Black hole and thin-shell wormhole solutions in Einstein-Hoffman-Born-Infeld theory

S. Habib Mazharimousavi,* M. Halilsoy,† and Z. Amirabi‡

Department of Physics, Eastern Mediterranean University,

G. Magusa, north Cyprus, Mersin 10, Turkey.

Abstract

We employ an old field theory model, formulated and discussed by Born, Infeld, Hoffman and Rosen during 1930s. Our method of cutting-gluing of spacetimes resolves the double-valuedness in the displacement vector $\vec{D}(\vec{E})$, pointed out by these authors. A characteristic feature of their model is to contain a logarithmic term, and by bringing forth such a Lagrangian anew, we aim to attract the interest of field theorists to such a Lagrangian. We adopt the Hoffman-Born-Infeld (HBI) Lagrangian in general relativity to construct black holes and investigate the possibility of viable thin-shell wormholes. In particular, the stability of thin-shell wormholes supported by normal matter in 5-dimensional Einstein-HBI-Gauss-Bonnet gravity is highlighted.

*Electronic address: habib.mazhari@emu.edu.tr

†Electronic address: mustafa.halilsoy@emu.edu.tr

‡Electronic address: zahra.amirabi@emu.edu.tr

I. INTRODUCTION

It is a well-known fact by now that non-linear electrodynamics (NED) with various formulations has therapeutic effects on the divergent results that arise naturally in linear Maxwell electrodynamics. The theory introduced by Born and Infeld (BI) in 1930s [1] constitutes the most prominent member among such class of viable NED theories. Apart from the healing power of singularities, however, drawbacks were not completely eliminated from the theory. One such serious handicap was pointed out by Born's co-workers shortly after the introduction of the original BI theory. This concerns the double-valued dependence of the displacement vector $\vec{D}\left(\vec{E}\right)$ as a function of the electric field $\vec{E}$ [2]. That is, for the common value of $\vec{E}$ the displacement $\overrightarrow{D}$ undergoes a branching which from physical grounds was totally unacceptable. To overcome this particular problem, Hoffman and Infeld [2] and Rosen[3], both published successive papers on this issue. Specifically, the model Lagrangian proposed by Hoffman and Infeld (HI) contained a logarithmic term with remarkable consequences. It removed, for instance, the singularity that used to arise in the Cartesian components of the $\vec{E}$. Being unaware of this contribution by HI, and after almost 70 years, we have rediscovered recently the ubiquitous logarithmic term of Lagrangian while in attempt to construct a model of elementary particle in Einstein-NED theory [4]. In our model the spacetime is divided into two regions: the inner region consists of the Bertotti-Robinson (BR) [5] spacetime while the outer region is a Reissner-Nordström (RN) type spacetime. The radius of our particle coincides with the horizon of the RN-type black hole solution whereas inner BR part represents a singularity-free uniform electric field region. The two regions and the NED are glued together at the horizon on which the appropriate boundary conditions gave not only a feasible geometrical model of a particle but remarkably resolved also the double-valued property of the displacement vector. In other words, with our technique, $\vec{D}\left(\vec{E}\right)$ turns automatically into a single-valued function.

In this paper we wish to make further use of the Hoffman-Born-Infeld (HBI) Lagrangian in general relativity, more specifically, in constructing 4-dimensional (4D) regular black holes and thin-shell wormholes. The wormholes in 4D requires, unfortunately, exotic matter to survive. We extend our model also to 5D-Gauss-Bonnet (GB) theory and search for the possibility of wormholes dominated by normal (i.e. satisfying the energy conditions) matter rather than exotic matter. Our analysis reveal that for the negative GB parameter ($\alpha < 0$)

5D thin-shell wormholes supported by normal matter exists, and they are stable against linear radial perturbations. Due to the intricate structure of the potential function, stability analysis is carried out numerically.

Organization of the paper is as follows. In Sec. II we adopt the HBI formalism to general relativity. Construction of regular black holes in EHBI gravity is presented in Sec. III. Thin-shell wormholes in EHBI theory follows in Sec. IV. In sec. V we give 5D black hole and wormholes in EHBIGB theory with emphasis on stability analysis of wormholes given in Sec. VI. We finalize the paper with Conclusion, which appears in Sec. VII.

II. REVIEW OF THE HBI APPROACH IN GENERAL RELATIVITY

Singularity for classical charged elementary particles leads to infinite self-electromagnetic energy. This should be removed from the Maxwell theory of charged particles and in this regard Born and Infeld (BI) introduced a non-linear electrodynamics such that successfully they solved the problem in some senses [1]. Briefly, we can summarize their proposal in curved spacetime by considering a spherically symmetric pure electrically charged particle described by the line element

