Some Properties of Hypergeometric Functions

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ABSTRACT

This thesis consists of five chapters. The first chapter gives brief information about the thesis. In the second chapter, we give some preliminaries and auxilary results which we will use in thesis.

In chapter three, the extension of beta function containing an extra parameter, which proved to be useful earlier, was used to extend Appell's hypergeometric functions of two variables and extend Lauricella's hypergeometric function of three variables. Furthermore, linear and bilinear generating relations for these extended hypergeometric functions are obtained by defining the extension of fractional derivative operator. Some properties of the extended fractional derivative operator are also presented.

In chapter four, we consider generalizations of gamma, beta and hypergeometric functions. Some recurrence relations, transformation formulas, operation formulas and integral representations are obtained for these new generalizations.

In chapter five, we present various families of generating functions for a class of polynomials in two variables. Furthermore, several general classes of bilinear, bilateral or mixed multilateral generating functions are obtained for these polynomials.

Keywords: Generating functions, Hypergeometric function, Fractional derivative operator, Gamma function, Beta function.

ÖZ

Bu tez beş bölümden oluşmuştur. Birinci bölümde tezin içeriği ile ilgili genel bilgiler verilmiştir. İkinci bölümde, tez boyunca kullanılacak olan temel bilgiler ve sonuçlar verilmiştir.

Üçüncü bölümde, daha önceden kullanışlı olduğu ıspatlanmış olan ve ekstra bir parametre içeren genişletilmiş beta fonksiyonu kullanılarak, iki değişkenli genişletilmiş Appell hipergeometrik fonksiyonları ve üç değişkenli genişletilmiş Lauricella hipergeometrik fonksiyonları verilmiştir. Yine bu bölümde, yeni bir kesirli türev operatörü tanımlanarak, genişletilmiş hipergeometrik fonksiyonlar için lineer ve bilineer doğurucu fonksiyon bağıntıları elde edilmiştir. Genişletilmiş kesirli türev operatörünün bazı özellikleri de sunulmuştur.

Dördüncü bölümde, genişletilmiş gamma, beta ve hipergeometrik fonksiyonlar ele alınmıştır. Bu yeni genelleşmeler için, bazı rekürans bağıntıları, dönüşüm formülleri ve integral gösterimler elde edilmiştir.

Beşinci bölümde, iki değişkenli polinom sınıfı için bir çok doğurucu fonksiyon aileleri sunulmuştur. Yine bu bölümde, bu polinomlar için daha geniş bilineer, bilateral ve karışık multilateral doğurucu fonksiyon sınıfları elde edilmiştir.

Anahtar Kelimeler: Doğurucu fonksiyon, Hipergeometrik fonksiyon, Kesirli türev operatörü, Gamma fonksiyonu, Beta fonksiyonu.

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LIST OF SYMBOLS

$\Gamma(z)$	Gamma function
$B(x,y)$	Beta function
${}_2F_1(a,b;c;z)$	Gauss hypergeometric function
$\phi(a;c;z)$	Confluent hypergeometric function
${}_pF_q(a_1,...,a_p;b_1,...,b_q;z)$	Generalized hypergeometric function
$F_1(a,b,c;d;x,y)$	First Appell's hypergeometric functions in two variables
$F_2(a,b,c;d,e;x,y)$	Second Appell's hypergeometric functions in two variables
$F_D^3\,(a,b,c,d;e;x,y,z)$	Lauricella's hypergeometric functions in three variables
$\Gamma_p^{(\alpha,\beta)}(z)$	generalization of gamma function
$B_p^{(\alpha,\beta)}(x,y)$	generalization of beta function
$F_p^{(\alpha,\beta)}(a,b;c;z)$	generalized Gauss hypergeometric function
$\phi_p^{(\alpha,\beta)}(a;c;z)$	generalized confluent hypergeometric function
$F_1(a,b,c;d;x,y;p)$	extended First Appell's hypergeometric functions in two variables
$F_2(a,b,c;d,e;x,y;p)$	extended Second Appell's hypergeometric functions in two variables
$F_{D,p}^3\,(a,b,c,d;e;x,y,z)$	extended Lauricella's hypergeometric functions in three variables
$g_n^{(\alpha,\beta)}(x,y)$	Lagrange polynomials in two variables
$h_n^{(\alpha,\beta)}(x,y)$	Lagrange-Hermite polynomials of two variables
$H_n(x,y)$	Hermite-Kampé de Feriét polynomials of two variables
$P_n^{(\alpha,\beta)}(z)$	Jacobi polynomials
D_z^{μ}	Riemann-Liouville fractional derivative of order μ
$D_z^{\mu,p}$	extended Riemann-Liouville fractional derivative of order μ
$\mathfrak{M}\{f:s\}$	Mellin transform of f

Chapter 1

INTRODUCTION

Many important functions in applied sciences are defined via improper integrals or series (or infinite products). The general name of these important functions are called special functions. The most famous among them is the gamma function. The gamma function was first introduced by the Swiss mathematician Leonhard Euler (1707-1783) in his goal to generalize the factorial function to non-integer numbers (real and complex numbers). Later, gamma function was studied by other famous mathematicians like Adrien-Marie Legendre (1752-1833), Carl Friedrich Gauss (1777-1855), Christoph Gudermann (1798-1852), Joseph Liouville (1809-1882), Karl Weierstrass (1815-1897), Charles Hermite (1822-1901).

For a complex number z with positive real part $(\mathrm{Re}\,(z) > 0)$, the *Gamma function* is defined by

$$\Gamma(z) := \int_0^\infty t^{z-1}\, \exp(-t)\, dt.$$

In studying the gamma function, Euler discovered another function, called the *beta function*,

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

$$(\mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0)$$

which is closely related to $\Gamma(z)$.

Gauss investigated hypergeometric series, which had in fact already been defined and named by Wallis in the 1650s. He noted that the ${}_2F_1$ or Gauss hypergeometric function actually covered a lot of known special functions. The *Gauss hypergeometric function* (GHF) is defined by

$$ {}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, $$

$$ (|z|<1;\ \mathrm{Re}(c)>\mathrm{Re}(b)>0,\ c\neq 0,-1,-2,...). $$

The *Confluent hypergeometric function* (CHF) $\phi(a;c;z)$ is also known as the *Kummer function* which is defined by

$$ \phi(a;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\frac{z^n}{n!}, $$

$$ (\mathrm{Re}(c)>\mathrm{Re}(a)>0). $$

In recent years, several extensions of the well known special functions (gamma, beta etc.) have been considered by several authors [8], [4], [7], [6], [21], [22]. In [8] and [4], M.A. Chaudhry et al. defined extension of gamma function and Euler's beta functions by

$$ \Gamma_p(x) := \int_0^{\infty} t^{x-1}\exp\left(-t-pt^{-1}\right)dt, $$

$$ (\mathrm{Re}\,(x)>0, \mathrm{Re}\,(p)>0) $$

and

$$ B_p\,(x,y) := \int_0^1 t^{x-1}\,(1-t)^{y-1}\exp\left[-\frac{p}{t(1-t)}\right]dt, $$

$$ (\mathrm{Re}(p)>0, \mathrm{Re}(x)>0, \mathrm{Re}(y)>0) $$

respectively. For which $p=0$ gives the original gamma and beta functions. Then they have been proved that this extension has connection with Macdonald, error and Whittaker's functions. Afterwards, in [5], M.A. Chaudhry et al. generalized the Gauss hypergeometric and confluent hypergeometric functions by

$$ F_p\,(a,b;c;z) = \sum_{n=0}^{\infty} \frac{B_p\,(b+n,c-b)}{B\,(b,c-b)}\,(a)_n\,\frac{z^n}{n!} $$

$$ (p\geq 0\,;\ \mathrm{Re}\,(c)>\mathrm{Re}\,(b)>0) $$

and

$$\phi_p\,(b;c;z) = \sum_{n=0}^{\infty} \frac{B_p\,(b+n, c-b)}{B\,(b, c-b)} \frac{z^n}{n!}$$

$$(p \geq 0\,;\ \mathrm{Re}\,(c) > \mathrm{Re}\,(b) > 0)$$

respectively. For which $p = 0$ gives the original Gauss hypergeometric and confluent hypergeometric functions. They gave the Euler type integral representations of the hypergeometric functions. Additionally, they have discussed the differentiation properties and Mellin transforms of $F_p\,(a,b;c;z)$ $(\phi_p\,(b;c;z))$ and obtained transformation formulas, recurrence relations, summation and asymptotic formulas for these functions.

We organize the thesis as follows:

In chapter 2, we give some preliminaries and auxilary results which are used throughout the thesis.

Chapter 3, 4 and 5 are the original parts of the thesis. It should be noted that the results obtained in these Chapter's were published in [27], [25] and [26] respectively.

In chapter 3, we obtain some linear and bilinear generating functions by means of the new defined extended Appell's functions $F_1\,(a,b,c;d;x,y;p)$, $F_2(a,b,c;d,e;x,y;p)$, and extended Lauricella's hypergeometric function $F_{D,p}^{3}\,(a,b,c,d;e;x,y,z)$ which are defined by

$$F_1(a,b,c;d;x,y;p) := \sum_{n,m=0}^{\infty} \frac{B_p(a+m+n, d-a)}{B(a, d-a)}\,(b)_n\,(c)_m\,\frac{x^n}{n!}\frac{y^m}{m!}$$

$$(\max\{|x|\,, |y|\} < 1),$$

$$F_2(a,b,c;d,e;x,y;p) := \sum_{n,m=0}^{\infty} \frac{(a)_{m+n}\,B_p\,(b+n, d-b)\,B_p\,(c+m, e-c)}{B\,(b, d-b)\,B\,(c; e-c)} \frac{x^n}{n!}\frac{y^m}{m!}$$

$$(|x| + |y| < 1),$$

and

$$F_{D,p}^{3}\left(a,b,c,d;e;x,y,z\right) := \sum_{m,n,r=0}^{\infty} \frac{B_{p}\left(a+m+n+r,e-a\right)\left(b\right)_{m}\left(c\right)_{n}\left(d\right)_{r}}{B\left(a,e-a\right)} \frac{x^{m}}{m!}\frac{y^{n}}{n!}\frac{z^{r}}{r!}$$

$$(\sqrt{|x|}+\sqrt{|y|}+\sqrt{|z|}<1),$$

respectively. Notice that the case $p=0$ gives the original functions. In obtaining these generating functions we consider a new fractional derivative operator [25], namely:

$$D_{z}^{\mu,p}\{f\left(z\right)\} = \frac{1}{\Gamma\left(-\mu\right)} \int_{0}^{z} f\left(t\right)\left(z-t\right)^{-\mu-1} \exp\left(\frac{-pz^{2}}{t(z-t)}\right) dt,$$

$$\left(\mathrm{Re}\left(\mu\right)<0, \mathrm{Re}\left(p\right)>0\right)$$

for which $p=0$ gives original fractional derivative operator.

In chapter 4, we present generalizations of gamma and beta functions [26] by

$$\Gamma_{p}^{(\alpha,\beta)}(x) := \int_{0}^{\infty} t^{x-1}\,{}_{1}F_{1}\left(\alpha;\beta;-t-\frac{p}{t}\right) dt$$

$$(\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(p)>0, \mathrm{Re}(x)>0)$$

and

$$B_{p}^{(\alpha,\beta)}\left(x,y\right) := \int_{0}^{1} t^{x-1}\left(1-t\right)^{y-1}\,{}_{1}F_{1}\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt,$$

$$(\mathrm{Re}(p)>0, \mathrm{Re}(x)>0, \mathrm{Re}(y)>0, \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0)$$

respectively. Then we use the new generalization of beta function to generalize the hypergeometric and confluent hypergeometric functions by

$$F_{p}^{(\alpha,\beta)}\left(a,b;c;z\right) := \sum_{n=0}^{\infty} \left(a\right)_{n} \frac{B_{p}^{(\alpha,\beta)}\left(b+n,c-b\right)}{B\left(b,c-b\right)} \frac{z^{n}}{n!}$$

$$(\mathrm{Re}(p)>0, \mathrm{Re}(c)>0, \mathrm{Re}(b)>0, \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0)$$

and

$$\phi_{p}^{(\alpha,\beta)}\left(b;c;z\right) := \sum_{n=0}^{\infty} \frac{B_{p}^{(\alpha,\beta)}\left(b+n,c-b\right)}{B\left(b,c-b\right)} \frac{z^{n}}{n!},$$

$$(\mathrm{Re}(p)>0, \mathrm{Re}(c)>0, \mathrm{Re}(b)>0, \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0)$$

respectively. The case $\alpha = \beta$ gives Chaudry's gamma, beta and hypergeometric functions. Then, we obtain some recurrence relations, transformation formulas, operation formulas and integral representations for these new generalizations.

In chapter 5, we consider the following family of bivariate polynomials,

$$S_n^{m,N}(x,y) := \sum_{k=0}^{\left[\frac{n}{N}\right]} A_{m+n,k} \frac{x^{n-Nk}}{(n-Nk)!} \frac{y^k}{k!}$$

$$(n, m \in \mathbb{N}_0; \quad N \in \mathbb{N}),$$

which was defined by Altın *et al.* [1]. We prove several general theorems involving various families of generating functions for the aforementioned polynomials in their *two-variable* notation $S_n^{m,N}(x,y)$ by applying the method which was used recently by Chen and Srivastava [12]. We also consider some applications and corollaries resulting from these theorems [27].

Chapter 2

PRELIMINARIES AND AUXILIARY RESULTS

2.1 Gamma, Beta Functions and their extended versions

In this section, we give definitions of the gamma and beta functions and their properties. Furthermore, we give extended forms of these special functions (see [2], [28]).

