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Colliding plane gravitational waves

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New exact solutions of the vacuum Einstein field equations are constructed which describe the collision of plane gravitational waves. These solutions generalize those of Szekeres by relaxing the requirement of collinear polarization.

I. INTRODUCTION

Penrose[1] discovered that in the field of plane gravitational waves null rays are focused on certain hypersurfaces where the Riemann tensor takes divergent values. Another situation where such focusing effects appear explicitly is the collision of two gravitational plane waves where each wave is focused by the field of the other and the resulting configuration possesses a space–time singularity. All these properties are verified by the exact solutions of Einstein equations given by Khan and Penrose[2] for colliding impulsive waves and Szekeres[3,4] for shock waves. These solutions, describing the collision between plane gravitational waves with constant linear polarization enable us to study the details of this focusing. It is natural to ask how the focusing properties and the resulting space–time singularity are modified when we introduce new degrees of freedom into the problem. For this purpose we have recently presented a new solution of the vacuum Einstein field equations which describes colliding impulsive gravitational waves with linear but not necessarily collinear polarizations.[5] This implies that the colliding plane waves are still linearly polarized but their directions of polarization are out of phase by a constant phase parameter. We have pointed out that certain features of the problem are modified; for example, the collision results in giving an angular momentum as well as a mass aspect to the gravitational field in the interaction region. The physical space–time singularity on the other hand, although undergoing minor modifications by this additional degree of freedom, is still present. Furthermore Szekeres' conclusion that the space–time singularities arise inevitably for arbitrarily weak incoming gravitational waves remains valid in this new situation as well. The general problem which takes into account the effect of arbitrary polarization has been considered by Sbytov[6] who showed without giving explicit solutions that the physical singularity appears even when the effect of arbitrary polarization is taken into account. The singularity in these solutions of Einstein's equations results from the assumptions of planar wave fronts as pointed out by Penrose[1] a long time ago.

In this paper we shall present a family of exact solutions which generalizes the family of Szekeres to the case of noncollinear polarizations. The first member of this family (i.e., impulsive waves) has already been given in Ref. 5. The plan for this paper is as follows: In Sec. II we shall review the Szekeres' solutions and cast them into a form where the colliding waves initially have a constant phase difference between them. Our method for obtaining the new solutions is based on the theory of harmonic mappings of Riemannian manifolds due to Eells and Sampson.[7] The application of this theory to general relativity[8–10] proved to be a useful technique that facilitates the solution of many problems. For the paper to be self-contained we shall briefly present the necessary tools for applying the theory of harmonic maps.

In Sec. III using harmonic maps we cast the basic field equations of this problem into a form similar to Ernst's[11] for axisymmetric fields. The solutions is then immediate, and we adapt a solution which involves two arbitrary constants. One of these constants which corresponds to the relative polarization angle of the incoming waves is an analog of Kerr's rotation parameter. The second constant on the other hand is a Taub–NUT like parameter which has no immediate physical interpretation for the colliding wave problem. Furthermore, there are other solutions of the field equations which include a Weyl–Tomimatsu–Sato parameter, but these solutions must be excluded as they do not reduce to the desired incoming and outgoing plane wave solutions. While in the family of Szekeres' solutions there are two independent parameters, we have been able to generalize them only for the case when these two parameters are equal. Finally in the Appendix we calculate the Newman–Penrose[12] curvature components which manifests the singularities of these solutions.

