\Omega_{d-2}^2 \tag{5}$$

where $\chi=0,\pm1$ and in which $H(r)$ is the solution (or solutions) of the following algebraic equation of order $\left[\frac{d-1}{2}\right]$

$$\sum_{k=1}^{\left[\frac{d-1}{2}\right]}\tilde{\alpha}_k H(r)^k=\frac{\Lambda}{3}+\frac{M}{r^{d-1}}-\frac{1}{(d-2)\,r^{d-1}}\int r^{d-2}\left(\mathcal{L}-2\mathcal{L}_{\mathcal{F}}\mathcal{F}\right)dr. \tag{6}$$

Here m is an integration constant, $\tilde{\alpha}_k=\prod_{i=3}^{2k}(d-i)\,\alpha_k$ and $\tilde{\alpha}_1=1$.

Proof: Variation of the action with respect to the metric tensor $g_{\mu\nu}$ yields the field equations in the form

$$\mathcal{G}_{\mu}^{\ \nu}\left(=\mathcal{G}_{\mu}^{\ \nu(EH)}+\sum_{k=2}^{\left[\frac{d-1}{2}\right]}\alpha_k\mathcal{G}_{\mu}^{\ \nu(k)}\right)+\frac{(d-2)(d-1)}{6}\Lambda\delta_{\mu}^{\ \nu}=T_{\mu}^{\ \nu}, \tag{7}$$

where $\mathcal{G}_{\mu}^{\ \nu(EH)}$ is the Einstein tensor, while

$$\mathcal{G}_{\mu}^{\ \nu(k)}=\frac{1}{2^{k+1}}\delta^{aa_1...a_kb_1...b_k}_{bc_1...c_kd_1...d_k}R^{c_1d_1}_{\ \ \ \ a_1b_1}...R^{c_kd_k}_{\ \ \ \ a_kb_k}. \tag{8}$$

The $T_{\mu}^{\ \nu}$ is given by

$$T_{\mu}^{\ \nu}=\frac{1}{2}\left(\mathcal{L}\delta_{\mu}^{\ \nu}-4\mathcal{L}_{\mathcal{F}}F_{\mu\lambda}F^{\nu\lambda}\right), \tag{9}$$

which clearly gives $T_{t}^{\ t}=T_{r}^{\ r}=\frac{1}{2}\mathcal{L}-\mathcal{L}_{\mathcal{F}}\mathcal{F}$, stating that $\mathcal{G}_{t}^{\ t}=\mathcal{G}_{r}^{\ r}$ and $T_{\theta_i}^{\ \theta_i}=\frac{1}{2}\mathcal{L}$. Now we introduce our metric as given by (5), where the choice of $g_{tt}=-\left(g_{rr}\right)^{-1}$ is a direct result of $\mathcal{G}_{t}^{\ t}=\mathcal{G}_{r}^{\ r}$ up to a constant coefficient where we set it to be one. By starting with the line element (5) one gets

$$\mathcal{G}_t^t=\mathcal{G}_r^r=-\frac{(d-2)}{2r^{d-2}}\left(r^{d-1}H(r)\right)'-\frac{(d-2)}{2r^{d-2}}\sum_{k=2}^{\left[\frac{d-1}{2}\right]}\tilde{\alpha}_k\left(r^{d-1}H(r)^k\right)', \tag{10}$$

and

$$\mathcal{G}_{\theta_i}^{\theta_i}=-\frac{1}{2r^{d-3}}\left(r^{d-1}H(r)\right)''-\frac{1}{2r^{d-3}}\sum_{k=2}^{\left[\frac{d-1}{2}\right]}\tilde{\alpha}_k\left(r^{d-1}H(r)^k\right)'', \tag{11}$$

where $(.)' = \frac{d}{dr}(.)$. Eq.s (10) and (11) admit

$$\left(\frac{r^{d-2}}{d-2}\mathcal{G}_t^{\ t}\right)' = r^{d-3}\mathcal{G}_{\theta_i}^{\ \theta_i}, \tag{12}$$

and imposing Einstein equations yield

$$\left(\frac{r^{d-2}}{d-2}\left(T_t^{\ t} - \frac{(d-2)(d-1)}{6}\Lambda\right)\right)' = r^{d-3}\left(T_{\theta_i}^{\ \theta_i} - \frac{(d-2)(d-1)}{6}\Lambda\right). \tag{13}$$

This is equivalent to,

$$\left(\frac{r^{d-2}}{d-2}\left(\frac{1}{2}\mathcal{L} - \mathcal{L}_{\mathcal{F}}\mathcal{F}\right)\right)' = r^{d-3}\left(\frac{1}{2}\mathcal{L}\right), \tag{14}$$

which integrates to

$$r^{d-2}\mathcal{L}_{\mathcal{F}}\sqrt{|\mathcal{F}|} = \text{constant}. \tag{15}$$

This result is in conform with the solution of the NLM equation (4) and the integration constant can be identified with the electric charge. Now recall from (7), that

$$\mathcal{G}_t^{\ t} + \frac{(d-2)(d-1)}{6}\Lambda = T_t^{\ t}, \tag{16}$$

or equivalently

$$-\frac{(d-2)}{2r^{d-2}}\left(r^{d-1}H(r)\right)' - \frac{(d-2)}{2r^{d-2}}\sum_{k=2}^{\left[\frac{d-1}{2}\right]}\tilde{\alpha}_k\left(r^{d-1}H(r)^k\right)' = \frac{1}{2}\mathcal{L} - \mathcal{L}_{\mathcal{F}}\mathcal{F} - \frac{(d-2)(d-1)}{6}\Lambda, \tag{17}$$

which implies

$$\sum_{k=1}^{\left[\frac{d-1}{2}\right]}\tilde{\alpha}_k H(r)^k = \frac{\Lambda}{3} + \frac{M}{r^{d-1}} - \frac{1}{(d-2)\,r^{d-1}}\int r^{d-2}\left(\mathcal{L} - 2\mathcal{L}_{\mathcal{F}}\mathcal{F}\right)dr, \tag{18}$$

for the integration constant $M = \frac{4m}{(d-2)}$. This completes the proof of our Theorem.

