AIP | Journal of Mathematical Physics

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Citation: Journal of Mathematical Physics **29**, 320 (1988); doi: 10.1063/1.528070
View online: http://dx.doi.org/10.1063/1.528070
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/29/2?ver=pdfcov
Published by the AIP Publishing

Studies in space-times admitting two spacelike Killing vectors

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(Received 25 June 1986; accepted for publication 30 September 1987)

Some properties of the space-times admitting two spacelike Killing vectors are studied. In particular, using harmonic maps the degree of freedom on the M' manifold is exploited to add scalar and electromagnetic fields to Bonnor's nonsingular solution. It is also shown that for vacuum space-times the noncommutativity of two spacelike Killing vectors is incompatible with the self-similarity requirement and such a self-similar vacuum space-time has no Taub–NUT equivalent extension.

I. INTRODUCTION

Static axially symmetric fields with two commuting Killing vectors were considered first by Weyl[1] who, under the assumed symmetry, presented a general solution to the Einstein field equations. Einstein and Rosen[2] studied later the intrinsically similar fields with two commuting spacelike Killing vectors which amounted to the cylindrically symmetrical gravitational waves. The imploding–exploding waves interpretation, as an example of scattering of such cylindrical waves, was given by Marder.[3] The same metric was considered independently both by Weber and Wheeler[4] and by Bonnor.[5] In particular, Bonnor gave an example of a nonsingular solution to the field equations that might be interpreted as a cosmological model of interest. The term "nonsingular," however, must be taken cautiously since, as explained in detail by Bonnor and by Weber and Wheeler, the fact that the metric tensor should behave like $1/r$ and the Riemann tensor like $1/r^2$ (as the requirements of asymptotic flatness) is not satisfied in such a solution. The well-known result that no nonsingular colliding plane wave (CGW) space-time exists[6] could be anticipated from the above case due to the inherent similarily between planar and cylindrical geometries. In other words both of these geometries are represented by the same metrics in a particular choice of coordinates but the boundary conditions differ and therefore the difference is a global one.

In this paper we make use of the same cylindrically symmetric metric to generate radiation sources, such as electromagnetic and massless scalar fields. The method we adopt in the solution generating technique was presented briefly earlier[7] and for the sake of completeness we shall review it here. In this new approach of harmonic maps we reduce the general relativistic problem to the one of classical field theory and it is our belief that this method adds considerable elegance and simplicity when compared to the other existing methods. An effective Lagrangian is introduced via the harmonic maps between the suitably chosen Riemannian manifolds. For a general review of the physics of harmonic maps we refer to the paper of Misner,[8] whereas for the mathematical aspects the paper of Eells and Sampson[9] provides the proper references to be consulted. To a certain extent we shall make use of the cylindrical wave line element with single polarization due to Einstein and Rosen,

$$ds^2 = e^{2(\gamma-\Psi)}(dt^2 - d\rho^2) - \rho^2 e^{-2\Psi}\, d\phi^2 - e^{2\Psi}\, dz^2, \tag{1}$$

where the metric functions Ψ and γ are only functions of t and ρ. The vacuum field equations, for later reference, are given by

$$\Psi_{\rho\rho} + (1/\rho)\Psi_\rho - \Psi_{tt} = 0, \tag{2}$$

$$\gamma_{\rho\rho} - \gamma_{tt} = \Psi_t^2 - \Psi_\rho^2, \tag{3}$$

whereas the integrability conditions are

$$\gamma_\rho = \rho(\Psi_\rho^2 + \Psi_t^2), \tag{4}$$

$$\gamma_t = 2\rho\Psi_\rho\Psi_t. \tag{5}$$

Solution of this set of equations is usually carried out by solving (2) first. This is the cylindrical wave equation that admits wave solutions. Following Bonnor, we solve (2) by the method of complex translation discovered first by Appell in 1887. Having known that $\Psi = (\rho^2 - t^2)^{-1/2}$ forms a solution, then Appell's theorem states that the real part of the complex function, $\Psi = [\rho^2 - (t - ic)^2]^{-1/2}$ ($c = \text{const}$), is also a solution to (2). Bonnor's final solution is expressed by

$$\Psi = [u + (u^2 + w^2)^{1/2}]^{1/2}[u^2 + w^2]^{-1/2},$$
$$\gamma = \frac{-\rho^2(u^2 - w^2)}{2(u^2 + w^2)^2} + \frac{1}{4c^2}\left[\frac{u}{(u^2 + w^2)^{1/2}} + 1\right], \tag{6}$$

where

$$u = \rho^2 - t^2 + c^2, \quad w = 2ct.$$

The regularity of this solution should be understood in the sense that no metric function or scalars from the Riemann tensor diverge for $\rho \to \infty$ and $t \to \pm\infty$, where the ranges are $0 \leqslant \rho < \infty$, $-\infty < t < +\infty$.