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{1}$$

They aimed to have a non-singular electric field (we use $c = \hbar = k_B = 8\pi G = \frac{1}{4\pi\epsilon_\circ} = 1$) with radial component

$$\begin{aligned} E_r &= \frac{q}{\sqrt{q^2b^2 + r^4}}, \\ (b &= \text{constant, the BI parameter, and } q = \text{constant charge}) \end{aligned} \tag{2}$$

which means that the Maxwell 2−form is of the form

$$\mathbf{F} = E_r dt \wedge dr. \tag{3}$$

The corresponding action is

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\mathcal{L}\left(F,^\star F\right), \tag{4}$$

in which $F = F_{\mu\nu}F^{\mu\nu}$, $^\star F = F_{\mu\nu}{}^\star F^{\mu\nu}$ and * stands for duality (here we only consider the static, spherically symmetric electric field such that $^\star F = 0$). Easily one finds the Maxwell equation modified into

$$d\left(\mathcal{L}_F\ {}^{\star}\mathbf{F}\right) = 0 \tag{5}$$

$$\left(\mathcal{L}_F = \frac{\partial \mathcal{L}}{\partial F}\right)$$

which reveals

$$d\left(\mathcal{L}_F\ E_r r^2 \sin\theta d\theta \wedge d\varphi\right) = 0, \tag{6}$$

or

$$\mathcal{L}_F\ E_r = \frac{c}{r^2}. \tag{7}$$

Since $F = F_{\mu\nu}F^{\mu\nu} = -2\ E_r{}^2$ and $r^2 = \sqrt{q^2\left(\frac{1-b^2 E_r{}^2}{E_r{}^2}\right)} = \sqrt{-q^2\left(\frac{2+b^2F}{F}\right)}$it yields

$$\mathcal{L}_F\ = c\sqrt{\frac{2}{2+b^2F}} \tag{8}$$

where c =constant of integration, which is identified as the charge q. Solution for the Lagrangian, after adjusting the constants, takes the form of

$$\mathcal{L} = \frac{4}{b^2}\left(1-\sqrt{1+\frac{b^2F}{2}}\right), \tag{9}$$

i.e., the BI Lagrangian.

This example gives an idea of how simple it is to find a Lagrangian which yields a non-singular electric field, but the question was whether this much was enough. Hoffman and Infeld [2] shortly after the BI non-linear Lagrangian, pointed this problem out and tried to get rid of any possible difficulties.

In [2] the authors remarked that although the electric field becomes finite at $r = 0$ it yields a discontinuity, for instance in the Cartesian component E_x. To quote from [2] "It is evident that any finite value for E_r at $r = 0$ will lead to a discontinuity of this type". Accordingly, their proposal alternative to the BI Lagrangian can be summarized as follows. The simplest non-singular electric field which takes zero value at $r = 0$ can be written as

$$E_r = \frac{qr^2}{(q^2b^2+r^4)}, \tag{10}$$

so that r^2 in terms of F is

$$r^2 = q\frac{1 \pm \sqrt{1-4b^2E_r^2}}{2E_r} = q\frac{1 \pm \sqrt{1+2b^2F}}{\sqrt{-2F}} \tag{11}$$

where $+$ and $-$ stand for $r^4 > q^2b^2$ and $r^4 < q^2b^2$, respectively. From (7) we find

$$\mathcal{L}_F = \frac{2c}{1 \pm \sqrt{1+2b^2F}} \tag{12}$$

where the positive branch leads to the Lagrangian

$$\mathcal{L}_+ = -\frac{2}{b^2}\left(k + \alpha\epsilon_+ - \ln\epsilon_+\right) \tag{13}$$

with $\alpha = 1$, $k = \ln 2 - 2$ and $\epsilon_+ = 1 + \sqrt{1+2b^2F}$. Let us note that we wrote the Lagrangian in this form to show consistency with [2]. Again we remind that the constant c has been chosen in such a way that $\lim_{b\to 0}\mathcal{L}_+ = -F$, which is the Maxwell limit. In analogy, the negative branch gives

$$\mathcal{L}_- = -\frac{2}{b^2}\left(k + \alpha\epsilon_- - \ln|\epsilon_-|\right) \tag{14}$$

where $\epsilon_- = 1 - \sqrt{1+2b^2F}$. It should be noted that, here one does not expect the Maxwell limit as b goes to zero. In fact, since $\mathcal{L}_-$ is defined for $r^4 < q^2b^2$, automatically b can not be zero unless r also goes to zero in which, the case $\mathcal{L}_-$ becomes meaningless.

Having $\mathcal{L}_+$ for $r^4 > q^2b^2$ and $\mathcal{L}_-$ for $r^4 < q^2b^2$ imposes $\left(\mathcal{L}_+ = \mathcal{L}_-\right)_{r^4=q^2b^2}$ which is satisfied, as it should. Also at $r^4 = q^2b^2$, one gets $E_r = \frac{1}{2b}$ which is the maximum value that E_r may take.

Based on the criticisms made in [2], as mentioned above, we see that this Lagrangian removes the discontinuity in, say, E_x. So, shall we adopt this Lagrangian for further results? The answer was given few years later by Rosen [3], which was not affirmative. The crux of the problem lies in the relation between E_r and D_r. Let us go back to the previous case (10) once more. It is known from non-linear electrodynamics [1–3] that

$$D_r = \mathcal{L}_F E_r = \frac{q}{r^2} \tag{15}$$

which is singular at $r = 0$. Of course, being singular for D_r does not matter; the problem arises once we consider D_r as a function of E_r. In this way at $r = 0$, $E_r = 0$ and $D_r = \infty$, and once $r = \infty$ again $E_r = 0$, but $D_r = 0$. This means that D_r in terms of E_r is double-valued(i.e., $D_r\left(E_r\left(r=0\right)=0\right) = \infty$ and $D_r\left(E_r\left(r=\infty\right)=0\right) = 0$). Concerning this objection Rosen suggested to reject this Lagrangian and instead he recommended that the Lagrangian should be a function of the potentials. For the detail of his work we suggest Ref. [3], but here we wish to draw attention to a recent paper we published [4] which gives a different solution to this problem. Before we give the detail of the solution we admit that during the time of working on [4] we were not aware about this problem, and we did not know the Hoffman-Infeld (HI) form of Lagrangian. In certain sense, we have rediscovered a Lagrangian of 70 years old anew, from the hard way.!

