A METHOD FOR GENERATING NEW SOLUTIONS IN GENERAL RELATIVITY

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We present another method of generating new solutions from old ones in general relativity. In particular we apply the method to some well-known classes of gravitational fields.

It is well known that the extremals of the harmonic map action between two riemannian manifolds provide genuine solutions for the Einstein field equations in the theory of general relativity [1]. Among particular classes of gravitational fields that have been extensively handled within the framework of harmonic maps are stationary axially symmetric [2,3] and colliding gravitational wave metrics [4,5]. The essential point in this approach is to consider two riemannian manifolds M (dimension n) and M' (dimension n') with a map, $f: M \to M'$ in such a way that the energy functional [6] of this map coincides with the Einstein–Hilbert action of the corresponding physical problem under consideration. In local coordinates this action reads

$$I(f) = \int g_{AB} \frac{\partial f^A}{\partial x^a} \frac{\partial f^B}{\partial x^b} g^{ab} |g|^{1/2} \, d^n x \, , \tag{1}$$

where $g_{ab}(x)$ and $g_{AB}(f)$ are the metrics of M and M', respectively, and the condition of the map to be harmonic is given by

$$\delta I(f) = 0 \, . \tag{2}$$

It is interesting to note that the equations obtained from this extremal condition of the harmonic map action suitable for the particular general relativistic problem does not produce all the Einstein equations obtained by the standard methods of calculation. However, this is not a handicap because the Einstein equations that are not involved in (2) turn out to be the integrability conditions for the equations obtained by (2) and therefore any solution to the set of harmonic map extremals provides a solution to the full set of equations automatically.

The method of solution which we want to present in this letter can be stated as

Theorem. Let f^A, for $A = 1, 2, ..., n'$, be a known solution to the field equations obtained by $\delta I(f) = 0$. Then there are new solutions $\tilde{f}^K$ for $K = 1, 2, ..., m'$, of the field equations resulting from $\delta I(\tilde{f}) = 0$, where $\tilde{f}^K$ is obtained from f^A either by isometry (for $m' = n'$), or imbedding (for $n' < m'$) of the metric g_{AB}.

Proof. Let $f^A = f^A(\tilde{f}^K)$ be a given transformation between the two sets of functions f and $\tilde{f}$. Substituting this into the action, one obtains

$$I(\tilde{f}) = \int g_{AB} \frac{\partial f^A}{\partial \tilde{f}^K} \frac{\partial f^B}{\partial \tilde{f}^L} \frac{\partial \tilde{f}^K}{\partial x^a} \frac{\partial \tilde{f}^L}{\partial x^b} g^{ab} |g|^{1/2} \, d^n x \, .$$

Defining now a new metric (whose dimensionality is $m' \neq n'$ in general)

$$\tilde{g}_{KL} = g_{AB} \frac{\partial f^A}{\partial \tilde{f}^K} \frac{\partial f^B}{\partial \tilde{f}^L} \, , \tag{3}$$

we see that the original action remains invariant and therefore the extremum condition $\delta I(\tilde{f}) = 0$, is satisfied. Thus $\tilde{f}^K$ serves as good as f^A does and constitutes a solution distinct from f^A. The proof is thus completed.

The interesting case however is the one for which g and $\tilde{g}$ have the same functional forms with equal dimensionalities. Such a problem is known as isometry

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and existence of such isometries is guaranteed by the existence of Killing vectors in the manifold M'. The imbedding case on the other hand arises whenever we want to generate solutions with sources from the vacuum ones [see example B(ii) below]. We should point out that the relation $f = f(\tilde{f})$, states a finite relation which is not the case for Bäcklund transformations and the two approaches are distinct. It should also be added that in analogy with gauge transformable solutions of group theoretical approaches there are trivial subclasses of isometries which do not generate new solutions. The first examples where the isometries of M' are employed were given by Matzner and Misner [7] and by Naugebeuer and Kramer [8]. In the following examples we present certain applications of the above-stated theorem.

(A) Consider the static spherically symmetric gravitational fields with static electric charge e described in the isotropic form by the line element

$$\mathrm{d}s^2 = B^{-2}\,\mathrm{d}t^2 - A^{-2}(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2)\,, \tag{4}$$

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\,,$$

where A and B are only functions of r. The solution for A and B is well known to represent uniquely the Reissner–Nordström (RN) solution. The equivalent lagrangian for this problem is

$$L = (\nabla R)^2 - R^2[(\nabla\psi)^2 - \mathrm{e}^{4\psi}(\nabla A_0)^2]\,, \tag{5}$$

where $A = R^{-2}\,\mathrm{e}^{-2\psi}$, $B = \mathrm{e}^{2\psi}$ and A_0 represents the only non-vanishing component of the electromagnetic vector potential. This lagrangian is identical with the one obtained by harmonic maps for the choices of riemannian manifolds

$$M\colon \mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\,,$$
$$M'\colon \mathrm{d}s'^2 = \mathrm{d}R^2 - R^2(\mathrm{d}\psi^2 - \mathrm{e}^{4\psi}\,\mathrm{d}A_0^2)\,. \tag{6}$$