II. METHOD FOR GENERATING NEW SOLUTIONS

It can easily be verified that Eqs. (2) and (3) follow from the variational principle of the action

$$I[\Psi,\gamma,\lambda] = \int [\gamma_\rho\lambda_\rho - \gamma_t\lambda_t - \lambda(\Psi_\rho^2 - \Psi_t^2)]\, d\rho\, dt, \tag{7}$$

where $\lambda = \rho$ is to be imposed as a coordinate condition subsequent to the variation. We recall from the theory of harmonic mappings of Riemannian manifolds that this action is in the form of an energy functional

$$E[f] = \int g'_{AB} \frac{\partial f^A}{\partial x^b} \frac{\partial f^B}{\partial x^b} g^{ab} |g|^{1/2}\, d^2x, \tag{8}$$

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where the manifolds M and M' are chosen, respectively, by

$$M:\ ds^2 = d\rho^2 - dt^2 + \lambda^2\, d\phi^2 = g_{ab}\, dx^a\, dx^b \quad (a,b\text{: } 1,2,3), \tag{9}$$

$$M':\ ds'^2 = (d\lambda/\lambda)d\gamma - d\Psi^2 = g'_{AB}\, dx'^A\, dx'^B \quad (A,B\text{: } 1,2,3). \tag{10}$$

Note that since $\lambda = \rho$ is a coordinate condition and ϕ is a cyclic variable, the effective dimensions of both M and M' are two. The map is represented by $f^A = \{\Psi,\gamma\}: M \to M'$, and the stationary conditions

$$\delta E\,[f^A] = 0 \tag{11}$$

are the statements that the maps are harmonic. As a side remark we note that in contrast to various maps in mathematics, harmonic maps explicitly can not be known *a priori* until Eqs. (11) are solved explicitly. Naturally all the information expected from a standard variational principle can also be extracted from the harmonic map action as well. For instance, besides the stationary requirement one may check the stability of a general relativistic manifold by studying the second variation due to Jacobi. In this paper we shall restrict ourselves only to the first variations. The theory of harmonic maps in general relativity can be summarized in the following: choose two Riemannian manifolds (9) and (10) in such a way that when the energy functional (8) is constructed from them, its stationary requirements (11) coincide exactly with the vacuum Einstein equations under consideration.[10]

Now, if f^A is a solution of the field equations (11), then a new solution f^K is obtained as a function of f^A from the isometry (invariance) of the line element of the M' manifold. This amounts to

$$ds'^2 = g'_{AB}(f)df^A\, df^B = \tilde{g}'_{KL}(\tilde{f})d\tilde{f}^K\, d\tilde{f}^L, \tag{12}$$

which yields the implicit relations

$$g'_{AB}(f)\frac{\partial f^A}{\partial \tilde{f}^K}\frac{\partial f^B}{\partial \tilde{f}^L} = \tilde{g}'_{KL}(\tilde{f}). \tag{13}$$

We note that such an isometry does not necessarily imply that the metric tensor $\tilde{g}'_{KL}$ has the same dimensions as that of $\tilde{g}'_{AB}$. In particular, we shall consider the case where the ranges of the indices K,L are larger than A,B and we shall interpret this as a problem of embedding. The mathematical details of embeddings are not our purpose here. We would like rather to make use of these concepts in order to yield tangible results that may prove useful in physics, and particularly in general relativity. We face embeddings in particular when we want to generate solutions with radiation sources from known solutions of vacuum. The method of isometry applies best to the two-dimensional problems and the reason for this may be connected with the existence of conformal techniques and analyticity in this particular dimension.

In order to obtain a new vacuum solution from a known solution we have to find Killing vectors of the corresponding M' manifold at hand. However, not every Killing vector yields a significantly new solution other than the original one. It is instructive at this point to mention a particularly well-known example. Stationary symmetrical gravitational fields (SAS) can be handled as a reduced formalism due to Ernst[11] or equivalently by the method due to Matzner and Misner.[12] In the latter method reduced Einstein equations are obtained from the variational principle of the map, $f^A = \{X,Y\}: M \to M'$, where

$$M: ds^2 = d\rho^2 + dz^2 + \rho^2\, d\phi^2, \tag{14}$$

$$M': ds'^2 = (dX^2 + dY^2)/X^2. \tag{15}$$

The Lagrangian density and the corresponding three Killing vectors of this M' are given by

$$L = \rho[(\nabla X)^2 + (\nabla Y)^2]/X^2, \tag{16}$$

and

$$\begin{aligned} \xi_1 &= \partial_Y, \\ \xi_2 &= 2XY\,\partial_X + (Y^2 - X^2)\partial_Y, \\ \xi_3 &= X\,\partial_X + Y\,\partial_Y, \end{aligned} \tag{17}$$

