Similarity solutions for the self-dual SU(2) fields

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(Received 20 December 1985)

We present new solutions to Yang's self-dual SU(2) equations. These solutions have the property that they are self-similar, together with some of their elliptical and transcendental extensions.

I. INTRODUCTION

Since the publication of Yang’s original work on the self-dual SU(2) gauge theory, much effort has been put forward to integrate these equations. Although this work was carried out in the four-dimensional Euclidean space—in connection with instantons—there is always freedom to suppress two of the coordinates and study the theory in a plane. Such a reduced formalism turns out to have much in common with Einstein’s equations that admit two Killing vectors, a topic to which much has been attained in the relativity community. By the same token, we integrate Yang’s equations once more in analogy with the similarity integral that we had obtained previously in the Einstein-Maxwell theory.

Static, axially symmetric self-dual Yang’s equations in the R gauge can be derived by the variation of the action

$$E[\phi, \Psi, \bar{\Psi}] = \int \frac{[\nabla \Psi]^2 + (\nabla \phi)^2}{\rho} \, d\rho \, dz \,,$$

where the real function $\phi$ and the complex function $\Psi$ depend only on $\rho$ and $z$. This action is in the form of an energy functional of the harmonic maps, $f: M \rightarrow M'$, where the respective manifolds are

$$M: ds^2 = dp^2 + dz^2 + \rho^2 \, dq^2 \,,$$

$$M': ds'^2 = \frac{|d\Psi|^2 + d\phi^2}{\phi^2} \,.$$

Our purpose is to consider composite maps, where a third manifold $M''$ is introduced between the manifolds $M$ and $M'$. By a corollary, the composition of a geodesic map and a harmonic map itself is harmonic; henceforth we consider the harmonic maps from $M''$ into $M'$, where

$$M'': ds''^2 = dv^2 \,,$$

is a one-dimensional manifold. The requirement that the maps from $M$ into $M''$ are harmonic restricts $v$ to be the arbitrary harmonic function in the $\rho$, $z$ coordinates, i.e.,

$$v_{\rho \rho} + \frac{1}{\rho} v_{\rho} + v_{zz} = 0 \,.$$

The energy functional of the maps from $M''$ into $M'$ reads

$$E[\phi, \Psi, \bar{\Psi}] = \int dv |\Psi|^2 + \phi^2 \,,$$

where prime denotes $d/dv$, so that Yang’s equations reduce to

$$\phi'' - \phi^2 + |\Psi|^2 = 0 \,,$$

$$\phi \Psi'' - 2\phi' \Psi' = 0 \,,$$

$$\phi \bar{\Psi}'' - 2\phi' \bar{\Psi}' = 0 \,.$$n

The integration of these equations is in the sequel.

II. COMPLETE SIMILARITY INTEGRAL

One notices first that Eqs. (6)–(8) are equivalent to

$$\Psi' = m_0 \phi^2 \,,$$

and

$$\phi'' - \phi^2 + |m_0|^2 \phi^4 = 0 \,,$$

where $m_0$ is a complex integration constant. Equation (10) is the typical Liouville equation for the function $\ln \phi$, which admits the solution

$$\phi = \frac{1}{\cosh(|m_0| \nu)} \,,$$

and, therefore,

$$\Psi = \frac{m_0}{|m_0|} \tanh(|m_0| \nu) + n_0 \,,$$

with $n_0$ another constant. We note that since a constant is harmonic, we have the freedom to omit an additive constant to the variable $v$.

The foregoing solution can be obtained alternatively by making use of the Hamilton-Jacobi (HJ) theory. The HJ functional is to be parametrized by $v$ and the HJ equation reads

$$\frac{\partial S}{\partial v} + H[\Psi, \phi, \frac{\partial S}{\partial \Psi}, \frac{\partial S}{\partial \phi}] = 0 \,,$$

where the Hamiltonian is defined by

$$H = \frac{1}{2} g^{-1/2} g'^{AB} \frac{\partial S}{\partial f^A} \frac{\partial S}{\partial f^B} \,,$$

and where $f^A$ are the coordinates of the $M'$ metric. Choosing $\Psi = \chi \rho^A$, the HJ equation becomes

$$\phi^{-2} \frac{\partial S}{\partial v} + \left( \frac{\partial S}{\partial \chi} \right)^2 + \left( \frac{\partial S}{\partial \phi} \right)^2 + \frac{1}{\chi} \left( \frac{\partial S}{\partial \lambda} \right)^2 = 0 \,.$$
whose separable solution can be expressed by
\[ S = -a_1 v + a_3 x + \int \left( a_2 - \frac{a_1^2}{x^2} \right)^{1/2} d\lambda + \int \left( \frac{a_1^2}{x^2} - a_2 \right)^{1/2} d\phi \ . \]

(16)

Here \( a_1, a_2, \) and \( a_3 \) are nontrivial constants and the self-dual similarity solution sought reduces then to the equations
\[ \frac{\partial S}{\partial a_1} = 0, \quad \frac{\partial S}{\partial a_2} = 0, \quad \frac{\partial S}{\partial a_3} = 0 . \]

(17)

Although this solution is a rather simple one, it has the feature that its independent variable occurs as an arbitrary harmonic function.

