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Particle Motion In Bell–Szekeres Space-Time[1]

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We solve the geodesics equation for a charged particle in Bell–Szekeres space-time. In the same geometry we give the test particle solution of Dirac's equation.

1. INTRODUCTION

It is a well-known fact of classical electrodynamics in flat space that electromagnetic (e.m.) waves do not scatter, whereas in general relativity the nonlinear character is manifested by scattering of e.m. waves in analogy with photon–photon scattering of quantum electrodynamics. The space-time arising from collisions of shock e.m. waves was discovered by Bell and Szekeres (BS) (1974). This nonnull e.m. solution to the Einstein–Maxwell equations is characterized by nonsingular behavior in contrast to the Einstein solution resulting from the colliding gravitational plane waves (Szekeres, 1972; Halil, 1979). Another aspect of the BS solution is that off the wave front it is conformally flat, therefore by a theorem of Tariq and Tupper (1974) it must be transformable to a Bertotti–Robinson (BR) (Bertotti, 1959; Robinson, 1959) solution. This latter solution of Einstein–Maxwell equations is known to represent an e.m. radiation filled universe and is connected with the Reissner–Nordström "throat" which is defined (Misner et al., 1973) for the case of charge $(Q)=$mass (M) and where $|Q-r|\ll Q$.

To our knowledge the solution of geodesics equations in BS geometry is absent and for BR is not without ambiguities (Lovelock, 1967) in the literature. From the cosmological point of view this problem is interesting since e.m. shocks produced by the astrophysical objects interact to develop BS regions. The only nonvanishing components of the e.m. field tensor

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admitted by the BS solution consist of E_x = const and B_y = const. It is known from the motion of charged particles in conductors that in the presence of a constant external magnetic field a transverse potential arises (Hall effect). For the present case we can identify the Hall potential similarly and observe that the electric fields associated with this potential are collinear with E_x but there is no chance that the two electric fields compensate each others.

We present the solution of geodesics equations in BS geometry and integrate the separable Hamilton–Jacobi functional completely. Since electron–positron pair creation is a frequently occurring phenomenon around pulsars, we investigate the solution of a Dirac particle in BS background. For this purpose we employ Chandrasekhar's (1976) treatment of Dirac's equation in the test particle approximation.

2. GEODESICS IN BS SPACE-TIME

Let us consider the head-on collision of shock e.m. waves with constant profile and characterized by the null-tetrad scalars, $\phi_0 = F_{\mu\nu} l^\mu m^\nu = k^{1/2} b =$ const and $\phi_2 = F_{\mu\nu} \bar{m}^\mu n^\nu = k^{1/2} a =$ const, respectively. Here $k = G/8c^4$ ($G =$ Newton's constant, $c =$ speed of light), a and b are real constants with our choice that $ab > 0$. For the detailed description of e.m. collision problem we refer to the article by Bell and Szekeres (1974).

If the null coordinates u and v represent the directions of propagations for e.m. shocks, we define new coordinates by

$$\begin{aligned} \xi &= au + bv \\ \eta &= au - bv \end{aligned} \tag{1}$$

which will prove to be suitable in the sequel. The coordinate lines $\xi =$ const ($\eta =$ const) represent families of elliptical (hyperbolic) curves. In these coordinates the BS solution is

$$ds^2 = \frac{1}{2ab}(d\xi^2 - d\eta^2) - \cos^2\eta\, dx^2 - \cos^2\xi\, dy^2 \tag{2}$$

while the e.m. vector potential has a single surviving component,

$$A_x \equiv A = (2/k)^{1/2} \sin\eta \tag{3}$$

Note that the factor $1/2ab$ in the line element is not significant but for reasons of correspondence with the null coordinates we shall keep it. The