respectively. The new solution that is generated from the isometry can be expressed by

$$\frac{df^A}{dt} = \alpha\xi_i f^A \quad (i = 1,2,3), \quad f^A = \{X,Y\} \quad (0 \leqslant t \leqslant 1), \tag{18}$$

where α is a constant, and t is a continuous parameter. For $t = 0$ we recover our old solution f^A, whereas for $t = 1$, the new solution $\tilde{f}^A$ is obtained. It turns out in this example that ξ_3 leads to a scale factor and therefore the isometry it generates results in the identity of two solutions (old and new). Similarly, ξ_1 also is not very interesting, but the vector ξ_2 results in a significant, new solution. Linear combinations of Killing vectors may lead to interesting results so that such cases should be investigated as well. The three Killing vectors in the example above in fact arise from the invariance under fractional linear transformations with three parameters when the harmonic map Lagrangian is expressed in Ernst's complex formulation. We have already stated that the two methods are equivalent.

The proposed method of generating new solutions can naturally be extended to cover the cases of different dimensional isometric transformations. We can imbed an M' manifold into a new manifold of higher dimensionality such that the new dimensions can be interpreted as the energy-momentum tensor due to some radiation sources. The idea of imbedding the configuration space into a larger dimension does not emerge here as a novel one, since the same procedure had been employed in particle physics long ago.[13] To conclude this section we would also like to add that the transitive property of isometric transformations provides us with additional means to find possible isometric solutions. Only when isometry is applied to a unique solution in a particular class of solutions (such as Schwarzschild, Kerr) does it fail to yield anything new.

III. RADIATION SOURCES WITHOUT SOURCES

In this section we shall exploit the degrees of freedom on the M' manifold to generate electromagnetic and massless scalar radiations as the source to a modified gravitational background. Let $\Gamma^\mu_{\nu\rho}$ be the Christoffel symbol of a pure gravitational space-time, so that the geodesic equation is given by

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0. \qquad (19)$$

When an Einstein–Maxwell (EM) solution is generated from a vacuum solution the new geodesic equation should read

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma'^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = \frac{q}{m}F^\mu_\alpha u^\alpha, \qquad (20)$$

where F^μ_α stands for the "induced" (as transmuted) electromagnetic (e.m.) field and Γ' is the new Christoffel symbol. The two geodesic equations should coincide, in reality, since the space-time has undergone a dual interpretation under which the actual physics of the overall process must remain invariant. Each of our results can be stated as a theorem.

Theorem 1: Given that the vacuum Einstein equations can be represented by the harmonic map between the metrics

$$ds^2 = d\rho^2 - dt^2 + \lambda^2\, d\phi^2, \qquad (21)$$

$$ds'^2 = (1/\lambda)d\lambda\, d\gamma - d\Psi^2, \qquad (22)$$

then a system of Einstein–Maxwell (EM) coupled equations can be represented by the harmonic map between

$$ds^2 = d\rho^2 - dt^2 + \lambda^2\, d\phi^2, \qquad (23)$$

and

$$ds'^2 = (1/\lambda)d\lambda\, d\gamma - (d\mu^2 + e^{-\mu}\, dA^2). \qquad (24)$$

The space-time metric reads

$$ds^2 = e^{2\gamma-\mu}(dt^2 - d\rho^2) - \rho^2e^{-\mu}\, d\phi^2 - e^\mu\, dz^2 \qquad (25)$$

and the isometry of the line elements (22) and (24) yields the constraint condition

$$d\mu^2 + e^{-\mu}\, dA^2 = 4\, d\Psi^2. \qquad (26)$$

(Note that we have introduced the factor 4 by scaling the functions μ and A by by $\frac{1}{2}$.)

Proof: The Lagrangian density obtained from the map between the metrics in (21) and (22) is given by

$$L_1 = (\lambda_\rho\gamma_\rho - \lambda_t\gamma_t) - \lambda(\Psi_\rho^2 - \Psi_t^2) \qquad (27)$$

which yields the vacuum equations (2) and (3). The Lagrangian density obtained from the map between (23) and (24) is given by

$$L_2 = (\lambda_\rho\gamma_\rho - \lambda_t\gamma_t) - \lambda\,[\mu_\rho^2 - \mu_t^2 + e^{-\mu}(A_\rho^2 - A_t^2)]. \qquad (28)$$

The Euler–Lagrange (EL) equation, $\delta L_2/\delta\gamma = 0$, holds true by virtue of the choice $\lambda = \rho$. Next is the equation $\delta L_2/\delta A = 0$, being equivalent to

$$(\rho e^{-\mu}A_t)_t - (\rho e^{-\mu}A_\rho)_\rho = 0, \qquad (29)$$

which stands for the only nontrivial Maxwell equation. To verify this, define the e.m. four-potential

$$A_\mu = (0,0,0,A), \qquad (30)$$

so that $F_{tz} = A_t$ and $F_{\rho z} = A_\rho$ are the nonvanishing components of the e.m. field tensor. It can be checked that the source-free Maxwell equation

$$\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0 \qquad (31)$$

coincides with (29), where the metric is (25).

