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Interacting Electromagnetic Shock Waves in General Relativity.

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Summary. — We study the interacting electromagnetic shock waves with nonconstant profiles in general relativity. It is shown that by modifying the metric functions of the Bell-Szekeres solution, such solutions can be obtained.

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1. – Introduction.

As a manifestation of the nonlinear feature of general relativity, two light waves scatter each other to develop a new region of space-time known as the interaction region. Although this problem, in analogy with photon-photon scattering of quantum electrodynamics is an important one, little has been achieved toward a complete understanding of it. The first and actually the only available solution so far was given by Bell and Szekeres (1) (henceforth, BS), describing the interaction of two constant-profile shock electromagnetic (e.m.) waves. A minor contribution to the BS solution was given later by showing that the number of incoming shock waves can be arbitrarily increased (2). In this case the interaction region emerges as a region of many BS cells whose exact number is determined by the number of incoming shocks. The main features of the BS solution however, such as gravitational impulse waves

(1) P. Bell and P. Szekeres: *Gen. Rel. Grav.*, **5**, 275 (1974).
(2) M. Gurses and M. Halilsoy: *Lett. Nuovo Cimento*, **34**, 588 (1982).

arising at the null boundaries, conformal properties and the removable singularities remain unchanged. The interaction region of the BS solution is isometric to the Bertotti-Robinson solution [3,4], which is known to be the unique conformally flat solution to Einstein-Maxwell (EM) equations for nonnull e.m. field.

In this paper we consider the interaction of nonconstant-profile e.m. shock waves and show that the solutions can be obtained by modifying the arguments of the metric functions in the BS solution. The conclusion is that locally we recover the BS line element with nonconstant e.m. field strength, whereas in the null co-ordinates the two solutions differ. We state our result depending on two functions restricted by two constraint conditions and the BS choice of null co-ordinates happens to be the simplest form satisfying those conditions.

2. – Interaction of electromagnetic waves.

The generic form of the line element describing the interaction region of the collinearly polarized [5] e.m. waves is given by [1]

$$\mathrm{d}s^2 = 2\exp[-M]\,\mathrm{d}u\,\mathrm{d}v - \exp[-U]\left(\exp[V]\,\mathrm{d}x^2 + \exp[-V]\,\mathrm{d}y^2\right), \tag{1}$$

where the metric functions depend on the null co-ordinates u and v. The incoming states (regions II and III) are characterized by nonflat metrics associated with incoming e.m. waves. Region I contains no e.m. waves and must naturally be flat. For the details of the space-time picture we refer to [5]. The Maxwell and Einstein-Maxwell (EM) field equations as derived in BS are given as follows (note that we adopt the same notations of BS):

$$2\varphi_{2,v} = U_v\varphi_2 - V_u\varphi_0, \tag{2}$$

$$2\varphi_{0,u} = U_u\varphi_0 - V_v\varphi_2, \tag{3}$$

$$\varphi_1 = 0, \tag{4}$$

$$U_{uv} = U_u U_v, \tag{5}$$

$$2U_{uu} - U_u^2 + 2U_u M_u = V_u^2 + 4k|\varphi_2|^2, \tag{6}$$

$$2U_{vv} - U_v^2 + 2U_v M_v = V_v^2 + 4k|\varphi_0|^2, \tag{7}$$

$$2M_{uv} + U_u U_v = V_u V_v, \tag{8}$$

$$2V_{uv} - U_u V_v - U_v V_u = 2k(\varphi_0\bar{\varphi}_2 + \bar{\varphi}_0\varphi_2) \tag{9}$$

$$\left(k = \text{const} = \frac{G}{8c^4}\right),$$

([3]) B. BERTOTTI: *Phys. Rev.*, **116**, 1331 (1959).

([4]) I. ROBINSON: *Bull. Acad. Pol. Sci., Ser. Sci. Math. Astron. Phys.*, **7**, 351 (1959).

([5]) P. SZEKERES: *J. Math. Phys.* (*N.Y.*), **13**, 286 (1972).

where the e.m. field amplitudes (*i.e.* the spinor components φ_0 and φ_2) are the « scale invariant » ones as defined in [5]. The solution to this set of equations as obtained by BS is

$$(10)\qquad \begin{cases} M=0\,, \quad \varphi_2=\bar{\varphi}_2=\dfrac{a}{\sqrt{k}}=\text{const}\,, \quad \varphi_0=\bar{\varphi}_0=\dfrac{b}{\sqrt{k}}=\text{const}\,, \\[2ex] \exp[-U]=\cos(au-bv)\cos(au+bv)\,, \quad \exp[V]=\dfrac{\cos(au-bv)}{\cos(au+bv)}\,. \end{cases}$$

The solution given in ref. [2], on the other hand, is obtained from BS by modifying the arguments of cosine terms, namely by making the substitutions

$$(11)\qquad \begin{cases} au\theta(u)\to\sum\limits_{i=1}^{m}a_i(u-u_i)\theta(u-u_i)\,, \\[2ex] bv\theta(v)\to\sum\limits_{i=1}^{m}b_i(v-v_i)\theta(v-v_i) \\[2ex] (a_i,\,b_i\ \text{constants, related to the fluxes of the shocks})\,. \end{cases}$$

The theta functions in the arguments guarantee the consistent matching of the disjoint space-time regions, albeit it restricts the metric to be of class C^0 and piecewise C^1. The fact that this substitution yields a nontrivial result is seen by computing the components of the Riemann tensor.

