

# Cosmic Strings Coupled with a Massless Scalar Field

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# Abstract

A scalar field generalization of Xanthopoulos's cylindrically symmetric solutions of the vacuum Einstein equations is obtained. The obtained solution preserves the properties of the Xanthopoulos solution, which are regular on the axis, asymptotically flat and free from the curvature singularities. The solution describes stable, infinite length of rotating cosmic string interacting with gravitational and scalar waves.

## I. INTRODUCTION

The first exact solution to the vacuum - Einstein equations in cylindrical geometry is dated back to 1925 by Guido Beck [1]. However, the well known solutions in this geometry belongs to Einstein and Rosen (ER) [2]. The solution presented by ER represents first exact radiative solutions. The solution is radiative because the waves carry away energy from the mass located at the axis of symmetry. Furthermore, the solution obtained by ER have two spacelike Killing vectors which are hypersurface orthogonal. The most general vacuum solution describing cylindrical waves was studied independently by Kompaneets and by Ehlers et al [3]. The cross polarized version of ER waves was given by Halilsoy [4].

Chandrasekhar has constructed rather different type of formalism for obtaining cylindrical waves with cross polarizations, similar to that used for the discussion of the collision of plane gravitational waves [5]. The main motivation for obtaining new cylindrical waves was to find applications in general relativity and astrophysics. One of the application arena for cylindrical spacetimes are cosmic strings. Cosmic strings are known to be topological defects that formed during the cosmological phase transitions as a result of spontaneous symmetry breaking of the Grand Unified Theory. Cosmic strings would have important consequences for astrophysics. To say the least, it is believed that the cosmic strings are responsible for galaxy formation and gravitational lensing.

A long time ago, by employing the method of Chandrasekhar, Xanthopoulos has obtained a family of three parameter and time dependent solutions both in vacuum- Einstein [6] (hereafter paper I) and Einstein - Maxwell theory[7] (hereafter paper II ). These solutions are cylindrically symmetric solutions with remarkable properties that, they are regular on the axis, asymptotically flat away from the axis and free from the curvature singularities. One of the parameters of the solution measures the azimuthal angle deficit that the space time exhibits and signals the presence of infinite length of a straight cosmic string.

One of the solution given in paper II, is just the Einstein-Maxwell extension of paper I. In that solution, it was shown that the deficit angle near the axis or away from that axis is independent of the electromagnetic parameter. For a particular value of deficit parameter  $\alpha$ , the deficit angle vanishes and the solution represents propagation of cylindrical gravitational and electromagnetic fields.

In this paper we present the Einstein-Scalar (ES) extension of paper I, with the properties outlined in the third paragraph. The solution describes the interaction of a spinning cosmic string with cylindrical gravitation and scalar fields. Our solution is one-parameter generalization of paper I. In addition to the parameters of paper I, we introduce a parameter

$\beta$  that controls the intensity of the scalar field, such that for  $\beta = 0$  the solution reduces to the one given in paper I. In our analysis we have shown that in contrast to the paper II, the deficit angle becomes dependent on the scalar field parameter  $\beta$ .

The paper is organised as follows; in section II we review the solution given in paper I. Section III deals with the construction of the Einstein-Scalar solution. In section IV we discussed the physical properties of the solution. We conclude the paper with a discussion in section V.

## II. REVIEW OF THE XANTHOPOULOS'S SOLUTION

This solution represents a three parameters time-dependent cylindrically symmetric solutions of the vacuum Einstein equations. The interesting properties of the solution are,

- (a) it is asymptotically flat.
- (b) it admits a regular axis and
- (c) it is free of curvature singularities and
- (d) it exhibits an angle deficit in going around the axis and it is interpreted as an infinitely long cosmic strings surrounded by gravitational field.

