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Exact Faraday rotation in the cylindrical Einstein-Maxwell waves

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We obtain the exact behavior of the cross-polarized cylindrical Einstein-Maxwell waves that generalizes the well-known Einstein-Rosen waves. In the presence of the second mode of polarization the outgoing waves interact with the incoming ones to exhibit an analogous effect of the Faraday rotation.

I. INTRODUCTION

Cylindrical gravitational waves were introduced first in a historical paper[1] by Einstein and Rosen (ER). In 1957 Weber and Wheeler[2] analyzed their properties and drew the conclusion that ER waves satisfy the physical reality requirements. Since then, the cylindrical gravitational waves of ER have continued to remain an important topic in the physics literature. By this token, in a recent study[3,4] a connection has been established between the Faraday rotation of the electromagnetic polarization vector in a plasma and an analogous effect in the cross-polarized waves of ER. The authors based their analysis on numerical integration of the field equations, leaving the exact correspondence between the two phenomena open. On this account, in this paper, we provide an exact solution to the Einstein-Maxwell (EM) equations that exhibits Faraday rotation and which reduce to the ER waves in a particular limit. Our line element that describes cross-polarized EM waves,

$$ds^2=e^{2(\gamma-\psi)}(dt^2-dr^2)-e^{2\psi}(dz+w\,d\phi)^2-r^2e^{-2\psi}d\phi^2\ , \tag{1}$$

involves the functions ψ, γ, and w to be determined as functions of t and r, through the EM equations. The EM equations are well known to reduce to the pair of Ernst equations[5]

$$\begin{aligned}(\xi\bar{\xi}+\eta\bar{\eta}-1)\nabla^2\xi&=2\nabla\xi\cdot(\bar{\xi}\nabla\xi+\bar{\eta}\nabla\eta)\ ,\\ (\xi\bar{\xi}+\eta\bar{\eta}-1)\nabla^2\eta&=2\nabla\eta\cdot(\bar{\xi}\nabla\xi+\bar{\eta}\nabla\eta)\ ,\end{aligned} \tag{2}$$

where ξ and η represent the gravitational and electromagnetic complex potentials, respectively. The differential operators here act on the cylindrically symmetrical flat space-time suitable to the description of the cylindrical-wave problem.

II. THE SOLUTION

Our solution makes use of a functional dependence between ξ and η, namely,[6]

$$\xi=a\xi_0,\quad \eta=(1-a^2)^{1/2}\xi_0\ , \tag{3}$$

where a is a real constant satisfying $0\le a\le 1$, and ξ_0 is the Ernst potential for the vacuum fields.[7]

By this choice, for the sake of an exact solution, we electrify gravity in such a manner that the total field energy (C energy) represented by the metric function γ remains the same with the γ of the ER waves. We adopt now the previously known vacuum solution[8] to the problem in the construction of EM fields.

For the reason, however, connected to the evolution of the waves, we shall express the solution in a different notation that is more appropriate to the initial-value problem for the cylindrical waves. Following Piran and Safier[3] and Piran, Safier, and Stark[4] we introduce the amplitudes

$$\begin{aligned}&I_+=2(\psi_t+\psi_r),\quad O_+=2(\psi_t-\psi_r)\ ,\\ &I_\times=\frac{e^{2\psi}}{r}(w_t+w_r),\quad O_\times=\frac{e^{2\psi}}{r}(w_t-w_r)\ ,\end{aligned} \tag{4}$$

which satisfy the following set of first-order equations in the ingoing $u=\frac{1}{2}(t-r)$ and outgoing $v=\frac{1}{2}(t+r)$ coordinates:

$$\begin{aligned}&I_{+,u}=\frac{I_+-O_+}{2r}+I_\times O_\times,\quad O_{+,v}=\frac{I_+-O_+}{2r}+I_\times O_\times\ ,\\ &I_{\times,u}=\frac{I_\times+O_\times}{2r}-I_+O_\times,\quad O_{\times,v}=-\frac{I_\times+O_\times}{2r}-I_\times O_+ .\end{aligned} \tag{5}$$