II. COLLIDING PLANE GRAVITATIONAL WAVES

Grvaitational plane waves are described by the metric for p-p waves[13]

$$ds^2 = 2du'dv' - dx'^2 - dy'^2 - 2H(x',y',u')du'^2, \tag{1}$$

where $H(x',y',u')$ is the real part of an analytic function in $x'+iy'$ and an arbitrary function of u'. For plane waves with constant linear polarization $H(x',y',u')$ takes the form

$$H(x',y',u') = h(u')(y'^2 - x'^2), \tag{2}$$

where $h(u')$ is given in the case of Szekeres' family of solutions by

$$h(u') = u'^{n-1}\delta(u') + \frac{n(1-n)(2-1/n)^{1/2}u(u')^{2(n-1)}\theta(u(u'))}{8(1-u^{2n}(u')\theta(u(u')))^2}, \tag{3}$$

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where u' is the harmonic coordinate appearing in the canonical form of the line element (1) while u is the Rosen coordinate whose relation to u' is given below. Here, θ denotes the Heaviside unit step function and the integer n satisfies the condition $n \geqslant 1$. We notice here that, for $n=1$, $h(u')=\delta(u')$ which corresponds to impulsive waves while for higher values of n it corresponds to shock waves. For discussing the problem of colliding waves it is necessary to obtain a C^0 form of the metric, we therefore transform to the Rosen form

$$ds^2 = 2e^{-M}dudv - e^{-U}[e^{V}\cosh W dx^2$$

$$+ e^{-V}\cosh W dy^2 - 2\sinh W dxdy\,], \tag{4}$$

where M, U, V, and W are functions of the null coordinates (u,v) only. For the case of Szekeres' family Rosen form is accomplished by the transformation

$$x' = (1-u^n\theta)^{1/2-\kappa/2}(1+u^n\theta)^{1/2+\kappa/2}X,$$

$$y' = (1+u^n\theta)^{1/2-\kappa/2}(1-u^n\theta)^{1/2+\kappa/2}Y,$$

$$u' = \int_0^u (1-u^{2n}\theta)^{(1-1/n)/2}du, \tag{5}$$

$$v' = v + \tfrac{1}{2}n\,u^{n-1}\theta(u)[1-u^{2n}\theta(u)]^{1/n-1/2}$$

$$\times\{X^2[\kappa-u^n\theta(u)][1+u^n\theta(u)]^{\kappa/2}$$

$$\times[1-u^n\theta(u)]^{-\kappa/2} - Y^2[\kappa+u^n\theta(u)]$$

$$\times[1+u^n\theta(u)]^{-\kappa/2}[1-u^n\theta(u)]^{\kappa/2}\},$$

where κ is a real parameter related to n by

$$\kappa^2 = 2 - 1/n \tag{6}$$

and the Rosen form of the metric is given as

$$ds^2 = 2[1-u^{2n}\theta(u)]^{(1-1/n)/2}dudv - [1-u^{2n}\theta(u)]$$

$$\times\{[1-u^n\theta(u)]^{-\kappa}[1+u^n\theta(u)]^{\kappa}dX^2$$

$$+[1+u^n\theta(u)]^{-\kappa}[1-u^n\theta(u)]^{\kappa}dY^2\}. \tag{7}$$

The metric (4) represents the most general form for plane waves with arbitrary polarization. In the case of linear polarization we have the simplifying feature that $W=0$, but in this paper we shall investigate the collision of linearly polarized plane waves with a relative phase difference which require two mutually nonorthogonal Killing vectors ξ_x and ξ_y. So we shall now introduce a new parameter which measures the angle of polarization of the gravitational wave within the coordinate system under consideration. For convenience we choose this parameter to be the angle of rotation of (X,Y) coordinates in accordance with