Definition 2.1.1. *(Euler Gamma function) For a complex number z with positive real part* $(\mathrm{Re}\,(z) > 0)$*, the gamma function is defined by*

$$\Gamma(z) := \int_0^{\infty} t^{z-1} \exp\left(-t\right) dt. \tag{2.1}$$

Using integration by parts, one can show that the following recurrance relation hold for $\Gamma(z)$:

$$\Gamma(z+1) = z\Gamma(z).$$

This functional equation generalizes the definition $n! = n(n-1)!$ of the factorial function. But also, evaluating $\Gamma\left(1\right)$ analytically we get

$$\Gamma\left(1\right) = 1.$$

Combining these two results we see that the factorial function is a special case of the gamma function:

$$\Gamma(n+1) = n\Gamma(n) = n\left(n-1\right)\Gamma(n-1) = ... = n!\Gamma(1) = n!.$$

Another useful function is a beta function.

Definition 2.1.2. *The beta function, (also called the Eulerian integral of the first kind) is defined by*

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$
$$(\mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0)$$

Equivalently,

$$B(x,y) := 2\int_0^{\frac{\pi}{2}} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\,d\theta,$$
$$(\mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0)$$

and

$$B(x,y) := \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}du.$$
$$(\mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0)$$

The beta function is symmetric, $B(x,y) = B(y,x)$ and it is related to the gamma function;

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.2}$$
$$(x, y \neq 0, -1, -2, ...)$$

In 1994, Chaudhry and Zubair [8] introduced the following extension of gamma function.

Definition 2.1.3. *(see [8], [10])The Extended Gamma function is defined by*

$$\Gamma_p(x) := \int_0^\infty t^{x-1}\,\exp\left(-t - pt^{-1}\right)dt, \tag{2.3}$$
$$(\mathrm{Re}(x) > 0, \mathrm{Re}(p) > 0).$$

In 1997, Chaudhry *et al.* [4] presented the following extension of Euler's beta function.

Definition 2.1.4. *(see [4]) The Extended Beta function is defined by*

$$B_p(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left[-\frac{p}{t(1-t)}\right]dt, \tag{2.4}$$

$$(\mathrm{Re}(p)>0, \mathrm{Re}(x)>0, \mathrm{Re}(y)>0).$$

It is clearly seem that $\Gamma_0(x) = \Gamma(x)$ and $B_0(x,y) = B(x,y)$.

2.2 Hypergeometric Functions and their extended versions

In this section, we give definition and some properties of the hypergeometric functions. The second order linear differential equation

$$z(1-z)\frac{d^2y}{dz^2} + [c-(a+b+1)z]\frac{dy}{dz} - aby = 0,$$

in which a, b and c are real or complex parameters, is called the hypergeometric equation. Series solutions of the hypergeometric equation valid in the neighborhoods of $z = 0, 1$ or ∞ can be developed by using Frobenious series method. Thus, if c is not an integer, the general solution of differential equation valid in a neighborhood of the origin is found to be

$$y = A\,{}_2F_1(a,b;c;z) + Bz^{1-c}\,{}_2F_1(a-c+1, b-c+1; 2-c; z),$$

where A and B are arbitrary constants, and

$$\begin{aligned} {}_2F_1(a,b;c;z) &= 1 + \frac{ab}{c.1}z + \frac{a(a+1)b(b+1)}{c(c+1).1.2}z^2 + ... \\ &= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \end{aligned}$$

$$(c \neq 0, -1, -2, ...)$$

where $(\lambda)_\nu$ denotes the Pochhammer symbol defined by

$$(\lambda)_0 \equiv 1 \text{ and } (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}.$$

Hence

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

is called Gauss hypergeometric function. This series is convergent for $|z| < 1$ where $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $|z| = 1$ where $\mathrm{Re}(c - a - b) > 0$.

The Gauss hypergeometric function can be given by Euler's integral representation as follows:

$$
{}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \, dt,
$$
$$
(|z| < 1; \ \mathrm{Re}(c) > \mathrm{Re}(b) > 0).
$$

Replacing $z = \dfrac{z}{b}$ and by letting $|b| \to \infty$, in Gauss's hypergeometric equation, we have

$$
z\frac{d^2y}{dz^2} + (c - z)\frac{dy}{dz} - ay = 0.
$$

This equation has a regular singularity at $z = 0$. The simplest solution of the equation is

$$
\begin{aligned}
\phi(a; c; z) &= 1 + \frac{a}{c.1}z + \frac{a(a+1)}{c(c+1).1.2}z^2 + ... \\
&= \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!},
\end{aligned}
$$
$$
(c \neq 0, -1, -2, ...).
$$

Hence, we get

$$
\phi(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!},
$$

which is called confluent hypergeometric function.

The confluent hypergeometric function can be given by an integral representation as follows:

$$
\phi(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} \exp(zt) \, dt,
$$
$$
(\mathrm{Re}(c) > \mathrm{Re}(a) > 0).
$$

A generalized form of the hypergeometric function is

$$
{}_pF_q(\alpha_1, ...\alpha_p; \gamma_1, ..., \gamma_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_p)_n}{(\gamma_1)_n ... (\gamma_q)_n} \frac{z^n}{n!}, \tag{2.5}
$$

$$
(p, q = 0, 1, ...).
$$

Setting $p = 2, q = 1$ in (2.5), we get the Gauss hypergeometric function,

$$
F(\alpha_1, \alpha_2; \gamma_1; z) := {}_2F_1(\alpha_1, \alpha_2; \gamma_1; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n}{(\gamma_1)_n} \frac{z^n}{n!}.
$$

Setting $p = q = 1$ in (2.5), we get confluent hypergeometric function,

$$
\phi(\alpha_1; \gamma_1; z) = {}_1F_1(\alpha_1; \gamma_1; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n}{(\gamma_1)_n} \frac{z^n}{n!}.
$$

For the convergence of the series ${}_pF_q$ (see [28], [29]);

case 1: If $p \leq q$, then the series converges for all z;

case 2: If $p = q+1$, then the series converges for $|z| < 1$ and otherwise series diverges;

case 3: If $p > q + 1$, the series diverges for $z \neq 0$.

If the series terminates, there is no question of convergence, and the conclusions (case 2) and (case 3) do not apply. If $p = q + 1$, then the series is absolutely convergent on the circle $|z| = 1$ if

$$
\mathrm{Re}\left(\sum_{j=1}^{q} \gamma_j - \sum_{i=1}^{p} \alpha_i\right) > 0.
$$

In 2004, Chaudhry *et al.* [5] used extended beta function $B_p(x, y)$ to extend the hypergeometric functions (and confluent hypergeometric functions) as follows:

$$
F_p(a, b; c; z) = \sum_{n=0}^{\infty} \frac{B_p(b + n, c - b)}{B(b, c - b)} (a)_n \frac{z^n}{n!}
$$

$$
(p \geq 0\,;\ \mathrm{Re}(c) > \mathrm{Re}(b) > 0),
$$

$$\phi_p(b;c;z) = \sum_{n=0}^{\infty} \frac{B_p(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}$$

$$(p \geq 0\,;\ \mathrm{Re}(c) > \mathrm{Re}(b) > 0),$$

and gave the Euler type integral representations

$$F_p(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \exp\left[-\frac{p}{t(1-t)}\right] dt \tag{2.6}$$

$$(p > 0\,;\ p = 0 \text{ and } |\arg(1-z)| < \pi < p;\ \mathrm{Re}(c) > \mathrm{Re}(b) > 0),$$

and

$$\phi_p(b;c;z) = \frac{\exp(z)}{B(b,c-b)} \int_0^1 t^{c-b-1}(1-t)^{b-1} \exp\left[-zt - \frac{p}{t(1-t)}\right] dt \tag{2.7}$$

$$(p > 0\,;\ p = 0 \text{ and } \mathrm{Re}(c) > \mathrm{Re}(b) > 0).$$

They called these functions as extended Gauss hypergeometric function (EGHF) and extended confluent hypergeometric function (ECHF), respectively, since $F_0(a,b;c;z) = {}_2F_1(a,b;c;z)$ and $\phi_0(b;c;z) = {}_1F_1(b;c;z)$. They have discussed the differentiation properties and Mellin transforms of $F_p(a,b;c;z)$ and $\phi_p(b;c;z)$. They obtained transformation formulas, recurrence relations, summation and asymptotic formulas for these functions.

2.3 Some Hypergeometric Functions of two and more variables

There are four Appell's hypergeometric functions of two variables (see [13], [31]). In this thesis we consider the first two Appell's hypergeometric functions. These functions are defined by

$$F_1(a,b,c;d;x,y) := \sum_{n,m=0}^{\infty} \frac{B(a+m+n,d-a)}{B(a,d-a)} (b)_n (c)_m \frac{x^n}{n!} \frac{y^m}{m!}$$

$$(\max\{|x|,|y|\} < 1),$$

and

$$F_2(a,b,c;d,e;x,y) := \sum_{n,m=0}^{\infty} \frac{(a)_{m+n}\, B\,(b+n,d-b)\, B\,(c+m,e-c)}{B\,(b,d-b)\, B\,(c;e-c)} \frac{x^n}{n!} \frac{y^m}{m!}$$

$$(|x|+|y|<1).$$

Lauricella functions of three variables are defined by

$$F_D^3\,(a,b,c,d;e;x,y,z) := \sum_{m,n,r=0}^{\infty} \frac{B\,(a+m+n+r,e-a)\,(b)_m\,(c)_n\,(d)_r}{B\,(a,e-a)} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^r}{r!}$$

$$(\sqrt{|x|}+\sqrt{|y|}+\sqrt{|z|}<1).$$

The Appell's hypergeometric functions F_1 and F_2 can be given an integral representation as follows:

$$F_1\,(a,b,c;d;x,y) = \frac{\Gamma\,(d)}{\Gamma\,(a)\,\Gamma\,(d-a)}$$

$$\times \int_0^1 t^{a-1}\,(1-t)^{d-a-1}\,(1-xt)^{-b}\,(1-yt)^{-c}\,dt,$$

$$(\ |arg\,(1-x)|<\pi, |arg\,(1-y)|<\pi; \mathrm{Re}\,(d)>\mathrm{Re}\,(a)>0).$$

and

$$F_2(a,b,c;d,e;x,y;p) = \frac{1}{B\,(b,d-b)\,B\,(c,e-c)} \int_0^1 \int_0^1 \frac{t^{b-1}\,(1-t)^{d-b-1}\,s^{c-1}\,(1-s)^{e-c-1}}{(1-xt-ys)^a}$$

$$(|x|+|y|<1);$$

$$(\mathrm{Re}\,(d)>\mathrm{Re}\,(b)>0, \mathrm{Re}\,(e)>\mathrm{Re}\,(c)>0, \mathrm{Re}\,(a)>0).$$

2.4 Generating Functions

In this section, we give definitions of linear, bilinear, bilateral, multivariable, multilinear, multilateral and multiple generating functions, from the book [32].

Definition 2.4.1. *(Linear generating functions) If two-variable function $F\,(x,t)$ can be expanded as a formal power series expansion in t as*

$$F\,(x,t) = \sum_{n=0}^{\infty} f_n\,(x)\,t^n$$

where each member of the coefficient set $\{f_n(x)\}_{n=0}^{\infty}$ *is independent of* t, *then the* $F(x,t)$ *is said to have generated the set* $\{f_n(x)\}$. *Therefore* $F(x,t)$ *is called a linear generating function for the set* $\{f_n(x)\}$.

Definition 2.4.2. *(Bilinear generating functions) If three-variable function* $F(x,y,t)$ *can be expanded as a formal power series expansion in* t *such that*

$$F(x,y,t) = \sum_{n=0}^{\infty} \gamma_n \, f_n(x) \, f_n(y) \, t^n$$

where the sequence $\{\gamma_n\}$ *is independent of x,y and t, then* $F(x,y,t)$ *is called a bilinear generating function for the set* $\{f_n(x)\}$.

Definition 2.4.3. *(Bilateral generating functions) If three-variable function* $H(x,y,t)$ *which is defined by a formal power series expansion in* t *as*

$$H(x,y,t) = \sum_{n=0}^{\infty} h_n \, f_n(x) \, g_n(y) \, t^n$$

where the sequence $\{h_n\}$ *is independent of x, y and t, and the sets of functions* $\{f_n(x)\}_{n=0}^{\infty}$ *and* $\{g_n(x)\}_{n=0}^{\infty}$ *are different, then* $H(x,y,t)$ *is called a bilateral generating function for the set* $\{f_n(x)\}$ *or* $\{g_n(x)\}$.

Definition 2.4.4. *(Multivariable generating functions) Suppose that* $G(x_1,...,x_r;t)$ *is a function of* $r+1$ *variables, which is defined by a formal power series expansion in* t *such that*

$$G(x_1,...,x_r;t) = \sum_{n=0}^{\infty} c_n \, g_n(x_1,...,x_r) \, t^n$$

where the sequence $\{c_n\}$ *is independent of the variables* $x_1,...,x_r$ *and t. Then* $G(x_1,...,x_r;t)$ *is called a multivariable generating function for the set* $\{g_n(x_1,...,x_r)\}_{n=0}^{\infty}$ *corresponding to the nonzero coefficients* c_n.

Definition 2.4.5. *(Multilinear and multilateral generating functions) A multivariable generating functions* $G(x_1,...,x_r;t)$ *which is defined in previous definition, is said to be a multilinear generating function if*

$$g_n(x_1,...,x_r) = f_{\alpha_1(n)}(x_1)...f_{\alpha_r(n)}(x_r)$$

where the sequence $\alpha_1(n), ..., \alpha_r(n)$ are functions of n which are not necessarily equal. If the functions $\left\{f_{\alpha_1(n)}(x_1)\right\}, ... \left\{f_{\alpha_r(n)}(x_r)\right\}$ are all different, the multivariable generating function $G(x_1, ..., x_r; t)$ is called a multilateral generating function.