Here, $I_{+(\times)}$ represents the incoming amplitude with linear (cross) polarization which reflects at $r=0$ and turns into the outgoing amplitude $O_{+(\times)}$. Our exact solution is

$$\begin{aligned}&I_+=-\frac{AC}{D},\quad O_+=\frac{BC}{D}\ ,\\ &I_\times=-\frac{A}{D}(1+a^2)\sin\alpha,\quad O_\times=-\frac{B}{D}(1+a^2)\sin\alpha\ ,\end{aligned} \tag{6}$$

where we have used the abbreviations

$$\begin{aligned}A&=J_1(r)\cos t+J_0(r)\sin t\ ,\\ B&=J_1(r)\cos t-J_0(r)\sin t\ ,\\ C&=2a\cosh[J_0(r)\cos t]\\ &\quad-(1+a^2)(1+\sin^2\alpha)^{1/2}\sinh[J_0(r)\cos t]\\ D&=(1+a^2)(1+\sin^2\alpha)^{1/2}\cosh[J_0(r)\cos t]\ ,\\ &\quad-2a\sinh[J_0(R)\cos t]+1-a^2\ ,\end{aligned} \tag{7}$$

in which J_0 and J_1 are Bessel functions of orders 0 and 1, and α is the cross-polarization angle. Note that we have chosen the amplitude and the separation constants that appear in the original ER solution[1] both to be unity. For $\alpha=0=a$ our solution reduces to the pure electromagnetic wave solution of ER type. It can be checked easily that in the above solution (6) the boundary conditions at the symmetry axis ($r=0$),

$$I_+ = O_+ , \quad I_\times = -O_\times , \tag{8}$$

are satisfied. Further, our solution satisfies, at $t=0$,

$$I_+ = -O_+ , \quad I_\times = O_\times , \tag{9}$$

and it has the property that for ($t=r=0$), as in the case of a nodal point, all amplitudes reduce to zero.

III. ROTATION OF THE POLARIZATION VECTOR

Figure 1 shows the linearly polarized ER amplitudes (I_+, O_+) which corresponds to the values ($\alpha=0, a=1$) in the solution (6). In Fig. 2 (for $\alpha=10°, a=0.5$) and Fig. 3 (for $\alpha=60°, a=0.5$) we display all the amplitudes $(I_+, O_+, I_\times, O_\times)$ for the indicated ranges of t and r. Similar to the cylindrically symmetrical soliton solutions[9] we define the relative polarization angle θ in accordance with

$$\tan\theta = \frac{I_\times}{I_+} = -\frac{O_\times}{O_+} = (1+a^2)\frac{\sin\alpha}{C} , \tag{10}$$

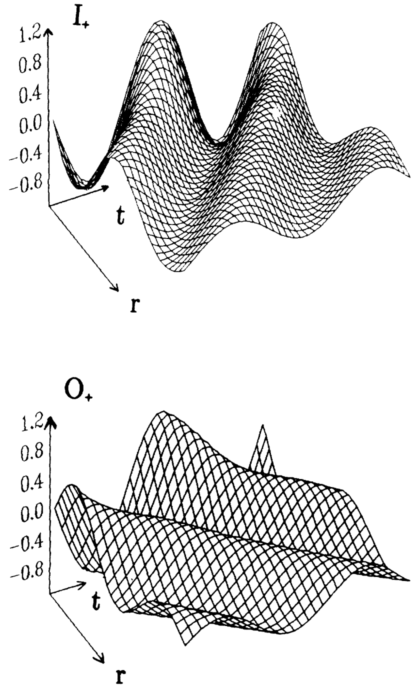

FIG. 1. The linearly independent polarized amplitudes (I_+, O_+) of Einstein and Rosen in the domain ($0 \le r \le 10, 0 \le t \le 10$).