$$X + iY = e^{i(\pi/2-\alpha\kappa)/2}(x+iy), \tag{8}$$

α being a real parameter. Now we obtain the metric (7) in the form

$$ds^2 = 2(1-p^2)^{(1-1/n)/2}dudv - \frac{(1-p^2)}{2P}$$

$$\times\{[1+P^2+(1-P^2)\sin\alpha\kappa]dx^2$$

$$+[1+P^2-(1-P^2)\sin\alpha\kappa]dy^2$$

$$+2\cos\alpha\kappa(P^2-1)dxdy\},$$

where

$$P = \left(\frac{1-p}{1+p}\right)^{\kappa}, \quad p = u^n\theta(u). \tag{9}$$

Let us note that with this choice of the rotation angle the choice $\alpha\kappa=\pi/2$ results in Eq. (7). In order to discuss collision of gravitational plane waves, it is convenient to consider space–time manifold in four disjoint patches as in Fig. 1. Let us consider two gravitational plane waves travelling in $+z$ and $-z$ directions. Prior to the collision of these waves the space–time region between them (region I) is Minkowski space while region II is given by the nonflat metric (9). We obtain region III from region II by replacing $u \leftrightarrow v$ and $\alpha \leftrightarrow \beta$ everywhere. In region II we shall employ the following null tetrad,

$$l_\mu = (1-p^2)^{(1-1/n)/2}\delta_\mu{}^0,$$

$$n_\mu = \delta_\mu{}^1,$$

$$m_\mu = 2^{-1/2}\,[(1-p)^{(1-\kappa)/2}(1+p)^{(1+\kappa)/2}\delta_\mu{}^2$$

$$+i(1+p)^{(1-\kappa)/2}(1-p)^{(1+\kappa)/2}\delta_\mu{}^3], \tag{10}$$

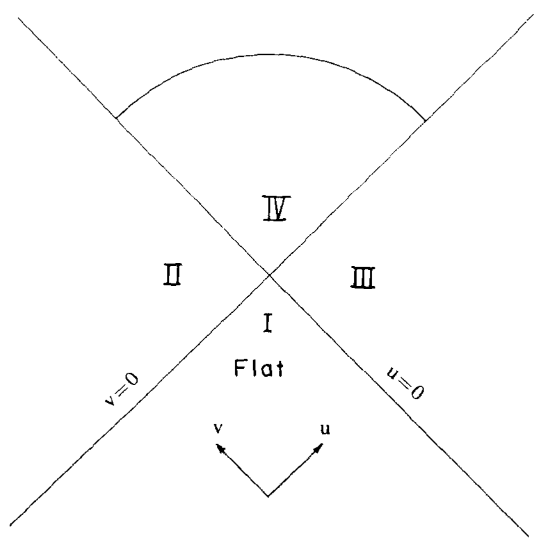

FIG. 1. Space–time diagram for colliding gravitational plane waves.

and we find that

$$\lambda = n\kappa\, u^{\,n-1}\theta(u)(1-p^2)^{(1/n-3)/2},$$

$$\mu = -n\, u^{\,2n-1}\theta(u)(1-p^2)^{(1/n-3)/2},$$

$$\psi_4 = \kappa n\,(1-p^2)^{1/n-3}[(1-n)\,u^{\,n-2} - u^{\,n-1}\delta(u)\,], \quad (11)$$

are the only nonvanishing Newman–Penrose (NP) quantities. The metric (7) represents a type N field. Similarly for the region III the nonvanishing NP quantities are σ, p, and ψ_0. We shall now consider the space–time geometry in the interaction region using these solutions as boundary conditions. The resulting space–time in the interaction region (region IV) becomes algebraically general.

The Einstein field equations for the metric (4) are well known, but as in Ref. 5 we shall use of Eells and Sampson's theory of harmonic mappings of Riemannian manifolds to cast the problem into a simple form. We consider two Riemannian manifolds (M,g) and (M',g') with dimensionalities n,n' respectively and a map $f{:}M \to M'$. Eells and Sampson's energy functional, which in local coordinates is given by

$$E(f) = \int g'_{AB}\frac{\partial f^A}{\partial x^i}\frac{\partial f^A}{\partial x^k}g^{ik}\,|g|^{1/2}d^{\,n}x, \quad (12)$$

defines an invariant functional of the mapping. We shall be interested in those maps for which the first variation vanishes

$$\delta E(f) = 0, \quad (13)$$

i.e., harmonic maps. We had shown earlier that the Einstein field equations for the metric (4) are obtained as harmonic maps where M is a flat two-dimensional manifold with the metric

$$ds^2 = 2du dv \quad (14)$$

and M' has metric

$$ds'^2 = e^{-U}(2dM\,dU + dU^2 - dW^2 - \cosh^2 W\,dV^2) \quad (15)$$

If we vary the energy functional formed from these two metrics, we obtain the Einstein field equations first obtained for this problem by Szekeres who used a different approach based on the Newman–Penrose formalism.