Finally, the remaining EL equation, $\delta L_2/\delta\mu = 0$, turns out to be identical with the EM equation,

$$\mu_{\rho\rho} - \mu_{tt} + (1/\rho) = \tfrac{1}{2}(A_t^2 - A_\rho^2)e^{-\mu}. \qquad (32)$$

This completes the proof that if L_1 describes a system of vacuum equations, then L_2 describes an EM system.

In order to see the significance of the foregoing theorem we generate some new EM solutions from the vacuum ones. To this end we solve first the constraint condition (26). It turns out that this equation, similar to taking the roots of unity, possesses a large class of solutions where μ and A are expressed as functions of Ψ. A particular integral of the constraint equation,

$$e^\mu = (\alpha + \tfrac{1}{2}\beta^2)\operatorname{sech}^2\Psi, \qquad A = 2(\alpha + \tfrac{1}{2}\beta^2)^{1/2}\tanh\Psi, \qquad (33)$$

where α,β are nonzero constants, was reported a long time ago by Misra.[14] In addition to this solution we present two more classes of solutions as follows.

(i) Let $A = 2b_0e^{\mu/2}$, where $b_0 = \text{const}$. The integration of the constraint equation yields

$$\mu = \pm 2(1 + b_0^2)^{-)1/2}\Psi.$$

In contrast to Misra's solution (33), this new solution has the feature that it has vacuum Einstein as a limit.

(ii) Letting $A = k\mu$ ($k = \text{const}$), we obtain from the constraint equation the transcendental relation

$$\frac{(1+k^2e^{-\mu})^{1/2}+1}{(1+k^2e^{-\mu})^{1/2}-1} = \exp[\pm 2\Psi + 2(1+k^2e^{-\mu})^{1/2}]. \qquad (34)$$

Being transcendental, this expression cannot be inverted for μ analytically in terms of Ψ. In fact, the constraint condition (26) possesses a large class of solutions sharing this transcendental nature.

Theorem 2: Given that the harmonic map between the manifolds (21) and (22) yields vacuum equations, then Einstein-massless scalar field equations can be generated from the map between the metrics

$$ds^2 = d\rho^2 - dt^2 + \lambda^2\, d\phi^2, \qquad (35)$$

$$ds'^2 = (1/\lambda)d\lambda\, d\gamma - (d\mu^2 + k\, d\phi^2), \qquad (36)$$

where k is the coupling constant.

Proof: The Lagrangian density for this map is given by

$$L_3 = (\lambda_\rho\gamma_\rho - \lambda_t\gamma_t) - \lambda\,[\mu_\rho^2 - \mu_t^2 + k(\phi_\rho^2 - \phi_t^2)]. \qquad (37)$$

We must show now that L_3 describes an Einstein-scalar system whereas the space-time metric is still (25). The constraint relation is expressed now by

$$d\mu^2 + kd\phi^2 = 4\, d\Psi^2. \qquad (38)$$

(Note here also that 4 is a scale factor.)

EL equations for the scalar field ϕ are given by

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\phi_\nu) = 0 \qquad (39)$$

or equivalently

$$(\rho\phi_t)_t - (\rho\phi_\rho)_\rho = 0. \qquad (40)$$

Einstein-scalar equations are obtained by the conditions $\delta L_3/\delta\mu = 0$ and $\delta L_3/\delta\lambda = 0$, and therefore L_3 forms a Lagrangian for an Einstein-scalar system.

By making use of the constraint condition it is not hard to obtain Einstein-scalar solutions. Choosing $\mu = 2\Psi \cos a_0$, $k^{1/2}\phi = 2\Psi \sin a_0$ ($a_0 = \text{const}$), by virtue of the vacuum Eqs. (2), (3), Einstein scalar equations are satisfied. The constant a_0 here plays the role of a phase constant which removes the scalar field for $a_0 = 0$. In the case of static spherically symmetric scalar fields, the corresponding solution obtained by similar means employed here is the Newman–Janis–Winicour (NJW) solution.[15] In a routine manner the uniqueness argument of NJW can be extended to the scalar solution obtained here in the cylindrically symmetric geometry.