Furthermore, the solution is Petrov type-D, and describes the propagation of non-radiating gravitational waves. The non-radiating property implies the stability of the cosmic string when it is interacting with gravitational waves. The technique used by Xanthopoulos in obtaining the solution is the one introduced by Chandrasekhar. In order to provide the regularity on the axis and asymptotic flatness behavior, he adopts the prolate coordinate system which is found very useful in the description of interacting plane gravitational waves. The following metric is obtained in these prolate coordinate system.

$$ds^2 = \alpha^2 X \left( \frac{d\tau^2}{\Delta} - \frac{d\sigma^2}{\delta} \right) - \frac{\Delta\delta X}{Y} d\varphi^2 - \frac{Y}{X} (dz - q_2 d\varphi)^2 \quad (1)$$

where,

$$\begin{aligned} X &= (1 - p\tau)^2 + q^2\sigma^2 \\ Y &= p^2\tau^2 + q^2\sigma^2 - 1 = p^2\Delta + q^2\delta \\ q_2 &= \frac{2q\delta(1 - p\tau)}{pY} \\ \Delta &= \tau^2 + 1 \\ \delta &= \sigma^2 - 1 \end{aligned} \quad (2)$$

such that  $\tau \in \Re$  and  $\sigma \geq 1$ . The parameters  $\alpha, p$  and  $q$  are constants with  $q^2 - p^2 = 1$ . Using the following transformation

$$\omega = \sqrt{\Delta\delta}, \quad t = \tau\sigma$$

line element (1) transforms into cylindrical coordinates

$$ds^2 = \frac{\alpha^2 X}{\tau^2 + \sigma^2} (dt^2 - d\omega^2) - \frac{\omega^2 X}{Y} d\varphi^2 - \frac{Y}{X} (dz - q_2 d\varphi)^2 \quad (3)$$

with

$$2\tau^2 = \sqrt{D} + t^2 - \omega^2 - 1, \quad 2\sigma^2 = \sqrt{D} - t^2 + \omega^2 + 1$$

where

$$\sqrt{D} = (\omega^2 - t^2 + 1)^2 + 4t^2 \geq 0.$$

Therefore the solution is analysed in the cylindrical coordinates while the field equations are solved in some other coordinates. With reference to the detailed analysis in paper I, the behavior of the metric functions near the axis ( $\omega \rightarrow 0^+$ ) and asymptotically ( $\omega \rightarrow +\infty$ ) are obtained by using the following relations.

Near the axis  $\omega \ll |t|$ ,  $t = \text{finite}$

$$\begin{aligned} \tau &\simeq t - \frac{\omega^2 t}{2(1+t^2)} + O(\omega^4) \\ \sigma &\simeq 1 + \frac{\omega^2}{2(1+t^2)} + O(\omega^4) \end{aligned} \quad (4)$$

Asymptotically  $\omega \gg |t|$

$$\begin{aligned} \tau &\simeq \frac{t}{\omega} + \frac{t(t^2-1)}{2\omega^3} + O(\omega^{-4}) \\ \sigma &\simeq \omega + \frac{(1-t^2)}{2\omega} + O(\omega^{-2}) \end{aligned} \quad (5)$$

In paper I, it was shown that the deficit angle which indicates the presence of cosmic strings obtained as

$$\delta\varphi = 2\pi(1 - e^{-C}) \quad (6)$$

where,  $C$  is the  $C$ -energy and it was also shown that this energy is non-radiating. The absence of radiation can also be justified by calculating the Weyl scalars, which are all vanishing in the asymptotic region.

### III. CONSTRUCTION OF THE EINSTEIN-SCALAR SOLUTION

Cylindrical gravitational waves with cross polarization are described in general by the line element of Jordan- Ehlers-Kundt- Kompaneetz [3] as

$$ds^2 = e^{2(\gamma-\psi)} (dt^2 - d\omega^2) - \omega \left[ \omega e^{-2\psi} d\varphi^2 + \frac{e^{2\psi}}{\omega} (dz + q_2 d\varphi)^2 \right] \quad (7)$$

In order to couple massless scalar field to the system we write the Lagrangian density as follows

$$L = (\lambda_\omega \gamma_\omega - \lambda_t \gamma_t) - \lambda [\psi_\omega^2 - \psi_t^2 + 2(\phi_\omega^2 - \phi_t^2)] - \frac{e^{4\psi}}{4\lambda} (q_{2\omega}^2 - q_{2t}^2) \quad (8)$$

where  $\phi$  is the scalar field and  $\gamma = \gamma(\omega, t)$ ,  $\psi = \psi(\omega, t)$ ,  $\phi = \phi(\omega, t)$ ,  $q_2 = q_2(\omega, t)$  and  $\lambda$  is a coordinate condition and for the present problem we choose it as  $\lambda = \omega$ .