III. NEW FAMILY OF EXACT SOLUTIONS

We shall now derive a new family of exact solutions of the Einstein's field equations which correspond to the collision of linearly polarized plane gravitational waves with different phase parameters. These will generalize exact solutions for collinear polarizations given by Khan–Penrose and Szekeres. For this purpose we shall consider the metric for M' manifold. As we noted earlier the 2-section of this manifold spanned by V and W coordinates is a space of constant curvature, but in order to change this line element into the normal form we first imbed this 2-section in a three-dimensional flat manifold. The imbedding coordinates are given by

$$\alpha = \cosh V\cosh W + \sinh W,$$

$$\beta = \cosh V\cosh W - \sinh W,$$

$$\gamma = \sinh V\cosh W, \quad (16)$$

subject to the constraint

$$\alpha\beta - \gamma^2 = 1. \quad (17)$$

The relevant part of the metric becomes $d\alpha d\beta - d\gamma^2$. Now let us choose a new parametrization which satisfies the constraint Eq. (17) by letting

$$\alpha = \cos\nu\,\sinh\omega + \cosh\omega,$$

$$\beta = -\cos\nu\,\sinh\omega + \cosh\omega,$$

$$\gamma = \sin\nu\sinh\omega, \quad (18)$$

the metric of M' then takes the form

$$ds'^2 = e^{-U}(2dM\,dU + dU^2 - d\omega^2 - \sinh^2\omega\,dv^2), \quad (19)$$

which is the required form. Once we have cast the metric of M' into this form we introduce a complex function η which is defined by

$$\eta = e^{\,i v/\kappa}\tanh\frac{\omega}{2\kappa}, \quad (20)$$

where κ is a constant so that the metric of M' becomes

$$ds'^2 = e^{-U}\left(2dM\,dU + dU^2 - 4\kappa^2\,\frac{d\eta d\bar\eta}{(1-\eta\bar\eta)^2}\right), \quad (21)$$

where the bar denotes complex conjugation. Varying the energy functional constructed from the metrics (14) and (21) with respect to M, U, and $\bar\eta$, we get the field equations

$$(e^{-U})_{uv} = 0, \quad (22)$$

$$2M_{uv} + U_{uv} = 2\kappa^2(\eta_u\bar\eta_v + \eta_v\bar\eta_u)(\eta\bar\eta - 1)^{-1}, \quad (23)$$

$$2\,\eta_{uv} - U_v\eta_u - U_u\eta_v = 4\bar\eta\eta_u\eta_v(\eta\bar\eta - 1)^{-1}. \quad (24)$$

There is an analogy between Eq. (24) and the Einstein's equation for stationary axisymmetric gravitational fields in Ernst's formulation

$$(\xi\bar\xi - 1)\nabla^2\xi = 2\bar\xi\nabla\xi\cdot\nabla\xi. \quad (25)$$

Note, however, that the definition of η in Eq. (20) is entirely different from Ernst's ξ. The crucial point here is the following: We want the coupled partial differential equations to be a familar set of equations so that we can directly write their solutions, but the choice of dependent as well as independent variables are further restricted by the requirement that the resulting solution should have the proper boundary conditions. These considerations suggest that we search for a coordinate transformation so that we can pass from the patch $\{u,v\}$ to another patch $\{\tau,\sigma\}$ which has properties analogous to prolate spheroidal coordinates. This transformation is given by