3. – Interacting electromagnetic waves with nonconstant field strength.

We provide now solutions to the above set of equations (2)-(9) for the case when φ_0 and φ_2 are not constants (with still $\varphi_1=0$). For this purpose we substitute $u\to f(u)$ and $v\to g(v)$, where f and g are functions to be determined below. The ansatz solution seeked is expressed in the Rosen form by

$$(12)\qquad \mathrm{d}s^2=2f'g'\,\mathrm{d}u\,\mathrm{d}v-\cos^2[af(u)\theta(u)-bg(v)\theta(v)[\,\mathrm{d}x^2-\\ -\cos^2[af(u)\theta(u)+bg(v)\theta(v)[\,\mathrm{d}y^2\,,$$

where $f'=\mathrm{d}f/\mathrm{d}u$, $g'=\mathrm{d}g/\mathrm{d}v$ and the e.m. spinor components are chosen as

$$(13)\qquad \varphi_2=\frac{a}{\sqrt{k}}f'\theta(u)\,,\qquad \varphi_0=\frac{b}{\sqrt{k}}g'\theta(v)\,.$$

The functions f and g are not arbitrary, but solutions of the EM equations consistently yield the following conditions:

$$f(u)\,\delta(u) = 0\,, \tag{14}$$

$$\left(\frac{\mathrm{d}f}{\mathrm{d}u}\right)\delta(u) \neq 0 \tag{15}$$

and similar conditions for the function g. It is readily observed that the simplest possible $f(u)$ satisfying these constraints is the one corresponding to the BS solution, namely, $f(u) = u$ (and $g(v) = v$). Other interesting values which satisfy the above constraints without giving rise to degeneracy in the space-time metric are the following:

$$\begin{cases} f(u) = \left\{\sin u,\ \sinh u,\ \mathrm{tgh}\, u,\ u \exp[u],\ u\cos u,\ \dfrac{u}{\cos u}, \dots\right\}, \\[2mm] g(v) = \left\{\sin v,\ \sinh v,\ \mathrm{tgh}\, v,\ v \exp[v],\ v\cos v,\ \dfrac{v}{\cos v}, \dots\right\}. \end{cases} \tag{16}$$

It is observed that the nonconstancy of e.m. spinor components serve to generate the metric function M, which vanishes in the case of BS. In the null tetrad

$$\begin{cases} l_\mu = \exp[-M/2]\,\delta^u_\mu\,, \qquad n_\mu = \exp[-M/2]\,\delta^v_\mu\,, \\ -\sqrt{2}\, m_\mu = \exp[-U/2](\exp[V/2]\,\delta^x_\mu + i \exp[V/2]\,\delta^y_\mu)\,, \end{cases} \tag{17}$$

the scale-invariant, nonvanishing Weyl components are given by

$$\begin{cases} \Psi_4 = -a\left(\dfrac{\mathrm{d}f}{\mathrm{d}u}\right)\delta(u)\,\mathrm{tg}\, b\, g(v)\theta(v)\,, \\[2mm] \Psi_0 = -b\left(\dfrac{\mathrm{d}g}{\mathrm{d}v}\right)\delta(v)\,\mathrm{tg}\, a\, f(u)\theta(u)\,. \end{cases} \tag{18}$$

Let us note that the constraint (15) is just the requirement to provide a nonflat metric.

In order to identify the incoming waves in the Brinkmann ([6]) co-ordinate system, which is a harmonic system, we express our solution in the BS form

$$\mathrm{d}s^2 = 2\,\mathrm{d}f\,\mathrm{d}g - \cos^2(af - bg)\,\mathrm{d}x^2 - \cos^2(af + bg)\,\mathrm{d}y^2\,, \tag{19}$$

that reduces in the region II ($g = 0$) to

$$\mathrm{d}s^2 = 2\,\mathrm{d}f\,\mathrm{d}g - \cos^2 af(\mathrm{d}x^2 + \mathrm{d}y^2)\,. \tag{20}$$

([6]) H. W. BRINKMANN: *Proc. Natl. Acad. Sci. U.S.A.*, **9**, 1 (1923).