Remark 1 *In the Theorem, we considered only a static electric field whose 2-form is $F = E(r)\,dt \wedge dr$, giving $\mathcal{F} = -2E^2$. Since $\mathcal{F}$ is only a function of r so is the Lagrangian $\mathcal{L}(\mathcal{F})$, this aided in the proof of the Theorem. We must add that the Theorem becomes of practical use whenever the expression on the right hand side is integrable. Otherwise we should resort to the multipole expansions for the asymptotically flat spacetimes.*

Remark 2 *In the case of pure magnetic field the Theorem is applicable only if* $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ *is only a function of* r*. This implies*

$$\left(\frac{r^{d-2}}{d-2}T_t^{\ t}\right)' = r^{d-3}T_{\theta_i}^{\ \theta_i}, \tag{19}$$

which leads to the same metric function in the form

$$\sum_{k=1}^{\left[\frac{d-1}{2}\right]} \tilde{\alpha}_k H(r)^k = \frac{\Lambda}{3} + \frac{M}{r^{d-1}} - \frac{2}{(d-2)\, r^{d-1}} \int r^{d-2} T_t^{\ t} dr. \tag{20}$$

Remark 3 *In the case of a general energy - momentum tensor*

$$T_\mu^\nu = diag\left(T_t^{\ t}, T_r^{\ r}, T_{\theta_1}^{\ \theta_1}, ...\right), \tag{21}$$

in which

$$T_t^{\ t} = T_r^{\ r}, T_{\theta_1}^{\ \theta_1} = T_{\theta_2}^{\ \theta_2} = ... \tag{22}$$

and

$$\left(\frac{r^{d-2}}{d-2}T_t^{\ t}\right)' = r^{d-3}T_{\theta_i}^{\ \theta_i}, \tag{23}$$

again, a solution in the form of (5) satisfies the Einstein equations with $H(r)$ *given by (20).*

Remark 4 *The case of* $\mathcal{L} - 2\mathcal{L}_{\mathcal{F}}\mathcal{F} = 0$ *must be excluded, since it implies a Lagrangian of the form* $\mathcal{L} = \sqrt{\mathcal{F}}$*, which fails to satisfy the energy and causality conditions [5]. This form of the Lagrangian lacks also the linear Maxwell limit.*

Example 1: As an application we consider the case of pure electric Einstein-Born-Infeld (EBI) black hole solution. The pure electric BI Lagrangian can be written as [6]

$$\mathcal{L}(\mathcal{F}) = 4\beta^2\left(1 - \sqrt{1 + \frac{\mathcal{F}}{2\beta^2}}\right), \tag{24}$$

where $\mathcal{F}$ is the electric field invariant given by $\mathcal{F} = 2F_{tr}F^{tr} = -2E(r)^2$. Note that for $\beta \to \infty$ we recover the standard, linear Maxwell Lagrangian. Consequently

$$\mathcal{L}_{\mathcal{F}} = -\frac{1}{\sqrt{1 + \frac{\mathcal{F}}{2\beta^2}}}, \tag{25}$$

and therefore upon solving the non-linear Maxwell equation one finds

$$E = \frac{q\beta}{\sqrt{q^2 + \beta^2 r^{2(d-2)}}}. \tag{26}$$

Finally we obtain

$$H(r)=\frac{\Lambda}{3}+\frac{M}{r^{d-1}}-\frac{2}{(d-2)\,r^{d-1}}\int r^{d-2}T_t^{\ t}dr==\frac{\Lambda}{3}+\frac{4m}{(d-2)\,r^{d-1}}- \tag{27}$$
$$\frac{4\beta^2}{(d-1)(d-2)}\left(1-\sqrt{1+\frac{q^2}{\beta^2r^{2(d-2)}}}\right)-\frac{4}{(d-1)(d-3)}\frac{q^2}{r^{2(d-2)}}\times$$
$${}_2F_1\left(\frac{1}{2},\frac{d-3}{2(d-2)},\frac{3d-7}{2(d-2)},-\frac{q^2}{\beta^2r^{2(d-2)}}\right),$$

in which ${}_2F_1$ stands for the hypergeometric function.

Example 2: Another example for the case of non-electric field is given by the Einstein–Yang-Mills (EYM) non-linear electrodynamics black hole solution. In fact in Ref. [9] we find that

$$T_t^{\ t}=T_r^{\ r}=2\beta^2\left(1-\sqrt{1+\frac{(d-2)(d-3)\,Q^2}{2\beta^2r^4}}\right), \tag{28}$$

and

$$T_{\theta_i}^{\ \theta_i}=2\beta^2\left(1-\sqrt{1+\frac{(d-2)(d-3)\,Q^2}{2\beta^2r^4}}\right)+\frac{2(d-3)\,Q^2}{r^4\sqrt{1+\frac{(d-2)(d-3)Q^2}{2\beta^2r^4}}}, \tag{29}$$

which clearly satisfies the conditions of the Theorem and the Einstein equations admit a black hole solution with

$$H(r)=\frac{\Lambda}{3}+\frac{4m}{(d-2)\,r^{d-1}}-\frac{4\beta^2}{(d-1)(d-2)}+$$
$$\frac{4\beta^2}{(d-2)\,r^{d-1}}\int dr r^{d-4}\sqrt{r^4+\frac{(d-2)(d-3)\,Q^2}{2\beta^2}}, \tag{30}$$

in conform with the solution given in Ref. [9].

Example 3: Our next example will be the general form of energy momentum tensor given by Salgado [10] which states that

$$T_{\mu(Diag.)}^{\nu}=\frac{C}{r^{n(1-k)}}\left[1,1,k,...,k\right],\qquad (C,k:\text{constants}),$$

admits solutions for Einstein equations. Now we show that this is a natural result for the case of Rem. 3. In Rem. 3 let's consider

$$T_t^{\ t}=T_r^{\ r},\qquad T_{\theta_i}^{\ \theta_i}=kT_t^{\ t}, \tag{31}$$

then, from equation (23) we obtain

$$\left(\frac{r^{d-2}}{d-2}T_t^{\ t}\right)'=kr^{d-3}T_t^{\ t}, \tag{32}$$

or, in a straightforward calculation one finds

$$T_t^{\ t} = \frac{C}{r^{n(1-k)}}, \tag{33}$$

where C is an integration constant. This verifies that the Theorem proved by Salgado [10] turns out to be a particular case of our more general Theorem.