$$\tau = u^n(1-v^{2n})^{1/2} + v^n(1-u^{2n})^{1/2},$$

$$\sigma = u^n(1-v^{2n})^{1/2} + v^n(1-u^{2n})^{1/2}, \tag{26}$$

where $n \geqslant 1$ is an integer. Under this change of coordinates the metric of M is transformed into

$$ds^2 = \Omega(\tau,\sigma)\left(\frac{d\tau^2}{1-\tau^2} - \frac{d\sigma^2}{1-\sigma^2}\right), \tag{27}$$

where the conformal factor Ω is irrelevant because it does not enter into the energy functional in Eq. (12). The usefulness of these new coordinates will appear when we rewrite the differential operators in the field equations using the $\{\tau,\sigma\}$ coordinate patch. First we note that in region IV

$$e^{-U} = 1 - u^{2n} - v^{2n} = (1-\tau^2)^{1/2}(1-\sigma^2)^{1/2} \tag{28}$$

and two useful identifies are given by

$$2\psi_{uv} - U_v\psi_u - U_u\psi_v$$
$$= \Omega(\tau,\sigma)\{[(1-\tau^2)\psi_\tau]_\tau - [(1-\sigma^2)\psi_\sigma]_\sigma\}, \tag{29}$$

$$\psi_u\chi_v + \psi_v\chi_u$$
$$= \Omega(\tau,\sigma)\{(1-\tau^2)\psi_\tau\chi_\tau - (1-\sigma^2)\psi_\sigma\chi_\sigma\}, \tag{30}$$

where ψ and χ are any two functions which are at least twice differentiable. It is straightforward to show that Eq. (24) in the coordinate patch $\{\tau,\sigma\}$ is given by

$$(\eta\bar{\eta}-1)\{[(1-\tau^2)\eta_\tau]_\tau - [(1-\sigma^2)\eta_\sigma]_\sigma\}$$
$$= 2\bar{\eta}\{(1-\tau^2)\eta_\tau^2 - (1-\sigma^2)\eta_\sigma^2\}, \tag{31}$$

which is the familiar Ernst's equation. It is well known that it admits a solution of the form

$$\eta = e^{i[(\alpha+\beta)/2]}\left[\tau\cos\left(\frac{\alpha-\beta}{2}\right) + i\sigma\sin\left(\frac{\alpha-\beta}{2}\right)\right], \tag{32}$$

where the arbitrary constants α and β are chosen to be polarization parameters in regions II and III respectively. Taking into considerations the boundary effects of the different space–time regions, we let $u \to u\theta(u)$ and $v \to v\theta(v)$ so that the solution (32) is equivalent to

$$\eta = e^{i\alpha}pw + e^{i\beta}qr, \tag{33}$$

where

$$p = u^n\theta(u), \quad q = v^n\theta(v), \quad r^2 = 1-p^2, \quad w^2 = 1-q^2.$$

Comparing the solution (32) with that given by Ernst for axisymmetric gravitational fields we immediately notice that $(\alpha-\beta)/2$ plays the role of a rotation parameter while $(\alpha+\beta)/2$ is the Taub–NUT parameter. Using this solution in the $\{u,v\}$ patch [i.e., Eq. (33)], we shall proceed to construct the space–time metric and show that it has the correct boundary values. This amounts to the determination of M, U, V, and W. From the definition (20) and (33) we read the solutions for ω and ν,

$$\sin\frac{\nu}{\kappa} = \frac{1}{|\eta|}(pw\sin\alpha + qr\sin\beta), \tag{34}$$

$$\sinh\frac{\omega}{\kappa} = \frac{2|\eta|}{1-|\eta|^2}. \tag{35}$$

The original metric functions V and W are given in terms of ω and ν by

$$e^{2V} = \frac{\cos\omega + \sin\nu\sinh\omega}{\cosh\omega - \sin\nu\sinh\omega}, \tag{36}$$