The Einstein-Scalar field equations, which are obtained by varying the Lagrangian described in equation (8), are

$$\psi_{tt} - \frac{\psi_\omega}{\omega} - \psi_{\omega\omega} = \frac{e^{4\psi}}{2\omega^2} (q_{2t}^2 - q_{2\omega}^2) \quad (9)$$

$$q_{2tt} + \frac{q_{2\omega}}{\omega} - q_{2\omega\omega} = 4(q_{2\omega}\psi_\omega - q_{2t}\psi_t) \quad (10)$$

$$\gamma_\omega = \omega(\psi_t^2 + \psi_\omega^2) + \frac{e^{4\psi}}{4\omega} (q_{2t}^2 + q_{2\omega}^2) + 2\omega(\phi_\omega^2 + \phi_t^2) \quad (11)$$

$$\gamma_t = 2\omega\psi_\omega\psi_t + \frac{e^{4\psi}}{2\omega} q_{2t}q_{2\omega} + 4\omega\phi_\omega\phi_t \quad (12)$$

$$\phi_{\omega\omega} + \frac{\phi_\omega}{\omega} - \phi_{tt} = 0 \quad (13)$$

The solution to these equations are obtained by employing the formalism of Chandrasekhar. Hence, we prefer to use the notation of Chandrasekhar [5].

If we set

$$\nu = \gamma - \psi \quad \text{and} \quad \chi = \omega e^{-2\psi} \quad (14)$$

Line element (7) becomes

$$ds^2 = e^{2\nu} (dt^2 - d\omega^2) - \omega [\chi d\varphi^2 + \chi^{-1} (dz - q_2 d\varphi)^2] \quad (15)$$

where  $\frac{\partial}{\partial z}$  and  $\frac{\partial}{\partial \varphi}$  are the axial and azimuthal Killing fields and  $\nu$ ,  $\chi$  and  $q_2$  are functions of  $\omega$  and  $t$ . Using equation (14) in the field equations (11) and (12) they transform into

$$4\nu_\omega = -\frac{1}{\omega} + \frac{\omega}{\chi} [\chi_t^2 + \chi_\omega^2 + q_{2t}^2 + q_{2\omega}^2] + 8\omega(\phi_\omega^2 + \phi_t^2) \quad (16)$$

$$\nu_t = \frac{\omega}{2\chi^2} [\chi_t\chi_\omega + q_{2t}q_{2\omega}] + 4\omega\phi_\omega\phi_t \quad (17)$$

respectively. In terms of the Ernst potentials ( $\Psi$  and  $\Phi$ ) defined in paper I equations (16) and (17) become

$$\left(\nu + \ln \sqrt{\Psi}\right)_t = \frac{\omega}{2\Psi^2} (\Psi_t\Psi_\omega + \Phi_t\Phi_\omega) + 4\omega\phi_\omega\phi_t \quad (18)$$

$$\left(\nu + \ln \sqrt{\Psi}\right)_\omega = \frac{\omega}{4\Psi^2} (\Psi_t^2 + \Psi_\omega^2 + \Phi_t^2 + \Phi_\omega^2) + 2\omega(\phi_\omega^2 + \phi_t^2) \quad (19)$$

The scalar field  $\phi$  is coupled to the system by shifting the metric function  $\nu$  in accordance with [8]

$$\nu = \nu_0 + \Gamma \quad (20)$$

where

$$\begin{aligned} \Gamma_t &= 4\omega\phi_\omega\phi_t \\ \Gamma_\omega &= 2\omega(\phi_\omega^2 + \phi_t^2) \end{aligned} \quad (21)$$

and  $\nu_0, \chi_0$  and  $q_{20}$  satisfy the vacuum Einstein equations and  $\Gamma$  is the additional metric function that arises due to the presence of the scalar field. Integrability condition for the equations (21) imposes the massless scalar field equation (13) as a constraint condition from which we can generate a large class of ES solution. The solution to the ES field equations will be obtained in the prolate coordinates as a requirement of the formalism of Chandrasekhar. If we set,