Definition 2.4.6. *(Multiple generating functions) An extension of the multivariable generating function is said to be a multiple generating function which is defined formally by*

$$\Psi(x_1, ..., x_r; t_1, ..., t_r) = \sum_{n_1,...,n_r=0}^{\infty} C(n_1, ...n_r)\, \Delta_{n_1,...,n_r}(x_1, ..., x_r)\, t_1^{n_1} ... t_r^{n_r}$$

where the multiple sequence $\{C(n_1, ...n_r)\}$ is independent of the variables $x_1, ..., x_r$ and $t_1, ...t_r$.

2.5 Mellin Transform and Riemann-Liouville Fractional Derivative

In this section, we give the definition of the Mellin transform and fractional derivative operator.

Definition 2.5.1. *Let $f(x)$ be a function defined on the positive real axis $0 < x < \infty$. The Mellin transformation $\mathfrak{M}$ is the operation, mapping the function f into the function F, defined on the complex plane by the relation:*

$$\mathfrak{M}\{f : s\} = F(s) = \int_0^\infty x^{s-1} f(x) dx.$$

The function $F(s)$ is called the Mellin transform of f.

Example 2.5.2. *The Mellin transform of the function $f(x) = \exp(-px)$ $(p > 0)$ is*

$$\mathfrak{M}\{f : s\} = p^{-s}\Gamma(s)$$
$$(\mathrm{Re}(s) > 0).$$

Indeed, in order to obtain the Mellin transform of $f(x) = \exp(-px)$, we multiply both sides of function $f(x)$ by p^{s-1} and integrate with respect to p over the interval $[0, \infty)$.

Thus

$$\mathfrak{M}\{f:s\}=F(s)=\int_0^\infty x^{s-1}\exp(-px)dx.$$

Using the definition of the Gamma function, we get

$$\mathfrak{M}\{f:s\}=p^{-s}\Gamma(s).$$

Definition 2.5.3. *(Mellin inversion formula) The inverse transform of the Mellin transform is given by*

$$\mathfrak{M}^{-1}\{F:x\}=f(x)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}x^{-s}F(s)ds.$$

Definition 2.5.4. *[16] The Riemann-Liouville fractional derivative of order μ is defined by*

$$D_z^\mu\{f(z)\}=\frac{1}{\Gamma(-\mu)}\int_0^z f(t)(z-t)^{-\mu-1}dt,$$
$$(\mathrm{Re}(\mu)<0)$$

where the integration path is a line from 0 to z in complex $t-$plane. For the case $m-1<\mathrm{Re}(\mu)<m\ \ (m=1,2,3...)$, it is defined by

$$\begin{aligned}D_z^\mu\{f(z)\}&=\frac{d^m}{dz^m}D_z^{\mu-m}\{f(z)\}\\&=\frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(-\mu+m)}\int_0^z f(t)(z-t)^{-\mu+m-1}dt\right\}.\end{aligned}$$

Example 2.5.5. *Let* $\mathrm{Re}(\lambda)>-1,\mathrm{Re}(\mu)<0$. *Then*

$$\begin{aligned}D_z^\mu\{z^\lambda\}&=\frac{1}{\Gamma(-\mu)}\int_0^z t^\lambda(z-t)^{-\mu-1}dt\\&=\frac{1}{\Gamma(-\mu)}\int_0^1(zu)^\lambda(z)^{-\mu-1}(1-u)^{-\mu-1}du\\&=\frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\int_0^1 u^\lambda(1-u)^{-\mu-1}du\\&=\frac{B(\lambda+1,-\mu)}{\Gamma(-\mu)}z^{\lambda-\mu}=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\mu+1)}z^{\lambda-\mu}.\end{aligned}$$

Example 2.5.6. *Let* $\mathrm{Re}(\lambda)>0,\mathrm{Re}(\mu)<0$ *and* $|z|<1$. *Then*

$$D_z^{\lambda-\mu}\{z^{\lambda-1}(1-z)^{-\alpha}\}=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}\,{}_2F_1(\alpha,\lambda;\mu;z).$$

Solution. Direct calculations yield

$$\begin{aligned}
D_z^{\lambda-\mu}\{z^{\lambda-1}(1-z)^{-\alpha}\} &= \frac{1}{\Gamma(\mu-\lambda)}\int_0^z t^{\lambda-1}(1-t)^{-\alpha}(z-t)^{\mu-\lambda-1}\,dt \\
&= \frac{z^{\mu-\lambda-1}}{\Gamma(\mu-\lambda)}\int_0^z t^{\lambda-1}(1-t)^{-\alpha}\left(1-\frac{t}{z}\right)^{\mu-\lambda-1}dt \\
&= \frac{z^{\mu-\lambda-1}z^{\lambda}}{\Gamma(\mu-\lambda)}\int_0^1 u^{\lambda-1}(1-uz)^{-\alpha}(1-u)^{\mu-\lambda-1}\,du \\
&= \frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}\,{}_2F_1(\alpha,\lambda;\mu;z)\,.
\end{aligned}$$

Hence the proof is completed. □

2.6 Elementary Series Identity

In this section, we give general series identities which are used throughout the thesis.

Lemma 2.6.1. *[28] The following series identities*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k) \tag{2.8}$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}C(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}C(k,n+k) \tag{2.9}$$

holds.

Proof. Consider the series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)t^{n+k} \tag{2.10}$$

in which t^{n+k} has been inserted for convenience and will be removed later by taking $t = 1$. Introducing new indices of summation j and m by

$$k=j, \qquad n=m-j, \tag{2.11}$$

so that the exponent $(n+k)$ in (2.10) becomes m. Since $n \geqslant 0$ and $k \geqslant 0$ in (2.11) then $j \geqslant 0$, $m-j \geqslant 0$ or $m \geqslant j \geqslant 0$. Thus we arrive at

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)t^{n+k}=\sum_{m=0}^{\infty}\sum_{j=0}^{m}A(j,m-j)t^{m}. \tag{2.12}$$

Finally, putting $t=1$ in the equality (2.12) and replacing j and m by k and n, we get (2.8). Similarly, equation (2.9) follows from (2.8). $\square$

Lemma 2.6.2. *For a bounded sequence $\{f(N)\}_{N=0}^{\infty}$ of essentially arbitrary complex numbers, we have*

$$\sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f(n+k)\frac{x^n}{n!}\frac{y^k}{k!}. \tag{2.13}$$

Proof. Using the Lemma 2.6.1, we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} f(n+k)\frac{x^n}{n!}\frac{y^k}{k!}(n+k)! &= \sum_{N=0}^{\infty}\sum_{k=0}^{N} f(N)\frac{x^{N-k}}{(N-k)!}\frac{y^k}{k!}N! \\
&= \sum_{N=0}^{\infty} f(N)\sum_{k=0}^{n}\binom{N}{k}x^{N-k}y^k \\
&= \sum_{N=0}^{\infty} f(N)\,(x+y)^N.
\end{aligned}$$

Whence the result. $\square$

2.7 Some Important Polynomials

In this section, we consider some important polynomials which will be used in Chapter 5.

Definition 2.7.1. *Lagrange polynomials in two variables are defined by*

$$g_n^{(\alpha,\beta)}(x,y) = \sum_{k=0}^{n}(\alpha)_{n-k}(\beta)_k\frac{x^{n-k}y^k}{(n-k)!k!}.$$

Definition 2.7.2. *Lagrange-Hermite polynomials of two variables are defined by*

$$h_n^{(\alpha,\beta)}(x,y) = \sum_{k=0}^{\left[\frac{n}{2}\right]}(\alpha)_{n-2k}(\beta)_k\frac{x^{n-2k}y^k}{(n-2k)!k!}.$$

Definition 2.7.3. *Hermite-Kampé de Feriét polynomials of two variables are defined by*

$$H_n(x,y) = \sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{x^{n-2k}y^k}{(n-2k)!k!}.$$

Definition 2.7.4. *[35] Jacobi polynomials are defined by*

$$P_n^{(\alpha,\beta)}(z) = \frac{n!}{(\alpha+\beta+n+1)_n}\sum_{k=0}^{n}\frac{(-1)^{n-k}(\beta+1)_n(\alpha+\beta+1)_{n+k}}{k!\,(n-k)!\,(\beta+1)_k(\alpha+\beta+1)_n}z^k.$$

Chapter 3

SOME GENERATING RELATIONS FOR EXTENDED HYPERGEOMETRIC FUNCTIONS VIA GENERALIZED FRACTIONAL DERIVATIVE OPERATOR

3.1 Introduction

Recently an extension of beta function containing an extra parameter, which proved to be useful earlier, was used to extend the hypergeometric functions [5]. The aim of this chapter is to obtain some generating functions for extended hypergeometric functions (EHF) by considering a new fractional derivative operator. We organize this chapter as follows.

In the second section, extensions of the first two Appell's hypergeometric functions of two variables, namely, $F_1\left(a,b,c;d;x,y;p\right)$ and $F_2(a,b,c;d,e;x,y;p)$, and extended Lauricella's hypergeometric function of three variables, $F_{D,p}^3\left(a,b,c,d;e;x,y,z\right)$, are defined and integral representations of $F_1\left(a,b,c;d;x,y;p\right)$ and $F_2(a,b,c;d,e;x,y;p)$ are obtained. In the third section, extended fractional derivative operator is defined and extended fractional derivatives of some elementary functions, which are needed in obtaining generating functions, are calculated in terms of extended Appell's hypergeometric functions and extended Lauricella's hypergeometric function. In the fourth section, some results related with Mellin transforms and extended fractional derivative operators are given. Last section contains the main results of the chapter. In this section, linear and bilinear generating functions for the extended hypergeometric functions are obtained via generalized fractional derivative operator by following the same method explained in [32].

3.2 The Extended Appell's Functions

In this section, we give extensions of the first two Appell's hypergeometric functions of two variables, $F_1\left(a,b,c;d;x,y;p\right)$ and $F_2(a,b,c;d,e;x,y;p)$, and extend Lauricella's hypergeometric function of three variables, $F_{D,p}^{3}\left(a,b,c,d;e;x,y,z\right)$. We further obtain integral representations of extended Appell's hypergeometric functions.

Let us define the extensions of the Appell's functions $F_1\left(a,b,c;d;x,y;p\right)$ and $F_2(a,b,c;d,e;x,y;p)$, and extended Lauricella's hypergeometric function $F_{D,p}^{3}\left(a,b,c,d;e;x,y,z\right)$ by

$$F_1(a,b,c;d;x,y;p) := \sum_{n,m=0}^{\infty} \frac{B_p(a+m+n,d-a)}{B(a,d-a)} \left(b\right)_n \left(c\right)_m \frac{x^n}{n!}\frac{y^m}{m!} \tag{3.1}$$

$$(\max\left\{|x|,|y|\right\}<1),$$

$$F_2(a,b,c;d,e;x,y;p) := \sum_{n,m=0}^{\infty} \frac{(a)_{m+n}\, B_p\left(b+n,d-b\right) B_p\left(c+m,e-c\right)}{B\left(b,d-b\right) B\left(c;e-c\right)} \frac{x^n}{n!}\frac{y^m}{m!} \tag{3.2}$$

$$(|x|+|y|<1),$$

and

$$F_{D,p}^{3}\left(a,b,c,d;e;x,y,z\right) := \sum_{m,n,r=0}^{\infty} \frac{B_p\left(a+m+n+r,e-a\right)\left(b\right)_m\left(c\right)_n\left(d\right)_r}{B\left(a,e-a\right)} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^r}{r!} \tag{3.3}$$

$$(\sqrt{|x|}+\sqrt{|y|}+\sqrt{|z|}<1),$$

respectively. Notice that the case $p=0$ gives the original functions.

Now we proceed by obtaining the integral representations of the functions $F_1(a,b,c;d;x,y;p)$ and $F_2(a,b,c;d,e;x,y;p)$.

Theorem 3.2.1. *For the extended Appell's functions $F_1(a,b,c;d;x,y;p)$, we have the following integral representation:*

$$F_1(a,b,c;d;x,y;p) = \frac{\Gamma(d)}{\Gamma(a)\Gamma(d-a)}$$
$$\times \int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\exp\left[-\frac{p}{t(1-t)}\right]dt,$$
$$(p>0; p=0 \textbf{ and } |arg(1-x)|<\pi, |arg(1-y)|<\pi; \mathrm{Re}(d)>\mathrm{Re}(a)>0)$$
$$(\mathrm{Re}(b)>0, \mathrm{Re}(c)>0).$$

Proof. Suppose that $|x|<1$, $|y|<1$, $\mathrm{Re}(b)>0$ and $\mathrm{Re}(c)>0$. Expanding $(1-xt)^{-b}$ and $(1-yt)^{-c}$, and considering the fact that the series involved are uniformly convergent and the integral

$$\int_0^1 t^{a+m+n-1}(1-t)^{d-a-1}\exp\left[-\frac{p}{t(1-t)}\right]dt$$

is absolutely convergent for $m,n\in\mathbb{N}_0 := \{0,1,2,3,...\}$, $\mathrm{Re}(d)>\mathrm{Re}(a)>0$ and $p\geq 0$ because of the fact that

$$\int_0^1 \left|t^{a+m+n-1}(1-t)^{d-a-1}\exp\left[-\frac{p}{t(1-t)}\right]\right|dt$$
$$\leq \int_0^1 t^{a+m+n-1}(1-t)^{d-a-1}\,dt,$$

we have a right to interchange the order summation and integration to get

$$\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\exp\left[-\frac{p}{t(1-t)}\right]dt$$
$$=\int_0^1 t^{a-1}(1-t)^{d-a-1}\exp\left[-\frac{p}{t(1-t)}\right]\sum_{n=0}^{\infty}(b)_n\frac{(xt)^n}{n!}\sum_{m=0}^{\infty}(c)_m\frac{(yt)^m}{m!}dt$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_0^1 t^{a+m+n-1}(1-t)^{d-a-1}\exp\left[-\frac{p}{t(1-t)}\right]dt\,(b)_n(c)_m\frac{x^n}{n!}\frac{y^m}{m!}.$$

Using the definition of extended beta function in (3.1), we get

$$\int_0^1 t^{a-1}(1-t)^{d-a-1}(1-xt)^{-b}(1-yt)^{-c}\exp\left[-\frac{p}{t(1-t)}\right]dt$$
$$=\frac{\Gamma(a)\Gamma(d-a)}{\Gamma(d)}F_1(a,b,c;d;x,y),$$

which proves the result for $|x|<1$ and $|y|<1$. Since the integral on the right side is analytic in the cut planes $|arg\,(1-x)|<\pi$, $|arg\,(1-y)|<\pi$, the proof is completed.