$$\sinh W = \cos\nu\sinh\omega. \tag{37}$$

In order to determine M, we integrate (23), so that the final solution for the metric functions is given as follows

$$e^{-U} = t^2 = 1-p^2-q^2, \tag{38}$$

$$e^{-M} = t^{-1}(rw)^{-\kappa^2}(1-|\eta|^2)^{\kappa^2}, \tag{39}$$

$$\sinh W = \frac{\eta^\kappa + \bar{\eta}^\kappa}{4|\eta|^\kappa}\left[\left(\frac{1+|\eta|}{1-|\eta|}\right)^\kappa - \left(\frac{1-|\eta|}{1+|\eta|}\right)^\kappa\right], \tag{40}$$

$$e^{2V} = \frac{(1+|\eta|)^{2\kappa}(2i|\eta|^\kappa + \eta^\kappa - \bar{\eta}^\kappa) + (1-|\eta|)^{2\kappa}(2i|\eta|^\kappa - \eta^\kappa + \bar{\eta}^\kappa)}{(1+|\eta|)^{2\kappa}(2i|\eta|^\kappa - \eta^\kappa + \bar{\eta}^\kappa) + (1-|\eta|)^{2\kappa}(2i|\eta|^\kappa + \eta^\kappa - \bar{\eta}^\kappa)} \tag{41}$$

where n and κ are related by (6). This solution may be expressed in terms of a null tetrad defined as

$$l_\mu = e^{-M/2}\delta_\mu^0,$$

$$n_\mu = e^{-M/2}\delta_\mu^1,$$

$$m_\mu = \tfrac{1}{2}e^{-U/2}[e^{V/2}(i\sinh\tfrac{1}{2}W - \cosh\tfrac{1}{2}W)\delta_\mu^2 + e^{-V/2}(\sinh\tfrac{1}{2}W - i\cosh\tfrac{1}{2}W)\delta_\mu^3]. \tag{42}$$

Now let us show that in the second region limit the solution (38)–(41) coincides with the Rosen form (9). For this purpose we set $q=0$ and obtain the solution

$$e^{-U} \equiv t^2 = 1-p^2, \tag{43}$$

$$e^{-M} = t^{\kappa^2-1}, \tag{44}$$

$$\sinh W=\tfrac{1}{2}\cos\alpha\kappa\left[\left(\frac{1+p}{1-p}\right)^{\kappa}-\left(\frac{1-p}{1+p}\right)^{\kappa}\right], \tag{45}$$

$$e^{2V}=\frac{(1+p)^{2\kappa}+(1-p)^{2\kappa}+\sin\alpha\kappa[(1+p)^{2\kappa}-(1-p)^{2\kappa}]}{(1+p)^{2\kappa}+(1-p)^{2\kappa}-\sin\alpha\kappa[(1+p)^{2\kappa}-(1-p)^{2\kappa}]}, \tag{46}$$

which gives the metric (9) so that the boundary conditions are satisfied. For $n=\kappa=1$ our solution (38)–(41) takes the form

$$e^{-U}\equiv t^2=1-p^2-q^2, \tag{47}$$

$$e^{-M}=\frac{1}{trw}\,[t^2+2p^2q^2-2pqrw\,\cos(\alpha-\beta)\,], \tag{48}$$

$$\sinh W=\frac{2(pw\cos\alpha+qr\cos\beta)}{t^2+2p^2q^2-2pqrw\,\cos(\alpha-\beta)}, \tag{49}$$

$$e^{2V}=\frac{1+p^2w^2+q^2r^2+2pqrw\cos(\alpha-\beta)+2(pw\sin\alpha+qr\sin\beta)}{1+p^2w^2+q^2r^2+2pqrw\cos(\alpha-\beta)-2(pw\sin\alpha+qr\sin\beta)}, \tag{50}$$

