On a Class of Transcendent Solutions in the Einstein-Maxwell Theory

1986 Europhys. Lett. 1 375

(http://iopscience.iop.org/0295-5075/1/8/001)

EUROPHYSICS LETTERS 15 April 1986

Europhys. Lett., **1** (8), pp. 375-376 (1986)

On a Class of Transcendent Solutions in the Einstein-Maxwell Theory.

M. HALILSOY

Nuclear Engineering Department King Abdulaziz University
P.O. Box 9027, *Jeddah*-21413 *Saudi Arabia*

(received 6 January 1986; accepted 11 February 1986)

PACS. 03.50. – Classical field theory.
PACS. 03.40K. – Waves and wave propagation: general mathematical aspects.

Abstract. – The role of the Bavinonic condition in a previously obtained solution of the Ernst equation is clarified.

Some time ago we had presented the complete self-similarity integral to Ernst equations in the Einstein-Maxwell theory [1]. Recently, in a search aiming at extending this procedure [2], we described a method to obtain solutions expressed in Painleve's fifth transcendents as solutions to Ernst equations [3],

$$\begin{cases} (\xi\bar{\xi}+\eta\bar{\eta}-1)\nabla^2\xi = 2(\bar{\xi}\nabla\xi+\bar{\eta}\nabla\eta)\nabla\xi\,, \\ (\xi\bar{\xi}+\eta\bar{\eta}-1)\nabla^2\eta = \ (\bar{\xi}\nabla\xi+\bar{\eta}\nabla\eta)\nabla\eta\,. \end{cases} \tag{1}$$

Here ξ and η correspond to the gravitational and electromagnetic complex potentials, respectively. In case that ξ and η happen to be geodesic with respect to both of the two harmonic functions v and $\tilde{v}$, separately, a theorem was proved showing that the foregoing equations are satisfied, provided the constraint condition

$$\nabla v\cdot\nabla\tilde{v}=0 \tag{2}$$

holds. In most of the exact solutions, however, metric functions fail to satisfy the geodesic requirement and, unless they are brought into such form by reparametrization, the theorem loses its meaning.

In this note we want to comment on our previous work where ξ and η were parametrized as

$$\begin{cases} \xi = y\cos\Psi\exp[i(\alpha+c\tilde{v})]\,, \\ \eta = y\sin\Psi\exp[i(\beta+c\tilde{v})]\,. \end{cases} \tag{3}$$

It can directly be checked that, while ξ and η are geodesic with respect to v, they fail to satisfy the same requirement with respect to $\tilde{v}$, hence the stated theorem does not apply in this case as well.

The Einstein-Maxwell (EM) Lagrangian has the form

$$L = (y^2 - 1)^{-2}\{y'^2 + y^2(1 - y^2)\Psi'^2 + y^2(\alpha'^2 \cos^2\Psi + \beta'^2 \sin^2\Psi) - \\ - y^4 \sin^2\Psi \cos^2\Psi(\alpha' - \beta')^2 + c^2 y^2 \exp[2v]\} \tag{4}$$

The fact that this Lagrangian bears no trace of $\tilde{v}$ prompts us to relax the harmonic requirement from $\tilde{v}$, however, this remains true as long as we choose our base manifold as

$$M_0 : \mathrm{d}s_0^2 = \exp[2v]\,\mathrm{d}v^2 + \mathrm{d}\tilde{v}^2 + \exp[2v]\,\mathrm{d}\varphi^2 \tag{5}$$

The configuration manifold (M') is as before [1],

$$M' : \mathrm{d}s'^2 = (\xi\bar{\xi} + \eta\bar{\eta} - 1)^{-2}\{\mathrm{d}\xi\,\mathrm{d}\bar{\xi}(1 - \eta\bar{\eta}) + \mathrm{d}\eta\,\mathrm{d}\bar{\eta}(1 - \xi\bar{\xi}) + \xi\bar{\eta}\,\mathrm{d}\eta\,\mathrm{d}\bar{\xi} + \eta\bar{\xi}\,\mathrm{d}\xi\,\mathrm{d}\bar{\eta}\} \tag{6}$$

and the energy functional of the harmonic map $f: M_0 \to M'$ yields the Lagrangian (4).

A basic question with utmost importance remaining yet is how to relate M_0, with the cylindrical metric M, with $\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathrm{d}z^2 + \rho^2\,\mathrm{d}\varphi^2$, which is essential to preserve the axial symmetry. We consider the harmonic map from M into M_0, however, it is known that the composition of two harmonic maps need not be harmonic. From this corollary [4] we deduce that once a composition is harmonic by construction, it does not guarantee that each joint map of the combination will be harmonic. It turns out that under the map between M and M_0, M_0 must reduce to M, implying that we are left with the unique choice $v = \log\rho$ and $\tilde{v} = z$. (Similarly, for space-times that admit two space-like Killing vectors, the choice $v = \log\rho$ and $\tilde{v} = t$, will yield a transcendental solution.)

In conclusion, the Lagrangian (4) describes transcendent e.m. fields on the base manifold M_0—at the cost of axial symmetry—where v is harmonic and $\tilde{v}$ is arbitrary (provided $J(v, \tilde{v})/(\rho, z) \neq 0$). But whenever we consider the base manifold to be cylindrical, then the transcendental class has a unique element, which is the one given by LEAUTE and MARCILHACY [5].

REFERENCES

[1] M. HALILSOY: *Lett. Nuovo Cimento*, **37**, 231 (1983).
[2] M. HALILSOY: *Lett. Nuovo Cimento*, **44**, 88 (1985).
[3] F. J. ERNST: *Phys. Rev.*, **168**, 1415 (1968).
[4] J. EELLS and J. H. SAMSON: *Am. J. Math.*, **86**, 109 (1964).
[5] B. LEAUTE and G. MARCILHACY: *Lett. Nuovo Cimento*, **40**, 102 (1984).