Weyl component of the curvature tensor takes divergent values for $\xi=\pi$ and $\eta=0$, but since such singularities correspond to the location of sources (i.e., wave front), they are not ambiguous. The geodesics equation reads

$$\frac{d^2x^\alpha}{ds^2}+\Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}=eF^\alpha{}_\beta\left(\frac{dx^\beta}{ds}\right) \tag{4}$$

where e is the charge and s is an affine parameter defined by $s=$(proper time)/(mass). From the field theoretical approach the same geodesics equations can be obtained from the Lagrangian density

$$\mathfrak{L}=\frac{1}{2ab}(\dot{\xi}^2-\dot{\eta}^2)-\cos^2\eta\,\dot{x}^2-\cos^2\xi\,\dot{y}^2+2e\left(\frac{2}{k}\right)^{1/2}\sin\eta\,\dot{x} \tag{5}$$

where the dot denotes d/ds. Since x and y are cyclic the corresponding equations yield the first integrals

$$\dot{y}\cos^2\xi=-\beta=\text{const} \tag{6}$$

$$\dot{x}\cos^2y-e(2/k)^{1/2}\sin\eta=-\alpha=\text{const} \tag{7}$$

The remaining two equations can be written in the following form:

$$\ddot{\xi}+abK_{,\xi}=0 \tag{8}$$

$$\ddot{\eta}-abK_{,\eta}=0 \tag{9}$$

where K represents the effective potential given by

$$K=\frac{-\beta^2}{\cos^2\xi}\cdot-\frac{\left[\alpha-e(2/k)^{1/2}\sin\eta\right]^2}{\cos^2\eta} \tag{10}$$

and is already separated in this coordinate system. The form of the effective potential suggests by comparison with the Newtonian potential a law "inverse cosine square" of attraction. By direct inspection of (8) and (9) one deduces a third constant of motion expressed by

$$(1/2ab)(\dot{\xi}^2-\dot{\eta}^2)+K=\gamma=\text{const} \tag{11}$$

which can be identified as the total energy of the system. Integration of (8) and (9) yields

$$\frac{1}{2}\dot{\xi}^2=\frac{ab\beta^2}{\cos^2\xi}+A \tag{12}$$

$$\frac{1}{2}\dot{\eta}^2=-\frac{ab\left[\alpha-e(2/k)^{1/2}\sin\eta\right]^2}{\cos^2\eta}+B \tag{13}$$

where the constants of integration are constrained by (11),

$$A-B=ab\gamma \tag{14}$$

All the foregoing expressions can be integrated and we give the results

$$\xi=\begin{cases}\sin^{-1}\left\{(1+1/q^2)^{1/2}\sin\left[(2|a'|)^{1/2}s+c'\right]\right\} & \text{for } A=|a'| \\ \sin^{-1}\left\{(1/p)\sinh\left[(2|a'|)^{1/2}s+c'\right]\right\} & \text{for } A=-|a'|\end{cases} \tag{15}$$

where $q^2=|a'|/ab\beta^2<1$, $p^2=q^2/(1-q^2)>1$ and c' is an integration constant. In order to provide the case $A=0$ we must choose $c'=0$:

$$\eta=\sin^{-1}\left\{\frac{(-\Delta)^{1/2}}{2C}\sin\left[(-2C)^{1/2}s+d'\right]-\frac{b'}{2C}\right\} \tag{16}$$

where

$$\Delta=-4B\left(B+\frac{2abe^2}{k}-ab\alpha^2\right),\qquad C=-\left(B+\frac{2abe^2}{k}\right)$$

$b'=2ab\alpha(2/k)^{1/2}$ and d' is an integration constant

$$x=(2ab)^{-1/2}\tan^{-1}\left\{(ab)^{-1/2}\frac{\left\{B\cos^2\eta-ab\left[\alpha-(2/k)^{1/2}e\sin\eta\right]^2\right\}^{1/2}}{\alpha\sin\eta-(2/k)^{1/2}e}\right\} \tag{17}$$

$$y=\frac{1}{2}(2ab)^{-1/2}\begin{cases}\cosh^{-1}\left[1+2/(1+q^2)\tan^2\xi\right], & A=|a'| \\ \cosh^{-1}\left[1+2(1+p^2)\tan^2\xi\right], & A=-|a'|\end{cases} \tag{18}$$

where p, q are given above. In the x, y integrations we neglected the unimportant additive constants.