$$\omega = \sqrt{\Delta\delta}, \quad t = \tau\sigma, \quad \Delta = \tau^2 + 1, \quad \delta = \sigma^2 - 1$$

we can express the theory of cylindrical gravitational and scalar waves in the  $(\tau, \sigma)$  coordinates whose range are

$$\tau \in \Re \quad \sigma \geq 1$$

note that  $\sigma = 1$  corresponds to the axis  $\omega = 0$ . We find that line element (15) becomes

$$ds^2 = (\tau^2 + \sigma^2) e^{2(\nu_0 + \Gamma)} \left[ \frac{d\tau^2}{\Delta} - \frac{d\sigma^2}{\delta} \right] - \sqrt{\Delta\delta} [\chi_0 d\varphi^2 + \chi_0^{-1} (dz - q_{20} d\varphi)^2] \quad (22)$$

In this paper we wish to couple a massless scalar field to the solution presented in paper I. Hence as an Einstein-vacuum solution  $\nu_0, \chi_0$  and  $q_{20}$ , we shall make use of the solution obtained in paper I that describes the cylindrical gravitational waves with interesting properties discussed. Therefore the resulting metric that describes ES solution is given by

$$ds^2 = \frac{\alpha^2 X}{\tau^2 + \sigma^2} e^{2\Gamma} (dt^2 - d\omega^2) - \frac{\omega^2 X}{Y} d\varphi^2 - \frac{Y}{X} (dz - q_2 d\varphi)^2 \quad (23)$$

where  $X, Y$  and  $q_2$  are given in equation (2). In terms of the new coordinates equation (21) becomes

$$\begin{aligned} \Gamma_\sigma &= \frac{2\Delta}{\tau^2 + \sigma^2} [\sigma\delta\phi_\sigma^2 + \sigma\Delta\phi_\tau^2 - 2\tau\delta\phi_\sigma\phi_\tau] \\ \Gamma_\tau &= \frac{2\delta}{\tau^2 + \sigma^2} [2\sigma\Delta\phi_\sigma\phi_\tau - \tau\delta\phi_\sigma^2 - \tau\Delta\phi_\tau^2] \end{aligned} \quad (24)$$

The massless scalar field equation in terms of  $(\tau, \sigma)$  may be written as

$$(\Delta\phi_\tau)_\tau - (\delta\phi_\sigma)_\sigma = 0 \quad (25)$$

This equation has many solutions. Among the others we wish to adopt as a scalar field, Bonnor's non-singular cylindrical wave solutions found long time ago [9]. The interesting property of this solution is that, it is regular near the axis and vanishes asymptotically. In terms of  $(\tau, \sigma)$  coordinates the scalar field is given by

$$\phi(\tau, \sigma) = \frac{\beta\sigma}{\tau^2 + \sigma^2} \quad (26)$$

where  $\beta$  is a constant parameter measuring the intensity of the scalar wave. Using equation (24) the additional metric function  $\Gamma$  is found as

$$\Gamma(\tau, \sigma) = \frac{\beta^2\Delta}{\tau^2 + \sigma^2} \left[ \frac{4\tau^4\Delta}{(\tau^2 + \sigma^2)^3} - \frac{4\tau^2(1 + 2\tau^2)}{(\tau^2 + \sigma^2)^2} + \frac{1 + 9\tau^2}{2(\tau^2 + \sigma^2)} - 1 \right]$$

Extending this to the Einstein-Maxwell-Scalar version of paper II is straightforward.

## IV. PHYSICAL PROPERTIES OF THE SOLUTION

### A. Boundary Conditions

In order to interpret the present solution physically acceptable, we should impose certain conditions on the behavior of the metric functions near the symmetry axis and asymptotically far away from the axis. These conditions, given in paper I, are as follows:

(a) The symmetry axis should be regular when  $\omega \rightarrow 0$ . This means that, the squared norm of the rotational (or azimuthal) Killing field  $\left(\frac{\partial}{\partial\varphi}\right)$

$$\left|\frac{\partial}{\partial\varphi}\right|^2 = -\omega\frac{(\chi^2 + q_2^2)}{\chi} \quad (27)$$

should approach zero like  $\omega^2$  near  $\omega = 0$ . This condition guarantees that the axis is regular.