□

Theorem 3.2.2. *For the function $F_2(a,b,c;d,e;x,y;p)$, we have the following integral representation:*

$$F_2(a,b,c;d,e;x,y;p)=\frac{1}{B\,(b,d-b)\,B\,(c,e-c)}$$
$$\times\int_0^1\int_0^1\frac{t^{b-1}\,(1-t)^{d-b-1}\,s^{c-1}\,(1-s)^{e-c-1}}{(1-xt-ys)^a}\exp\left[-\frac{p}{t(1-t)}-\frac{p}{s(1-s)}\right]dtds.$$
$$(p>0;p=0\ \textit{and}\ |x|+|y|<1);$$
$$(\mathrm{Re}\,(d)>\mathrm{Re}\,(b)>0,\mathrm{Re}\,(e)>\mathrm{Re}\,(c)>0,\mathrm{Re}\,(a)>0).$$

Proof. Let $|x|+|y|<1$ and $\mathrm{Re}\,(a)>0$. Expanding $(1-xt-ys)^{-a}$ we have

$$\int_0^1\int_0^1\frac{t^{b-1}\,(1-t)^{d-b-1}\,s^{c-1}\,(1-s)^{e-c-1}}{(1-xt-ys)^a}\exp\left[-\frac{p}{t(1-t)}-\frac{p}{s(1-s)}\right]dtds$$
$$=\int_0^1\int_0^1 t^{b-1}\,(1-t)^{d-b-1}\exp\left[-\frac{p}{t(1-t)}\right]s^{c-1}\,(1-s)^{e-c-1}\exp\left[-\frac{p}{s(1-s)}\right]$$
$$\times\sum_{N=0}^{\infty}(a)_N\frac{(xt+ys)^N}{N!}dtds.$$

Taking into account the Lemma 2.6.2, we get

$$\int_0^1\int_0^1\frac{t^{b-1}\,(1-t)^{d-b-1}\,s^{c-1}\,(1-s)^{e-c-1}}{(1-xt-ys)^a}\exp\left[-\frac{p}{t(1-t)}-\frac{p}{s(1-s)}\right]dtds$$
$$=\frac{1}{B\,(b,d-b)\,B\,(c;e-c)}\int_0^1\int_0^1 t^{b-1}\,(1-t)^{d-b-1}\exp\left[-\frac{p}{t(1-t)}\right]$$
$$\times\,s^{c-1}\,(1-s)^{e-c-1}\exp\left[-\frac{p}{s(1-s)}\right]\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(a)_{m+n}\frac{(xt)^n}{n!}\frac{(ys)^m}{m!}dtds.$$

Since the series

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(a)_{m+n}\frac{(xt)^n}{n!}\frac{(ys)^m}{m!}$$

is uniformly convergent for $|x|+|y|<1$ and the integrals

$$\int_0^1 t^{b-1}\,(1-t)^{d-b-1}\exp\left[-\frac{p}{t(1-t)}\right]dt,$$
$$\int_0^1 s^{c-1}\,(1-s)^{e-c-1}\exp\left[-\frac{p}{s(1-s)}\right]ds$$

are absolutely convergent for $p \geq 0,\ \operatorname{Re}(d) > \operatorname{Re}(a) > 0$ and $p \geq 0,\ \operatorname{Re}(e) > \operatorname{Re}(c) > 0$ respectively, we have a right to interchange the order of of summation and integration to obtain

$$\int_0^1 \int_0^1 \frac{t^{b-1}(1-t)^{d-b-1} s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a} \exp\left[-\frac{p}{t(1-t)} - \frac{p}{s(1-s)}\right] dtds$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} (a)_{m+n} \frac{x^n}{n!}\frac{y^m}{m!} \int_0^1 t^{b+n-1}(1-t)^{d-b-1} \exp\left[-\frac{p}{t(1-t)}\right] dt$$
$$\times \int_0^1 s^{c+m-1}(1-s)^{e-c-1} \exp\left[-\frac{p}{s(1-s)}\right] ds.$$

Finally by (2.4) and (3.2), we get

$$\int_0^1 \int_0^1 \frac{t^{b-1}(1-t)^{d-b-1} s^{c-1}(1-s)^{e-c-1}}{(1-xt-ys)^a} \exp\left[-\frac{p}{t(1-t)} - \frac{p}{s(1-s)}\right] dtds$$
$$= B(b, d-b)\, B(c, e-c)\, F_2(a, b, c; d, e; x, y; p).$$

Whence the result. $\square$

3.3 Extended Riemann-Liouville Fractional Derivative

The classical Riemann-Liouville fractional derivative of order μ is defined in Chapter 2. Fractional calculus has become an active research field since it has various applications in different areas of science and engineering, such as fluid flow, electrical networks and probability. Systematic account of the investigations of various authors in the field of fractional calculus and its applications has well presented in [34]. The use of fractional derivative in the generating function theory has explained by Srivastava and Manocha [32].

Now, adding a new parameter, we consider the following generalization of the extended Riemann-Liouville fractional derivative operator:

$$D_z^{\mu,p}\{f(z)\} = \frac{1}{\Gamma(-\mu)} \int_0^z f(t)\,(z-t)^{-\mu-1} \exp\left(\frac{-pz^2}{t(z-t)}\right) dt, \tag{3.4}$$
$$(\operatorname{Re}(\mu) < 0, \operatorname{Re}(p) > 0)$$

and for $m-1<\mathrm{Re}(\mu)<m \ \ (m=1,2,3...)$,

$$\begin{aligned}
D_z^{\mu,p}\{f(z)\} &= \frac{d^m}{dz^m} D_z^{\mu-m}\{f(z)\} \\
&= \frac{d^m}{dz^m}\left\{\frac{1}{\Gamma(-\mu+m)}\int_0^z f(t)(z-t)^{-\mu+m-1}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt\right\}
\end{aligned}$$

where the path of integration is a line from 0 to z in complex $t-$plane. For the case $p=0$ we obtain the classical Riemann-Liouville fractional derivative operator.

We start our investigation by calculating the extended fractional derivatives of some elementary functions.

Example 3.3.1. *Let* $\mathrm{Re}(\lambda)>-1, \mathrm{Re}(\mu)<0$. *Then*

$$D_z^{\mu,p}\{z^\lambda\} = \frac{B(\lambda+1,-\mu;p)}{\Gamma(-\mu)} z^{\lambda-\mu}.$$

Solution. Using (3.4) and (2.4), we get

$$\begin{aligned}
D_z^{\mu,p}\{z^\lambda\} &= \frac{1}{\Gamma(-\mu)}\int_0^z t^\lambda (z-t)^{-\mu-1}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt \\
&= \frac{1}{\Gamma(-\mu)}\int_0^1 (zu)^\lambda (z)^{-\mu-1}(1-u)^{-\mu-1}\exp\left(\frac{-pz^2}{uz(z-uz)}\right)zdu \\
&= \frac{z^{\lambda-\mu}}{\Gamma(-\mu)}\int_0^1 u^\lambda (1-u)^{-\mu-1}\exp\left(\frac{-p}{u(1-u)}\right)du \\
&= \frac{B_p(\lambda+1,-\mu)}{\Gamma(-\mu)} z^{\lambda-\mu}.
\end{aligned}$$

Whence the result. □

Example 3.3.2. *Let* $\mathrm{Re}(\lambda)>0, \mathrm{Re}(\alpha)>0, \mathrm{Re}(\mu)<0$ *and* $|z|<1$. *Then*

$$D_z^{\lambda-\mu,p}\{z^{\lambda-1}(1-z)^{-\alpha}\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_p(\alpha,\lambda;\mu;z).$$

Solution. Direct calculations yield

$$
\begin{aligned}
& D_z^{\lambda-\mu,p}\{z^{\lambda-1}(1-z)^{-\alpha}\} \\
& = \frac{1}{\Gamma(\mu-\lambda)} \int_0^z t^{\lambda-1}(1-t)^{-\alpha} \exp\left(\frac{-pz^2}{t(z-t)}\right)(z-t)^{\mu-\lambda-1}\, dt \\
& = \frac{z^{\mu-\lambda-1}}{\Gamma(\mu-\lambda)} \int_0^z t^{\lambda-1}(1-t)^{-\alpha}\left(1-\frac{t}{z}\right)^{\mu-\lambda-1} \exp\left(\frac{-pz^2}{t(z-t)}\right) dt \\
& = \frac{z^{\mu-\lambda-1}z^{\lambda}}{\Gamma(\mu-\lambda)} \int_0^1 u^{\lambda-1}(1-uz)^{-\alpha}(1-u)^{\mu-\lambda-1} \exp\left(\frac{-p}{u(1-u)}\right) du.
\end{aligned}
$$

By (2.6), we can write

$$
\begin{aligned}
D_z^{\lambda-\mu,p}\{z^{\lambda-1}(1-z)^{-\alpha}\} &= \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)} B(\lambda,\mu-\lambda) F_p(\alpha,\lambda;\mu;z) \\
&= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_p(\alpha,\lambda;\mu;z).
\end{aligned}
$$

Hence the proof is completed. □

The more general form of the above result is given in the following example.

Example 3.3.3. *Let* $\operatorname{Re}(\mu) > \operatorname{Re}(\lambda) > 0, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0$; $|az| < 1$ *and* $|bz| < 1$. *Then*

$$
D_z^{\lambda-\mu,p}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_1(\lambda,\alpha,\beta;\mu;az,bz;p).
$$

More generally, letting $\operatorname{Re}(\mu) > \operatorname{Re}(\lambda) > 0, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0, \operatorname{Re}(\gamma) > 0$, $|az| < 1$, $|bz| < 1$ *and* $|cz| < 1$ *we have*

$$
\begin{aligned}
& D_z^{\lambda-\mu,p}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma}\} \\
& = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_{D,p}^3(\lambda,\alpha,\beta,\gamma;\mu;az,bz,cz).
\end{aligned}
$$

Solution. Considering Example 3.2.1, we get

$$
\begin{aligned}
&D_z^{\lambda-\mu,p}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}\} \\
&=\frac{1}{\Gamma(\mu-\lambda)}\int_0^z t^{\lambda-1}(1-at)^{-\alpha}(1-bt)^{-\beta}\exp\left(\frac{-pz^2}{t(z-t)}\right)(z-t)^{\mu-\lambda-1}\,dt \\
&=\frac{z^{\mu-\lambda-1}}{\Gamma(\mu-\lambda)}\int_0^z t^{\lambda-1}(1-at)^{-\alpha}(1-bt)^{-\beta}\left(1-\frac{t}{z}\right)^{\mu-\lambda-1}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt \\
&=\frac{z^{\mu-\lambda-1}z^{\lambda}}{\Gamma(\mu-\lambda)}\int_0^1 u^{\lambda-1}(1-auz)^{-\alpha}(1-buz)^{-\beta}(1-u)^{\mu-\lambda-1}\exp\left(\frac{-p}{u(1-u)}\right)du \\
&=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_1(\lambda,\alpha,\beta;\mu;az,bz;p).
\end{aligned}
$$

Using Example 3.3.1 and (3.3), we obtain

$$
\begin{aligned}
&D_z^{\lambda-\mu,p}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}(1-cz)^{-\gamma}\} \\
&=\frac{z^{\mu-1}}{\Gamma(\mu-\lambda)}\sum_{m,n,r=0}^{\infty}\frac{(\alpha)_m(\beta)_n(\gamma)_r}{m!n!r!}a^m b^n c^r B_p(\lambda+m+n+r,\mu-\lambda)\,z^{m+n+r} \\
&=\frac{B(\lambda,\mu-\lambda)}{\Gamma(\mu-\lambda)}z^{\mu-1}\sum_{m,n,r=0}^{\infty}\frac{B_p(\lambda+m+n+r,\mu-\lambda)}{B(\lambda,\mu-\lambda)}\frac{(\alpha)_m(\beta)_n(\gamma)_r}{m!n!r!}(az)^m(bz)^n(cz)^r \\
&=\frac{\Gamma(\lambda)}{\Gamma(\mu)}z^{\mu-1}F_{D,p}^{3}(\lambda,\alpha,\beta,\gamma;\mu;az,bz,cz).
\end{aligned}
$$

Whence the result. □

Example 3.3.4. *For* $\mathrm{Re}(\mu)>\mathrm{Re}(\lambda)>0, \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(\gamma)>0; \left|\frac{x}{1-z}\right|<1$ *and* $|x|+|z|<1$, *we have*

$$
\begin{aligned}
&D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_p\left(\alpha,\beta;\gamma;\frac{x}{1-z}\right)\right\} \\
&=\frac{1}{B(\beta,\gamma-\beta)\Gamma(\mu-\lambda)}z^{\mu-1}F_2(\alpha,\beta,\lambda;\gamma,\mu;x,z;p).
\end{aligned}
$$

Solution. Using Example 3.3.1 and (3.2), we get

$$
\begin{aligned}
&D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}(1-z)^{-\alpha}F_p\left(\alpha,\beta;\gamma;\frac{x}{1-z}\right)\right\}\\
&=D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}(1-z)^{-\alpha}\frac{1}{B(\beta,\gamma-\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_n B_p(\beta+n,\gamma-\beta)}{n!}\left(\frac{x}{1-z}\right)^n\right\}\\
&=\frac{1}{B(\beta,\gamma-\beta)}D_z^{\lambda-\mu,p}\left\{z^{\lambda-1}\sum_{n=0}^{\infty}(\alpha)_n B_p(\beta+n,\gamma-\beta)\frac{x^n}{n!}(1-z)^{-\alpha-n}\right\}\\
&=\frac{1}{B(\beta,\gamma-\beta)}\sum_{m,n=0}^{\infty}B_p(\beta+n,\gamma-\beta)\frac{x^n}{n!}\frac{(\alpha)_n(\alpha+n)_m}{m!}D_z^{\lambda-\mu,p}\left\{z^{\lambda-1+m}\right\}\\
&=\frac{1}{B(\beta,\gamma-\beta)}\sum_{m,n=0}^{\infty}B_p(\beta+n,\gamma-\beta)\frac{x^n}{n!}\frac{(\alpha)_{n+m}}{m!}\frac{B_p(\lambda+m,\mu-\lambda)}{\Gamma(\mu-\lambda)}z^{\mu+m-1}\\
&=\frac{1}{B(\beta,\gamma-\beta)\Gamma(\mu-\lambda)}z^{\mu-1}F_2(\alpha,\beta,\lambda;\gamma,\mu;x,z;p).
\end{aligned}
$$

The result is proved. □

The next theorem determines the extended fractional integral of an analytic function.

