PP-WAVES IN THE GENERALIZED EINSTEIN THEORIES

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Received 5 July 1978

We present pp-wave solutions to the generalized Einstein–Maxwell field theory introduced by Horndeski and to Mansouri-Chang theory of gravitation.

In general relativity a plane-fronted wave with parallel rays (pp-wave) is described by the fundamental form [1]

$$ds^2 = 2\mathrm{d}u\,\mathrm{d}v + \mathrm{d}z\,\mathrm{d}\bar{z} - 2V(u,x,y)\,\mathrm{d}u^2, \tag{1}$$

where $z = x + iy$ and a bar denotes complex conjugation. The components of this metric tensor is of the Kerr–Schild form [2]

$$g_{\mu\nu} = \eta_{\mu\nu} - 2V n_\mu n_\nu, \tag{2}$$

where $\eta_{\mu\nu}$ is the usual Minkowski metric and n_μ is a null vector with respect to both $g_{\mu\nu}$ and $\eta_{\mu\nu}$. Furthermore, n_μ satisfies

$$n_\mu = \partial_\mu u, \tag{3}$$

and

$$n_{\mu;\nu} = n_{\mu,\nu} = 0. \tag{4}$$

The corresponding Ricci tensor is given by

$$R_{\mu\nu} = n_\mu n_\nu \Box V, \tag{5}$$

where

$$\Box = \eta^{\mu\nu}\,\partial_\mu\partial_\nu. \tag{6}$$

Gravitational and electromagnetic pp-wave solutions have been examined some time ago by several authors (e.g. refs. [1, 3]). Here we present the pp-wave solutions to the two recent theories of gravitation which involve certain modifications to the usual Einstein's theory of general relativity. One of these theories is due to Horndeski [4] who has shown that in four dimensions the Einstein–Maxwell field theory (EMFT) is not the only vector–tensor field theory of electromagnetism and gravitation. He showed that any second order, source-free, vector–tensor field theory of electromagnetism and gravitation may be expressed as:

$$G_{\mu\nu} = 8\pi(T_{\mu\nu} + \lambda_{\mu\nu}) \equiv 8\pi\,\tau_{\mu\nu} \tag{7}$$

and

$$F_\mu{}^\nu{}_{;\nu} + \tfrac{1}{2}\lambda F_{\alpha\beta;\gamma}\,{}^*R^*{}_\mu{}^{\gamma\alpha\beta} = 0, \tag{8}$$

where star denotes duality, λ is an arbitrary constant,

$$T_{\mu\nu} = (1/4\pi)\,(F_\mu{}^\alpha F_{\nu\alpha} - \tfrac{1}{4} g_{\mu\nu} F^{\alpha\beta}F_{\alpha\beta}), \tag{9a}$$

and

$$A_{\mu\nu} = (1/8\pi)\,({}^*F_\mu{}^{\alpha;\beta}\,{}^*F_{\nu\beta;\alpha} + F_{\alpha\gamma}F_\beta{}^{\gamma}\,{}^*R^{*\mu\alpha\nu\beta}). \tag{9b}$$

This theory, which is called the generalized Einstein–Maxwell field theory (GEMFT), is derivable from a variational principle and is consistent with the notion of charge conservation. Furthermore it is compatible with Maxwell's equations in flat space–time and is in agreement with Einstein's theory in the absence of an electromagnetic field. However, in contrast to EMFT, the solutions to the GEMFT involve some features which are not shared by the solutions of EMFT. For example in Minkowski space–time in the vicinity of point charges the energy momentum tensor predicts regions of negative energy [5] and magnetic monopole solutions are obtained when the space–time geometry possesses spherical symmetry [6]. Since the total energy momentum tensor has a non-zero trace the theory is not conformally invariant. In this note, we shall point out another property of GEMFT which is absent in EMFT: In GEMFT vanishing of the total energy momentum tensor $\tau_{\mu\nu}$ does not imply the vanishing of the field strength tensor $F_{\mu\nu}$ ‡1. We start with metric (2) and take the electromagnetic vector potential as $A_\mu = n_\mu \phi(u, x, y)$. Hence we have

$$F_{\mu\nu} = n_\mu k_\nu - n_\nu k_\mu, \tag{10}$$

with

$$k_\mu = \partial_\mu \phi, \tag{11}$$

which yields by virtue of the Maxwell's equation

$$\Box\phi = 0. \tag{12}$$

For our special metric the curvature couplings in the field equations (7) and (8) vanish as in the flat space–time. The total electromagnetic energy momentum tensor reduces to

$$\tau_{\mu\nu} = (1/8\pi)\,(\tfrac{1}{2}\lambda\,\Box\psi + 2\psi)\,n_\mu n_\nu, \tag{13}$$

where

$$\psi = k_\mu k^\mu \equiv \eta^{\mu\nu}(\partial_\mu\phi)\,(\partial_\nu\phi). \tag{14}$$