(b) Asymptotically far away from the symmetry axis, i.e. in all directions such that  $\omega \rightarrow \infty$ , the solution should become asymptotically flat. It should be noted that this solution is not asymptotically simple in the sense of Penrose. The reason is the spatial infinity in the direction  $z \rightarrow \infty$  cannot be carried out, because,  $\frac{\partial}{\partial z}$  is a Killing vector and the metric functions are independent of  $z$ .

### B. Behaviour Near The Axis : $\omega \ll t$ ( $\omega \rightarrow 0$ )

Using the expressions given in equation (4) we find the behavior of the metric functions near the axis ( $\omega \rightarrow 0^+$ ) as follows

$$\begin{aligned} 1 - p\tau &\simeq 1 - pt + O(\omega^2) \\ \delta &\simeq \frac{\omega^2}{t^2 + 1} + O(\omega^4) \end{aligned} \quad (28)$$

$$\begin{aligned}
X &\simeq [(1-pt)^2 + q^2] + O(\omega^2) \\
Y &\simeq p^2(1+t^2) + O(\omega^2) \\
\tau^2 + \sigma^2 &\simeq 1 + t^2 + O(\omega^2) \\
e^{2\Gamma} &= e^{-\beta^2}
\end{aligned}$$

so that:

$$\begin{aligned}
\chi &\simeq \frac{A}{p^2}\omega + O(\omega^3) \\
q_2 &\simeq \frac{2q(1-pt)}{p^3(1+t^2)^2}\omega^2 + O(\omega^3) \\
\frac{\chi^2 + q_2^2}{\chi} &\simeq \frac{A}{p^2}\omega + O(\omega^2)
\end{aligned} \tag{29}$$

where

$$A = \frac{(1-pt)^2 + q^2}{1+t^2}$$

The norms of the two Killing fields behave like

$$\begin{aligned}
\left| \frac{\partial}{\partial \varphi} \right|^2 &= -\omega \frac{\chi^2 + q_2^2}{\chi} \simeq -\frac{A}{p^2}\omega^2 + O(\omega^3) \\
\left| \frac{\partial}{\partial z} \right|^2 &= -\frac{\omega}{\chi} \simeq \frac{p^2}{A} + O(\omega)
\end{aligned} \tag{30}$$

We show that in the limit  $\omega \rightarrow 0^+$  is a regular surface of the spacetime. Note that, the coupling of the scalar field does not play a role in the behavior of the two Killing fields. Using expressions (28 -29 ), we find the metric near the axis;

$$ds^2 \simeq \alpha^2 A e^{-\beta^2} \left[ dt^2 - d\omega^2 - \frac{\omega^2 e^{\beta^2}}{p^2 \alpha^2} d\varphi^2 \right] - \frac{p^2}{A} dz^2 + \frac{4q(1-pt)\omega^2}{p(1+t^2)^2 A} dz d\varphi + O(\omega^3) \tag{31}$$

### C. Asymptotic Behaviour : $\omega \gg |t|$ ( $\omega \rightarrow \infty$ )

Using the expressions (5), we find that

$$\begin{aligned}
X &\simeq q^2\omega^2 + (1 + q^2 - q^2t^2) + O(\omega^{-1}) \\
Y &\simeq q^2\omega^2 + (p^2 - q^2t^2) + O(\omega^{-1}) \\
\delta &\simeq \omega^2 - t^2 + O(\omega^{-1}) \\
q_2 &\simeq \frac{2}{pq} \left[ 1 - \frac{pt}{\omega} \right] + O(\omega^{-2}) \\
\tau^2 + \sigma^2 &\simeq \omega^2 + (1 - t^2) + O(\omega^{-1}) \\
\Gamma &\simeq 0
\end{aligned} \tag{32}$$



Line element (23) , asymptotically becomes;

$$ds^2 \simeq \alpha^2 q^2 \left[ dt^2 - d\omega^2 - \frac{\omega^2}{\alpha^2 q^2} d\varphi^2 \right] - \left( dz - \frac{2}{pq} d\varphi \right)^2 \quad (33)$$

by letting  $z \rightarrow \tilde{z} = z - \frac{2}{pq}\varphi$ , the last term in the metric is changed and metric (23) becomes asymptotically flat.

