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Cross-Polarized Cylindrical Gravitational Waves of Einstein and Rosen.

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Summary. — Using Einstein-Rosen's linearly polarized waves as the seed solution, we derive an interesting solution for the Einstein's equations that describes the evolution of such waves with the second polarization.

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1. – Introduction.

Cylindrical gravitational waves with cross polarization are described by the line element

$$ds^2 = \exp[2(\gamma - \Psi)](dt^2 - d\rho^2) - \exp[2\Psi](dz + \omega\, d\phi)^2 - \rho^2 \exp[-2\Psi]\, d\phi^2, \tag{1}$$

due originally to Jordan, Ehlers, Kundt (1) and Kompaneetz (2). Metric functions Ψ, γ and ω are functions of ρ and t alone and the particular case ($\omega = 0$), describing waves with single polarization, was studied first in a historical paper by Einstein and Rosen (ER) (3). The vacuum Einstein equations are equivalent to the

(1) P. JORDAN, J. EHLERS and W. KUNDT: *Abh. Akad. Wiss. Mainz Math. Naturwiss.-Kl.*, 2 (1960).
(2) A. S. KOMPANEETZ: *Ž. Èksp. Teor. Fiz.*, 34, 953 (1958) [*Sov. Phys. JETP*, 7, 659 (1958)].
(3) A. EINSTEIN and N. ROSEN: *J. Franklin Inst.*, 223, 43 (1937).

following set of equations:

$$\Psi_{tt} - \frac{1}{\rho}\,\Psi_\rho - \Psi_{\rho\rho} = \frac{\exp[4\Psi]}{2\rho^2}\,(\omega_t^2 - \omega_\rho^2)\,, \tag{2}$$

$$\omega_{tt} + \frac{1}{\rho}\,\omega_\rho - \omega_{\rho\rho} = 4(\omega_\rho \Psi_\rho - \omega_t \Psi_t)\,, \tag{3}$$

$$\gamma_\rho = \rho(\Psi_t^2 + \Psi_\rho^2) + \frac{\exp[4\Psi]}{4\rho}\,(\omega_t^2 + \omega_\rho^2)\,, \tag{4}$$

$$\gamma_t = 2\rho\Psi_\rho\Psi_t + \frac{\exp[4\Psi]}{2\rho}\,\omega_t\omega_\rho\,. \tag{5}$$

A constrained Lagrangian describing this system of equations is

$$\mathscr{L} = (\gamma_\rho\lambda_\rho - \gamma_t\lambda_t) - \lambda(\Psi_\rho^2 - \Psi_t^2) - \frac{\exp[4\Psi]}{4\lambda}\,(\omega_\rho^2 - \omega_t^2)\,, \tag{6}$$

where $\lambda = \rho$ is to be imposed as a coordinate condition subsequent to the variation. The (Ψ, ω) part of this Lagrangian is equivalent to the one introduced by Ernst [4] in connection with stationary fields, namely

$$L_0 = \frac{|\nabla\xi|^2}{(1-|\xi|^2)^2}\,, \tag{7}$$

where

$$\xi = \frac{(1-i\omega)^2 - \lambda^2\exp[-4\Psi]}{(1+\lambda\exp[-2\Psi])^2 + \omega^2}\,. \tag{8}$$

Equations (2) and (3) are equivalent, now, to the Ernst equation

$$(|\xi|^2 - 1)\,\nabla^2\xi = 2\bar{\xi}(\nabla\xi)^2\,, \tag{9}$$

where the gradient and the Laplacian are defined on the geometry

$$ds_0^2 = d\rho^2 - dt^2 + \rho^2 d\phi^2\,, \tag{10}$$

in which ϕ is a cyclic variable.

In the following section we proceed to derive a solution with a nontrivial cross-term in the metric ($\omega \neq 0$) and interpret it to describe the self-interacting gravitational waves.

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[4] F. J. Ernst: *Phys. Rev.*, **167**, 1175 (1968).

2. – The solution.