$$p=u\theta(u),\quad q=v\theta(v),$$

which is the solution reported in Ref. 5. In the limit $\alpha=\beta=\pi/2$ this solution reduces to the solution by Khan and Penrose,

$$e^{-U}\equiv t^2=1-p^2-q^2, \tag{51}$$

$$e^{-M}=t^3\,\frac{r^{-1}w^{-1}}{(pq+rw)^2}, \tag{52}$$

$$W=0, \tag{53}$$

$$e^{V}=\frac{r+q}{r-q}\,\frac{w+p}{w-p}. \tag{54}$$

Finally, in the limit $\kappa\alpha=\beta\kappa=\pi/2$ for $n=2$, $\kappa=(3/2)^{1/2}$ the solution (38)–(41) reduces to

$$e^{-U}\equiv t^2=1-p^2-q^2, \tag{55}$$

$$e^{-M}=t^5\,\frac{(rw)^{-3/2}}{(pq+rw)^3}, \tag{56}$$

$$W=0, \tag{57}$$

$$e^{V}=\left[\left(\frac{w+p}{w-p}\right)\left(\frac{r+q}{r-q}\right)\right]^{(3/2)^{1/2}}, \tag{58}$$

which corresponds to the solution given by Szekeres. We have therefore generalized Szekeres' family to the case of linear but noncollinearly polarized plane gravitational waves for the case when Szekeres' parameters n_1 and n_2 are equal. In another publication we shall show that gravitational wave and stationary axially symmetric fields can be treated in a unified manner,[14] where the solution of one class enables us to derive solutions to the other class and vice-versa. This procedure can be extended to Einstein–Maxwell fields as well.

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APPENDIX: SINGULARITIES

In order to see the physical singularities of our solutions, we calculate the curvature invariants which are as follows:

$$\mathrm{Re}\psi_2=n^2\,\frac{u^{n-1}v^{n-1}}{rw}\,\theta(u)\theta(v)\left(\frac{-pqrw}{t^4}+\kappa^2\,\frac{(t^2+2p^2q^2)\,\cos(\alpha-\beta)-2\,pqrw}{(1-|\eta|^2)^2}\right),$$

$$\mathrm{Im}\psi_2=-\,\frac{\kappa^2n^2u^{n-1}v^{n-1}\theta(u)\theta(v)}{rw|\eta|^3(1-|\eta|^2)}\,(\,|\eta\,|^2-8p^2q^2r^2w^2)\cosh\omega\,\sin(\alpha-\beta),$$

$$\mathrm{Re}\psi_4=\frac{-\kappa n}{2r|\eta|(1+\cos^2\nu\sinh^2\omega)^{1/2}}\Bigg[\bigg([(n-1)u^{n-2}\theta(u)+u^{n-1}\delta(u)]\Big(\frac{qw}{|\eta|}\cos\nu\sinh\omega\cdot\cosh\omega\cdot\sin(\alpha-\beta)+\frac{2\sin\nu Z}{1-|\eta|^2}\Big)$$

$$+nu^{2(n-1)}\theta(u)\Big\{\frac{qw}{|\eta|}\cos\nu\cdot\sinh\omega\cdot\cosh\omega\cdot\sin(\alpha-\beta)\Big(\frac{p(1-\kappa^2)}{r^2}-\frac{3p}{t^2}+\frac{(\kappa^2+1)|\eta|^2-1}{|\eta|^2(1-|\eta|^2)}+\frac{4\cosh\omega Z}{\sinh\omega r|\eta|(1-|\eta|^2)}\Big)$$

$$+\frac{2\sin\nu}{1-|\eta|^2}\Big[r(1-2q^2)-\frac{pqw}{r^2}(1+2r^2)\cos(\alpha-\beta)-\frac{1}{r}Z^2\Big(\frac{-3p}{t^2}-\frac{\kappa^2p}{r^2}+\frac{(2\kappa^2-3)n^2+1}{|\eta|^2(1-|\eta|^2)}\Big)\Big]$$