Example 4: One may notice that a proper choice of $\tilde{\alpha}_k$ leads to a Chern-Simons (CS) [11] gravity in odd dimensions. To do so we set

$$\tilde{\alpha}_k = \frac{\bar{\alpha}_k}{\bar{\alpha}_1}, \text{ for } k \geq 2 \text{ and } -\frac{\Lambda}{3} = \frac{\bar{\alpha}_0}{\bar{\alpha}_1}, \tag{34}$$

and we rewrite (18) as

$$\sum_{k=0}^{\left[\frac{d-1}{2}\right]} \bar{\alpha}_k H(r)^k = \bar{\alpha}_1 \left(\frac{M}{r^{d-1}} - \frac{1}{(d-2)\, r^{d-1}} \int r^{d-2} \left(\mathcal{L} - 2\mathcal{L}_{\mathcal{F}} \mathcal{F} \right) dr \right). \tag{35}$$

Now we choose

$$\bar{\alpha}_k = \binom{\left[\frac{d-1}{2}\right]}{k} \ell^{2k-d}, \tag{36}$$

where

$$-\frac{\Lambda}{3} = \frac{\bar{\alpha}_0}{\bar{\alpha}_1} = \frac{\ell^{-2}}{\left[\frac{d-1}{2}\right]}, \tag{37}$$

to get from the binomial expansion

$$\left(1 + \ell^2 H(r)\right)^{\left[\frac{d-1}{2}\right]} = \ell^d \bar{\alpha}_1 \left(\frac{M}{r^{d-1}} - \frac{1}{(d-2)\, r^{d-1}} \int r^{d-2} \left(\mathcal{L} - 2\mathcal{L}_{\mathcal{F}} \mathcal{F} \right) dr \right). \tag{38}$$

This implies that

$$H(r) = -\frac{1}{\ell^2} + \frac{\sigma}{\ell^2} \left[\ell^d \bar{\alpha}_1 \left(\frac{M}{r^{d-1}} - \frac{1}{(d-2)\, r^{d-1}} \int r^{d-2} \left(\mathcal{L} - 2\mathcal{L}_{\mathcal{F}} \mathcal{F} \right) dr \right) \right]^{\frac{1}{\left[\frac{d-1}{2}\right]}}, \tag{39}$$

where $\sigma = +1$ if $\left[\frac{d-1}{2}\right]$ is an odd integer and $\sigma = \pm 1$ if $\left[\frac{d-1}{2}\right]$ is an even integer.

The latter equation for $d =$odd, $\chi = 1$ admits ($\ell^2 > 0$)

$$f(r) = 1 + \frac{r^2}{\ell^2} - \sigma \left[\ell \bar{\alpha}_1 \left(M - \frac{2B(r)}{(d-2)} \right) \right]^{\frac{2}{d-1}} \tag{40}$$

where $f(r) = \left(\chi - r^2 H(r)\right)$ is the metric function and $B(r) = \int^r z^{d-2} T_t^t(z)\, dz$. This metric function by using (39) and (40) becomes

$$f(r) = 1 + \frac{r^2}{\ell^2} - \sigma \left[m + 1 - \frac{(d-1)\, B(r)}{(d-2)\, \ell^{(d-3)}} \right]^{\frac{2}{d-1}}. \tag{41}$$

in which the new integration constant m is related to M [11]. In the sequel we investigate some thermodynamic properties of this solution.

1. With $\sigma = +1$ or $\frac{d-1}{2}$ is an odd integer ($d = 7, 11, 15, ...$)

In this case one finds the Hawking's temperature as

$$T_H = \frac{1}{4\pi} f'(r_+) = \frac{1}{2\pi}\left(\frac{r_+}{\ell^2} + \frac{r_+^3 T_t^t(r_+)}{(d-2)\left(\ell^2 + r_+^2\right)^{\left(\frac{d-3}{2}\right)}}\right). \tag{42}$$

The specific heat capacity of the black hole for constant charge is defined by

$$C_q = T_H\left(\frac{\partial S}{\partial T_H}\right)_q, \qquad S = \frac{(d-1)\,\pi^{\frac{d-1}{2}}}{4\left(\frac{d-1}{2}\right)!} r_+^{d-2}, \tag{43}$$

and is obtained as

$$C_q = \frac{(d-1)(d-2)\,\pi^{\left(\frac{d-1}{2}\right)} r_+^{d-2}}{\Gamma\left(\frac{d+1}{2}\right)} \frac{\Upsilon}{\Psi}, \tag{44}$$

where

$$\Upsilon = (d-2) + \frac{r_+^2 \ell^2}{\left(\ell^2 + r_+^2\right)^{\left(\frac{d-3}{2}\right)}} T_t^t(r_+), \tag{45}$$

$$\Psi = (d-2) + \frac{\ell^2 r_+^2}{\left(\ell^2 + r_+^2\right)^{\left(\frac{d-1}{2}\right)}}\left[r_+\left(\ell^2 + r_+^2\right)\frac{\partial}{\partial r} T_t^t(r_+) + T_t^t(r_+)\left(3\ell^2 - r_+^2(d-6)\right)\right]. \tag{46}$$

2. With $\sigma = \pm 1$ or $\frac{d-1}{2}$ is an even integer ($d = 5, 9, 13, ...$)

It is clear that in this case $\sigma = -1$ does not claim any horizon and therefore it is out of our interest but for $\sigma = 1$ branch the Hawking temperature and the specific heat capacity are given as (42) and (44) but there exists an additional constraint on the free parameter in order to have a real metric function.