As the solution of Ernst equation we adopt

$$\xi = y(X)\exp[i\beta(X)]\,, \tag{11}$$

where y and β are both functions of a single function X, that satisfies the cylindrical wave equation

$$X_{tt} - \frac{1}{\rho}X_\rho - X_{\rho\rho} = 0\,. \tag{12}$$

Complete integral of this system is given (5) (without electromagnetism) by

$$y^2 = \frac{\cosh\alpha\,\cosh 2X - 1}{\cosh\alpha\,\cosh 2X + 1}\,, \tag{13}$$

$$\operatorname{tg}(\beta - \beta_0) = -\sinh\alpha\,\operatorname{ctgh} 2X\,, \tag{14}$$

in which β_0 and α are both constants of integration. For our later convenience we shall make the choice $\beta_0 = 0$, since this can be justified by a coordinate transformation. Make now the parametrization (6)

$$\xi = \frac{\exp[2\Psi] - 1 + i\Phi}{\exp[2\Psi] + 1 + i\Phi}\,, \tag{15}$$

where the auxiliary potential Φ is related to ω by the pair of equations

$$\begin{cases} \rho\Phi_t = \exp[4\Psi]\,\omega_\rho\,, \\ \rho\Phi_\rho = \exp[4\Psi]\,\omega_t\,. \end{cases} \tag{16}$$

Comparing the foregoing expressions we obtain

$$\exp[2\Psi] = \frac{1 - y^2}{1 + y^2 - 2y\cos\beta}\,, \qquad \Phi = \frac{2y\sin\beta}{1 + y^2 - 2y\cos\beta}\,. \tag{17}$$

Since we are interested in the Einstein-Rosen waves, we would like to choose a particular seed function given by $X = \frac{1}{2}AJ_0(\rho\sigma)\cos\sigma t$, where J_0 is Bessel's function of order 0, and A and σ are constants. As a result of integrating ω from (16) and the quadrature equation for γ, we obtain the following solution for the

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(5) M. HALILSOY: *Lett. Nuovo Cimento*, **37**, 231 (1983).

(6) S. CHANDRASEKHAR: *Proc. R. Soc. London, Ser. A*, **408**, 209 (1986).

metric functions:

$$(18)\qquad \begin{cases} \exp[-2\Psi] = \exp[AJ_0\cos\sigma t]\sinh^2\dfrac{\alpha}{2} + \exp[-AJ_0\cos\sigma t]\cosh^2\dfrac{\alpha}{2}\,, \\ \omega = -(A\sinh\alpha)\,\rho\, J_1(\rho\sigma)\sin\sigma t\,, \\ \gamma = \dfrac{1}{8}A^2[\sigma^2\rho^2(J_0^2+J_1^2) - 2\sigma\rho J_0 J_1\cos^2\sigma t] = \gamma_{\mathrm{ER}}\,, \end{cases}$$

where $J_1(\rho\sigma)$ is the Bessel's function of order 1. It is observed that the metric function γ remains invariant under the addition of cross polarization. This is connected with the fact that γ represents the energy of the waves, as suggested by various authors (6).

In the limit $\alpha = 0$, our solution obviously reduces to the solution of Einstein and Rosen. We would like to note also that if one adopts the parametrization (8), without integrating ω from the auxiliary potential Φ, then the metric that one obtains will be diagonalizable.

The problem of interacting cylindrical gravitational waves can be cast into a suitable characteristic form, where the ingoing and outgoing field strengths are denoted by $(I_+, I_\times)$ and $(O_+, O_\times)$ respectively. The notations + and × stand for the two different polarization states (*i.e.* linear and cross, respectively). The field equations (2)-(3) in these new amplitudes take the following first-order forms (7):

$$(19)\qquad I_{+,u} = \frac{I_+ - O_+}{2\rho} + I_\times O_\times\,,$$

$$(20)\qquad O_{+,v} = \frac{I_+ - O_+}{2\rho} + I_\times O_\times\,,$$

$$(21)\qquad I_{\times,u} = \frac{I_\times + O_\times}{2\rho} - I_+ O_\times\,,$$

$$(22)\qquad O_{\times,v} = \frac{I_\times + O_\times}{2\rho} - O_+ I_\times\,,$$

where the null coordinates are defined by $2u = t - \rho$ and $2v = t + \rho$ and the amplitudes are defined by

$$(23)\qquad \begin{cases} I_+ = 2(\Psi_t + \Psi_\rho)\,, \qquad O_+ = 2(\Psi_t - \Psi_\rho)\,, \\ I_\times = \dfrac{\exp[2\Psi]}{\rho}(\omega_t + \omega_\rho)\,, \qquad O_\times = \dfrac{\exp[2\Psi]}{\rho}(\omega_t - \omega_\rho)\,. \end{cases}$$

(7) T. PIRAN and P. N. SAFIER: *Nature (London)*, 318, 271 (1985).