Returning to the problem, we see that in the case of the HI Lagrangian they used two different forms for inside and outside of the typical particle in order to keep the spacetime spherically symmetric, static Reissner-Nordström (RN) type. This is understandable since in 1930s RN solution was one of the best known solution whereas the Bertotti-Robinson (BR) [5] solution was yet unknown. The latter, i.e., BR, constitutes a prominent inner substitute to (RN) as far as Einstein-Maxwell solutions are concerned, and resolves the singularity at $r = 0$, which caused HI to worry about [2]. As we gave the detail of such a choice in Ref. [4], one can choose $\mathcal{L}_+ = -\frac{2}{b^2}\left(k + \alpha\epsilon_+ - \ln\epsilon_+\right)$ for all regions (i.e., $r \geq \sqrt{qb} =$ the radius of our particle, and $r \leq \sqrt{qb}$). For outside we adopted a RN type spacetime while for inside we had to choose a BR type spacetime. Accordingly one finds

$$E_r = \begin{cases} \frac{1}{2b}, & r \leq \sqrt{qb} \\ \frac{qr^2}{(q^2b^2+r^4)}, & r \geq \sqrt{qb} \end{cases} \tag{16}$$

and consequently

$$D_r = \begin{cases} \frac{1}{b}, & r \leq \sqrt{qb} \\ \frac{q}{r^2}, & r \geq \sqrt{qb} \end{cases} \tag{17}$$

which clearly reveals that D_r is not a double valued function of E_r any more. We note that in matching the two spacetimes the Lanczos energy-momentum tensor [6] was employed. Let us add further that this is not the unique choice, so that the opposite choice also is possible. That is, a RN type spacetime for $r \leq \sqrt{qb}$ and a BR type spacetime for $r \geq \sqrt{qb}$. In this latter choice the Lagrangian is $\mathcal{L}_- = -\frac{2}{b^2}\left(k + \alpha\epsilon_- - \ln|\epsilon_-|\right)$ everywhere, which yields

$$E_r = \begin{cases} \frac{qr^2}{(q^2b^2+r^4)}, & r \leq \sqrt{qb} \\ \frac{1}{2b}, & r \geq \sqrt{qb} \end{cases} \tag{18}$$

and

$$D_r = \begin{cases} \frac{q}{r^2}, & r \leq \sqrt{qb} \\ \frac{1}{b}, & r \geq \sqrt{qb} \end{cases} \tag{19}$$

is again not double-valued. In Ref. [4] we studied in detail the first case alone. Obviously, the second case also can be developed into a model of elementary particle.

III. A DIFFERENT ASPECT OF THE EHBI SPACETIME

Once more, we start with the EHBI Lagrangian

$$\mathcal{L}=\begin{cases}\mathcal{L}_{-}, & r\leq\sqrt{qb}\\ \mathcal{L}_{+}, & r\geq\sqrt{qb}\end{cases}. \tag{20}$$

where b is a free parameter such that

$$\lim_{b\to 0}\mathcal{L}=\lim_{b\to 0}\mathcal{L}_{+}=-F \tag{21}$$

and

$$\lim_{b\to\infty}\mathcal{L}=\lim_{b\to\infty}\mathcal{L}_{-}=0 \tag{22}$$

which are the RN and Schwarzschild (S) limits, respectively. Our action is chosen now as

$$S=\frac{1}{2}\int d^4x\sqrt{-g}\left(R+\mathcal{L}\left(F\right)\right) \tag{23}$$

where R is the Ricci scalar for the line element (1) and $\mathcal{L}(F)$ is the NED Lagrangian described hitherto. The Einstein-NED equations are

$$G_{\mu}^{\nu}=T_{\mu}^{\nu}=\frac{1}{2}\left[\mathcal{L}\left(F\right)\delta_{\mu}^{\nu}-4\mathcal{L}_{F}F_{\mu\lambda}F^{\nu\lambda}\right] \tag{24}$$

whereas the electromagnetic field $F_{\mu\lambda}$ satisfies (5). A solution to the Einstein equations which gives all the correct limits is

$$\begin{gathered}f\left(r\right)=1-\tfrac{2m}{r}+\tfrac{q^2}{3r_\circ^4}r^2\ln\left(\tfrac{r^4}{r^4+r_\circ^4}\right)-\\ \tfrac{q^2\sqrt{2}}{3rr_\circ}\left[\tan^{-1}\left(\tfrac{\sqrt{2}r}{r_\circ}+1\right)+\tan^{-1}\left(\tfrac{\sqrt{2}r}{r_\circ}-1\right)\right]-\tfrac{q^2\sqrt{2}}{6rr_\circ}\ln\left[\tfrac{r^2+r_\circ^2-\sqrt{2}rr_\circ}{r^2+r_\circ^2+\sqrt{2}rr_\circ}\right]+\tfrac{\sqrt{2}q^2\pi}{3rr_\circ},\end{gathered} \tag{25}$$

where $r_\circ=\sqrt{qb}$ and m is the corresponding mass of S (and RN) source. One can easily show that

$$\lim_{b\to 0}f\left(r\right)=1-\frac{2m}{r}+\frac{q^2}{r^2} \tag{26}$$

and

$$\lim_{b\to\infty}f\left(r\right)=1-\frac{2m}{r}. \tag{27}$$

It is interesting to observe that although the ADM mass of EHBI solution is still m, the effective mass depends on charge and HBI parameter, i.e.,

$$m_{eff}=m-\frac{\sqrt{2}q^2\pi}{6r_\circ}. \tag{28}$$

Here one may set the effective mass to zero (note that the ADM mass of the EHBI is not zero and survives in the metric indirectly) i.e.,

$$m_{ADM} = \frac{\sqrt{2}q^2\pi}{6r_\circ} \tag{29}$$

to get a regular metric function whose Kretschmann and Ricci scalars are finite at any point. It should be noted that this is not the case for the regular solution mentioned in [2], i.e. in contrast to [2], our EHBI black hole is not massless.