To complete this section we give an example for the CSBI black hole ($\chi = 1, \sigma = 1$) in 5-dimensions. To do so we recall that in 5-dimensions

$$B(r) = \int^r z^3 T_t^t(z)\,dz = \frac{\beta^2 r^4}{2}\left(1 - \sqrt{1 + \frac{q^2}{\beta^2 r^6}}\right) + \frac{3}{4}\frac{q^2}{r^2}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{3}, \frac{4}{3}, -\frac{q^2}{\beta^2 r^6}\right) \tag{47}$$

and therefore

$$f(r) = 1 + \frac{r^2}{\ell^2} - \left\{m + 1 - \left[\frac{2\beta^2 r^4}{3\ell^2}\left(1 - \sqrt{1 + \frac{q^2}{\beta^2 r^6}}\right) + \frac{q^2}{\ell^2 r^2}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{3}, \frac{4}{3}, -\frac{q^2}{\beta^2 r^6}\right)\right]\right\}^{\frac{1}{2}}. \tag{48}$$

In Fig. 1 we plot $f(r)$ for the specific choices of mass, charge and the cosmological constant. The corresponding temperature and specific heat capacity plots are given in Fig.s 2 and 3.

Example 5 (Clouds of strings as source): We consider another application of the Theorem that incorporates energy-momentum tensor representing clouds of string type matter fields [12]. More recently [13], clouds of string type energy-momentum is considered in Einstein-Gauss-Bonnet (EGB) gravity. In Rem. 2 and 3, we considered the case for a general energy - momentum tensor and its corresponding solution. Clouds of string type matter fields obey the condition imposed on the energy - momentum tensor stated in Remark 3, such that,

$$T_t^t = T_r^r = \frac{a}{r^{d-2}}, \qquad (49)$$
$$a = \text{real constant},$$

which leads

$$T_{\theta_i}^{\theta_i} = 0. \qquad (50)$$

Therefore the general solution for the metric function is obtained from the algebraic equation

$$\sum_{k=1}^{\left[\frac{d-1}{2}\right]} \tilde{\alpha}_k H(r)^k = \frac{\Lambda}{3} + \frac{M}{r^{d-1}} - \frac{2a}{(d-2)\, r^{d-2}}. \qquad (51)$$

The solution for $H(r)$ generalizes the solution obtained in Ref.[13] to higher order Lovelock gravity. And hence, our general solution includes the solution obtained in [13] if we restrict the spacetime dimension to $d = 5$. For this particular case the solution is obtained from

$$H(r) + \tilde{\alpha}_2 H(r)^2 = \frac{\Lambda}{3} + \frac{M}{r^4} - \frac{2a}{3r^3}, \qquad (52)$$

or equivalently

$$f(r) = \chi + \frac{r^2}{4\alpha_2}\left(1 \pm \sqrt{1 + 8\alpha_2\left(\frac{\Lambda}{3} + \frac{M}{r^4} - \frac{2a}{3r^3}\right)}\right). \qquad (53)$$

This solution is nothing but the black hole in EGB gravity in the presence of the string cloud type matter fields [12]. In this theory one can easily check that the corresponding energy momentum tensor in d-dimensions is given by

$$T_\mu^\nu = \text{diag}\left(\frac{a}{r^{d-2}}, \frac{a}{r^{d-2}}, 0, 0, ...\right). \qquad (54)$$

At this stage we wish to go a step further to relate our solution to the CS solution in odd d-dimensions as $(\chi = 1)$,

$$f(r) = 1 + \frac{r^2}{\ell^2} - \sigma\left[m + 1 - \frac{(d-1)\, ar}{(d-2)\, \ell^{(d-3)}}\right]^{\frac{2}{d-1}}, \qquad (55)$$

which in 5-dimensions becomes

$$f\left(r\right) = 1+\frac{r^{2}}{\ell^{2}} \pm \sqrt{m+1-\frac{4ar}{3\ell^{2}}}. \tag{56}$$

The thermodynamic properties of this solution is also investigated. The Hawking temperature (T_H) and heat capacity C_a at constant string parameter a is calculated at the location of the event horizon (r_h) are given by,

$$T_H = \frac{l^2(3r_h + a) + 3r_h^3}{3\pi l^2(l^2 + r_h^2)}, \tag{57}$$

$$C_a = \frac{3\pi r_h^2(l^2 + r_h^2)\left[l^2\left(3r_h + a\right) + 3r_h^3\right]}{2\left[l^2\left(6r_h^2 + 3l^2 - 2ar_h\right) + 3r_h^4\right]} \tag{58}$$

We also analyzed the thermodynamic stability which is indicated by the positive heat capacity C_a. If the heat capacity has unbounded discontinuity at particular points of r_h, this implies possible phase change from stable to unstable black hole solution. As illustrated in Fig. 4 the transitions from stable to unstable black hole solution is not continuous and therefore possible Hawking-Page type phase transition occurs [15]. The occurrence of the phase transition crucially depends on the ratio of a/l. In Fig. 4, the values of this ratio that creates phase transitions is depicted.

III. A GENERALIZATION TO THE RADIATING METRICS

Following Bonnor and Vaidya (BV) [8], we consider a non-linear electrodynamic Lagrangian $\mathcal{L}\left(\mathcal{F}\right)$, a null current $\mathbf{J}$ and a coupling term $A_\mu J^\mu$ added to the original Lagrangian such that

$$d\left({}^{\star}\mathbf{F}\mathcal{L}_{\mathcal{F}}\right) = {}^{\star}\mathbf{J}. \tag{59}$$