Theorem 3: The two foregoing theorems (1) and (2) can be combined to yield a Lagrangian for the EM-scalar field system. The harmonic map will now be between the manifolds,

$$ds^2 = d\rho^2 - dt^2 + \lambda^2 d\phi^2, \tag{41}$$

$$ds'^2 = (1/\lambda)d\lambda\, d\gamma - [d\mu^2 + e^{-\mu} dA^2 + k\, d\phi^2]. \tag{42}$$

Proof: The effective Lagrangian density of the map between the given manifolds (41) and (42) will be

$$L_4 = \lambda_\rho \gamma_\rho - \lambda_t \gamma_t - \lambda\, [\mu_\rho^2 - \mu_t^2 + e^{-\mu}(A_\rho^2 - A_t^2) + k(\phi_\rho^2 - \phi_t^2)] \tag{43}$$

and the constraint condition will be given by

$$d\mu^2 + e^{-\mu} dA^2 + k\, d\phi^2 = 4\, d\Psi^2. \tag{44}$$

EL equations for L_4 with respect to each function will yield all EM-scalar field equations. The proof follows therefore from the foregoing theorems.

The following solution, for example, solves the constraint condition (44) and therefore constitutes also a solution for the EM-scalar system,

$$\begin{aligned} &\mu = 2\Psi \cos b_0, \\ &A = 4 \exp(\Psi \cos b_0)\cdot \cos c_0 \tan b_0, \\ &k^{1/2}\phi = 2\Psi \sin b_0 \sin c_0 \\ &\qquad (b_0, c_0\text{: constants}). \end{aligned} \tag{45}$$

One observes simply that $c_0 = 0$ implies that only the e.m. field exists and $b_0 = 0$ leaves only the scalar field. Vacuum is recovered for $b_0 = 0 = c_0$.

Finally we would like to note that the e.m. field adopted in the foregoing solutions was of the form $A_\mu = \delta_A^x A = (0,0,0,A)$. This may be extended to the case with two nonvanishing components, given as $A_\mu = \delta_\mu^x A + \delta_\mu^y B = (0,0,B,A)$. By this latter choice, however, the constraint condition to be solved becomes

$$d\mu^2 + e^{-\mu} dA^2 + (1/\rho)e^{\mu} dB^2 = 4\, d\Psi^2, \tag{46}$$

whose particular integrals are rather involved compared with the former case where $B = 0$.

IV. TWO REMARKS ON BONNOR'S SOLUTION

(1) In this section we derive an equation for the timelike geodesics where the space-time element is being projected onto the (ρ,t) plane. In other words we simplify the general geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{47}$$

for the particular case of $\phi = z = 0$, and where the cylindrical radius is to be parametrized by t. For this purpose we choose the following variational principle to yield directly the projected geodesic equation:

$$I = \int ds = \int e^Z (1 - \dot\rho^2)^{1/2}\, dt, \tag{48}$$

where $\dot\rho = d\rho/dt$, and $Z = \gamma - \Psi$. As it is already implied by this reduced action principle we can study the cases for $\dot\rho < 1$, i.e., the timelike geodesics. The resulting equation for geodesics is obtained as

$$\ddot\rho = (\dot\rho^2 - 1)\Big(\dot\rho \frac{\partial Z}{\partial t} + \frac{\partial Z}{\partial \rho}\Big). \tag{49}$$

Unfortunately, the relative simplicity of this equation does not help in the search for an analytic solution for ρ as a function of time. The difficulty originates from the rather complicated form of $Z \equiv \gamma - \Psi$, in Bonnor's solution. A numerical solution, however, can be achieved by assigning values for $\dot\rho$ in the interval $0 < \dot\rho < 1$ and plotting the resulting ρ for arbitrary values of the running time. In this way we find the trajectory of a particle in the nonsingular cosmological model given by Bonnor.

(2) Our second remark concerns the physical meaning of the nonzero constant c in Bonnor's solution (6). (Note that we have fixed the other constant b that appears in the original solution[5] by $b = 1$.) We want to explain that this constant c (and b) is not connected with the topology of the cosmological model. The degree of harmonic maps for the case of S^2 into S^2, as had been shown by Eells and Sampson, turns out to be finite and gives the number of windings that the base manifold is being wrapped. The energy of the map also emerges as proportional to the same topological integer. The integer property of the map arises from the uniqueness requirements of the rotational components of the map. All such nice topological features, however, can hardly find room in general relativity. The reason can be attributed to the noncompact, hyperbolic nature of Riemannian manifolds. To see the inherent difference between the compact and noncompact manifolds, from the physics point of interest, we refer to the analysis of Hirayama *et al.*[16] In this reference it is explained that for Heisenberg's ferromagnet the number of slips of the spin vector equals the degree of the harmonic map. The same analysis, on the other hand, when applied to the Weyl (or TS) class of gravitational fields, yields a divergent result. Having learned also from the two-dimensional field theories[17] that the topological class does not change in the course of time, we can handle Bonnor's cosmological model as a one-dimensional field theory on a flat background. An index can be defined for Bonnor's Ψ field by an expression proportional to $\int_0^\infty \rho \Psi_\rho\, d\rho = 1 \sinh^{-1}(\infty)$, which diverges unless an infinite factor is subtracted.