By employing the solution (25) now we proceed to investigate some thermodynamical properties of the EHBI black hole. To do so we find the horizon of the BH by equating the metric function to zero, which gives the effective mass in terms of the horizon radius

$$\begin{aligned} m_{eff} &= \tfrac{r_h}{2}\left(1+\tfrac{q^2}{3r_\circ^4}r_h^2\ln\left(\tfrac{r_h^4}{r_h^4+r_\circ^4}\right)-\right.\\ &\left.\tfrac{q^2\sqrt{2}}{3r_hr_\circ}\left[\tan^{-1}\left(\tfrac{\sqrt{2}r_h}{r_\circ}+1\right)+\tan^{-1}\left(\tfrac{\sqrt{2}r_h}{r_\circ}-1\right)\right]-\tfrac{q^2\sqrt{2}}{6r_hr_\circ}\ln\left[\tfrac{r_h^2+r_\circ^2-\sqrt{2}r_hr_\circ}{r_h^2+r_\circ^2+\sqrt{2}r_hr_\circ}\right]\right), \quad r_h > r_\circ. \end{aligned} \tag{30}$$

Hawking temperature in terms of the event horizon radius is given accordingly by

$$T_H = \frac{1}{4\pi r_h}\left(1-\frac{q^2r_h^2}{r_\circ^4}\ln\left(1+\frac{r_\circ^4}{r_h^4}\right)\right). \tag{31}$$

Also the heat capacity, which is defined as

$$C_q = T_H\left(\frac{\partial S(r)}{\partial T_H}\right)_q, \tag{32}$$

is given by

$$C_q = \frac{\pi r_h^2\left(r_h^4+r_\circ^4\right)\left(q^2r_h^2\ln\left(\frac{r_h^4}{r_h^4+r_\circ^4}\right)+r_\circ^4\right)}{q^2r_h^2\left(r_h^4+r_\circ^4\right)\ln\left(\frac{r_h^4}{r_h^4+r_\circ^4}\right)-r_\circ^8+\left(4q^2-r_h^2\right)r_h^2r_\circ^4} \tag{33}$$

whose zeros of the denominator indicate possible phase transitions.

IV. THIN-SHELL WORMHOLES IN $4D$

Following the establishment of black hole solutions in the EHBI action (23) with line element (1) our next venture is to investigate the possibility of thin-shell wormholes in the same theory. Here we follow the standard method of constructing a thin-shell wormhole [7]. To do so, we take two copies of EHBI spacetimes, and from each manifold we remove the following $4D$ submanifold

$$\Omega_{1,2} \equiv \left\{r_{1,2} \leq a \,\middle|\, a > \sqrt{qb}\right\} \tag{34}$$

in which a is a constant and b is the HBI parameter introduced before. In addition, we restrict our free parameters to keep our metric function non-zero and positive for $r > \sqrt{qb}$. In order to have a complete manifold we define a manifold $\mathcal{M} = \Omega_1 \cup \Omega_2$ whose boundary is given by the two timelike hypersurfaces

$$\partial\Omega_{1,2} \equiv \left\{ r_{1,2} = a \middle| a > \sqrt{qb} \right\}. \tag{35}$$

After identifying the two hypersurfaces, $\partial\Omega_1 \equiv \partial\Omega_2 = \partial\Omega$, the resulting manifold will be geodesically complete [7] with two asymptotically flat regions connected by a traversable Lorantzian wormhole. The throat of the wormhole is at $\partial\Omega$ and the induced metric on $\mathcal{M}$ with coordinate $\{X^i\}$ and induced metric h_{ij}, takes the form

$$ds^2_{ind} = -d\tau^2 + a(\tau)^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \tag{36}$$

where τ represents the proper time on the hypersurface $\partial\Omega$. Lanczos equations [7] read

$$S^i_j = -\frac{1}{8\pi} \left(\left[K^i_j \right] - \delta^i_j \left[K \right] \right), \tag{37}$$

where the extrinsic curvature K_{ij} (with trace K) is defined by

$$K_{ij} = -n_k \left(\frac{\partial^2 X^k}{\partial\xi^i \partial\xi^j} + \Gamma^k_{mn} \frac{\partial X^m}{\partial\xi^i} \frac{\partial X^n}{\partial\xi^j} \right), \tag{38}$$

in which n_k is normal to $\mathcal{M}$, so that $h_{ij} = g_{ij} - n_i n_j$ and $\xi^i = (\tau, \theta, \phi)$. Upon substitution into (37) we obtain the surface stress-energy tensor in the form

$$S^i_j = diag\left(-\sigma, p_\theta, p_\phi \right). \tag{39}$$

Here σ, and $p_\theta = p_\phi$ are the surface-energy density and the surface pressures, respectively. A detailed study shows [8] that

$$\sigma = -\frac{1}{2\pi a} \sqrt{f(a) + \dot{a}^2} \tag{40}$$

and

$$p_\theta = p_\phi = -\frac{1}{2}\sigma + \frac{1}{8\pi} \frac{2\ddot{a} + f'(a)}{\sqrt{f(a) + \dot{a}^2}}. \tag{41}$$