Now, we consider the d-dimensional version of the BV metric

$$ds^2 = -(\chi - r^2 H\left(r,u\right))du^2 + 2\epsilon dr du + r^2 d\Omega_{d-2}^2, \tag{60}$$

with the outgoing null coordinate u and field 2-form

$$\mathbf{F} = E\left(u,r\right) dr \wedge du. \tag{61}$$

This gives

$${}^{\star}\mathbf{F} = E\left(u,r\right)\sqrt{-g} d\theta_1 \wedge d\theta_2 ... \wedge d\theta_{d-2}, \tag{62}$$

in which $g = \det(g_{\mu\nu})$ and the Einstein equation is given by

$$\mathcal{G}_\mu^\nu = T_\mu^{\nu(em)} + T_\mu^{\nu(fluid)}. \tag{63}$$

Here

$$\mathcal{G}_\mu^\nu = \mathcal{G}_\mu^{\ \nu EH} + \sum_{k=2}^{\left[\frac{d-1}{2}\right]} \alpha_k \mathcal{G}_\mu^{\ \nu(k)},$$
$$T_\mu^{\nu(em)} = \frac{1}{2}\left(\mathcal{L}\delta_\mu^{\ \nu} - 4\mathcal{L}_\mathcal{F} F_{\mu\lambda}F^{\nu\lambda}\right) \tag{64}$$

and

$$T_\mu^{\nu(fluid)} = -V^\nu V_\mu. \tag{65}$$

for a null vector V_μ. We start now with NLM equation (50) which leads to

$$\begin{aligned} &d\left({}^*\mathbf{F}\mathcal{L}_\mathcal{F}\right) = d\left(E(u,r)\mathcal{L}_\mathcal{F}\sqrt{-g}d\theta_1 \wedge d\theta_2 ... \wedge d\theta_{d-2}\right) = \\ &\left[\left(E(u,r)\mathcal{L}_\mathcal{F}\sqrt{-g}\right)_r dr + \left(E(u,r)\mathcal{L}_\mathcal{F}\sqrt{-g}\right)_u du\right] \wedge d\theta_1 \wedge d\theta_2 ... \wedge d\theta_{d-2} =^* \mathbf{J}. \end{aligned} \tag{66}$$

By using the relation between 1-form current $\mathbf{J}$ and its dual i.e., $\mathbf{J} = (-\mathbf{1})^d \ {}^{**}\mathbf{J}$ one finds

$$\begin{aligned} \mathbf{J} = (-\mathbf{1})^d &\left[\left(E(u,r)\mathcal{L}_\mathcal{F}\sqrt{-g}\right)_r \ {}^\star\left(dr \wedge d\theta_1 \wedge d\theta_2 ... \wedge d\theta_{d-2}\right) + \right. \\ &\left.\left(E(u,r)\mathcal{L}_\mathcal{F}\sqrt{-g}\right)_u \ {}^\star\left(du \wedge d\theta_1 \wedge d\theta_2 ... \wedge d\theta_{d-2}\right)\right]. \end{aligned} \tag{67}$$

From the metric we find

$${}^\star\left(dr \wedge d\theta_1 \wedge d\theta_2 ... \wedge d\theta_{d-2}\right) = \frac{(-1)^{d-1}}{\sqrt{-g}}\left(dr - \epsilon f du\right), \tag{68}$$

$${}^\star\left(du \wedge d\theta_1 \wedge d\theta_2 ... \wedge d\theta_{d-2}\right) = \frac{(-1)^{d-1}}{\sqrt{-g}} du, \tag{69}$$

and therefore

$$\mathbf{J} = \left(E(u,r)\mathcal{L}_\mathcal{F} r^{d-2}\right)_r \frac{1}{r^{d-2}}\left(dr - \epsilon f du\right) - \left(E(u,r)\mathcal{L}_\mathcal{F}\right)_u \ du. \tag{70}$$

This current is going to be null i.e., $J_\mu = (J_u, 0, ..., 0) = J_u \delta_\mu^u$ which means that $\left(E(u,r)\mathcal{L}_\mathcal{F} r^{d-2}\right)_r = 0$ integrates to

$$E(u,r)\mathcal{L}_\mathcal{F} r^{d-2} = Q(u) \tag{71}$$

or

$$E(u,r)\mathcal{L}_\mathcal{F} = \frac{Q(u)}{r^{d-2}}, \tag{72}$$

for a u dependent charge $Q(u)$. After considering these results we find

$$\mathbf{F} = \frac{Q(u)}{r^{d-2}\mathcal{L}_{\mathcal{F}}} dr \wedge du, \tag{73}$$

$$\mathbf{J} = -(-1)^{d-1}\left(E(u,r)\mathcal{L}_{\mathcal{F}}\right)_u du = (-1)^d \frac{\dot{Q}(u)}{r^{d-2}} du. \tag{74}$$

where $\dot{Q}(u) = \frac{dQ(u)}{du}$. The explicit form of the Maxwell field can be expressed by

$$F = F_{\mu\nu}F^{\mu\nu} = 2F_{ru}F^{ru} = 2F_{ru}\left(g^{\alpha r}g^{\beta u}F_{\alpha\beta}\right). \tag{75}$$

while our metric tensor $g_{\mu\nu}$ and $g^{\mu\nu}$ are

$$g_{\mu\nu} = \begin{pmatrix} -f & \epsilon & 0 & 0 & . & . \\ \epsilon & 0 & 0 & 0 & . & . \\ 0 & 0 & r^2 & 0 & . & . \\ 0 & 0 & 0 & r^2\sin^2\theta & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{\epsilon} & 0 & 0 & . & . \\ \frac{1}{\epsilon} & f & 0 & 0 & . & . \\ 0 & 0 & \frac{1}{r^2} & 0 & . & . \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix} \tag{76}$$

giving

$$\mathcal{F} = -2\left(F_{ru}\right)^2. \tag{77}$$

The energy - momentum tensor components are given by

$$T_u^{u(em)} = \frac{1}{2}\left(\mathcal{L} - 4\mathcal{L}_{\mathcal{F}}F_{u\lambda}F^{u\lambda}\right) = \frac{1}{2}\left(\mathcal{L} - 2\mathcal{L}_{\mathcal{F}}\mathcal{F}\right), \tag{78}$$

$$T_r^{r(em)} = \frac{1}{2}\left(\mathcal{L} - 4\mathcal{L}_{\mathcal{F}}F_{r\lambda}F^{r\lambda}\right) = \frac{1}{2}\left(\mathcal{L} - 2\mathcal{L}_{\mathcal{F}}\mathcal{F}\right) = T_u^{u(em)}, \tag{79}$$

$$T_{\theta_i}^{\theta_i(em)} = \frac{1}{2}\mathcal{L}, \tag{80}$$

$$T_u^{r(em)} = T_r^{u(em)} = 0. \tag{81}$$

The null-fluid current vector is $V_\mu = (V_u, 0, ..., 0) = V_u\delta_\mu^u$ and therefore

$$V^\mu = \epsilon\delta_r^\mu V_u, \tag{82}$$

implying that

$$g_{\mu\nu}V^\mu V^\nu = g_{rr}\left(V^r\right)^2, \tag{83}$$

which obviously vanishes and

$$T_\mu^{\nu(fluid)} = -V^\nu V_\mu = -V^r V_u \delta_r^\nu \delta_\mu^u = -\epsilon\left(V_u\right)^2 \delta_r^\nu \delta_\mu^u. \tag{84}$$