#### D. Existence of Cosmic Strings

Comparing the two metrics obtained for near the axis ( $\omega \rightarrow 0$ ) and asymptotic case ( $\omega \rightarrow \infty$ ), we observe that exact metric (23) allows an angle deficit measured by  $\frac{e^{\beta^2}}{\alpha^2 p^2}$  near the axis and  $\frac{1}{\alpha^2 q^2}$  asymptotically far away from the axis. This can be shown as follows.

For any metric in the form of

$$ds^2 = f [d\omega^2 + k^2 \omega^2 d\varphi^2] + \dots \quad (34)$$

with constant  $k$  and  $f$  independent of  $\omega$ , the angle deficit near the axis ( $\omega \rightarrow 0$ ) or asymptotically ( $\omega \rightarrow \infty$ ), can be obtained from the definition:

$$\delta\varphi = 2\pi - \lim_{\omega \rightarrow 0(\infty)} \frac{\int_0^{2\pi} \sqrt{g_{\varphi\varphi}} d\varphi}{\int_0^\omega \sqrt{g_{\omega\omega}} d\omega} \quad (35)$$

Using the above definition, we find the deficit angle as,

$$\delta\varphi = 2\pi (1 - k) \quad (36)$$

Exact metric (23) displays a conical singularity on the axis, measured by the angle deficit, which signals the existence of the cosmic string on the axis.

$$(\delta\varphi)_{ax} = 2\pi \left( 1 - \frac{e^{\frac{\beta^2}{2}}}{|\alpha| |p|} \right) \quad (37)$$

For  $\alpha = e^{\frac{\beta^2}{2}} |p|^{-1}$  it removes the angle deficit and implies the absence of the cosmic string. The axis becomes locally flat, and the metric (23) represents the propagation of coupled cylindrical gravitational and scalar waves.

In references [10, 11, 12, 13] it has been shown that the mass per unit length of the string  $\mu_0$  is related to the angle deficit by

$$8\pi\mu_0 = \delta\varphi \quad (38)$$

Hence, the mass density  $\mu_0$  for the present paper becomes

$$\mu_0 = \frac{1}{4} \left( 1 - \frac{e^{\frac{\beta^2}{2}}}{|\alpha| |p|} \right) \quad (39)$$

Note that the mass per unit length  $\mu_0$  is constant, does not depend on time  $t$ . Asymptotically the spacetime exhibits an angle deficit given by

$$(\delta\varphi)_{asy} = 2\pi \left( 1 - \frac{1}{|\alpha||q|} \right) \quad (40)$$

For the value of  $\alpha = e^{\frac{\beta^2}{2}} |p|^{-1}$  eliminates the angle deficit near the axis while asymptotically the angle deficit becomes

$$(\delta\varphi)_{asy} = 2\pi \left( 1 - \frac{p}{qe^{\frac{\beta^2}{2}}} \right) \quad (41)$$

Since  $q^2 - p^2 = 1$ , this implies that  $|q| > |p|$  and gives a larger angle deficit compared to the result obtained in paper I.

The choice  $\alpha = |q|^{-1}$ , removes the asymptotic angle deficit and the angle deficit near the axis becomes

$$(\delta\phi)_{ax} = 2\pi \left( 1 - \frac{qe^{\frac{\beta^2}{2}}}{p} \right) \quad (42)$$

This choice causes a larger negative angle surplus near the axis compared with the paper I. The choice  $\alpha > e^{\frac{\beta^2}{2}} |p|^{-1}$  imposes deficit angle both in asymptotically and near the axis. Note that; for this particular case asymptotic angle deficit is greater than the angle deficit near the axis. This property is also encountered in the analysis of paper I and paper II. It has been shown in paper II that, the angle deficit near the axis and asymptotically, are independent of electromagnetic parameter. However, in our case the angle deficit near the axis depends on the constant parameter  $\beta$ , which measures the intensity of the scalar wave but asymptotically the angle deficit is independent of the scalar field. This is natural and expected because the intensity of the scalar field is maximum near the axis and approaches zero for far away from the axis. Hence, in contrast to the electromagnetic case (paper II), intervening gravitational and scalar waves contribute to the angle deficit.