In terms of these new amplitudes our solution reads as follows:

$$I_+ = \sigma A \left(\frac{M}{N}\right)(J_0 \sin \sigma t + J_1 \cos \sigma t)\,, \tag{24}$$

$$I_\times = -\frac{\sigma A \sinh \alpha}{N}(J_0 \sin \sigma t + J_1 \cos \sigma t)\,, \tag{25}$$

$$O_+ = \sigma A \left(\frac{M}{N}\right)(J_0 \sin \sigma t - J_1 \cos \sigma t)\,, \tag{26}$$

$$O_\times = \frac{\sigma A \sinh \alpha}{N}(J_0 \sin \sigma t - J_1 \cos \sigma t)\,, \tag{27}$$

in which we have abbreviated $M = \exp[AJ_0 \cos \sigma t] \sinh^2(\alpha/2) - \exp[-AJ_0 \cos \sigma t] \cdot \cdot \cosh^2(\alpha/2)$ and $N = \exp[-2\Psi]$.

We can study further the asymptotic behaviour of these fields by making use of the Bessel's functions and the expansion

$$\exp[AJ_0 \cos \sigma t] \underset{\rho \to \infty}{\simeq} 1 + AJ_0 \cos \sigma t\,. \tag{28}$$

The asymptotic values can be expressed in a compact form by

$$\boldsymbol{I} = I_+ + iI_\times = A\left(\frac{2\sigma}{\pi\rho}\right)^{1/2} \sin\left(\frac{\pi}{4} - 2v\sigma\right)\exp[i\theta]\,, \tag{29}$$

$$\boldsymbol{O} = O_+ + iO_\times = -A\left(\frac{2\sigma}{\pi\rho}\right)^{1/2} \sin\left(\frac{\pi}{4} + 2u\sigma\right)\exp[-i\theta]\,, \tag{30}$$

in which u and v are the null coordinates and we have redefined our second polarization parameter by $\operatorname{tg}\theta = \sinh \alpha$. The expression $\boldsymbol{I} \cdot \boldsymbol{O}$ as can readily be observed is asymptotically independent of the second polarization.

Similar to the recently published solutions ([6,8]), our solution is regular everywhere. This feature is decided after one studies the components of the Riemann tensor. For this purpose we have calculated the only nonzero Weyl scalars Ψ_2, Ψ_0 and Ψ_4 in the null tetrad of Szekeres ([9]). Among these, Ψ_2 is the

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([8]) T. Piran, P. N. Safier and J. Katz: *Phys. Rev. D*, 34, 331 (1986).

([9]) P. Szekeres: *J. Math. Phys.* (*N.Y.*), 13, 286 (1972).

most compact one and we give it here:

$$\Psi_2 = \frac{\sigma^2 A^2 (J_0^2 \sin^2 \sigma t - J_1^2 \cos^2 \sigma t)}{8(\cosh\alpha \cosh 2X - \sinh 2X)^2} \cdot \tag{31}$$

$$\cdot [(\cosh\alpha \sinh 2X - \cosh 2X + 2i \sinh\alpha)^2 + 3\sinh^2\alpha] +$$

$$+ \sigma A J_1 \cos\sigma t \, \frac{\cosh\alpha \sinh 2X - \cosh 2X + 2i\sinh\alpha}{4\rho(\cosh\alpha\cosh 2X - \sinh 2X} \,.$$