In the Weyl case, the scalar field propagating on flat space is given by $\Psi = \tanh^{-1}\xi = \delta \tanh^{-1} x$. Here, ξ is the real version of the Ernst potential, x is one of the prolate spheroidal coordinates ($1 < x < \infty$), and δ is the Weyl parameter. A topological index could be defined from Ψ, provided $\Psi(\infty) - \Psi(1)$ is a finite number. It turns out that before one accepts δ as the topological degree one has to divide (or subtract) by an infinite factor, since $\Psi(1)$ diverges.

Comparing the two cases it seems that Bonnor's solution is the first member of a larger family, yet to be discovered and the corresponding parameter of Weyl's δ will characterize the topological class, albeit in some ambiguous way.

V. THERE IS NO HYPERSURFACE NONORTHOGONAL SELF-SIMILAR COSMOLOGICAL VACUUM MODEL

The general space-time geometry that describes cylindrical gravitational waves with the cross polarization term is given by[18]

$$ds^2 = e^{2(\gamma - \Psi)}(dt^2 - d\rho^2) - e^{2\Psi}(dz + w\, d\phi)^2 - \rho^2 e^{-2\Psi}\, d\phi^2, \tag{50}$$

which is considered as a generalization of the Einstein–Rosen metric. From the inherent identity between cylindrical and planar geometries this metric can be transformed into the metric that describes colliding plane gravitational waves. This latter metric due to Szekeres[19] is given by

$$ds^2 = 2e^{-M}\, du\, dv - e^{-U}[e^V \cosh W\, dx^2 + e^{-V}\cosh W\, dy^2 - 2\sinh W\, dx\, dy]. \tag{51}$$

(1) It is our purpose to show now that this metric admits no self-similar solutions, simply because whenever it does, it turns out to be diagonalized. By the self-similar solution, here we imply that all metric functions depend functionally on a single harmonic function $\sigma = e^{-U}$, where $\sigma_{uv} = 0$, or in the case of the metric (50), σ satisfies $\sigma_{\rho\rho} + (1/\rho)\sigma_\rho - \sigma_{tt} = 0$. Let us note that although the choice of harmonic variables is not an imperative one, the structure of Einstein equations suggests that such a choice facilitates the formalism to a great extent.[20]

The self-similar vacuum Einstein equations are obtained from the harmonic map between the manifolds

$$M\colon\ ds^2 = d\sigma^2, \tag{52}$$

$$M'\colon\ ds'^2 = dW^2 + \cosh^2 W\, dV^2. \tag{53}$$

The metric function M, which does not appear in the map, turns out to satisfy a quadrature equation that, as a requirement of complete integrability, must admit a solution. The self-similar Lagrangian and equations are given in the following:

$$L = W'^2 + \cosh W \cdot V'^2, \tag{54}$$

$$V' \cosh^2 W = a_0 = \text{const}, \tag{55}$$

$$W'' = a_0^2 (\sinh W)/(\cosh^3 W) \tag{56}$$

$$\left(' \equiv \frac{d}{d\sigma}\right).$$

Solutions for V and W take the form

$$e^{2V} = \frac{b_0 + a_0 \tanh b_0\sigma}{b_0 - a_0 \tanh b_0 \sigma}, \tag{57}$$

$$\sinh W = \left[1 - \left(\frac{a_0}{b_0}\right)^2\right]^{1/2} \sinh b_0\sigma \tag{58}$$

$$(b_0 = \text{const}).$$

However, it can be observed by the coordinate transformation

$$x = \bar{x}\cos(\alpha/2)(+ \bar{y}\sin(\alpha/2), \quad y = \bar{x}\sin(\alpha/2) + \bar{y}\cos(\alpha/2) \tag{59}$$

that the metric function W can be set to zero. The choice of α that accomplishes this task is

$$\alpha = \tan^{-1}[(b_0/a_0)^2 - 1]. \tag{60}$$

(2) As the second point we would like to check whether the space-time metric with two spacelike Killing vectors admits a Taub–NUT-like solution. To this end we consider the Ernst equation in the coordinates[21,22]

$$\tau = u(1-v^2)^{1/2} + v(1-u^2)^{1/2}, \quad \sigma = u(1-v^2)^{1/2} - v(1-u^2)^{1/2}. \tag{61}$$