## E. Discussion of Energy

One of the important elements in cylindrically symmetric systems is the  $C$ -energy introduced by Thorne [14]. This  $C$ -energy represents the total gravitational and scalar energy per unit length between  $\omega = 0$  and  $\omega$  at any time  $t$ . The  $C$ -energy in the present paper is described by the quantity.

$$C = \nu + \ln \sqrt{\Psi} + \Gamma \quad (43)$$

which is equivalent to

$$C = \frac{1}{2} \ln \left[ \frac{\alpha^2 (p^2 \tau^2 + q^2 \sigma^2 - 1)}{\tau^2 + \sigma^2} \right] + \frac{\beta^2 \Delta}{\tau^2 + \sigma^2} \left[ \frac{4\tau^4 \Delta}{(\tau^2 + \sigma^2)^3} - \frac{4\tau^2 (1 + 2\tau^4)}{(\tau^2 + \sigma^2)^2} + \frac{1 + 9\tau^2}{2(\tau^2 + \sigma^2)} - 1 \right] \quad (44)$$

Near the axis (when  $\sigma = 1$  or  $\omega \rightarrow 0$ ), equation (44) gives

$$C \simeq \ln |\alpha p| - \frac{\beta^2}{2} \quad (\omega \rightarrow 0) \quad (45)$$

while asymptotically it becomes

$$C \simeq \ln |\alpha q| + \ln \left| 1 - \frac{1}{2q^2\omega^2} \right| \quad (w \rightarrow \infty) \quad (46)$$

Using equation (43) in equations (18) and (19) we obtained

$$C_t = \frac{\omega}{2\Psi^2} (\Psi_t\Psi_\omega + \Phi_t\Phi_\omega) + 4\omega\phi_\omega\phi_t \quad (47)$$

$$C_\omega = \frac{\omega}{4\Psi^2} (\Psi_t^2 + \Phi_t^2 + \Psi_\omega^2 + \Phi_\omega^2) + 2\omega (\phi_\omega^2 + \phi_t^2) \quad (48)$$

We introduce the null coordinates

$$\begin{aligned} u &= t + \omega \\ v &= t - \omega \end{aligned} \quad (49)$$

so that future null infinity corresponds to  $u \rightarrow \infty$  with finite  $v$  while past null infinity corresponds to  $v \rightarrow -\infty$  with finite  $u$ . Equations (18) and (19) take the form

$$C_u = \frac{1}{2} (C_t + C_\omega) > 0 \quad (50)$$

$$C_v = \frac{1}{2} (C_t - C_\omega) \quad (51)$$

where equation (51) measures the rate of radiation of the  $C$  energy. For the present paper, equation (51) in terms of  $(\tau, \sigma)$  coordinates takes the form

$$C_v = -\frac{(\sigma\sqrt{\Delta} + \tau\sqrt{\delta})^2}{(\tau^2 + \sigma^2)^2} \left\{ \frac{\sqrt{\Delta\delta}}{2Y} + \frac{\omega\beta^2}{(\tau^2 + \sigma^2)^4} \left[ \sqrt{\delta} (\tau^2 - \sigma^2) + 2\tau\sigma\sqrt{\Delta} \right]^2 \right\} \quad (52)$$

Using the following asymptotic behavior

$$\begin{aligned} \sqrt{\Delta} &\rightarrow 1 & \sqrt{\delta} &\rightarrow \omega & \tau^2 + \sigma^2 &\simeq \omega^2 - t^2 + 1 \\ \tau^2 - \sigma^2 &\simeq -\omega^2 + t^2 - 1 & Y &\simeq q^2\omega^2 + (p^2 - q^2t^2) \\ \sigma &\simeq \omega + \frac{1-t^2}{2\omega} & \tau &\simeq \frac{t}{\omega} + \frac{t(t^2-1)}{2\omega^3} \end{aligned} \quad (53)$$

we obtain that

$$C_v \simeq -\frac{1}{2\omega^3} \left( 1 + \frac{2t}{\omega} \right) \left( \frac{1}{q^2} + 2\beta^2 \right) + O(\omega^{-5}) \quad (54)$$

It is clear to observe that  $\lim_{\omega \rightarrow \infty} C_v = 0$  and the solution is non-radiating