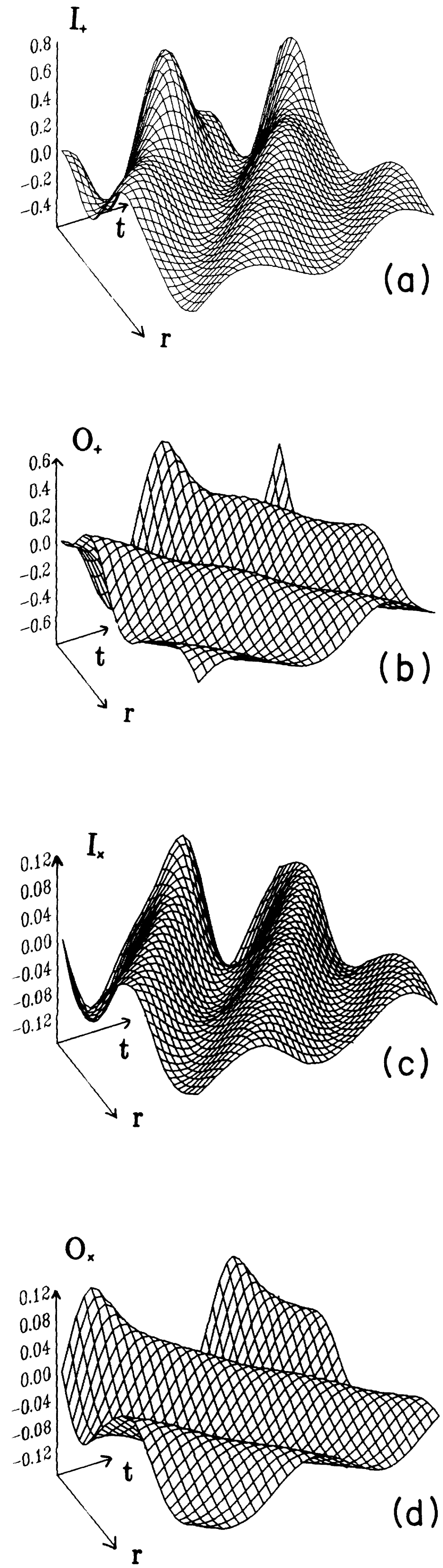

FIG. 2. The amplitudes $(I_+, O_+, I_\times, O_\times)$ for the particular parameters $\alpha=10°$ and $a=0.5$ in the domain ($0 \le r \le 10, 0 \le t \le 10$).