Also the conservation equation gives

$$\frac{d}{d\tau}\left(\sigma a^2 \right) + p \frac{d}{d\tau}\left(a^2 \right) = 0 \tag{42}$$

or

$$\dot{\sigma}+2\frac{\dot{a}}{a}\left(p+\sigma\right)=0. \tag{43}$$

For the static structure, one gets

$$\sigma_0=-\frac{1}{2\pi a_0}\sqrt{f\left(a_0\right)},\quad p_0=\frac{\sqrt{f\left(a_0\right)}}{4\pi a_0}\left(1+\frac{a}{2}\frac{f'\left(a_0\right)}{f\left(a_0\right)}\right). \tag{44}$$

The total amount of exotic matter for constructing such a thin-shell wormhole is given by

$$\Omega=\int\left(\rho+p\right)\sqrt{-g}d^3x. \tag{45}$$

Here $\rho=\delta\left(r-a\right)\sigma\left(a\right)$ where $\delta\left(.\right)$ is the Dirac delta function, radial pressure p is negligible because of the thin shell, and therefore

$$\Omega=4\pi a^2\sigma\left(a\right)=-2a\sqrt{f\left(a\right)}. \tag{46}$$

The EHBI metric function now takes the form

$$f\left(r\right)=1-\frac{2m}{r}+\frac{q^2}{3r_\circ^4}r^2\ln\left(\frac{r^4}{r^4+r_\circ^4}\right)- \\ \frac{q^2\sqrt{2}}{3rr_\circ}\left[\tan^{-1}\left(\frac{\sqrt{2}r}{r_\circ}+1\right)+\tan^{-1}\left(\frac{\sqrt{2}r}{r_\circ}-1\right)\right]-\frac{q^2\sqrt{2}}{6rr_\circ}\ln\left[\frac{r^2+r_\circ^2-\sqrt{2}rr_\circ}{r^2+r_\circ^2+\sqrt{2}rr_\circ}\right]+\frac{\sqrt{2}q^2\pi}{3rr_\circ},\quad r>r_\circ, \tag{47}$$

in which $r_\circ=\sqrt{qb}$. The equation of motion for the thin-shell wormhole can be extracted from (40)

$$\dot{a}^2+V\left(a\right)=0, \tag{48}$$

in which the thin shell's potential is given by

$$V\left(a\right)=f\left(a\right)-\left(2\pi a\sigma\left(a\right)\right)^2. \tag{49}$$

In order to investigate the radial perturbation around an equilibrium radius (a_0) we assume a linear relation between the pressure and density

$$p=p_0+\beta^2\left(\sigma-\sigma_0\right), \tag{50}$$

in which p_0, σ_0 and β are constants. Upon expansion around a_0 (which requires $V\left(a_0\right)=V'\left(a_0\right)=0$) up to the second order yields

$$V\left(a\right)\cong\frac{1}{2}V''\left(a_0\right)\left(a-a_0\right)^2. \tag{51}$$

By considering (43) and using $\sigma'=\frac{\dot{\sigma}}{\dot{a}}$ one gets

$$V''\left(a_0\right)=f_0''-\frac{f_0'^2}{2f_0}-\frac{1+2\beta^2}{a_0^2}\left(2f_0-a_0f_0'\right) \tag{52}$$

in which $f_0 = f(a_0)$. The stability conditions $V''(a_0) \geq 0$ leads to

$$\text{for } 2f_0 \gtrless a_0 f_0', \qquad 1+2\beta^2 \lesseqgtr \frac{a_0^2}{2f_0}\left(\frac{2f_0''f_0 - f_0'^2}{2f_0 - a_0 f_0'}\right). \tag{53}$$

Since its source is already exotic matter we shall not be interested in this particular wormhole any further in the present paper. Instead, we shall go to 5D, in which the black hole and wormhole constructions render it possible to make normal matter, stable wormholes. This is the main strategy in the following chapters.

V. 5−DIMENSIONAL EHBI BLACK HOLE

In order to extend the $4D$ EHBI black hole solution to $5D$ with a cosmological constant Λ we choose our action as

$$S = \frac{1}{2}\int dx^5 \sqrt{-g}\left\{-4\Lambda + R + \mathcal{L}(\mathcal{F})\right\}, \tag{54}$$

where

$$\mathcal{L} = \begin{cases} \mathcal{L}_-, & r \leq \sqrt{qb} \\ \mathcal{L}_+, & r \geq \sqrt{qb} \end{cases} \tag{55}$$

and the nonlinear Maxwell equation (5) in 5D leads to the radial electric field

$$E_r = \frac{qr^3}{(q^2b^2 + r^6)}. \tag{56}$$

Variation of the action (54) yields the field equations as

$$G_\mu^{\ \nu} + 2\Lambda\delta_\mu^{\ \nu} = T_\mu^{\ \nu}, \tag{57}$$

$$T_\mu^{\ \nu} = \frac{1}{2}\left(\mathcal{L}\delta_\mu^{\ \nu} - 4\mathcal{L}_\mathcal{F} F_{\mu\lambda}F^{\nu\lambda}\right),$$

which clearly gives $T_t^{\ t} = T_r^{\ r} = \left(\frac{1}{2}\mathcal{L} - \mathcal{L}_\mathcal{F}\mathcal{F}\right)$, stating also that $G_t^{\ t} = G_r^{\ r}$ and $T_{\theta_i}^{\ \theta_i} = \frac{1}{2}\mathcal{L}$. Now, we introduce our ansatz line element ($\chi = \pm 1, 0$)

$$ds^2 = -(\chi - r^2 H(r))dt^2 + \frac{1}{(\chi - r^2 H(r))}dr^2 + r^2 d\Omega_3^2 \tag{58}$$

in which $H(r)$ is a function to be determined, to cover both the topological and non-topological black hole solutions [9]. Our choice of $g_{tt} = -(g_{rr})^{-1}$ is a direct result of