Theorem 3.3.5. *Let $f(z)$ be an analytic function in the disc $|z|<\rho$ and has the power series expansion $f(z)=\sum\limits_{n=0}^{\infty}a_n z^n$. Then*

$$
\begin{aligned}
D_z^{\mu,p}\{z^{\lambda-1}f(z)\}&=\sum_{n=0}^{\infty}a_n D_z^{\mu,p}\{z^{\lambda+n-1}\}\\
&=\frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty}a_n B_p(\lambda+n,-\mu)\,z^n
\end{aligned}
$$

provided that $\operatorname{Re}(\lambda)>0,\operatorname{Re}(\mu)<0$ *and* $|z|<\rho$.

Proof. We have

$$
\begin{aligned}
&D_z^{\mu,p}\{z^{\lambda-1}f(z)\}=D_z^{\mu,p}\left\{z^{\lambda-1}\sum_{n=0}^{\infty}a_n z^n\right\}\\
&=\frac{1}{\Gamma(-\mu)}\int_0^z t^{\lambda-1}\sum_{n=0}^{\infty}a_n t^n(z-t)^{-\mu-1}\exp\left(\frac{-pz^2}{t(z-t)}\right)dt\\
&=\frac{1}{\Gamma(-\mu)}\int_0^1(z\xi)^{\lambda-1}z^{-\mu-1}(1-\xi)^{-\mu-1}\exp\left(\frac{-p}{\xi(1-\xi)}\right)\sum_{n=0}^{\infty}a_n(z\xi)^n z\,d\xi\\
&=\frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)}\int_0^1\xi^{\lambda-1}(1-\xi)^{-\mu-1}\exp\left(\frac{-p}{\xi(1-\xi)}\right)\sum_{n=0}^{\infty}a_n(z\xi)^n d\xi.
\end{aligned}
$$

Since the series $\sum_{n=0}^{\infty} a_n z^n \xi^n$ is uniformly convergent in the disc $|z| < \rho$ for $0 \leq \xi \leq 1$ and the integral $\int_0^1 \left| \xi^{\lambda-1} (1-\xi)^{-\mu-1} \exp\left(\frac{-p}{\xi(1-\xi)}\right) \right| d\xi$ is convergent provided that $\mathrm{Re}(\lambda) > 0$, $\mathrm{Re}(\mu) < 0$ and $\mathrm{Re}(p) > 0$, we can change the order of integration and summation and obtain

$$
\begin{aligned}
& D_z^{\mu,p}\{z^{\lambda-1} f(z)\} \\
& = \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n z^n \int_0^1 \xi^{\lambda+n-1} (1-\xi)^{-\mu-1} \exp\left(\frac{-p}{\xi(1-\xi)}\right) d\xi \\
& = \sum_{n=0}^{\infty} a_n \frac{z^{\lambda+n-1-\mu}}{\Gamma(-\mu)} B_p(\lambda+n, -\mu) \\
& = \frac{z^{\lambda-\mu-1}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} a_n B_p(\lambda+n, -\mu) z^n.
\end{aligned}
$$

Whence the result. □

3.4 Mellin Transforms and Extended Riemann-Liouville Fractional Derivative Operator

The main result of this section is the following:

Example 3.4.1. *Let the Extended Riemann-Liouville fractional derivative be defined by (3.4). Then we have*

$$\mathfrak{M}\left[D_z^{\mu,p}\left\{z^{\lambda}\right\} : s\right] = \frac{\Gamma(s)}{\Gamma(-\mu)} B(\lambda+s+1, s-\mu) z^{\lambda-\mu}$$

where $\mathrm{Re}(\lambda) > -1, \mathrm{Re}(\mu) < 0, \mathrm{Re}(s) > 0.$

Solution. Using the definition of the Mellin transform, we get

$$
\begin{aligned}
& \mathfrak{M}\left[D_z^{\mu,p}\left\{z^{\lambda}\right\} : s\right] = \int_0^{\infty} p^{s-1} D_z^{\mu,p}\left\{z^{\lambda}\right\} dp \\
& = \frac{1}{\Gamma(-\mu)} \int_0^{\infty} p^{s-1} \int_0^z t^{\lambda} (z-t)^{-\mu-1} \exp\left(\frac{-pz^2}{t(z-t)}\right) dt dp \\
& = \frac{z^{-\mu-1}}{\Gamma(-\mu)} \int_0^{\infty} p^{s-1} \int_0^z t^{\lambda} \left(1-\frac{t}{z}\right)^{-\mu-1} \exp\left(\frac{-pz^2}{t(z-t)}\right) dt dp \\
& = \frac{z^{-\mu-1}}{\Gamma(-\mu)} \int_0^{\infty} p^{s-1} \int_0^1 u^{\lambda} z^{\lambda} (1-u)^{-\mu-1} \exp\left(\frac{-p}{u(1-u)}\right) z du dp.
\end{aligned}
$$

Now, letting $\operatorname{Re}(\lambda) > -1, \operatorname{Re}(\mu) < 0, \operatorname{Re}(s) > 0$, then the integrals

$$\int_0^1 u^{\lambda}(1-u)^{-\mu-1} \exp\left(\frac{-p}{u(1-u)}\right) du$$

and

$$\int_0^{\infty} p^{s-1} \exp\left(\frac{-p}{u(1-u)}\right) dp$$

are absolutely convergent and therefore the order of the integration can be interchanged to yield

$$\begin{aligned}
&\mathfrak{M}\left[D_z^{\mu,p}\left\{z^{\lambda}\right\} : s\right] \\
&= \frac{z^{\lambda-\mu}}{\Gamma(-\mu)} \int_0^1 u^{\lambda}(1-u)^{-\mu-1} \int_0^{\infty} p^{s-1} \exp\left(\frac{-p}{u(1-u)}\right) dpdu.
\end{aligned}$$

Making the substition $r = \frac{p}{u(1-u)}$, we get

$$\begin{aligned}
&\mathfrak{M}\left[D_z^{\mu,p}\left\{z^{\lambda}\right\} : s\right] \\
&= \frac{z^{\lambda-\mu}}{\Gamma(-\mu)} \int_0^1 u^{\lambda}(1-u)^{-\mu-1} \int_0^{\infty} u^{s-1}(1-u)^{s-1} r^{s-1} e^{-r} u(1-u)\, drdu \\
&= \frac{z^{\lambda-\mu}}{\Gamma(-\mu)} \int_0^1 u^{\lambda+s}(1-u)^{s-\mu-1} \int_0^{\infty} r^{s-1} e^{-r} drdu \\
&= \frac{\Gamma(s)}{\Gamma(-\mu)} z^{\lambda-\mu} \int_0^1 u^{\lambda+s}(1-u)^{s-\mu-1} du = \frac{\Gamma(s)}{\Gamma(-\mu)} B(\lambda+s+1, s-\mu)\, z^{\lambda-\mu},
\end{aligned}$$

which completes the proof. □

As an application of the above example we have the following:

Example 3.4.2. *Let the extended Riemann-Liouville fractional integral be defined by (3.4). Then we have*

$$\mathfrak{M}\left[D_z^{\mu,p}\left\{(1-z)^{-\alpha}\right\} : s\right] = \frac{\Gamma(s)\, z^{-\mu}}{\Gamma(-\mu)\, B(s+1, s-\mu)} F(\alpha, s+1; 2s-\mu+1; z)$$

where $\operatorname{Re}(\mu) < 0, \operatorname{Re}(s) > 0, \operatorname{Re}(\alpha) > 0$ *and* $|z| < 1$.

Solution. Letting $\operatorname{Re}(\mu) < 0, \operatorname{Re}(s) > 0, \operatorname{Re}(\alpha) > 0$ and $|z| < 1$ and then using

Example 3.4.1 with $\lambda = n$, we can write that

$$\begin{aligned}
\mathfrak{M}\left[D_z^{\mu,p}\left\{(1-z)^{-\alpha}\right\}:s\right] &= \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\mathfrak{M}\left[D_z^{\mu,p}\left\{z^n\right\}:s\right] \\
&= \frac{\Gamma(s)}{\Gamma(-\mu)}\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}B(n+s+1,s-\mu)\,z^{n-\mu} \\
&= \frac{\Gamma(s)\,z^{-\mu}}{\Gamma(-\mu)}\sum_{n=0}^{\infty}B(n+s+1,s-\mu)\frac{(\alpha)_n\,z^n}{n!}.
\end{aligned}$$

Whence the result. □

3.5 Generating Functions

In this section, we obtain linear and bilinear generating relations for the extended hypergeometric functions $F_p(a,b;c;z)$ by following the methods described in [32]. We start with the following theorem:

Theorem 3.5.1. *For the extended hypergeometric functions we have*

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}F_p(\lambda+n,\alpha;\beta;x)\,t^n = (1-t)^{-\lambda}F_p\left(\lambda,\alpha;\beta;\frac{x}{1-t}\right) \tag{3.5}$$

where $|x| < \min\{1,|1-t|\}$ *and* $\mathrm{Re}(\lambda) > 0, \mathrm{Re}(\beta) > \mathrm{Re}(\alpha) > 0$.

Proof. Considering the elementary identity

$$[(1-x)-t]^{-\lambda} = (1-t)^{-\lambda}\left[1-\frac{x}{1-t}\right]^{-\lambda}$$

and expanding the left hand side we have, for $|t| < |1-x|$ that

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-x)^{-\lambda}\left(\frac{t}{1-x}\right)^n = (1-t)^{-\lambda}\left[1-\frac{x}{1-t}\right]^{-\lambda}.$$

Now, multiplying both sides of the above equality by $x^{\alpha-1}$ and applying the extended fractional derivative operator $D_x^{\alpha-\beta,p}$ on both sides, we can write

$$\begin{aligned}
&D_x^{\alpha-\beta,p}\left\{\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}(1-x)^{-\lambda}\left(\frac{t}{1-x}\right)^n x^{\alpha-1}\right\} \\
&= (1-t)^{-\lambda}D_x^{\alpha-\beta,p}\left\{x^{\alpha-1}\left[1-\frac{x}{1-t}\right]^{-\lambda}\right\}.
\end{aligned}$$

Interchanging the order, which is valid for $\text{Re}(\alpha) > 0$ and $|t| < |1-x|$, we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_x^{\alpha-\beta,p} \left\{ x^{\alpha-1} (1-x)^{-\lambda-n} \right\} t^n$$
$$= (1-t)^{-\lambda} D_x^{\alpha-\beta,p} \left\{ x^{\alpha-1} \left[1 - \frac{x}{1-t} \right]^{-\lambda} \right\}.$$

Using Example 3.3.2, we get the desired result. □

The following theorem gives another linear generating relation for the extended hypergeometric functions.

Theorem 3.5.2. *For the extended hypergeometric functions we have*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_p(\rho - n, \alpha; \beta; x) t^n = (1-t)^{-\lambda} F_1 \left(\alpha, \rho, \lambda; \beta; x, \frac{-xt}{1-t}; p \right)$$

where $\text{Re}(\beta) > \text{Re}(\alpha) > 0, \text{Re}(\rho) > 0, \text{Re}(\lambda) > 0; |t| < \dfrac{1}{1+|x|}$.