In order to obtain null geodesics we set $\gamma=0$ or $A=B$ in the above notation. We must also take $e=0$ since no charged particle moves on the null geodesics.

The covariant component of the force is given by

$$f_\mu = eF_{\mu\nu}\left(\frac{dx^\nu}{ds}\right)$$

whose x component in explicit form is

$$f_x = -e\left(\dot{t}E_x - \dot{z}B_y\right)$$

where $E_x = k^{1/2}(a-b)$, and $B_y = k^{-1/2}(a+b)$. The second term in the parentheses for f_x can be identified as the Hall electric field, where $\dot{z}$ is to be substituted from the geodesics. Let us note that, $2^{1/2}2ab\dot{z}=(a-b)\dot{\xi}-(a+b)\dot{\eta}$ and $2^{1/2}2ab\dot{t}=(a+b)\dot{\xi}-(a-b)\dot{\eta}$, so that in order to get $f_x=0$ we must have $(a+b)\dot{z}=(a-b)\dot{t}$ or equivalently $ab\dot{\eta}=0$. Since $a\neq 0 \neq b$ and $\dot{\eta}\neq 0$ by (13), we conclude that the Hall potential does not compensate the effect of the already existing electric potential due to E_x.

3. HAMILTON–JACOBI EQUATION

We will give a complete integral of the Hamilton–Jacobi equation

$$g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu} - eA_\mu\right)\left(\frac{\partial S}{\partial x^\nu} - eA_\nu\right) = \pm m^2 \tag{19}$$

where $g^{\mu\nu}$ correspond to the BS line element (2). This is equivalent to

$$2ab\left(S_\xi^2 - S_\eta^2\right) - \frac{1}{\cos^2\eta}\left[\mu - e\left(\frac{2}{k}\right)^{1/2}\sin\eta\right]^2 - \frac{\nu^2}{\cos^2\xi} = \pm m^2 \tag{20}$$

where μ and ν are constants associated with the Killing coordinates x and y. Further separation of S is assumed in the form

$$S = X(\xi) + E(\eta) + \mu x + \nu y \tag{21}$$

After separating ξ and η by the fourth constant l we obtain the following ordinary differential equations:

$$2ab\left(\frac{dX}{d\xi}\right)^2 - \frac{\nu^2}{\cos^2\xi} + m^2 = l \tag{22}$$

$$2ab\left(\frac{dE}{d\eta}\right)^2 + \frac{1}{\cos^2\eta}\left[\mu - e\left(\frac{2}{k}\right)^{1/2}\sin\eta\right]^2 = l \tag{23}$$

We readily observe from (23) that the fourth constant l is a positive definite quantity. The solutions for these ordinary differential equations are given as follows:

$$-\left(\frac{2ab}{\nu^2+l\pm m^2}\right)^{1/2} X(\xi) = -P\sin^{-1}(P\sin\xi)$$
$$+\frac{1}{2}(1-P^2)^{1/2}\log\left[\frac{\sin\xi-1}{\sin\xi+1}\,\frac{(1-P^2)^{1/2}(1-P^2\sin^2\xi)^{1/2}+1+P^2\sin\xi}{(1-P^2)^{1/2}(1-P^2\sin^2\xi)^{1/2}+1-P^2\sin\xi}\right] \tag{24}$$