${G_t}^t = {G_r}^r$ up to a constant coefficient, which is chosen to be one. The Einstein tensor components are given by

$$\begin{aligned} {G_t}^t &= {G_r}^r = -\frac{3}{2r^3}\left(r^4 H(r)\right)' \\ G_{\theta_i}^{\theta_i} &= -\frac{1}{2r^2}\left(r^4 H(r)\right)'' \end{aligned} \tag{59}$$

from which, one obtains a general class of $H(r)$ functions depending on the choice of T_t^t,

$$H(r) = \frac{\Lambda}{3} + \frac{4m}{(d-2)\, r^{d-1}} - \frac{2}{(d-2)\, r^{d-1}} \int r^{d-2} T_t^t dr. \tag{60}$$

Now, with the particular choice of the energy-momentum tensor component as

$$T_t^t = -\frac{1}{b^2} \ln\left(1 + \frac{b^2 q^2}{r^6}\right) \tag{61}$$

the metric function is found to be

$$\begin{aligned} f(r) = \chi - \tfrac{\Lambda}{3} r^2 - \tfrac{4m}{3r^2} - \tfrac{q^2\sqrt{3}}{6r^2 r_\circ^2} \tan^{-1}\left(\tfrac{1}{\sqrt{3}}\left[\tfrac{2r^2}{r_\circ^2} - 1\right]\right) - \\ \tfrac{q^2}{12 r^2 r_\circ^2} \ln\left|\tfrac{r^4 + r_\circ^4 - r^2 r_\circ^2}{r^4 + r_\circ^4 + 2r^2 r_\circ^2}\right| + \tfrac{1}{6}\tfrac{r^2 q^2}{r_\circ^6} \ln\left(\tfrac{r^6}{r^6 + r_\circ^6}\right) + \tfrac{\sqrt{3} q^2 \pi}{12 r^2 r_\circ^2}, \end{aligned} , \tag{62}$$

where $r_\circ^6 = b^2 q^2$ and m is the ADM mass of the black hole. One observes that this solution in two extremal limits for b yields

$$\begin{aligned} \lim_{b\to 0} f(r) &= \chi - \frac{\Lambda}{3} r^2 - \frac{4m}{3r^2} + \frac{q^2}{3r^4}, \\ \lim_{b\to \infty} f(r) &= \chi - \frac{\Lambda}{3} r^2 - \frac{4m}{3r^2}. \end{aligned} \tag{63}$$

Further, in the sense of usual ADM mass, even if one adjusts

$$m_{ADM} = \frac{\sqrt{3} q^2 \pi}{16 r_\circ^2} \tag{64}$$

unlike the case of 4D, the metric remains singular at origin.

VI. 5D STABLE, NORMAL MATTER THIN-SHELL WORMHOLE IN EHBIGB THEORY

Our action and metric ansatz in 5D EHBIGB theory of gravity are given respectively by

$$S = \frac{1}{2} \int dx^5 \sqrt{-g}\left\{-4\Lambda + R + \alpha L_{GB} + \mathcal{L}(\mathcal{F})\right\} \tag{65}$$

and

$$ds^2 = -f(r)\,dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\left(d\phi^2 + \sin^2\phi d\psi^2\right)\right) \tag{66}$$

where $L_{GB} = R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ and α is the GB parameter. The inclusion of the GB term modifies (57) and (58), which can be expressed as an algebraic equation for $H(r)$ given by

$$H(r) + 4\alpha H(r)^2 = \frac{\Lambda}{3} + \frac{4m}{3r^4} - \frac{2}{3r^4}\int r^3 T_t^t dr. \tag{67}$$

Upon insertion of (61) for T_t^t one obtains $H(r)$, and as a result

$$\begin{aligned} &f_\pm(r) = \chi + \tfrac{r^2}{8\alpha} \times \left\{1 \pm \sqrt{1 + 16\alpha H}\right\}, \\ &H = \tfrac{\Lambda}{3} + \tfrac{4m_{eff}}{3r^4} + \tfrac{q^2\sqrt{3}}{6r^4 r_\circ^2}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}\left[\tfrac{2r^2}{r_\circ^2} - 1\right]\right) + \tfrac{q^2}{12r^4 r_\circ^2}\ln\left|\tfrac{r^4 + r_\circ^4 - r^2 r_\circ^2}{r^4 + r_\circ^4 + 2r^2 r_\circ^2}\right| - \tfrac{1}{6}\tfrac{q^2}{r_\circ^6}\ln\left(\tfrac{r^6}{r^6 + r_\circ^6}\right) \end{aligned} \tag{68}$$

in which $r_\circ^6 = b^2q^2$ and $m_{eff} = m - \frac{\sqrt{3}q^2\pi}{16r_\circ^2}$. One can easily check the following limits