By virtue of (10), (11) and (12) the field equation (8) is identically satisfied. Using (5) and (13) the field equation (7) becomes

$$\Box V = \tfrac{1}{2}\lambda + 2\psi. \tag{15}$$

Hence, any simultaneous solution of (12) and (15) describes the space–time geometry and gives the field strength $F_{\mu\nu}$. All possible solutions of ϕ are of the form

$$\phi = h(u)\,\mathrm{Re}\,f(z), \tag{16}$$

where $h(u)$ is an arbitrary function of u and $f(z)$ is an analytic function of z. Hence, any choice of $f(z)$ leads us to a solution for V. In particular, choosing

$$f(z) = z^{-\alpha}, \tag{17}$$

where α is an arbitrary real constant, yields the solution

$$V = \frac{h^2(u)}{4\rho^{2\alpha}}\left(1 + \frac{2\lambda\alpha^2}{\rho^2}\right), \tag{18}$$

with

$$\rho^2 = x^2 + y^2. \tag{19}$$

The only non-vanishing Weyl spinor component ψ_4 is singular at $\rho = 0$ and $\rho = \infty$ (depending on the choice of α) which are true singularities. For this particular solution and for $\alpha > 0$ the total energy momentum tensor becomes negative in the vicinity of the singularity $\rho = 0$ provided $\lambda < 0$. In EMFT vanishing of the energy momentum tensor implies the vanishing of electromagnetic field tensor $F_{\mu\nu}$, but in GEMFT vanishing of $\tau_{\mu\nu}$ in (13) does not require $F_{\mu\nu}$ to be zero. As an example for such a solution we take (12) and (13) with

$$\tfrac{1}{2}\lambda\,\Box\psi + 2\psi = 0, \tag{20}$$

to be satisfied simultaneously. A particular solution is

$$\phi = A(u)\,\mathrm{e}^{\mu x}\cos[\mu y + \beta(u)], \tag{21}$$

where $A(u)$ and $\beta(u)$ are arbitrary functions of u and the real parameter μ is given by

$$\mu^2 + 1/\lambda = 0. \tag{22}$$

Hence, for $\lambda < 0$ we have solutions with $\tau_{\mu\nu} = 0$. Here the space–time metric is the vacuum gravitational pp-metric with

$$\Box V = 0. \tag{23}$$

Since there is no contribution of the geometry in eqs. (12) and (20), any solution to these equations are

‡1 G.W. Horndeski has kindly informed us that he has found a solution for $\tau_{\mu\nu} = 0$ where the resulting space–time is of Petrov type III.

also the solutions of GEMFT in flat space–time with $\tau_{\mu\nu} = 0$.

In a recent work Mansouri and Chang [7] (MC) have proposed a new theory of gravitation where the total action emerges as the sum of the usual Einstein's and Yang's actions [8]. The Einstein's theory is obtained when the so called "bundle parameter" Q is set to zero. Pavelle [9] examined the solutions of the vacuum MC field equations and observed that the predictions of this theory are indistinguishable from those of Einstein's theory. The metric forms that were used in this work did not result in a non-einsteinian solution of MC theory. Here we give an example to such a solution by using the Kerr–Schild form of the pp-wave metric. Using (5) as the Ricci tensor, vacuum field equations reduce to

$$\Box(4Q\Box V + V) = 0. \tag{24}$$

Choosing

$$V = \mathrm{Re}\, g(u, z) + q(u, x, y), \tag{25}$$

where $g(u, z)$ is a complex analytic function of z then eq. (24) reduces to the following equation for $q(u, x, y)$:

$$q_{xx} + q_{yy} + (1/4Q)\, q = 0. \tag{26}$$

The general solution of (26) may be expressed [10] in terms of trigonometric or hyperbolic functions depending upon the sign of Q and the integration constants must be treated as functions of u. Here we remark that the first and the second terms in V behave like massless and massive scalar fields respectively. It may be easily seen that in contrast to the einsteinian solutions, there are no non-einsteinian solutions of MC fields equations which describe linearly polarized gravitational plane-waves [‡2].

We wish to thank R. Güven and G.W. Horndeski for stimulating discussions.

‡2 For definition of linearly polarized plane waves see ref. [1].

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