The isometry implied in the theorem above is given by

$$\mathrm{e}^{2\tilde{\psi}} = (2A_0 + c)^2\,\mathrm{e}^{2\psi} - \mathrm{e}^{-2\psi}\,,$$
$$\tilde{A}_0 = -(2A_0 + c)/[(A_0 + c)^2 - \mathrm{e}^{-4\psi}]\,, \tag{7}$$
$$\tilde{R} = R, \quad c = \text{const.}$$

Under this isometry the resulting solution is definitely again RN, however, charge and mass have been changed in accordance with

$$m \to M = m(1 + c^2) + 2ec\,,$$
$$e \to Q = e(1 - c^2) + 2c(m + ec)\,, \tag{8}$$

so that the relation

$$M^2 - Q^2 = (c^2 - 1)^2(m^2 - e^2)$$

holds. Let us note that in the original RN, as $m \to 0$, the space–time is not flat whereas no particle with $m = 0$, $e \neq 0$ is known. With the new choices (8), if we set $e = 0$, $c \neq 0$, we see that the resulting M and Q have the property that both vanish in the limit $m \to 0$. Note also that the isometry (7) is known as Ehlers transformation [9] and corresponds to a subgroup of SL(2,R) transformations on the configuration manifold.

(B) The space–time representing colliding plane gravitational waves is characterized by the effective lagrangian

$$L = \mathrm{e}^{-U}[2\nabla M \cdot \nabla U + (\nabla U)^2 - (\nabla W)^2 - \cosh^2 W\,(\nabla V)^2]\,,$$

which results via (1) in the riemannian manifolds

$$M\colon \mathrm{d}s^2 = 2\,\mathrm{d}u\,\mathrm{d}v\,,$$
$$M'\colon \mathrm{d}s'^2 = \mathrm{e}^{-U}[2\,\mathrm{d}U\,\mathrm{d}M + \mathrm{d}U^2 - \mathrm{d}W^2 - \cosh^2 W\,\mathrm{d}V^2]\,. \tag{9}$$

Any isometry generated solution in this problem is valid only within the interaction region of the colliding waves and becomes relevant to the cosmological models. Out of many such solutions obtained by isometry (or imbedding) let us present only two explicitly.

(i) The isometry given by

$$U' = U, \quad W = 0\,, \quad V' = V + \alpha U\,,$$
$$M' = M + \alpha V + \tfrac{1}{2}\alpha^2 U\,, \quad \alpha = \text{const.}\,, \tag{10}$$

is a new solution whenever (U, V, M) is a known solution.

(ii) We can generate also a solution with a scalar field source by imbedding the metric M' into higher dimension (let $W = 0$). The metric of the new M' reads now

$$\mathrm{d}s'^2 = \mathrm{e}^{-U}[2\,\mathrm{d}U\,\mathrm{d}M + \mathrm{d}U^2 - \mathrm{d}V^2 - k\,\mathrm{d}\phi^2]\,, \tag{11}$$

where k is the coupling constant and the new dimension represents the scalar field. Then, if (U, M, V) is a solution to the vacuum equations (U', M', V', ϕ) represents a solution to the Einstein scalar field equations where the imbedding relations are given by

$$U' = U\,, \quad M' = M + [1 + (1-\beta^2)^{1/2}](U+V)\,,$$
$$V' = U + (1-\beta^2)^{1/2}(U+V)\,, \quad k^{1/2}\phi = \beta(U+V)\,,$$
$$\beta = \text{const.} \qquad (12)$$

(C) Using the transitive character of isometry, that "isometry of an isometry is still an isometry" generates further solutions from any two isometry generated ones.

Finally we want to mention a particular class of solutions arising when the harmonic map lagrangian vanishes (or equivalently, the metric of M' becomes null). As an example for such a case consider the M' metric of example (B) with $W = 0$. Since $ds'^2 = 0$, we have

$$2\,dU\,dM + dU^2 - dV^2 = 0\,.$$

Taking $V = aU, M = \frac{1}{2}(a^2 - 1)U$, with a = const., then the vacuum field equations reduce to

$$(e^{-U})_{uu} = (e^{-U})_{vv} = (e^{-U})_{uv} = 0\,. \qquad (13)$$

The particular solution $e^{-U} = 1 \pm u \pm v$ was reported [10] as a non-singular solution to the wave collision problem. However, by a coordinate transformation this choice is realized as Kasner's cosmology [11] and furthermore in order to match at the boundaries in a consistent way one must find a global expression for which the only candidate seems to be $e^{-U} = 1 \pm u\theta(u) \pm v\theta(v)$. But such a choice unfortunately fails to satisfy the vacuum equations (13) at the boundaries and therefore is not an acceptable solution for the problem of colliding plane gravitational waves.

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