This line element is obtained from the Brinkmann metric

$$ds^2 = 2\,dU\,dV - dX^2 - dY^2 - \theta(U)(X^2 + Y^2)\,dU^2 \tag{21}$$

by the following co-ordinate transformation (which is equivalent to a null rotation):

$$\begin{cases} U = f\,, \quad X = xF\,, \quad Y = yF\,, \\ V = g + \frac{1}{2}(x^2 + y^2)FF_{,f}\,, \end{cases} \tag{22}$$

where $F = \cos af$. It is observed that f and g correspond to the null co-ordinates of regions II and III, respectively, when the waves are expressed in harmonic co-ordinates.

4. – Electromagnetic-potential approach.

The basic EM equations (5), (8) and (9) are obtained by the variational principle of the Lagrangian

$$L = \exp[-U](M_u U_v + M_v U_u + U_u U_v - V_u V_v) - 2k\exp[-V]A_u A_v\,, \tag{23}$$

where the e.m. 4-potential is given by $A_\mu = A\delta^x_\mu$. The e.m. spinor components are defined by

$$\begin{cases} \varphi_2 = \dfrac{1}{\sqrt{2}} A_u \exp\left[\dfrac{U-V}{2}\right], \\ \varphi_0 = -\dfrac{1}{\sqrt{2}} A_v \exp\left[\dfrac{U-V}{2}\right]. \end{cases} \tag{24}$$

The variational equation, $\delta L/\delta A = 0$, yields the equation satisfied by the potential function A, which is equivalent to the Maxwell equations given in eqs. (2) and (3). This is given by

$$(\exp[-V]A_u)_v + (\exp[-V]A_v)_u = 0\,, \tag{25}$$

which admits the solution

$$A = \sqrt{\frac{2}{k}}\sin\left(af\theta(u) - bg\theta(v)\right). \tag{26}$$

We introduce now the following co-ordinates:

$$\tau = \sin(af - bg)\,, \qquad \sigma = \sin(af + bg)\,, \tag{27}$$

so that the space-time describing interacting e.m. waves is expressed by

$$\mathrm{d}s^2 = \frac{1}{2ab}\left(\frac{\mathrm{d}\tau^2}{1-\tau^2} - \frac{\mathrm{d}\sigma^2}{1-\sigma^2}\right) - (1-\tau^2)\,\mathrm{d}x^2 - (1-\sigma^2)\,\mathrm{d}y^2 \tag{28}$$

and the e.m. potential becomes $A = \sqrt{(2/k)}\,\sigma$.

Another useful co-ordinate system is provided by the choice

$$\xi = af + bg\,, \qquad \eta = af - bg\,. \tag{29}$$

This co-ordinate system proves to be useful in studying geodesics motion and the Dirac equation in the interaction region (7). The space-time metric reads (we scaled x and y for obvious reason in the sequel)

$$\mathrm{d}s^2 = \frac{1}{2ab}\,[\mathrm{d}\eta^2 - \mathrm{d}\xi^2 - \cos^2\eta\,\mathrm{d}y'^2 - \cos^2\xi\,\mathrm{d}x'^2]\,. \tag{30}$$

This is easily transformed into Bertotti-Robinson solution by the following transformations:

$$\sin\eta = \frac{t}{r}\,, \quad x' = \varphi\,, \quad 2y' = \ln(r^2 - t^2)\,, \quad \xi = \frac{\pi}{2} - \theta\,, \quad \frac{1}{2ab} = e^2\,. \tag{31}$$

5. – Discussion.

The proper solution for the problems of interacting (colliding) waves in general relativity should go from region I (flat space), through regions II and III (incoming regions) into region IV (the interaction region). The reverse order, namely from region IV to regions II and III, although happens to be the simpler route, results mostly in nonphysical incoming states. The choice of realistic wave forms yield unfortunately set of coupled systems of partial differential equations whose exact solutions become almost impossible.

For e.m. case, we have shown that changing the incoming waves only serves to modify the metric functions in the BS metric. No matter how the profile of the incoming waves is chosen from a set that satisfies certain constraints, the e.m. waves continue into the interaction region unchanged. This may be a general feature of interacting e.m. waves, and for this reason we prefer to name the problem interaction, rather than collision. Locally, in the co-ordinates (τ, σ) or (ξ, η) all solutions are expressed in BS form, but in the null co-ordinates the details of incoming e.m. waves modify the arguments.

(7) M. Halil: *Int. J. Theor. Phys.*, **20**, 911 (1981).

As a final remark, we would like to point out an interesting property of the Riemann tensor components for the interacting e.m. waves. The periodic nature of Ψ_0 and Ψ_4 in eq. (18) remids a sequence of dark (bright) amplitudes in analogy with the interacting beams in a double slit experiment. Within coming decades experimental general relativity may develop enough to check this aspect empirically.