$$\begin{aligned} &\lim_{b\to 0} f_\pm(r) = \chi + \tfrac{r^2}{8\alpha}\left\{1 \pm \sqrt{1 + 16\alpha\left(\tfrac{\Lambda}{3} + \tfrac{4m}{3r^4} - \tfrac{q^2}{3r^6}\right)}\right\}, \\ &\lim_{b\to\infty} f_\pm(r) = \chi + \tfrac{r^2}{8\alpha}\left\{1 \pm \sqrt{1 + 16\alpha\left(\tfrac{\Lambda}{3} + \tfrac{4m}{3r^4}\right)}\right\}, \\ &\lim_{\alpha\to 0} f_-(r) = \chi - \tfrac{\Lambda}{3}r^2 - \tfrac{4m_{eff}}{3r^2} - \tfrac{q^2\sqrt{3}}{6r^2 r_\circ^2}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}\left[\tfrac{2r^2}{r_\circ^2} - 1\right]\right) - \\ &\tfrac{q^2}{12r^2 r_\circ^2}\ln\left|\tfrac{r^4 + r_\circ^4 - r^2 r_\circ^2}{r^4 + r_\circ^4 + 2r^2 r_\circ^2}\right| + \tfrac{1}{6}\tfrac{r^2 q^2}{r_\circ^6}\ln\left(\tfrac{r^6}{r^6 + r_\circ^6}\right), \end{aligned} \tag{69}$$

as expected. With the solution (68), (66) represents a 5D black hole in EHBIGB gravity and now we shall proceed to construct a thin-shell wormhole solution in the same spacetime. For this process it is necessary to remove the regions

$$M_{1,2} = \{r_{1,2} \leq a, \quad a > r_h\} \tag{70}$$

from the underlying spacetime. Here r_h is the event horizon and subsequently we paste the remaining regions of spacetime to provide geodesic completeness. The time-like boundary surface $\Sigma_{1,2}$ on $M_{1,2}$ are glued such that

$$\Sigma_{1,2} = \{r_{1,2} = a, \quad a > r_h\}. \tag{71}$$

From the Darmois-Israel formalism [10], in terms of the original coordinates $x^\gamma = (t, r, \theta, \phi, \psi)$, we define the new set of coordinates $\xi^i = (\tau, \theta, \phi, \psi)$, with τ the proper time. Following the generalized Darmois-Israel junction conditions apt for the GB gravity [11] a surface energy-momentum tensor is defined by $S_j^i = diag(\sigma, p_\theta, p_\phi, p_\psi)$, which has already been defined in terms of the extrinsic curvature of induced metric in 4D in Sec. IV. By employing this formalism, Richarte and Simeone [12] established a thin shell wormhole in EMGB gravity supported by normal matter. The thin-shell geometry whose radius is assumed a function of proper time is given by

$$\Sigma : f\left(r,\tau\right) = r - a\left(\tau\right) = 0. \tag{72}$$

The generalized Darmois-Israel conditions on Σ determines the surface energy-momentum tensor. S_{ab} which is expressed by [11]

$$-\frac{1}{8}S_{ab} = 2\left\langle K_{ab} - Kh_{ab}\right\rangle + 4\alpha\left\langle 3J_{ab} - Jh_{ab} + 2P_{acdb}K^{cd}\right\rangle. \tag{73}$$

Here a bracket implies a jump across Σ, and $h_{ab} = g_{ab} - n_a n_b$ is the induced metric with coordinate set $\{X^a\}$ which helps to define the extrinsic curvature introduced in Sec. IV. The divergence-free part of the Riemann tensor P_{abcd} and the tensor J_{ab} (with trace $J = J_a^a$) are given by [11]

$$P_{abcd} = R_{abcd} + \left(R_{bc}h_{da} - R_{bd}h_{ca}\right) - \left(R_{ac}h_{db} - R_{ad}h_{cb}\right) + \frac{1}{2}R\left(h_{ac}h_{db} - h_{ad}h_{cb}\right), \tag{74}$$

$$J_{ab} = \frac{1}{3}\left[2KK_{ac}K_b^c + K_{cd}K^{cd}K_{ab} - 2K_{ac}K^{cd}K_{ab} - K^2K_{ab}\right]. \tag{75}$$

By employing these expressions through (73) we find the energy density and surface pressures for a generic metric function $f\left(r\right)$, with $r = a\left(\tau\right)$. The results are given by [13]

$$\sigma \;=\; -S_\tau^\tau = -\frac{\Delta}{4\pi}\left[\frac{3}{a} - \frac{4\alpha}{a^3}\left(\Delta^2 - 3\left(1+\dot{a}^2\right)\right)\right], \tag{76}$$

$$S_\theta^\theta \;=\; S_\phi^\phi = S_\psi^\psi = p = \frac{1}{4\pi}\left[\frac{2\Delta}{a} + \frac{\ell}{\Delta} - \frac{4\alpha}{a^2}\left(\ell\Delta - \frac{\ell}{\Delta}\left(1+\dot{a}^2\right) - 2\ddot{a}\Delta\right)\right], \tag{77}$$

where $\ell = \ddot{a} + f'\left(a\right)/2$ and $\Delta = \sqrt{f\left(a\right) + \dot{a}^2}$ in which

$$f\left(a\right) = \left. f_-\left(r\right)\right|_{r=a}. \tag{78}$$

We note that in our notation a 'dot' denotes derivative with respect to the proper time τ and a 'prime' with respect to the argument of the function. For simplicity, we set the cosmological constant to zero. It can be checked that by substitution from (76) and (77) the conservation equation

$$\frac{d}{d\tau}\left(\sigma a^3\right) + p\frac{d}{d\tau}\left(a^3\right) = 0. \tag{79}$$

holds true. For the static configuration of radius a_0 we have the constant values

$$\sigma_0 = -\frac{\sqrt{f(a_0)}}{4\pi}\left[\frac{3}{a_0} - \frac{4\alpha}{a_0^3}\left(f(a_0) - 3\right)\right], \tag{80}$$

$$p_0 = \frac{\sqrt{f(a_0)}}{4\pi}\left[\frac{2}{a_0} + \frac{f'(a_0)}{2f(a_0)} - \frac{2\alpha}{a_0^2}\frac{f'(a_0)}{f(a_0)}\left(f(a_0) - 1\right)\right]. \tag{81}$$