Proof. Considering

$$[1-(1-x)t]^{-\lambda} = (1-t)^{-\lambda} \left[1 + \frac{xt}{1-t} \right]^{-\lambda}$$

and expanding the left hand side we have, for $|t| < |1-x|$ that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1-x)^n t^n = (1-t)^{-\lambda} \left[1 - \frac{-xt}{1-t} \right]^{-\lambda}.$$

Now multiplying both sides of the above equality by $x^{\alpha-1}(1-x)^{-\rho}$ and applying the extended fractional derivative operator $D_x^{\alpha-\beta,p}$ on both sides, we get

$$D_x^{\alpha-\beta,p} \left\{ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} x^{\alpha-1} (1-x)^{-\rho+n} t^n \right\}$$
$$= (1-t)^{-\lambda} D_x^{\alpha-\beta,p} \left\{ x^{\alpha-1} (1-x)^{-\rho} \left[1 - \frac{-xt}{1-t} \right]^{-\lambda} \right\}.$$

Interchanging the order, which is valid for $\text{Re}(\alpha) > 0$ and $|xt| < |1-t|$, we get

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_x^{\alpha-\beta,p} \left\{ x^{\alpha-1} (1-x)^{-(\rho-n)} \right\} t^n$$
$$= (1-t)^{-\lambda} D_x^{\alpha-\beta,p} \left\{ x^{\alpha-1} (1-x)^{-\rho} \left[1 - \frac{-xt}{1-t} \right]^{-\lambda} \right\}.$$

Using Example 3.3.2 and Example 3.3.3, we get the desired result. □

Finally we have the following bilinear generating relation for the extended hypergeometric functions.

Theorem 3.5.3. *For the extended hypergeometric functions we have*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_p(\gamma, -n; \delta; y) F_p(\lambda+n, \alpha; \beta; x) t^n$$
$$= (1-t)^{-\lambda} F_2\left(\lambda, \alpha, \gamma; \beta, \delta; \frac{x}{1-t}, \frac{-yt}{1-t}; p\right)$$

where $\mathrm{Re}(\delta) > \mathrm{Re}(\gamma) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\lambda) > 0, \mathrm{Re}(\beta) > 0; |t| < \dfrac{1-|x|}{1+|y|}$ *and* $|x| < 1$.

Proof. Replacing $t \longrightarrow (1-y)t$ in (3.5), multiplying the resulting equality by $y^{\gamma-1}$ and then applying the extended fractional derivative operator $D_y^{\gamma-\delta,p}$, we get

$$D_y^{\gamma-\delta,p}\left\{\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} y^{\gamma-1} F_p(\lambda+n, \alpha; \beta; x)(1-y)^n t^n\right\}$$
$$= D_y^{\gamma-\delta,p}\left\{(1-(1-y)t)^{-\lambda} y^{\gamma-1} F_p\left(\lambda, \alpha; \beta; \frac{x}{1-(1-y)t}\right)\right\}.$$

Interchanging the order of the above equation, which is valid for $|x| < 1, \left|\frac{1-y}{1-x}t\right| < 1$ and $\left|\frac{x}{1-t}\right| + \left|\frac{yt}{1-t}\right| < 1$, we can write that

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_y^{\gamma-\delta,p}\left\{y^{\gamma-1}(1-y)^n\right\} F_p(\lambda+n, \alpha; \beta; x) t^n$$
$$= (1-t)^{-\lambda} D_y^{\gamma-\delta,p}\left\{y^{\gamma-1}\left(1-\frac{-yt}{1-t}\right)^{-\lambda} F_p\left(\lambda, \alpha; \beta; \frac{\frac{x}{1-t}}{1-\frac{-yt}{1-t}}\right)\right\}.$$

Using Example 3.3.2 and Example 3.3.4, we get the result. □

Chapter 4

EXTENSION OF GAMMA, BETA AND HYPERGEOMETRIC FUNCTIONS

4.1 Introduction

In this chapter, we consider the following generalizations of gamma and Euler's beta functions

$$\Gamma_p^{(\alpha,\beta)}(x) := \int_0^\infty t^{x-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)dt \tag{4.1}$$

$$(\mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \mathrm{Re}(p)>0, \mathrm{Re}(x)>0),$$

$$B_p^{(\alpha,\beta)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt, \tag{4.2}$$

$$(\mathrm{Re}(p)>0, \mathrm{Re}(x)>0, \mathrm{Re}(y)>0, \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0).$$

respectively. It is obvious by (2.3), (4.1) and (2.4), (4.2) that, $\Gamma_p^{(\alpha,\alpha)}(x) = \Gamma_p(x)$, $\Gamma_0^{(\alpha,\alpha)}(x) = \Gamma(x)$, $B_p^{(\alpha,\alpha)}(x,y) = B_p(x,y)$ and $B_0^{(\alpha,\beta)}(x,y) = B(x,y)$.

This chapter is organized as follow:

In section 4.2, different integral representations and some properties of new generalized gamma and Euler's beta function are obtained. Additionally, relations of new generalized gamma and beta functions are discussed. In the third section, we generalize the hypergeometric function and confluent hypergeometric function by using $B_p^{(\alpha,\beta)}(x,y)$. Then we obtain the integral representations of this new generalized Gauss hypergeometric functions. Furthermore we discuss the differentiation properties, Mellin transforms, transformation formulas, recurrence relations, summation formulas for this new hypergeometric functions.

4.2 Some Properties of Gamma and Beta Functions

It is important and useful to obtain different integral representations of the new generalized beta function, for later use. Also it is useful to discuss the relationships between classical gamma functions and new generalizations. We start with the following integral representation for $\Gamma_p^{(\alpha,\beta)}(x)$.

Theorem 4.2.1. *For the new generalized gamma function, we have*

$$\Gamma_p^{(\alpha,\beta)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \Gamma_{p\mu^2}(s)\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu$$

where $\Gamma_p(s)$ is the Chaudhry's gamma function.

Proof. Using the integral representation of confluent hypergeometric function, we have

$$\Gamma_p^{(\alpha,\beta)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^\infty \int_0^1 u^{s-1}e^{-ut-\frac{pt}{u}}t^{\alpha-1}(1-t)^{\beta-\alpha-1}dtdu.$$

Now using the one-to-one transformation (except possibly at the boundaries and maps the region onto itself) $\nu = ut, \mu = t$ in the above equality and considering that the Jacobian of the transformation is $J = \frac{1}{\mu}$, we get

$$\Gamma_p^{(\alpha,\beta)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^\infty \int_0^1 v^{s-1}e^{-v-\frac{p\mu^2}{v}}dv\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu.$$

From the uniform convergence of the integrals, the order of integration can be interchanged to yield that

$$\begin{aligned}\Gamma_p^{(\alpha,\beta)}(s) &= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \left[\int_0^\infty v^{s-1}e^{-v-\frac{p\mu^2}{v}}dv\right] \mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu \\ &= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 \Gamma_{p\mu^2}(s)\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu.\end{aligned}$$

Whence the result. □

The case $p = 0$ in the above Theorem gives (see [2], p.192)

$$\Gamma^{(\alpha,\beta)}(s) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1 \Gamma(s)\mu^{\alpha-s-1}(1-\mu)^{\beta-\alpha-1}d\mu = \frac{\Gamma(\beta)\Gamma(\alpha-s)\Gamma(s)}{\Gamma(\alpha)\Gamma(\beta-s)}. \tag{4.3}$$

The next theorem gives the Mellin transform representation of the function $B_p^{(\alpha,\beta)}(x,y)$ in terms of the ordinary beta function and $\Gamma^{(\alpha,\beta)}(s)$.

Theorem 4.2.2. *Mellin transform representation of the new generalized beta function is given by*

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta)}(x,y)\,dp = B(s+x,y+s)\Gamma^{(\alpha,\beta)}(s), \tag{4.4}$$

$$(\mathrm{Re}\,(s) > 0, \mathrm{Re}\,(x+s) > 0, \mathrm{Re}\,(y+s) > 0),$$

$$(\mathrm{Re}(p) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0).$$

Proof. Multiplying (4.2) by p^{s-1} and integrating with respect to p from $p = 0$ to $p = \infty$, we get

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta)}(x,y)\,dp = \int_0^\infty p^{s-1}\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1(\alpha;\beta;\frac{-p}{t(1-t)})dtdp. \tag{4.5}$$

From the uniform convergence of the integral, the order of integration in (4.5) can be interchanged. Therefore, we have

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta)}(x,y)\,dp = \int_0^1 t^{x-1}(1-t)^{y-1}\int_0^\infty p^{s-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dpdt. \tag{4.6}$$

Now using the one-to-one transformation (except possibly at the boundaries and maps the region onto itself) $\nu = \frac{p}{t(1-t)}$, $\mu = t$ in (4.6), we get,

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta)}(x,y)\,dp = \int_0^1 \mu^{(s+x)-1}(1-\mu)^{(y+s)-1}\,d\mu\int_0^\infty \nu^{s-1}\,{}_1F_1(\alpha;\beta;-\nu)d\nu.$$

Therefore, using (4.3), we have

$$\int_0^\infty p^{s-1}B_p^{(\alpha,\beta)}(x,y)\,dp = B(s+x,y+s)\Gamma^{(\alpha,\beta)}(s).$$

Hence the proof is completed. □

Corollary 4.2.3. *By Mellin inversion formula, we have the following complex integral representation for* $B_p^{(\alpha,\beta)}(x,y)$:

$$B_p^{(\alpha,\beta)}(x,y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} B(s+x,y+s)\Gamma^{(\alpha,\beta)}(s)\,p^{-s}ds.$$

Remark 4.2.4. *Putting* $s=1$ *and considering that* $\Gamma^{(\alpha,\beta)}(1) = \frac{\Gamma(\beta)\Gamma(\alpha-1)}{\Gamma(\alpha)\Gamma(\beta-1)}$ *in (4.4), we get*

$$\int_0^\infty B_p^{(\alpha,\beta)}(x,y)\,dp = B(x+1,y+1)\frac{\Gamma(\beta)\Gamma(\alpha-1)}{\Gamma(\alpha)\Gamma(\beta-1)}.$$

Letting $B_p^{(\alpha,\alpha)}(x,y) = B_p(x,y)$*, it reduces to Chaudhry's [4] interesting relation*

$$\int_0^\infty B_p(x,y)\,dp = B(x+1,y+1),$$

$$(\mathrm{Re}\,(x) > -1, \mathrm{Re}\,(y) > -1)$$

between the classical and the extended beta functions.

Theorem 4.2.5. *For the new generalized beta function, we have the following integral representations:*

$$B_p^{(\alpha,\beta)}(x,y) = 2\int_0^{\pi/2} \cos^{2x-1}\theta\sin^{2y-1}\theta\,{}_1F_1\left(\alpha;\beta;-p\sec^2\theta\csc^2\theta\right)d\theta,$$
$$B_p^{(\alpha,\beta)}(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,{}_1F_1\left(\alpha;\beta;-2p-p(u+\frac{1}{u})\right)du.$$

Proof. Letting $t=\cos^2\theta$ in (4.2), we get

$$\begin{aligned} B_p^{(\alpha,\beta)}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\;{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt \\ &= 2\int_0^{\pi/2}\cos^{2x-1}\theta\sin^{2y-1}\theta\,{}_1F_1(\alpha;\beta;-p\sec^2\theta\csc^2\theta)d\theta. \end{aligned}$$

On the other hand, letting $t=\frac{u}{1+u}$ in (4.2), we get

$$\begin{aligned} B_p^{(\alpha,\beta)}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1}\;{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt \\ &= \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,{}_1F_1\left(\alpha;\beta;-2p-p(u+\frac{1}{u})\right)du. \end{aligned}$$

Hence the proof is completed. □

Theorem 4.2.6. *For the new generalized beta function, we have the following functional relation:*

$$B_p^{(\alpha,\beta)}(x,y+1)+B_p^{(\alpha,\beta)}(x+1,y)=B_p^{(\alpha,\beta)}(x,y).$$

Proof. Direct calculation yield

$$\begin{aligned}
&B_p^{(\alpha,\beta)}(x,y+1)+B_p^{(\alpha,\beta)}(x+1,y)\\
&=\int_0^1 t^x(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt+\int_0^1 t^{x-1}(1-t)^y\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt\\
&=\int_0^1\left[t^x(1-t)^{y-1}+t^{x-1}(1-t)^y\right]{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt\\
&=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt=B_p^{(\alpha,\beta)}(x,y).
\end{aligned}$$

Whence the result. □

Theorem 4.2.7. *For the product of two new generalized gamma function, we have the following integral representation:*

$$\Gamma_p^{(\alpha,\beta)}(x)\Gamma_p^{(\alpha,\beta)}(y)=4\int_0^{\frac{\pi}{2}}\int_0^\infty r^{2(x+y)-1}\cos^{2x-1}\theta\sin^{2y-1}\theta \tag{4.7}$$
$$\cdot{}_1F_1\left(\alpha;\beta;-r^2\cos^2\theta-\frac{p}{r^2\cos^2\theta}\right){}_1F_1\left(\alpha;\beta;-r^2\sin^2\theta-\frac{p}{r^2\sin^2\theta}\right)drd\theta.$$

Proof. Substituting $t=\eta^2$ in (4.1), we get

$$\Gamma_p^{(\alpha,\beta)}(x)=2\int_0^\infty \eta^{2x-1}\,{}_1F_1\left(\alpha;\beta;-\eta^2-\frac{p}{\eta^2}\right)d\eta.$$

Therefore

$$\begin{aligned}
\Gamma_p^{(\alpha,\beta)}(x)\Gamma_p^{(\alpha,\beta)}(y)=4\int_0^\infty\int_0^\infty \eta^{2x-1}\xi^{2y-1}\,{}_1F_1\left(\alpha;\beta;-\eta^2-\frac{p}{\eta^2}\right)\\
\cdot\,{}_1F_1\left(\alpha;\beta;-\xi^2-\frac{p}{\xi^2}\right)d\eta d\xi.
\end{aligned}$$

Letting $\eta=r\cos\theta$ and $\xi=r\sin\theta$ in the above equality,

$$\Gamma_p^{(\alpha,\beta)}(x)\Gamma_p^{(\alpha,\beta)}(y)=4\int_0^{\frac{\pi}{2}}\int_0^\infty r^{2(x+y)-1}\cos^{2x-1}\theta\sin^{2y-1}\theta$$
$$\cdot{}_1F_1\left(\alpha;\beta;-r^2\cos^2\theta-\frac{p}{r^2\cos^2\theta}\right){}_1F_1\left(\alpha;\beta;-r^2\sin^2\theta-\frac{p}{r^2\sin^2\theta}\right)drd\theta.$$

Whence the result. □

Remark 4.2.8. *Putting $p = 0$ and $\alpha = \beta$ in (4.7), we get the classical relation between the gamma and beta functions:*

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 4.2.9. *For the new generalized beta function, we have the following summation relation:*

$$B_p^{(\alpha,\beta)}(x, 1-y) = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B_p^{(\alpha,\beta)}(x+n, 1),$$

$$(\mathrm{Re}(p) > 0).$$

Proof. From the definition of the new generalized beta function, we get

$$B_p^{(\alpha,\beta)}(x, 1-y) = \int_0^1 t^{x-1}(1-t)^{-y} \; {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) dt.$$

Using the following binomial series expansion

$$(1-t)^{-y} = \sum_{n=0}^{\infty} (y)_n \frac{t^n}{n!},$$

$$(|t| < 1),$$

we obtain

$$B_p^{(\alpha,\beta)}(x, 1-y) = \int_0^1 \sum_{n=0}^{\infty} \frac{(y)_n}{n!} t^{x+n-1} \; {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) dt.$$

Therefore, interchanging the order of integration and summation and then using (4.2), we obtain

$$\begin{aligned} B_p^{(\alpha,\beta)}(x, 1-y) &= \sum_{n=0}^{\infty} \frac{(y)_n}{n!} \int_0^1 t^{x+n-1} \; {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) dt, \\ &= \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B_p^{(\alpha,\beta)}(x+n, 1). \end{aligned}$$

Hence the proof is completed. □

Now we obtain differential recurrence relations for generalized gamma and generalized beta functions defined by (4.1) and (4.2), respectively.