$$-\kappa q^2w^2\frac{\sin\nu\sinh\omega\cdot\cosh\omega}{1+\cos^2\nu\sinh^2\omega}\cdot\frac{\sin^2(\alpha-\beta)}{r|\eta|^3}\Big\}\bigg)\Bigg],$$

$$\mathrm{Im}\psi_4=-\frac{\kappa n}{2r|\eta|(1+\cos^2\nu\sinh^2\omega)}\Bigg[\bigg([(n-1)u^{n-2}\theta(u)+u^{n-1}\delta(u)]\Big(\frac{2\cos\nu\cosh\omega}{1-|\eta|^2}Z$$

$$-\frac{qw\sin\nu\sinh\omega\sin(\alpha-\beta)}{|\eta|}\Big)+nu^{2(n-1)}\theta(u)\Big\{\frac{2\cos\nu\cdot\cosh\omega}{1-|\eta|^2}\Big[r(1-2q^2)-\frac{pqw}{r^2}$$

$$\times(1+2r^2)\cos(\alpha-\beta)+\frac{2(\kappa^2+2)|\eta|^2-3|\eta|^4-1}{r|\eta|^2(1-|\eta|^2)^2}Z^2-\frac{p}{1-|\eta|^2}\Big(\frac{3}{t^2}+\frac{\kappa^2}{r^2}\Big)\Big]$$

$$-\frac{qw}{|\eta|}\sin\nu\sinh\omega\sin(\alpha-\beta)\Big[p\Big(\frac{1-\kappa^2}{r^2}-\frac{3}{t^2}\Big)+\frac{2[(\kappa^2+1)|\eta|^2-1]}{r|\eta|^2(1-|\eta|^2)}Z\Big]$$

$$-\frac{\kappa q^2w^2}{r|\eta|^3}\cos\nu\sinh\omega\cdot\cosh^2\omega\sin^2(\alpha-\beta)-\frac{4\kappa qw}{r|\eta|^2}\sin\nu\cdot\cosh\omega\sin(\alpha-\beta)Z\Big\}\bigg)\Bigg],$$

$$\psi_1=\psi_3=0,\quad \psi_0=\psi_4\quad(u\longleftrightarrow v,\alpha\longleftrightarrow\beta),$$

where

$$Z=pr(1-2q^2)+qw(1-2p^2)\cos(\alpha-\beta)\quad\text{and}\quad\cos\nu=\frac{\eta^\kappa+\bar\eta^\kappa}{2|\eta|^\kappa},\quad 2\cosh\omega=\Big(\frac{1+|\eta|}{1-|\eta|}\Big)^\kappa-\Big(\frac{1-|\eta|}{1+|\eta|}\Big)^\kappa$$

are to be substituted into these expressions. We observe that $r=w=0$ are singular surfaces expected from the focusing properties of the incoming waves. Same singularities arise from the roots of $|\eta|=0$. This is equivalent to $p^2+q^2-2p^2q^2=2pqrw\cos(\alpha-\beta)$, other roots of which depend on $(\alpha-\beta)$. The spacelike singularity $t^2=1-u^{2n}-v^{2n}=0$ reappears in the above invariants as well. We notice further that another singularity is provided by $1-|\eta|^2=0$, which is equivalent to $t^2=2pq[rw\cos(\alpha-\beta)-pq]$, which gives additional singularities depending on the values of α and β. For example, the choice $\alpha-\beta=(2n-1)\pi/2$ gives $t^2=-2p^2q^2$ which is satisfied for two symmetric hyperbolic branches starting at $(u=1,\ v=0)$ and $(v=1,\ u=0)$ and going in the increasing u,v directions so that it lies beyond the main singularity $t^2=0$. The singularity $t^2=0$ seems to be the essential feature of colliding plane gravitational waves.

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