We would like to add that linear superposition of the waves, as sums or integrals with suitable amplitude factors, can be obtained easily. We mention, as an example, the form of the waves considered by Bonnor ([10]), which is obtained from the ER waves in the way described by Weber and Wheeler ([11]). The seed solution in this particular case is to be chosen by $Y = y/(x^2 + y^2)$, where the coordinates are defined by

$$\rho = (x^2+1)^{1/2}(y^2-1)^{1/2}\,, \qquad t = xy\,, \tag{32}$$

where the ranges of these coordinates are $-\infty < x < +\infty$ and $1 \leqslant y < \infty$. In this coordinate system Laplace equation, $\nabla^2 Y = 0$ is given by

$$[(x^2+1)\,Y_x]_x - [(y^2-1)\,Y_y]_y = 0\,. \tag{33}$$

The next step is to employ the solution for Ψ:

$$\exp[-2\Psi] = \exp[-2Y]\sinh^2\frac{a}{2} + \exp[2Y]\cosh^2\frac{a}{2}\,, \tag{34}$$

where a is a constant, and integrate ω from the pair of equations

$$\begin{cases} \omega_x = 2\sinh a \dfrac{(y^2-1)(y^2-x^2)}{(x^2+y^2)^2}\,, \\[2ex] \omega_y = 4\sinh a \dfrac{xy(x^2+1)}{(x^2+y^2)^2}\,. \end{cases} \tag{35}$$

After this, it remains to integrate for the metric function γ from the quadrature equations and, as in the Einstein-Rosen case, γ turns out to be an

([10]) W. B. Bonnor: *J. Math. Mech.*, **6**, 203 (1957).
([11]) J. Weber and J. A. Wheeler: *Rev. Mod. Phys.*, **29**, 509 (1957).

invariant. In conclusion, the solution is

$$(36)\qquad \begin{cases} \exp[-2\Psi] = \sinh^2\dfrac{a}{2}\exp\left[\dfrac{-2y}{x^2+y^2}\right] + \cosh^2\dfrac{a}{2}\exp\left[\dfrac{2y}{x^2+y^2}\right], \\[2ex] \omega = 2\sinh a\,\dfrac{x(y^2-1)}{x^2+y^2}, \\[2ex] \gamma = \gamma_B = \dfrac{(x^2+1)(y^2-1)}{4(x^2+y^2)^4}(6x^2y^2-x^4-y^4) + \dfrac{y^2-x^2-2}{8(x^2+y^2)}. \end{cases}$$

In the limit $a=0$, we obtain the solution given long ago by Bonnor (5) and therefore our solution generalizes Bonnor's nonsingular fields in general relativity.

3. – Discussion of energy.

Eells and Sampson (12) define an invariant energy functional from the harmonic maps between the two given Riemannian manifolds by

$$(37)\qquad E(f) = \frac{1}{2}\int_M g'_{AB}(f)\,\frac{\partial f^A}{\partial x^a}\,\frac{\partial f^B}{\partial x^b}\,g^{ab}\sqrt{g}\,\mathrm{d}^n x = \frac{1}{2}\int_M (\text{Lagrangian})\,\mathrm{d}\rho\,\mathrm{d}t\,,$$

It was shown that Einstein equations admitting two killing vectors can be cast into the mathematical formulation of harmonic maps (13). For the problem of cylindrical waves, the two Riemannian manifolds are chosen by

$$(38)\qquad \begin{cases} M: \mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b = \mathrm{d}\rho^2 - \mathrm{d}t^2 + \lambda^2\mathrm{d}\phi^2\,, \\[2ex] M': \mathrm{d}s'^2 = g'_{AB}(f)\,\mathrm{d}f^A\,\mathrm{d}f^B = \mathrm{d}\gamma\,\dfrac{\mathrm{d}\lambda}{\lambda} - \mathrm{d}\Psi^2 - \dfrac{\exp[4\Psi]}{4\lambda^2}\,\mathrm{d}\omega^2\,. \end{cases}$$

Here $f^A = \{\Psi, \lambda, \omega, \gamma\}$ represents the harmonic maps such that the integrand of $E(f)$ coincides with Lagrangian (6), and the variational principle $\delta E(f) = 0$ yields the Einstein equations.