Theorem 4.2.10. *For generalized gamma function, we have the following recurrence relation:*

$$\frac{d^2\left(\Gamma_p^{(\alpha,\beta)}(x+5)\right)}{dp^2}+p\frac{d^2\left(\Gamma_p^{(\alpha,\beta)}(x+3)\right)}{dp^2}-\beta\frac{d\left(\Gamma_p^{(\alpha,\beta)}(x+2)\right)}{dp}$$
$$-\frac{d\left(\Gamma_p^{(\alpha,\beta)}(x+3)\right)}{dp}-p\frac{d\left(\Gamma_p^{(\alpha,\beta)}(x+1)\right)}{dp}+\alpha\Gamma_p^{(\alpha,\beta)}(x)=0.$$

Proof. Taking derivatives under the integral symbol by using the Leibnitz rule, we get

$$\frac{d^2\left(\Gamma_p^{(\alpha,\beta)}(x+5)\right)}{dp^2}+p\frac{d^2\left(\Gamma_p^{(\alpha,\beta)}(x+3)\right)}{dp^2}-\beta\frac{d\left(\Gamma_p^{(\alpha,\beta)}(x+2)\right)}{dp}$$
$$-\frac{d\left(\Gamma_p^{(\alpha,\beta)}(x+3)\right)}{dp}-p\frac{d\left(\Gamma_p^{(\alpha,\beta)}(x+1)\right)}{dp}+\alpha\Gamma_p^{(\alpha,\beta)}(x)$$
$$=\int_0^\infty t^{x-1}\left[\left(t^3+pt\right)\frac{d^2z}{dp^2}+\left(t^2+\beta t+p\right)\frac{dz}{dp}+\alpha z\right]dt,$$

where $z={}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)$. On the other hand, since $z={}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)$ is a solution of the equation

$$\left(t^3+pt\right)\frac{d^2z}{dp^2}+\left(t^2+\beta t+p\right)\frac{dz}{dp}+\alpha z=0,$$

we get the result. □

Theorem 4.2.11. *For generalized beta function, we have the following recurrence relation:*

$$p\frac{d^2\left(B_p^{(\alpha,\beta)}(x+3,y+3)\right)}{dp^2}+\beta\frac{d\left(B_p^{(\alpha,\beta)}(x+2,y+2)\right)}{dp}$$
$$+p\frac{d\left(B_p^{(\alpha,\beta)}(x+1,y+1)\right)}{dp}+\alpha B_p^{(\alpha,\beta)}(x,y)=0.$$

Proof. Let $\mathcal{S}$ denotes the left handside of the above assertion. Taking derivatives under the integral symbol in (4.2) by using the Leibnitz rule, we get

$$\mathcal{S}=\int_0^1 t^{x-1}(1-t)^{y-1}\left[pt(1-t)\frac{d^2z}{dp^2}+(\beta t(1-t)+p)\frac{dz}{dp}+\alpha z\right]dt,$$

where $z = {}_1F_1\left(\alpha;\beta;\dfrac{-p}{t(1-t)}\right)$. Since $z = {}_1F_1\left(\alpha;\beta;\dfrac{-p}{t(1-t)}\right)$ is a solution of the equation

$$pt(1-t)\frac{d^2z}{dp^2} + \left(\beta t(1-t) + p\right)\frac{dz}{dp} + \alpha z = 0,$$

we get the result. □

4.3 Generalized Gauss Hypergeometric and Confluent Hypergeometric Functions

In this section we use the new generalization (4.2) of beta function to generalize the hypergeometric and confluent hypergeometric functions defined by

$$F_p^{(\alpha,\beta)}(a,b;c;z) := \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}$$

and

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z) := \sum_{n=0}^{\infty} \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!},$$

respectively.

We call the $F_p^{(\alpha,\beta)}(a,b;c;z)$ by generalized Gauss hypergeometric function (GGHF) and ${}_1F_1^{(\alpha,\beta;p)}(b;c;z)$ by generalized confluent hypergeometric function (GCHF).

Observe that [5],

$$F_p^{(\alpha,\alpha)}(a,b;c;z) = F_p(a,b;c;z),$$
$$F_0^{(\alpha,\beta)}(a,b;c;z) = {}_2F_1(a,b;c;z),$$

and

$${}_1F_1^{(\alpha,\alpha;p)}(b;c;z) = {}_1F_1^{(p)}(b;c;z) = \phi_p(b;c;z),$$
$${}_1F_1^{(\alpha,\beta;0)}(b;c;z) = {}_1F_1(b;c;z).$$

4.4 Integral Representations of the GGHF and GCHF

The GGHF can be provided with an integral representation by using the definition of the new generalized beta function (4.2). We get

Theorem 4.4.1. *For the GGHF, we have the following integral representations:*

$$F_p^{(\alpha,\beta)}(a,b;c;z) := \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \cdot {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)(1-zt)^{-a}\,dt, \tag{4.8}$$

$$(\operatorname{Re}(p) > 0;\ p = 0 \text{ and } |arg(1-z)| < \pi; \operatorname{Re}(c) > \operatorname{Re}(b) > 0)$$

$$F_p^{(\alpha,\beta)}(a,b;c;z) := \frac{1}{B(b,c-b)} \int_0^\infty u^{b-1}(1+u)^{a-c}[1+u(1-z)]^{-a} \cdot {}_1F_1\left(\alpha;\beta;-2p-p\left(u+\frac{1}{u}\right)\right)du,$$

$$F_p^{(\alpha,\beta)}(a,b;c;z) := \frac{2}{B(b,c-b)} \int_0^{\frac{\pi}{2}} \sin^{2b-1}\nu\cos^{2c-2b-1}\nu\left(1-z\sin^2\nu\right)^{-a} \cdot {}_1F_1\left(\alpha;\beta;\frac{-p}{\sin^2\nu\cos^2\nu}\right)d\nu.$$

Proof. Direct calculations yield

$$\begin{aligned}
F_p^{(\alpha,\beta)}(a,b;c;z) &:= \sum_{n=0}^{\infty}(a)_n \frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!} \\
&= \frac{1}{B(b,c-b)}\sum_{n=0}^{\infty}(a)_n \int_0^1 t^{b+n-1}(1-t)^{c-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\frac{z^n}{n!}dt \\
&= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\sum_{n=0}^{\infty}(a)_n\frac{(zt)^n}{n!}dt \\
&= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)(1-zt)^{-a}\,dt.
\end{aligned}$$

Setting $u = \frac{t}{1-t}$ in (4.8), we get

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^\infty u^{b-1}(1+u)^{a-c}[1+u(1-z)]^{-a} \cdot {}_1F_1\left(\alpha;\beta;-2p-p\left(u+\frac{1}{u}\right)\right)du.$$

On the other hand, substituting $t = \sin^2 \nu$ in (4.8), we have

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{2}{B(b,c-b)} \int_0^{\frac{\pi}{2}} \sin^{2b-1}\nu \cos^{2c-2b-1}\nu \left(1 - z\sin^2\nu\right)^{-a} \cdot {}_1F_1\left(\alpha;\beta;\frac{-p}{\sin^2\nu\cos^2\nu}\right) d\nu.$$

Hence the proof is completed. □

A similar procedure yields integral representation of the GCHF by using the definition of the new generalized beta function.

Theorem 4.4.2. *For the GCHF, we have the following integral representations:*

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z) := \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} e^{zt}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt, \tag{4.9}$$

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z) := \int_0^1 \frac{(1-u)^{b-1} u^{c-b-1}}{B(b,c-b)} e^{z(1-u)}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{u(1-u)}\right) du.$$

$$(p \geq 0;\ \textit{and}\ \operatorname{Re}(c) > \operatorname{Re}(b) > 0)$$

Remark 4.4.3. *Putting $p = 0$ in (4.8) and (4.9), we get the integral representations of the classical GHF and CHF.*

4.5 Differentiation Formulas for the New GGHF's and New GCHF's

In this section, by using the formulas $B(b,c-b) = \frac{c}{b}B(b+1,c-b)$ and $(a)_{n+1} = a(a+1)_n$, we obtain new formulas including derivatives of GGHF and GCHF with respect to the variable z .

Theorem 4.5.1. *For GGHF, we have the following differentiation formula:*

$$\frac{d^n}{dz^n}\left\{F_p^{(\alpha,\beta)}(a,b;c;z)\right\} = \frac{(b)_n (a)_n}{(c)_n} F_p^{(\alpha,\beta)}(a+n,b+n;c+n;z).$$

Proof. Taking the derivative of $F_p^{(\alpha,\beta)}(a,b;c;z)$ with respect to z, we obtain

$$\begin{aligned}\frac{d}{dz}\left\{F_p^{(\alpha,\beta)}(a,b;c;z)\right\} &= \frac{d}{dz}\left\{\sum_{n=0}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^n}{n!}\right\}\\ &= \sum_{n=1}^{\infty}(a)_n\frac{B_p^{(\alpha,\beta)}(b+n,c-b)}{B(b,c-b)}\frac{z^{n-1}}{(n-1)!}.\end{aligned}$$

Replacing $n\to n+1$, we get

$$\begin{aligned}\frac{d}{dz}\left\{F_p^{(\alpha,\beta)}(a,b;c;z)\right\} &= \frac{ba}{c}\sum_{n=0}^{\infty}(a+1)_n\frac{B_p^{(\alpha,\beta)}(b+n+1,c-b)}{B(b+1,c-b)}\frac{z^n}{n!}\\ &= \frac{ba}{c}F_p^{(\alpha,\beta)}(a+1,b+1;c+1;z).\end{aligned}$$

Recursive application of this procedure gives us the general form:

$$\frac{d^n}{dz^n}\left\{F_p^{(\alpha,\beta)}(a,b;c;z)\right\} = \frac{(b)_n(a)_n}{(c)_n}F_p^{(\alpha,\beta)}(a+n,b+n;c+n;z).$$

Whence the result. □

Theorem 4.5.2. *For GCHF, we have the following differentiation formula:*

$$\frac{d^n}{dz^n}\,{}_1F_1^{(\alpha,\beta;p)}(b;c;z) = \frac{(b)_n}{(c)_n}\,{}_1F_1^{(\alpha,\beta;p)}(b+n;c+n;z).$$

4.6 Mellin Transform Representation of the GGHF's and GCHF's

In this section, we obtain the Mellin transform representations of the GGHF and GCHF.