The simplest Ernst potential $\xi = \tau$ turns out to be the Khan–Penrose (KP)[23] solution for the CGW. From the experience of SAS space-times one obtains, by taking $\xi = e^{i\alpha}\tau$ ($\alpha = \text{const}$), the Taub–NUT solution. If the same replacement is carried out here, for the space-time with two spacelike Killing vectors, the resulting solution turns out to be diagonalizable. Thus the Taub–NUT type solution does not exist for the metric under investigation. For the cylindrically symmetrical line element the same proof can be done by employing the similar type of coordinates to (61),

$$2\tau = [(1+t)^2 - \rho^2]^{1/2} + [(1-t)^2 - \rho^2]^{1/2}, \quad 2\sigma = [(1+t)^2 - \rho^2]^{1/2} - [(1-t)^2 - \rho^2]^{1/2}. \tag{62}$$

As a matter of fact, a more general result can be proved in this line: whenever the real and the imaginary parts of the Ernst potential are functionally related (i.e., one can be expressed in terms of the other) then the metric reduces to a diagonal one.

Finally, we explore the possible self-similar cosmological vacuum model in the presence of two commuting Killing vectors. In the metric above we take $W = 0$, and express the remaining metric functions as functions of a common harmonic function. Since the proof is rather simple, we shall just content ourselves by stating the result that such a self-similar cosmology happens to be the Kasner[24] cosmology. Any other form of solution must be transformable into Kasner solution by a coordinate transformation.

VI. KILLING VECTORS OF THE M' MANIFOLD

Obviously the metric used by Bonnor [Eq. (6)] can be transformed into the plane wave space-time by the following identifications:

$$2^{1/2}u = t - \rho, \quad M = 2(\Psi - \gamma), \quad V = 2\Psi - \ln\rho, \quad 2^{1/2}v = t + \rho, \quad e^{-U} = \rho, \quad z \to x, \quad \phi \to y, \tag{63}$$

so that the resulting space-time metric reads

$$ds^2 = 2e^{-M}\,du\,dv - e^{-U}(e^V dx^2 + e^{-V} dy^2). \quad (64)$$

This is the particular case of the Szekeres metric (51) when the cross polarization term is suppressed. The vacuum Einstein equations for this more general line element are obtained from the harmonic maps[21,22] between the manifolds

$$ds^2 = 2\,du\,dv, \quad (65)$$

$$ds' = e^{-U}[2\,dU\,dM + dU^2 - dW^2 - dV^2 \cosh^2 W]. \quad (66)$$

From the metric functions, U is chosen as a coordinate condition and the determination of M is reduced to quadratures. By setting $W=0$, first, the metric of M' takes the form

$$ds'^2 = e^{-U}[2\,dU\,dM + dU^2 - dV^2]. \quad (67)$$

The problem now is to determine the nontrivial Killing vectors of this line element which will aid in generating a new solution from the old one. It can be verified that this geometry admits a nontrivial Killing vector

$$\xi = \xi^A \frac{\partial}{\partial X^A} = V\frac{\partial}{\partial M} + U\frac{\partial}{\partial V}. \quad (68)$$

We shall proceed now to obtain the new solution $(\widetilde{U}, \widetilde{M}, \widetilde{V})$ generated from a known solution (U, M, V) by the isometry of this Killing vector. The isometry equation is given by

$$\dot{X}_A = \alpha \xi X_A, \quad (69)$$

where α is a new parameter. Upon substituting ξ one obtains

$$\dot{U} = 0, \quad \dot{M} = \alpha V, \quad \dot{V} = \alpha U. \quad (70)$$

Imposing now the initial $(t=0)$ and the image $(t=1)$ conditions of the isometry, the new solution is expressed by

$$\widetilde{U} = U, \quad \widetilde{V} = V + \alpha U, \qquad \widetilde{M} = M + \alpha V + \tfrac{1}{2}\alpha^2 U. \quad (71)$$

Choosing as (U, M, V) the nonsingular solution of Bonnor, by this isometry we obtain a new solution with an additional parameter. The same isometry has been employed elsewhere to generate new scalar plane waves.[25]

The foregoing method of isometries can equivalently be handled in the Ernst formalism. Defining the complex potential by

$$\eta = \frac{\sinh V \cosh W - i \sinh W}{\cosh V \cosh W + 1}, \quad (72)$$

the following equality holds true:

$$\frac{4\,d\eta\,d\bar{\eta}}{(1-\eta\bar{\eta})^2} = dW^2 + \cosh^2 W \cdot dV^2. \quad (73)$$