## F. The Weyl and Ricci Scalars

The description of the solution in terms of the Weyl and Ricci scalars are obtained by Newman-Penrose formalism. We introduce new coordinates  $(\theta, \psi)$  by

$$\begin{aligned}\tau &= \sinh \psi \\ \sigma &= \cosh \theta\end{aligned}\tag{55}$$

The line element (23) describing ES solution takes the form

$$ds^2 = U^2 (d\psi^2 - d\theta^2) - \frac{V^2}{1 - \varepsilon\varepsilon^*} [(1 - \varepsilon) dz + i(1 + \varepsilon) d\varphi]\tag{56}$$

where

$$U^2 = \alpha^2 X e^{2\Gamma}$$

and a suitable null basis for the metric is

$$\begin{aligned}l &= \frac{U}{\sqrt{2}} (d\psi - d\theta) \\ n &= \frac{U}{\sqrt{2}} (d\psi + d\theta) \\ m &= -\frac{1}{\sqrt{2}} (VL_-^* dz - iVL_+^* d\varphi)\end{aligned}\tag{57}$$

Here

$$L_{\pm} = \frac{1 \pm \varepsilon}{\sqrt{1 - |\varepsilon|^2}}$$

$$V^2 = \cosh \psi \sinh \theta$$

$$\varepsilon = \frac{Z - 1}{Z + 1} \quad \text{where} \quad Z = \chi + iq_2$$

and  $*$  denotes the complex conjugate. The exact form of the Weyl and Ricci scalars are too long and complicated. Hence, we prefer to give their asymptotic forms

$$\psi_2 \simeq -\frac{i}{2\alpha^2 q^3 \omega^3}\tag{58}$$

$$\psi_0 \simeq -\frac{3i}{2\alpha^2 q^3 \omega^3} + \frac{\beta^2}{2\alpha^2 q^2 \omega^4} + O(\omega^{-5})\tag{59}$$

$$\psi_4 \simeq -\frac{3i}{2\alpha^2 q^3 \omega^3} + \frac{\beta^2}{2\alpha^2 q^2 \omega^4} + O(\omega^{-5}) \quad (60)$$

$$\phi_{00} \simeq \frac{\beta^2}{2\alpha^2 q^2 \omega^4} \left(1 - \frac{1}{\omega^2}\right) + O(\omega^{-7}) \quad (61)$$

$$\phi_{22} \simeq \frac{\beta^2}{2\alpha^2 q^2 \omega^4} \left(1 - \frac{1}{\omega^2}\right) + O(\omega^{-7}) \quad (62)$$

$$\phi_{11} \simeq \frac{\beta^2}{2\alpha^2 q^2 \omega^4} \left(1 + \frac{q-11}{2\omega^2}\right) + O(\omega^{-7}) \quad (63)$$

$$\Lambda \simeq -\frac{\beta^2}{6\alpha^2 q^2 \omega^4} \left(1 + \frac{q-11}{2\omega^2}\right) + O(\omega^{-7}) \quad (64)$$

The asymptotic dying-off of the Weyl (like  $\omega^{-3}$ ) and Ricci (like  $\omega^{-4}$ ) scalars indicates that the present solution is non-radiating. This is totally in agreement with the  $C$ -energy discussion.

The non-radiating character of the present solution indicates the stability of the cosmic string in the presence of a scalar field as well.

It should be noted that the inclusion of scalar field changes the geometric interpretation of the resulting spacetime from Petrov type-D to Petrov type-I.

## V. DISCUSSION

In this paper, we have given the scalar field generalization of the vacuum Einstein solution of paper I, that describes the interaction of rotating cosmic strings with gravitational waves. The obtained solution is a kind of special solution that preserves the properties of the background solution; that is, regular on the axis, asymptotically flat and free from the curvature singularities. It is shown that the presence of a scalar field contributes to the angle deficit near the axis. However in the asymptotic case, the angle deficit is produced completely from the gravitational waves. Our analysis shows that, the effect of the scalar field tends to increase the angle deficit. We also note that the inclusion of the scalar field did not change the stability character of the cosmic string. This is not surprising because the character of the inserted scalar field is well-behaved and asymptotically vanished. Addition of a different scalar field which does not behave well will change all these nice features completely.

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