Finally, we give the explicit form of the energy momentum tensor as

$$T^{\nu}_{\ \mu} = \begin{pmatrix} \frac{1}{2}\left(\mathcal{L}-2\mathcal{L}_{\mathcal{F}}\mathcal{F}\right) & 0 & 0 & 0 & . & . \\ -\epsilon\left(V_u\right)^2 & \frac{1}{2}\left(\mathcal{L}-2\mathcal{L}_{\mathcal{F}}\mathcal{F}\right) & 0 & 0 & . & . \\ 0 & 0 & \frac{1}{2}\mathcal{L} & 0 & . & . \\ 0 & 0 & 0 & \frac{1}{2}\mathcal{L} & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix}. \tag{85}$$

Similarly the Einstein tensor can be expressed by

$$G^{\nu}_{\ \mu} = \begin{pmatrix} -\frac{d-2}{2r^{d-2}}\left(r^{d-1}\sum_{i=1}^{\left[\frac{d-1}{2}\right]}\tilde{\alpha}_i H^i\right)' & 0 & 0 & . & . \\ \frac{d-2}{2}r\frac{\partial}{\partial u}\sum_{i=1}^{\left[\frac{d-1}{2}\right]}\tilde{\alpha}_i H^i & -\frac{d-2}{2r^{d-2}}\left(r^{d-1}\sum_{i=1}^{\left[\frac{d-1}{2}\right]}\tilde{\alpha}_i H^i\right)' & 0 & . & . \\ 0 & 0 & -\frac{1}{2r^{d-3}}\left(r^{d-1}\sum_{i=1}^{\left[\frac{d-1}{2}\right]}\tilde{\alpha}_i H^i\right)'' & . & . \\ 0 & 0 & 0 & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{pmatrix} \tag{86}$$

where

$$f\left(u,r\right) = 1 - r^2 H\left(u,r\right). \tag{87}$$

After all these arrangements we are ready now to solve the Einstein equations. We start with the uu component

$$-\frac{d-2}{2r^{d-2}}\left(r^{d-1}\sum_{i=1}^{\left[\frac{d-1}{2}\right]}\tilde{\alpha}_i H^i\right)' = -\left(\mathcal{L}-2\mathcal{L}_{\mathcal{F}}\mathcal{F}\right) \rightarrow \sum_{i=1}^{\left[\frac{d-1}{2}\right]}\tilde{\alpha}_i H^i = \frac{M(u)}{r^{d-1}} - \frac{2}{\left(d-2\right)r^{d-1}}\int r^{d-2}T^r_r dr. \tag{88}$$

One can easily check that the rr component gives the same result while the angular components give

$$-\frac{1}{2r^{d-3}}\left(r^{d-1}\sum_{i=1}^{\left[\frac{d-1}{2}\right]}\tilde{\alpha}_i H^i\right)'' = \frac{1}{2}\mathcal{L} \rightarrow \left(\frac{r^{d-2}}{d-2}\left(\mathcal{L}-2\mathcal{L}_{\mathcal{F}}\mathcal{F}\right)\right)' = r^{d-3}\mathcal{L} \tag{89}$$

which is nothing but the NLM equation (14), already satisfied. In the last step we work on $\mathcal{G}_r^u = T_r^u$ giving

$$\frac{d-2}{2} r \frac{\partial}{\partial u} \sum_{i=1}^{\left[\frac{d-1}{2}\right]} \tilde{\alpha}_i H^i = -\epsilon \left(V_u\right)^2 \tag{90}$$

and therefore

$$\frac{d-2}{2} r \left(\frac{\dot{M}(u)}{r^{d-1}} - \frac{2}{(d-2)\, r^{d-1}} \int r^{d-2} \dot{T}_r^r dr \right) = -\epsilon \left(V_u\right)^2 \tag{91}$$

which finally determines the null-fluid current vector by

$$V_u^2 = -\epsilon \left(\frac{\dot{M}(u)}{r^{d-1}} - \frac{1}{(d-2)\, r^{d-1}} \partial_u \int dr r^{d-2} \left(\mathcal{L} - 2\mathcal{L}_{\mathcal{F}} \mathcal{F}\right) \right). \tag{92}$$

IV. RADIATING POWER-MAXWELL SOURCE IN LOVELOCK GRAVITY

In this section our choice for the NLM field consists of a particular kind, namely

$$\mathcal{L}\left(\mathcal{F}\right) = -\mathcal{F}^q \tag{93}$$

in which q stands for a constant parameter and $\mathcal{F} = F_{\mu\nu} F^{\mu\nu}$, as before. Let us note that this particular version of non-linearity attracted considerable interest in recent years [14]. Although the parameter q (i.e. the power) may be assumed arbitrary, imposition of the energy - conditions with other requirements restrict q to a limited set of values. The rest of the gravitational action will be chosen as in the previous sections. Similarly, our choice of the line element follows that of (60). The electromagnetic field 2-form turns out to be

$$F = \frac{Q(u)}{r^{d-2} \mathcal{L}_{\mathcal{F}}} dr \wedge du, \tag{94}$$

and energy-momentum component $T_r{}^r$

$$T_r{}^r = \frac{1}{2} \left(\mathcal{L} - 2\mathcal{L}_{\mathcal{F}} \mathcal{F}\right) = \left(q - \frac{1}{2}\right) \mathcal{F}^q\ , \qquad \left(q \neq \frac{1}{2}\right), \tag{95}$$

so that

$$\mathcal{F}^q = (-1)^q \left(\frac{2Q^2(u)}{q^2 r^{2(d-2)}} \right)^{\frac{q}{2q-1}}. \tag{96}$$