The left-hand side of this equality coincides exactly with the M' manifold of the Ernst Lagrangian. Thus any isometry of the rhs corresponds to an isometry of the lhs and vice versa. For instance, the isometry

$$\eta \to \eta' = [1 + \eta(i\beta - 1)]/[1 + \eta(i\beta + 1)] \quad (74)$$

with the real parameter β, which is known as the Ehlers[26] transformation, can directly be adopted in the generation of a new cosmological model. However, our line of search will follow an alternative route, rather than employing well-known results. Once a pair (V, W) of solutions is known we shall proceed to generate a new pair $(\widetilde{V}, \widetilde{W})$ by employing the isometry

$$d\widetilde{W}^2 + d\widetilde{V}^2 \cosh^2 \widetilde{W} = dW^2 + dV^2 \cosh W. \quad (75)$$

This can be achieved by making use of the general Killing vector, namely,

$$\xi = (c_1 e^V + c_2 e^{-V})\frac{\partial}{\partial W} + [c_3 - (c_1 e^V - c_2 e^{-V})\tanh W]\frac{\partial}{\partial V}, \quad (76)$$

where c_1, c_2, and c_3 are arbitrary constants. As an example, we shall obtain the new solution corresponding to the linear combination of the two Killing vectors

$$\xi_{(1)} = e^{-V}\left(\frac{\partial}{\partial W} + \tanh W \frac{\partial}{\partial V}\right), \quad (77)$$

$$\xi_{(2)} = e^{V}\left(\frac{\partial}{\partial W} - \tanh W \frac{\partial}{\partial V}\right), \quad (78)$$

in accordance with the relation

$$\dot{X}_A = (\alpha_0 \xi_{(1)} + \beta_0 \xi_{(2)}) X_A, \quad (79)$$

where α_0 and β_0 are constants. We obtain equivalently the pair

$$\dot{V} = \tanh W(\alpha_0 e^{-V} - \beta_0 e^V), \quad (80)$$

$$\dot{W} = \alpha_0 e^{-V} + \beta_0 e^V. \quad (81)$$

After tedious calculations one obtains the new solution

$$\sinh \widetilde{W} = \cosh\alpha \sinh W + \tfrac{1}{2}\sinh\alpha \cosh W(\beta e^V + \beta^{-1} e^{-V}), \quad (82)$$

$$\beta \cosh \widetilde{W} e^{\widetilde{V}} = \sinh\alpha \sinh \widetilde{W} + \tfrac{1}{2}\cosh\widetilde{W}[\beta e^V(\cosh\alpha + 1) \times \beta^{-1} e^{-V}(\cosh\alpha - 1)], \quad (83)$$

where the new parameters α, β are defined by

$$\alpha = 2(\alpha_0\beta_0)^{1/2}, \quad \beta = (\beta_0/\alpha_0)^{1/2}.$$

It is readily observed that in the limit $\alpha = 0$ we recover the old solution (V, W), but otherwise we have a new solution $(\widetilde{V}, \widetilde{W})$ generated from the isometry. The constant β emerges as a scale parameter for the functions e^V and $e^{\widetilde{V}}$ and therefore it can be washed out from the solution.

At this stage we can also check whether a solution with $W \neq 0$ can be generated from a known solution with $W = 0$.[27] For this purpose our isometry takes the form

$$\sinh \widetilde{W} = \sinh\alpha \cosh V, \quad (84)$$

$$\cosh \widetilde{W} e^{\widetilde{V}} = \cosh\alpha \cosh V + \sinh V. \quad (85)$$

After some simple algebra it can be observed that the corresponding space-time metric diagonalizes under the hyperbolic rotation

$$x' = x\cosh(\alpha/2) + y\sinh(\alpha/2), \qquad y' = x\sinh(\alpha/2) + y\cosh(\alpha/2), \quad (86)$$

and as a result such a solution does not exist. The method of isometries fails to add a cross term to a diagonal metric but it maps a given solution into a new one.

VII. CONCLUSION

We have shown that space-times admitting two spacelike Killing vectors admit dual interpretations and to this end, the method of harmonic maps proves to be a useful technique. What seems a more important question however, is whether such dual properties of the vacuum fields have any physical significance beyond mathematics. Consider, for instance, a proton and a neutron in a given vacuum field that admits the e.m. field via dual interpretation. The apparent paradox between the geodesics equations of proton and neutron will be resolved provided their mass difference is attributed to an e.m. origin.

The method of isometries in the M' manifold provides a promising feature and a useful alternative to already existing methods in general relativity. As a matter of fact, the method of harmonic maps applies to any theory whose Lagrangian is expressed in pure kinetic form. Self-dual SU(N) field equations and instantons in classical field theory provide such examples, to mention a few. Further, in the instanton problem the base manifold is the four-dimensional Euclidean manifold with definite metric that can be mapped onto a sphere and the degree of harmonic maps results in a topologically significant number.

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