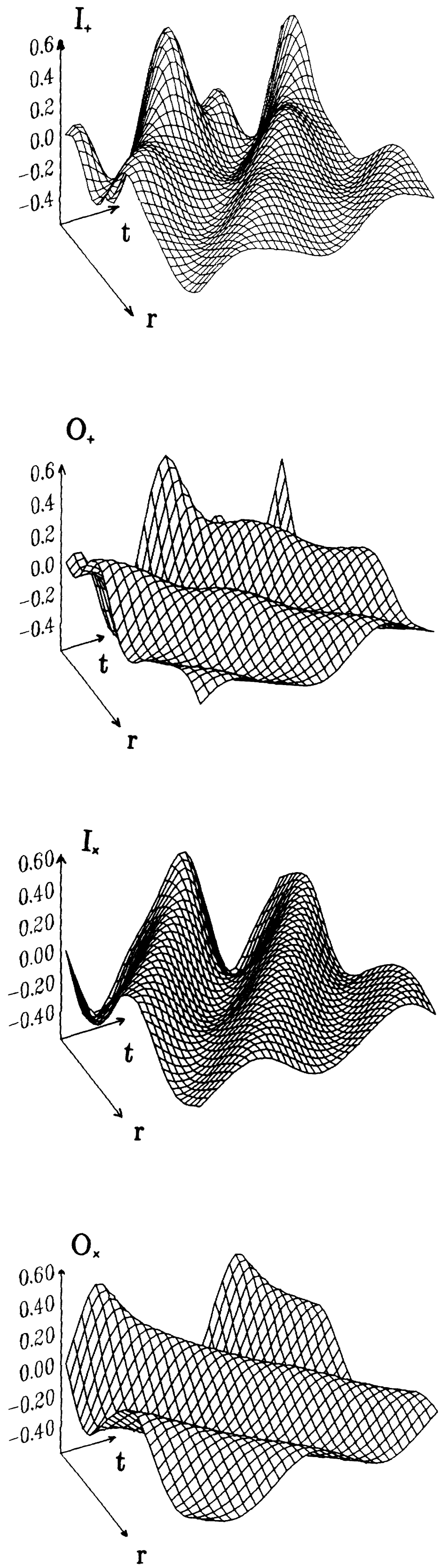

FIG. 3. The amplitudes $(I_+, O_+, I_\times, O_\times)$ for the parameters values $\alpha = 60°$ and $a = 0.5$ in the domain $(0 \le r \le 10, 0 \le t \le 10)$.

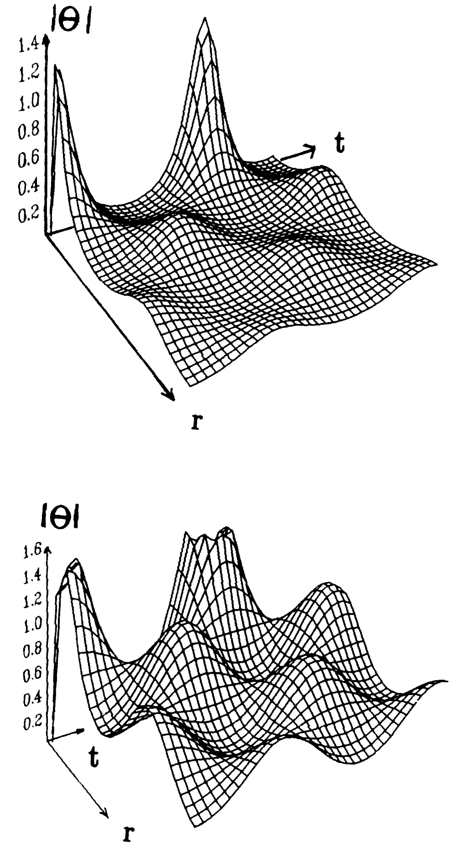

FIG. 4. The absolute value of θ for the special parameters $\alpha = 10°, a = 0.5$ and $\alpha = 60°, a = 0.5$, respectively, in the domain $(0 \le r \le 10, 0 \le t \le 10)$.

where, the function C is given in (7). The absolute change of this angle (i.e., Faraday rotation) as a function of t and r is shown in Fig. (4).

Finally, in order to see the ratio of the different modes of polarizations we define the total linear and the total cross polarized amplitudes by

$$T_+ = (I_+^2 + O_+^2)^{1/2}, \quad T_\times = (I_\times^2 + O_\times^2)^{1/2}, \tag{11}$$

respectively. The energy density γ_r is expressed in terms of these amplitudes by

$$\gamma_r = \tfrac{1}{8} r (T_+^2 + T_\times^2)^{1/2}. \tag{12}$$

By studying the ratio

$$\frac{T_\times}{T_+} = |\tan\theta| \tag{13}$$

we conclude that the dominant contribution to the resultant polarization alternates between the linear and the cross modes.

IV. CONCLUSION

In contrast with the plane-wave counterparts where a mutual focusing property results in an eventual singularity, unbounded amplitudes do not develop in the cylindrical-wave problem. Instead, the amplitude patterns extend from the symmetry axis to infinity smoothly and with decreasing magnitudes in asymptotic regions. Although the labeling may imply the opposite, it can be observed from the general solution (6) that the linearly polarized amplitudes (I_+, O_+) are not independent from the cross-polarization angle α. For this reason the graphs of $(I_+, I_\times, O_+, O_\times)$ are not exactly the same as the numerical graphs of Piran and Safier.[3] We recall that they had chosen particular input pulses in their analysis.

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