Following the procedure of the previous section further, which led us to the condition (88) we obtain now, in analogy

$$\sum_{i=1}^{\left[\frac{d-1}{2}\right]} \tilde{\alpha}_i H^i = \frac{M(u)}{r^{d-1}} - h_d(u) r^{\frac{2(q-d+1)}{2q-1}} \tag{97}$$

in which

$$h_d(u) = (-1)^q \left(\frac{2q-1}{d-2}\right) \left(\frac{2Q^2(u)}{q^2}\right)^{\frac{q}{2q-1}}. \tag{98}$$

It is observed that an arbitrary q does not guarantee the reality of the metric function, which enforces us to choose q appropriately.

The particular dimensionality $d = 5$ brings in significant simplicity to the foregoing expressions such as

$$\tilde{\alpha}_1 H + \tilde{\alpha}_2 H^2 = \frac{M(u)}{r^4} - h_5(u) r^{\frac{2(q-4)}{2q-1}}, \tag{99}$$

where $\tilde{\alpha}_1 = 1$ and $\tilde{\alpha}_2 = 2\alpha$. The quadratic equation for H can easily be solved and the corresponding metric function $f(r,u)$ is determined by

$$f(r,u) = \chi + \frac{r^2}{4\alpha} \left(1 \pm \sqrt{1 + 8\alpha \left(\frac{M(u)}{r^4} - h_5(u) r^{\frac{2(q-4)}{2q-1}}\right)}\right). \tag{100}$$

In analogy to Eq. (82), the null-current component for the present case turns out to be

$$V_u^2 = -\epsilon \frac{3}{2} r \left(\frac{\dot{M}(u)}{r^4} - \dot{h}_5(u) r^{\frac{2(q-4)}{2q-1}}\right) \tag{101}$$

in which, as usual, a 'dot' represents $\frac{d}{du}$.

To be able to proceed further with the thermodynamical properties in the present choice of $d = 5$ (and $\chi = +1$), we must determine the apparent horizon through $f(r,u) = 0$. This leads us to the algebraic relation

$$2\alpha - M(u) + r_h^2 + h_5(u) r_h^k = 0 \tag{102}$$

in which r_h denotes apparent horizon (if any) and we have abbreviated

$$k = 2\left(\frac{5q-6}{2q-1}\right). \tag{103}$$

Tab. 1 shows the relation between q and k for certain leading numbers of interest. With reference to Table 1, we can find a finely-tuned set of q powers so that $h_5(u)$ will be real.

Clearly, any odd/even integer q will do the job whereas non-integer q's will not serve the purpose. For instance, $q=4$ is a good choice. Accordingly, we find

$$h_5(u) = \frac{7}{3}\left(\frac{Q^2(u)}{8}\right)^{4/7} > 0, \tag{104}$$

and

$$\begin{aligned} r_h(u) &= \frac{1}{2}\left(\sqrt{1+4h_5(u)\,(M(u)-2\alpha)}-1\right) \\ M(u) &> 2\alpha. \end{aligned} \tag{105}$$

The corresponding metric function takes the form

$$f(r,u) = 1 + \frac{r^2}{4\alpha}\left(1 \pm \sqrt{1+8\alpha\frac{M(u)}{r^4} - h_5(u)}\right). \tag{106}$$

Different choices of q values from Table 1 can be treated in a similar manner to obtain the corresponding $f(r,u)$ function, which we shall not go any further in this paper.

V. CONCLUSION

Methods to generate new solutions for various energy-momenta are available in the literature. In this regard, recently we have proved a theorem that generalizes Salgado's theorem [7]. The whole issue in such a problem is the physical significance of the energy-momentum under consideration. Scalar field source, for instance, is known to work only in $d=4$. Linear Maxwell electrodynamics has already been incorporated in higher dimensional Lovelock gravity [2, 4]. Our example 5 which is a particular version of the Theorem concerns a cloud of strings without Maxwell field. We explore thermodynamic properties of such a cloud for a particular solution in $d=5$ spacetime dimension, representing Chern-Simon black holes which undergoes a Hawking-Page phase transition (Fig. 4). In this paper we extended the problem to cover a non-linear Maxwell (NLM) source. Among this class of theories Born-Infeld (BI) is the most familiar one, which arises as a particular example to the Theorem. The Chern-Simons-Born-Infeld (CSBI) metrics that we found by using our Theorem turn out to be thermodynamically stable. Next, we address to dynamical problem that covers time dependent metrics and NLM fields. Stated otherwise, we have generalized the well-known dynamical Bonnor-Vaidya (BV) metric to the general Lovelock theory with NLM sources. A radiating null current source naturally accompanies the radiating energy - momentum of such metrics which lose mass and charge. It is needless to remark, finally, that the opposite problem of the 'shining star', namely, the collapse of time dependent energy - momentum and null radiation current is also solved in the same theory.

[1] D. Lovelock, J. Math. Phys. **12**, 498 (1971); N. Deruelle and L. Farina-Busto, Phys. Rev. D **41**, 3696 (1990); G. A. Mena Marugan, ibid. **46**, 4320 (1992); **46**, 4340 (1992).

[2] D. Konikowska and M. Olechowski, Phys. Rev. D **76**, 124020 (2007); F. Canfora and A. Giacomini, Phys. Rev. D **78**, 084034 (2008); A. Chakrabarti and D. H. Tchrakian, Phys. Rev. D **65**, 024029 (2001).

[3] D. G. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985); J. T. Wheeler, Nucl. Phys. B **268**, 737 (1986); R. C. Myers and J. Z. Simon, Phys. Rev. D **38**, 2434 (1988).