Let us show first that the Hamiltonian constructed from the Lagrangian density (6) turns out to be zero. For this purpose we define the conjugate momenta by $P_\psi = \partial\mathscr{L}/\partial\dot{\psi}$, etc., where the dot stands for time derivative. The Hamiltonian density $\mathscr{H}_0$ is defined then by

$$(39)\qquad \mathscr{H}_0 = P_\psi\dot{\psi} + P_\omega\dot{\omega} + P_\gamma\dot{\gamma} - \mathscr{L}\,,$$

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(12) J. Eells jr. and J. H. Sampson: *Am. J. Math.*, **86**, 109 (1964).
(13) Y. Nutku: *Ann. Inst. H. Poincaré A*, **21**, 175 (1974).

which leads, after substitutions, to

$$\mathcal{H}_0 = \lambda(\psi_t^2 + \psi_\rho^2) + \frac{\exp[4\psi]}{4\lambda}(\omega_t^2 + \omega_\rho^2) - \lambda_\rho \gamma_\rho . \tag{40}$$

By virtue of eq. (4) and the fact that $\lambda = \rho$, this expression for $\mathcal{H}_0$ vanishes. One possible way to overcome this difficulty is to consider only the unconstrained (ψ, ω) part and neglect the γ-term in the Lagrangian. This reduced part of the Lagrangian is well known to be identical with the Ernst Lagrangian in which γ does not appear. Once this choice is made, our reduced Lagrangian density is

$$\mathcal{L}_0 = -\lambda(\psi_\rho^2 - \psi_t^2) - \frac{\exp[4\psi]}{4\lambda}(\omega_\rho^2 - \omega_t^2) , \tag{41}$$

which yields the positive definite Hamiltonian density

$$\mathcal{H} = \lambda(\psi_\rho^2 + \psi_t^2) + \frac{\exp[4\psi]}{4\lambda}(\omega_\rho^2 + \omega_t^2) . \tag{42}$$

Comparing this with eq. (4) we observe that

$$\mathcal{H} = \gamma_\rho . \tag{43}$$

An energy can thus be defined by integrating this density:

$$E = \int \mathcal{H} \mathrm{d}\rho = \int^{\rho} \gamma_\rho \mathrm{d}\rho = \gamma . \tag{44}$$

This energy is called «C»-energy and it represents the total gravitational energy per unit length between $\rho = 0$ and ρ at time t. (Note that «C» stands for the word cylindrical.) It was introduced first by Thorne [14] in 1965 from a different line of thought. Our derivation of «C»-energy here is due to Chandrasekhar [6].

We remark that, in order to have a conserved energy, we must have $\dot{\mathcal{H}} = \gamma_{\rho t} = 0$. The transcendental cylindrical waves found by Chandrasekhar satisfy this criterion. For the ER waves (and also in this paper) on the other hand we have

$$\gamma_t = \frac{1}{4} A^2 \sigma^2 \rho J_0 J_1 \sin 2\sigma t \neq 0 , \tag{45}$$

which implies that $\gamma_{t\rho} \neq 0$.

The energy per unit length in the z-direction confined in the cylindrical

[14] K. THORNE: *Phys. Rev. B*, 138, 251 (1965).

annulus between ρ_1 and ρ_2 ($>\rho_1$) is given for the ER waves by

$$E = \int_{\rho_1}^{\rho_2} \gamma_\rho \, d\rho = \frac{A^2}{8} \left[\sigma^2 \rho^2 (J_0^2 + J_1^2) - 2\sigma\rho J_0 J_1 \cos^2 \sigma t\right] \Big|_{\rho_1}^{\rho_2}, \tag{46}$$

which is a positive definite quantity. However, due to condition (45), the waves represented by the ER solution are not stationary.

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I thank Prof. H. H. Aly for valuable discussions.

RIASSUNTO (*)

Usando le onde linearmente polarizzate di Einstein-Rosen come soluzione seme, si deduce una soluzione interessante per le equazioni di Einstein che descrive l'evoluzione di tali onde con la seconda polarizzazione.

(*) *Traduzione a cura della Redazione.*

Кросс-поляризованные цилиндрические гравитационные волны Эйнштейна и Розена.

Резюме (*). — Используя линейно поляризованные волны Эйнштейна и Розена, как затравочное решение, мы выводим интересное решение для уравнений Эйнштейна, которое описывает эволюцию таких волн с второй поляризацией.

(*) *Переведено редакцией.*