### III. AN ELLIPTICAL SOLUTION

We reparametrize the foregoing functions by
\[ \phi = y \cos \Omega, \]
\[ \Omega = y \sin \Omega e^{iA + iB} \quad (b \text{ is a real constant}), \]
and consider the harmonic map between
\[ M'': d\xi^2 = dv^2 + d\tilde{v}^2 \]
and
\[ M': d\xi^2 = |\Psi|^2 + d\phi^2 . \]

(19)

(20)

As is observed, we make \( M'' \) a two-dimensional manifold, where \( \tilde{v} \) is a new function whose Jacobian with \( v \) must not vanish everywhere. The functions \( \psi, \Psi, \) and \( \lambda \) are still only functions of \( v \). The energy functional constructed from \( M'' \) into \( M' \) will yield the Lagrangian
\[ L = \left( \frac{y'}{y \cos \Omega} \right)^2 + \left( \frac{\Omega'}{\cos \Omega} \right)^2 + \tan^2 \Omega (\lambda^2 + b^2) . \]

(21)

The Yang equations resulting from the variational principles admit the first integrals
\[ \tan^2 \Omega \lambda' = c_0, \]
\[ y' = a_0 y \cos^2 \Omega, \]
where \( c_0 \) and \( a_0 \) are both real integration constants. The equation for \( \Omega \) turns out to be nontrivial:
\[ \Omega'' - \sin \Omega \cos^2 \Omega a_0^2 + \tan \Omega \left( \Omega'' + c_0 \cot^2 \Omega - b^2 \right) = 0 . \]

(23)

Defining a new function by \( M = \arctanh (\sin \Omega) \), this equation is transformed into
\[ M'' - a_0^2 \sinh M \cosh M - c_0 \sinh^2 M - b^2 \sinh M \cosh M = 0 , \]
which is equivalent to the expression
\[ \int \frac{dR}{(b^2 R^2 + (b^2 + 1) R^2 + (1 - a_0^2 - c_0^2) R - c_0^2)^{1/2}} = 2v . \]

(24)

Note that we have redefined \( R = \sinh^2 M \) and \( l \) is a new constant of integration. It is known that for \( b \neq 0 \) this can be transformed into the standard elliptical forms by the proper choice of the constants \( A \) and \( B \) in the transformation \( R = (A + Bx)/(1 + x) \); however, we shall not pursue it further here. Assuming this has been carried out, the final solution is
\[ \lambda = \text{const} + c_0 \int \frac{dv}{R} \]
and
\[ y = \text{const} \exp \left( a_0 \int \frac{dv}{1 + R} \right) . \]

### IV. A TRANSCENDENT SOLUTION

As a final class of solutions we show that the self-dual Yang-Mills equations admit solutions expressible in terms of Painleve’s fifth transcendent. Although this class was discovered before, \( 7 \) we shall rederive it by an alternative method.

We choose the \( M'' \) manifold to be in one of the following forms:

(i) \( ds''^2 = e^{2x} dv^2 + d\tilde{v}^2 + e^{2x} d\phi^2 \),
(26)

(ii) \( ds''^2 = dv^2 + d\tilde{v}^2 - v + e^{2x} d\phi^2 \),
(27)

and \( \Psi \) is chosen as in Sec. III, \( \Psi = y \sin \Omega e^{i(A + iB)} \), where \( \beta \) is a real constant. The Lagrangian of the new map takes the form
\[ L = \left( \frac{y'}{y \cos \Omega} \right)^2 + \left( \frac{\Omega'}{\cos \Omega} \right)^2 + \tan^2 \Omega (\lambda^2 + \beta^2 e^{2x}) . \]
(28)

and the \( \Omega \) equation is modified to
\[ d^2 w/dv^2 - \left( \frac{dw}{dv} \right)^2 \left( \frac{1}{2w} + \frac{1}{w - 1} \right) - 2(1 - w)^2 \left( a_0^2 w + \frac{c_0^2}{w} \right) - 2\beta^2 e^{2x} w = 0 . \]
(29)

where \( w = \sin^2 \Omega \). Changing the independent variable by \( x = e^{2x} \), this equation becomes
\[ w' + \frac{1}{x} w - w^2 \left( \frac{1}{2w} + \frac{1}{w - 1} \right) \]
\[ - \frac{(1 - w)^2}{2x^2} \left( a_0^2 w + \frac{c_0^2}{w} \right) - \frac{\beta^2}{2x} w = 0 , \]
which is a particular Painleve’s fifth transcendent, whose general form is
\[ w' + \frac{1}{x} w - w^2 \left( \frac{1}{2w} + \frac{1}{w - 1} \right) \]
\[ + \frac{(1 - w)^2}{x^2} \left( a_0 w + \frac{c_0}{w} \right) + \frac{x}{w} + \delta w^2 + \frac{1}{w - 1} = 0 . \]
(30)

It is crucial that either \( \delta \neq 0 \) (as in Einstein-Maxwell theory) or \( \gamma \neq 0 \) (in this case) in order that the transcendental nature remains. Let us note however that we have yet to meet the case where both \( \gamma \neq 0 \) and \( \delta \neq 0 \). The final
solution is
\[
\lambda = \text{const} + \int (w^{-1} - 1) \, dv, \quad \phi = \text{const}(1 - w)^{1/2}\exp\left[ a_0 \int (1 - w) \, dv \right], \quad e^{-i\lambda}\Psi = \frac{w^{1/2}}{(1 - w)^{1/2}} \phi e^{i\theta_0},
\]
where \(v\) is harmonic and \(\tilde{v}\) is an arbitrary function. We must add, however, that once we want to recover axial symmetry in the problem, we are bound to make the choices \(v = \ln p\) and \(\tilde{v} = z\) for the base manifold (26), which will result in the particular solution already given in Ref. 7.

\begin{enumerate}
\item L. Witten, Phys. Rev. D 19, 718 (1979).
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