In order to investigate the radial perturbation around an equilibrium radius (a_0) we assume a linear relation between the pressure and density [13], as in the 4D case

$$p = p_0 + \beta^2 (\sigma - \sigma_0). \tag{82}$$

Here the constant σ_0 and p_0 are given by (80) and (81) whereas β^2 is a constant parameter which can be identified with the speed of sound. By virtue of the latter equation we express the energy density in the form

$$\sigma(a) = \left(\frac{\sigma_{0+}p_0}{\beta^2+1}\right)\left(\frac{a_0}{a}\right)^{3\left(\beta^2+1\right)} + \frac{\beta^2\sigma_{0-}p_0}{\beta^2+1}. \tag{83}$$

This, together with (79) lead us to the equation of motion for the radius of throat, which reads

$$-\frac{\sqrt{f(a)+\dot{a}^2}}{4\pi}\left[\frac{3}{a} - \frac{4\alpha}{a^3}\left(f(a) - 3 - 2\dot{a}^2\right)\right] = \left(\frac{\sigma_{0+}p_0}{\beta^2+1}\right)\left(\frac{a_0}{a}\right)^{3\left(\beta^2+1\right)} + \frac{\beta^2\sigma_{0-}p_0}{\beta^2+1}. \tag{84}$$

After some manipulation this can be cast into

$$\dot{a}^2 + V(a) = 0, \tag{85}$$

where

$$V(a) = f(a) - \left(\left[\sqrt{A^2+B^3} - A\right]^{1/3} - \frac{B}{\left[\sqrt{A^2+B^3} - A\right]^{1/3}}\right)^2 \tag{86}$$

involves the root of a third order algebraic equation with

$$A = \frac{\pi a^3}{4\alpha}\left[\left(\frac{\sigma_{0+}p_0}{\beta^2+1}\right)\left(\frac{a_0}{a}\right)^{3\left(\beta^2+1\right)} + \frac{\beta^2\sigma_{0-}p_0}{\beta^2+1}\right], \tag{87}$$

$$B = \frac{a^2}{8\alpha} + \frac{1-f(a)}{2}. \tag{88}$$

It is a simple exercise to show that $V(a)$, and $V'(a)$, both vanish at $a = a_0$. The stability requirement for the wormhole reduces to the determination of $V''(a_0) > 0$. Due to the complicated structure of the potential we shall proceed through numerical analysis in order to explore the stability regions, if there is any at all. In doing this, we shall concentrate mainly on the case $\alpha < 0$, since after all, the case $\alpha > 0$ does not have a good record in the context of normal matter thin-shell wormholes.

Fig (1) displays the $V''(a) > 0$ as stability region upon projection in the two-dimensional variables β and a_0 with $\alpha < 0$. The metric function $f(r)$ and the reality of $\sigma > 0$ are also visible in the inscribed plots. Beyond these regions and for the associated variables, in particular for the choice $\alpha > 0$, stable wormhole construction in EHBIGB theory doesn't seem possible.

VII. CONCLUSION

The original non-linear BI electrodynamics aimed at removing point-like singularities and resulting divergences. This, however, didn't resolve the dauble-valuedness in the displacement vector $\vec{D}(\vec{E})$ as a function of the electric field. This was the main motivation for emergence of Hoffman's version of the BI type Lagrangian, which contained an ubiquitous logarithmic term [14]. We have shown that such a supplementary term in the Lagrangian has benefits also when employed in general relativity. Firstly, it removes the double valuedness in $\vec{D}(\vec{E})$, as observed / proposed seven decades before. Secondly, by cutting and gluing (pasting) method we construct black hole spacetime. This may be developed into a finite, geometrical model of elementary particle as addressed in [4]. Lastly, as we have emphasized in the present paper, the HBI type Lagrangian can be used in wormhole construction. These wormholes have the attractive features of being supported only by normal matter. By exploiting the boundary conditions while gluing the inner and outer parts we remove the divergent part arising from the solution in 4D which, however, doesn't seem possible in the Gauss-Bonnet augmented Lagrangian in higher dimensions. Further, the thin-shell wormhole obtained in the EHBIGB gravity the wormhole can be made stable [15]. This is upon finely-tuned parameters and an intricate potential function which is required to have positive second derivative. From these feats it is hoped that the logarithmic Lagrangian will draw attention from various circles of field theorists for further applications.

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Figure caption:

Fig. 1: The stability region (i.e. $V''(a) > 0$) for the chosen parameters, $r_0 = 1.00$, $q = 0.75$ and $m_{eff} = 0$ (Eq. (28)). This is given as a projection into the plane with axes β and $\frac{a_0}{|\alpha|}$. The plot of the metric function $f(r)$ and energy density σ are also inscribed in the figure.