[4] M. H. Dehghani and M. Shamirzaie, Phys. Rev. D **72**, 124015 (2005); M. H. Dehghani, N. Alinejadi and S. H. Hendi, Phys. Rev. D **77**, 104025 (2008); S. H. Mazharimousavi and M. Halilsoy, Phys. Rev. D **76**, 087501 (2007); S. H. Mazharimousavi and M. Halilsoy, Phys. Lett. B **659**, 471 (2008); S. H. Mazharimousavi and M. Halilsoy, Phys. Lett. B **681**, 190471 (2009); S. H. Mazharimousavi and M. Halilsoy, J. Cosmol. Astropart. Phys. **12,** 005 (2008);

[5] S. H. Mazharimousavi and M. Halilsoy, Phys. Lett. B **681,** 190 (2009).

[6] M. Born and L. Infeld, *Foundations of the New Field Theory.* Proc. Roy. Soc, A **144**, 425 (1934); B. Hoffmann and L. Infeld, Phys. Rev. **51,** 765 (1937); N. Rosen, Phys. Rev. **55**, 94 (1938); Gibbons and Rasheed Nucl. Phys. B 454 (1995); S. H. Mazharimousavi and M. Halilsoy, Phys. Lett. B **678,** 407 (2009).

[7] J. Crisostomo, R. Troncoso, and J. Zanelli, Phys. Rev. D **62**, 084013 (2000); R. G. Cai, Phys. Lett. B **582**, 237 (2004); S. H. Mazharimousavi, O. Gurtug, and M. Halilsoy, arXiv:0809.3649, to appear in IJMPD.

[8] P. C. Vaidya, Proc. Indian Acad. Sci. A **33**, 264 (1951); P. C. Vaidya, Gen. Relativ. Gravit. **31**, 119 (1999); T. Kobayashi, Gen. Relativ. Gravit. **37**, 1869 (2005); H. Maeda, Class. Quant. Grav. **23**, 2155 (2006); H. Maeda and N. Dadhich, Phys. Rev. D **75**, 044007 (2007); S. Yang and D.Y. Chen, Int. J. Theor. Phys. **46,** 2923 (2007); S. Wanglin, L. Xingye, W. Li and Z.

Qingxiang, Nuc. Phy. B **166,** 270 (2007); W. B. Bonnor and P. C. Vaidya, Gen. Relativ. Gravit. 1 **127,** (1970); T. Jacobson, Phys. Rev. D **57**, 4890 (1998); M. H. Dehghani and N. Farhangkhah, Phys. Rev. D **78**, 064015 (2008); S. G. Ghosh and D. W. Deshkar, Phys. Rev. D **77**, 047504 (2008); R. G. Cai, L.M Cao, Y. P. Hu and S. P. Kim, Phys. Rev. D **78**, 124012 (2008); A. E. Dominguez and E. Gallo, Phys. Rev. D **73**, 064018 (2006); J. Podolský and O. Svítek, Phys. Rev. D **71**, 124001 (2005); F. Girotto and A. Saa, Phys. Rev. D **70**, 084014 (2004); T. Kobayashi, Gen. Relativ. Gravit. **37**, 11 (2005).

[9] S. H. Mazharimousavi, M. Halilsoy and Z. Amirabi, Phys. Rev. D **78**, 064050 (2008).

[10] M. Salgado, Class. Quant. Grav. **20,** 4551 (2003); E. Gallo, Gen. Rel. Grav. **36,** 1463 (2004); A. K. Dawood and S. G. Ghosh, Phys. Rev. D **70,** 104010 (2004); A.E. Dominguez and E. Gallo, Phys. Rev. D **73,** 064018 (2006); S. G. Ghosh and A. K. Dawood, Gen. Rel. Grav. **40,** 9 (2008).

[11] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. D **49,** 975 (1994); R. G. Cai and N. Ohta, Phys. Rev. D **74,** 064001 (2006); G. Allemandi, M. Francaviglia and M. Raiteri, Class. Quant. Grav. **20,** 5103 (2003); G. A. S. Dias, S. Gao and J. P. S. Lemos, Phys. Rev. D **75**, 024030 (2007).

[12] P. S. Letelier, Phys. Rev. D **20**, 1294 (1979).

[13] E. Herscovich and M. G. Richarte, Phys. Lett. B **689**, 192 (2010).

[14] H. Maeda, M. Hassaïne, C. Martínez, Phys. Rev. D **79,** 044012 (2009); M. Hassaïne, C. Martínez, Class. Quantum Grav. **25,** 195023 (2008); M. Hassaïne, C. Martínez, Phys. Rev. D **75,** 027502 (2007).

[15] S. W. Hawking and D. N. Page, Commun. Math. Phys. **87**, 577 (1983).

Captions:

Table 1: The relation between the parameters k and q according to Eq. (103).

Figure 1: The plot of $f(r)$ (Eq. (48)), for the specific parameters $m=\ell=q=1$, as β ranges from 0 to ∞. It is seen that black holes solutions are available only for $\beta\leq\beta_{critical}=0.2276$.

Figure 2: Hawking temperature T_H versus event horizon radius r_+ for the specific parameters $\ell=q=1$ and for $\beta\in(0,\beta_{critical}]$. By taking the absolute value of T_H automatically deletes the negative temperatures as non-physical.

Figure 3: specific heat capacity C_q versus r_+ for versus r_+ for $\ell = q = 1$ and $\beta \in (0, \beta_{critical}]$. This plot, (together with Fig. 1), reveals that our CSBI black hole solution is thermodynamically stable.

Figure 4: Heat Capacity C_a versus the horizon radius r_h for specific parameters $a = 5$ and $\ell = 1$. The singularity in C_a and therefore occurrence of Hawking-Page phase transition is clearly seen. The dash region in the inscribed figure depicts $\frac{a}{\ell}$ versus r_h for which such a transition occurs.

Table 1:

k	0	1	2	3	4	6	.	.	.
q	$\frac{6}{5}$	$\frac{11}{8}$	$\frac{5}{3}$	$\frac{9}{4}$	4	-3	.	.	.