$$-(2ab)^{1/2}E(\eta)$$
$$=\frac{1}{2}\left[\mu+e\left(\frac{2}{k}\right)^{1/2}\right]\sin^{-1}\left\{\frac{l-\mu^2-\mu e(2/k)^{1/2}+\sin\eta\left[R+\mu e(2/k)^{1/2}\right]}{N(\sin\eta+1)}\right\}$$
$$-\frac{1}{2}\left[\mu-e\left(\frac{2}{k}\right)^{1/2}\right]\sin^{-1}\left\{\frac{l-\mu^2+\mu e(2/k)^{1/2}+\sin\eta\left[-R+\mu e(2/k)^{1/2}\right]}{N(\sin\eta-1)}\right\}$$
$$+R^{1/2}\sin^{-1}\left[\frac{\mu e(2/k)^{1/2}-R\sin\eta}{N}\right] \tag{25}$$

where

$$R=l+\frac{2e^2}{k} \quad \text{and} \quad N=l^{1/2}\left(l-\mu^2+\frac{2e^2}{k}\right)^{1/2}=l^{1/2}(R-\mu^2)^{1/2}$$

4. DIRAC'S EQUATION IN BS SPACE-TIME

Dirac's equation in Newman–Penrose (Newman and Penrose, 1962) spin coefficient formalism is given by the coupled equations (Chandrasekhar, 1976)

$$\begin{aligned}
(D+\varepsilon-\rho)F_1+(\bar{\delta}+\pi-\alpha)F_2&=i\mu_e G_1\\
(\Delta+\mu-\gamma)F_2+(\delta+\beta-\tau)F_1&=i\mu_e G_2\\
(D+\bar{\varepsilon}-\bar{\rho})G_2-(\delta+\bar{\pi}-\bar{\alpha})G_1&=i\mu_e F_2\\
(\Delta+\bar{\mu}-\bar{\gamma})G_1-(\bar{\delta}+\bar{\beta}-\bar{\tau})G_2&=i\mu_e F_1
\end{aligned} \tag{26}$$

where $D = l^{\mu}\partial_{\mu}$, $\Delta = n^{\mu}\partial_{\mu}$ and $\delta = m^{\mu}\partial_{\mu}$ are the directional derivatives while $\alpha, \beta, \gamma, \ldots$ denote the spin coefficients. The complex functions F_1, F_2, G_1, and G_2 are the spinor components of the Dirac's wave function and $2^{1/2}\mu_e$ denotes the mass of the Dirac particle. The generic form of the BS metric can be taken as

$$ds^2 = 2du\,dv - e^{-U}(e^{V}dx^2 + e^{-V}dy^2) \tag{27}$$

let us choose the null tetrad by the set

$$\begin{aligned} &l_{\mu} = \delta_{\mu}^{\ 0}, \qquad n_{\mu} = \delta_{\mu}^{\ 1} \\ &m_{\mu} = 2^{-1/2}e^{-U/2}\left(e^{V/2}\delta_{\mu}^{\ 2} + ie^{-V/2}\delta_{\mu}^{\ 3}\right) \end{aligned} \tag{28}$$

In this tetrad the nonvanishing spin coefficients are

$$\begin{aligned} &\lambda = \tfrac{1}{2}V_u, \qquad \rho = \tfrac{1}{2}U_v \\ &\sigma = -\tfrac{1}{2}V_v, \qquad \mu = -\tfrac{1}{2}U_u \end{aligned} \tag{29}$$

We shall assume in the following a dependence on the Killing coordinates given by $e^{i(\mu x + \nu y)}$, $\mu = \text{const (real)}$, $\nu = \text{const (real)}$. Substituting all these expressions into (26) and scaling the spinors by

$$e^{-U/2}F_i = f_i, \qquad e^{-U/2}G_i = g_i, \qquad i = 1,2$$

we obtain in the ξ, η coordinates the following system of equations:

$$\begin{aligned} &b(\partial_{\xi} - \partial_{\eta})f_1 + 2^{-1/2}\left(\frac{\nu}{\cos\xi} + \frac{i\mu}{\cos\eta}\right)f_2 = i\mu_e g_1 \\ &b(\partial_{\xi} - \partial_{\eta})g_2 - 2^{-1/2}\left(\frac{\nu}{\cos\xi} - \frac{i\mu}{\cos\eta}\right)g_1 = i\mu_e f_2 \\ &a(\partial_{\xi} + \partial_{\eta})f_2 + 2^{-1/2}\left(\frac{\nu}{\cos\xi} - \frac{i\mu}{\cos\eta}\right)f_1 = i\mu_e g_2 \\ &a(\partial_{\xi} + \partial_{\eta})g_1 - 2^{-1/2}\left(\frac{\nu}{\cos\xi} + \frac{i\mu}{\cos\eta}\right)g_2 = i\mu_e f_1 \end{aligned} \tag{30}$$

We observe from these equations that the choices $f_2 = (b/a)^{1/2}f_1$ and $g_2 = (a/b)^{1/2}g_1$ decouple at the second order in the following form:

$$\begin{aligned} &Qf_1 = qf_1 \\ &Qg_1 = -\bar{q}g_1 \end{aligned} \tag{31}$$

where

$$Q=\partial_{\xi\xi}-\partial_{\eta\eta}-\frac{1}{2ab}\left(\frac{\nu^2}{\cos^2\xi}+\frac{\mu^2}{\cos^2\eta}\right)+\frac{\mu_e^2}{ab}$$

and

$$q=-(2ab)^{-1/2}\left(\nu\frac{\sin\xi}{\cos^2\xi}+i\mu\frac{\sin\eta}{\cos^2\eta}\right)$$

The separable ξ,η dependences of the solutions are given by the expressions

$$f_1\sim g(\eta)\tan\left(\frac{\pi}{4}-\frac{\xi}{2}\right)^{\nu(2ab)-1/2}\tan\left(\frac{\pi}{4}-\frac{\eta}{2}\right)^{-i\mu(2ab)-1/2}$$

$$g_1\sim g(\eta)\tan\left(\frac{\pi}{4}-\frac{\xi}{2}\right)^{-\nu(2ab)-1/2}\tan\left(\frac{\pi}{4}-\frac{\eta}{2}\right)^{i\mu(2ab)-1/2}$$

where the function $g(\eta)$ is required to satisfy the ordinary differential equation

$$(1-x^2)g_{xx}+(\alpha_1-x)g_x+\beta_1 g=0 \tag{32}$$

with $x=\sin\eta$, $\alpha_1=-2i\mu(2ab)^{-1/2}$, and $\beta_1=-\mu_e^2/ab$. In case we have a massless particle then $\beta_1=0$, which implies $g=\text{const}$. Such a differential equation (32) is the usual price one has to pay in incorporating mass into the problem. The final solution can be expressed in the following form:

$$\begin{aligned}
F_1&=A_1 g(\eta)(\cos\xi)^{-1/2}\left[\tan\left(\frac{\pi}{4}-\frac{\xi}{2}\right)\right]^{\nu(2ab)-1/2}\\
&\quad\times(\cos\eta)^{-1/2}\left[\tan\left(\frac{\pi}{4}-\frac{\eta}{2}\right)\right]^{-i\mu(2ab)-1/2}e^{i(\mu x+\nu y)}\\
G_1&=B_1 g(\eta)(\cos\xi)^{-1/2}(\cos\eta)^{-1/2}\left[\tan\left(\frac{\pi}{4}-\frac{\xi}{2}\right)\right]^{-\nu(2ab)-1/2}\\
&\quad\times\left[\tan\left(\frac{\pi}{4}-\frac{\eta}{2}\right)\right]^{i\mu(2ab)-1/2}e^{i(\mu x+\nu y)}
\end{aligned} \tag{33}$$

$F_2=(b/a)^{1/2}F_1, G_2=(a/b)^{1/2}G_1$ and where A_1, B_1 are arbitrary complex constants. Unless μ and ν are specified the location of singularities (if any) cannot be observed, but we can definitely choose regions where the foregoing spinor components are regular.

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