Theorem 4.6.1. *For the GGHF, we have the following Mellin transform representation:*

$$\mathfrak{M}\left\{F_p^{(\alpha,\beta)}(a,b;c;z):s\right\} := \frac{\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\,{}_2F_1(a,b+s;c+2s;z).$$

Proof. To obtain the Mellin transform, we multiply both sides of (10) by p^{s-1} and integrate with respect to p over the interval $[0,\infty)$. Thus we get

$$\begin{aligned}&\mathfrak{M}\left\{F_p^{(\alpha,\beta)}(a,b;c;z):s\right\} := \int_0^{\infty}p^{s-1}F_p^{(\alpha,\beta)}(a,b;c;z)\,dp\\ &= \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}.\left[\int_0^{\infty}p^{s-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dp\right]dt.\end{aligned} \tag{4.10}$$

Substituting $u = \frac{p}{t(1-t)}$ in (4.10), we obtain

$$\begin{aligned}\int_0^\infty p^{s-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dp &= \int_0^\infty u^{s-1}t^s(1-t)^s\,{}_1F_1\left(\alpha;\beta;-u\right)du\\ &= t^s(1-t)^s\int_0^\infty {}_1F_1\left(\alpha;\beta;-u\right)du\\ &= t^s(1-t)^s\Gamma^{(\alpha,\beta)}(s).\end{aligned}$$

Thus we get

$$\begin{aligned}\mathfrak{M}\left\{F_p^{(\alpha,\beta)}\left(a,b;c;z\right):s\right\} &= \frac{1}{B(b,c-b)}\int_0^1 t^{b+s-1}\left(1-t\right)^{c+s-b-1}\left(1-zt\right)^{-a}\Gamma^{(\alpha,\beta)}(s)dt\\ &= \frac{\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\frac{1}{B(b+s,c+s-b)}\\ &\cdot\int_0^1 t^{b+s-1}\left(1-t\right)^{c+2s-(b+s)-1}\left(1-zt\right)^{-a}dt\\ &= \frac{\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\,{}_2F_1\left(a,b+s;c+2s;z\right).\end{aligned}$$

Hence the proof is completed. □

Corollary 4.6.2. *By Mellin inversion formula, we have the following complex integral representation for* $F_p^{(\alpha,\beta)}$ *:*

$$F_p^{(\alpha,\beta)}\left(a,b;c;z\right) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\,{}_2F_1\left(a,b+s;c+2s;z\right)p^{-s}ds.$$

Theorem 4.6.3. *For the new GCHF, we have the following Mellin transform representation:*

$$\mathfrak{M}\left\{{}_1F_1^{(\alpha,\beta;p)}\left(b;c;z\right):s\right\} := \frac{\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\,{}_1F_1\left(b+s;c+2s;z\right).$$

Corollary 4.6.4. *By Mellin inversion formula, we have the following complex integral representation for* ${}_1F_1^{(\alpha,\beta;p)}$ *:*

$${}_1F_1^{(\alpha,\beta;p)}\left(b;c;z\right) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma^{(\alpha,\beta)}(s)B(b+s,c+s-b)}{B(b,c-b)}\,{}_1F_1\left(b+s;c+2s;z\right)p^{-s}ds.$$

4.7 Transformation Formulas

Theorem 4.7.1. *For the new GGHF, we have the following transformation formula:*

$$F_p^{(\alpha,\beta)}\left(a,b;c;z\right) = \left(1-z\right)^{-a}F_p^{(\alpha,\beta)}\left(a,c-b;b;\frac{z}{z-1}\right),$$

$$(|\arg(1-z)| < \pi).$$

Proof. By writing

$$[1 - z(1-t)]^{-a} = (1-z)^{-a}\left(1 + \frac{z}{1-z}t\right)^{-a}$$

and replacing $t \to 1-t$ in (4.8), we obtain

$$\begin{aligned} F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{(1-z)^{-a}}{B(b,c-b)} \int_0^1 (1-t)^{b-1}\, t^{c-b-1}\left(1 - \frac{z}{z-1}t\right)^{-a} \\ \cdot {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt \end{aligned}$$

$$(\mathrm{Re}\,(p) > 0;\ p = 0 \text{ and } |z| < \pi; \mathrm{Re}\,(c) > \mathrm{Re}\,(b) > 0).$$

Hence,

$$F_p^{(\alpha,\beta)}(a,b;c;z) = (1-z)^{-a}\, F_p^{(\alpha,\beta)}\left(a, c-b; b; \frac{z}{z-1}\right).$$

□

Remark 4.7.2. *Note that, replacing z by $1 - \dfrac{1}{z}$ in Theorem 4.7.1, one easily obtains the following transformation formula*

$$F_p^{(\alpha,\beta)}\left(a,b;c;1-\frac{1}{z}\right) = z^{\alpha} F_p^{(\alpha,\beta)}(a, c-b; b; 1-z)$$

$$(|\arg(z)| < \pi).$$

Furthermore, replacing z by $\dfrac{z}{1+z}$ in Theorem 4.7.1, we get the following transformation formula

$$F_p^{(\alpha,\beta)}\left(a,b;c;\frac{z}{1+z}\right) = (1+z)^{a}\, F_p^{(\alpha,\beta)}(a, c-b; b; z)$$

$$|\arg(1+z)| < \pi.$$

Theorem 4.7.3. *For the new GCHF, we have the following transformation formula:*

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z) = \exp(z)\,{}_1F_1^{(\alpha,\beta;p)}(c-b;c;-z).$$

Remark 4.7.4. *Setting $z = 1$ in (4.8), we have the following relation between new defined hypergeometric and beta functions:*

$$\begin{aligned} F_p^{(\alpha,\beta)}(a,b;c;1) &= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-a-b-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt \\ &= \frac{B_p^{(\alpha,\beta)}(b, c-a-b)}{B(b,c-b)}. \end{aligned}$$

4.8 Differential Recurrence Relations for GGHF's and GCHF's

In this section we obtain some differential recurrence relations for GGHF's and GCHF's. We start with the following theorem:

Theorem 4.8.1. *For GGHF's we have the following recurrence relation:*

$$pB(b+3,c-b+3)\ \frac{d^2F_p^{(\alpha,\beta)}\ (a,b+3;c+6;z)}{dp^2}$$
$$-\beta B(b+2,c-b+2)\frac{dF_p^{(\alpha,\beta)}\ (a,b+2;c+4;z)}{dp}$$
$$-pB(b+1,c-b+1)\frac{dF_p^{(\alpha,\beta)}\ (a,b+1;c+2;z)}{dp}+\alpha F_p^{(\alpha,\beta)}\ (a,b;c;z)=0.$$

Proof. Let $\mathcal{S}$ denotes the left handside of the above assertion. Taking derivatives under the integral symbol in (4.8) by using the Leibnitz rule, we get

$$\mathcal{S}=\int_0^1 t^{x-1}\,(1-t)^{y-1}\left[pt(1-t)\frac{d^2z}{dp^2}+(\beta t(1-t)+p)\,\frac{dz}{dp}+\alpha z\right]dt,$$

where $z=\ {}_1F_1\left(\alpha;\beta;\dfrac{-p}{t(1-t)}\right)$. Since $z=\ {}_1F_1\left(\alpha;\beta;\dfrac{-p}{t(1-t)}\right)$ is a solution of the equation

$$pt(1-t)\frac{d^2z}{dp^2}+(\beta t(1-t)+p)\,\frac{dz}{dp}+\alpha z=0,$$

we get the result. $\square$

In a similar manner, we have the following for GCHF's:

Theorem 4.8.2. *For GCHF's we have the following recurrence relation:*

$$pB(b+3,c-b+3)\ \frac{d_1^2F_1^{(\alpha,\beta;p)}\ (b+3;c+6;z)}{dp^2}$$
$$-\beta B(b+2,c-b+2)\frac{d_1F_1^{(\alpha,\beta;p)}\ (b+2;c+4;z)}{dp}$$
$$-pB(b+1,c-b+1)\frac{d_1F_1^{(\alpha,\beta;p)}\ (b+1;c+2;z)}{dp}+\alpha_1F_1^{(\alpha,\beta;p)}\ (b;c;z)=0.$$

Chapter 5

SOME FAMILIES OF GENERATING FUNCTIONS FOR A CLASS OF BIVARIATE POLYNOMIALS

5.1 Introduction

Over three decades ago, Srivastava [30] considered the following family of polynomials,

$$S_n^N(z) := \sum_{k=0}^{\left[\frac{n}{N}\right]} \frac{(-n)_{Nk}}{k!} A_{n,k} z^k \tag{5.1}$$

$$(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\,; N \in \mathbb{N}),$$

where $\{A_{n,k}\}_{n,k=0}^{\infty}$ is a bounded double sequence of real or complex numbers, $[a]$ denotes the greatest integer in $a \in \mathbb{R}$, and $(\lambda)_\nu$, denotes the Pochhammer symbol. The Srivastava polynomials $S_n^N(z)$ in (5.1) and their such interesting variants as follows:

$$S_{n,m}^N(z) := \sum_{k=0}^{\left[\frac{n}{N}\right]} \frac{(-n)_{Nk}}{k!}\, A_{m+n,k}\, z^k \quad (n, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\,; N \in \mathbb{N}), \tag{5.2}$$

were investigated rather extensively by Gonzaléz *et al.* [15] and (more recently) by Lin *et al.* [18]. Clearly, we have

$$S_{n,0}^N(z) = S_n^N(z).$$

Motivated essentially by the definitions (5.1) and (5.2), the following family of bivariate polynomials was studied by Altın *et al.* [1]:

$$S_n^{m,N}(x, y) := \sum_{k=0}^{\left[\frac{n}{N}\right]} A_{m+n,k} \frac{x^{n-Nk}}{(n-Nk)!} \frac{y^k}{k!} \qquad (n, m \in \mathbb{N}_0; \quad N \in \mathbb{N}), \tag{5.3}$$

who showed that the *two-variable* polynomials $S_n^{m,N}(x, y)$ include, as their particular cases, such well known polynomials as Lagrange-Hermite polynomials, Lagrange polynomials and Hermite-Kampé de Feriét polynomials (see, for details, [1]). However, by comparing the definitions (5.2) and (5.3), it is easily observe that

$$S_n^{m,N}(x,y) = \frac{x^n}{n!} S_{n,m}^N \left(\frac{y}{(-x)^N} \right), \tag{5.4}$$

and

$$S_{n,m}^N \left(\frac{y}{(-x)^N} \right) = \frac{n!}{x^n} S_n^{m,N}(x,y), \tag{5.5}$$

which exhibit the fact that the *two-variable* polynomials $S_n^{m,N}(x,y)$ are substantially the same as the *one-variable* polynomials $S_{n,m}^N(z)$ which were introduced and investigated earlier by Gonzaléz et al. [15]. Lately, the following family of polynomials in three variables was studied by Srivastava *et al.* [33]

$$S_n^{m,M,N}(x,y,z) := \sum_{k=0}^{\left[\frac{n}{N}\right]} \sum_{l=0}^{\left[\frac{k}{M}\right]} A_{m+n,k,l} \frac{x^l}{l!} \frac{y^{k-Ml}}{(k-Ml)!} \frac{z^{n-Nk}}{(n-Nk)!}$$

$$(n,m \in \mathbb{N}_0; \quad M,N \in \mathbb{N}),$$

where$\{A_{m,n,k}\}_{m,n,k\in\mathbb{N}_0}$ is a suitably bounded triple sequence of real or complex numbers. Finally, it should be mentioned a very recent paper related with the above families of polynomials [19].

In this chapter, we present various families of generating functions for a class of polynomials in *two-variables* defined by (5.3). Furthermore, several general classes of bilinear, bilateral or mixed multilateral generating functions are obtained for these polynomials.

5.2 First Set of Main Results and Their Consequences

In this section, by applying a method which was used recently by Chen and Srivastava [12] for obtaining several general double series identities, we first aim to obtaining various families of generating functions for the polynomials $S_n^{m,N}(x,y)$, given by (5.3). We then consider some interesting applications and corollaries resulting from these general theorems on generating functions.

Theorem 5.2.1. *Let $\{\Omega(n)\}_{n=0}^{\infty}$ be a bounded sequence of complex numbers. Then*

$$\sum_{n,m=0}^{\infty} \frac{\Omega(2m+n)}{\left(\nu+\frac{1}{2}\right)_m} S_n^{2m,N}(x,y) \frac{z^{2m}}{m!} w^n = \sum_{m,n,k=0}^{\infty} \Omega(m+n+Nk)$$

$$.A_{m+n+Nk,k} \frac{(\nu)_m}{(2\nu)_m} \frac{(-4z)^m}{m!} \frac{(xw+2z)^n}{n!} \frac{\left(yw^N\right)^k}{k!} \tag{5.6}$$

$$\left(\nu+\frac{1}{2}, 2\nu \notin \mathbb{Z}_0^- = \{0,-1,-2,...\}\right),$$

provided that each member of the series identity (5.6) exists.

Proof. For the sake of convenience, we denote the left hand side of (5.6) by $\Delta_{\nu,N}(x,y,z,w)$. Then using the definition (5.3) of $S_n^{m,N}(x,y)$ in the left hand side of (5.6) and setting $n \to n+Nk$, we have

$$\Delta_{\nu,N}(x,y,z,w) = \sum_{m,n,k=0}^{\infty} \frac{\Omega(2m+n+Nk)}{\left(\nu+\frac{1}{2}\right)_m} A_{2m+n+Nk,k} \frac{z^{2m}(xw)^n(yw^N)^k}{m!n!k!}. \tag{5.7}$$

Now setting

$$n \to n-2m \qquad (0 \leq m \leq \left[\frac{n}{2}\right]; n,m \in \mathbb{N}_0)$$

in (5.7) and then using the following elementary identity:

$$(n-2m)! = \frac{n!}{2^{2m}(-\frac{n}{2})_m(-\frac{n}{2}+\frac{1}{2})_m} \qquad (0 \leq m \leq \left[\frac{n}{2}\right]; n,m \in \mathbb{N}_0),$$

we get

$$\begin{aligned}
\Delta_{\nu,N}(x,y,z,w) &= \sum_{n,k=0}^{\infty} \Omega(n+Nk) A_{n+Nk,k} \frac{(xw)^n(yw^N)^k}{n!k!} \\
&\quad . \left(\sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{(-\frac{n}{2})_m(-\frac{n}{2}+\frac{1}{2})_m}{m!\left(\nu+\frac{1}{2}\right)_m} \left(\frac{2z}{xw}\right)^{2m} \right) \\
&= \sum_{n,k=0}^{\infty} \Omega(n+Nk) A_{n+Nk,k} \frac{(xw)^n(yw^N)^k}{n!k!} \\
&\quad ._2F_1\left[-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}; \nu+\frac{1}{2}; \left(\frac{2z}{xw}\right)^2\right].
\end{aligned} \tag{5.8}$$

Finally, by making use of the following quadratic transformation for Gauss hypergeometric function ${}_2F_1$:

$$ {}_2F_1\left(\alpha,\alpha+\frac{1}{2};\gamma;z^2\right)=(1+z)^{-2\alpha}\,{}_2F_1\left(2\alpha,\gamma-\frac{1}{2};2\gamma-1;\frac{2z}{1+z}\right) \tag{5.9}$$

$$\left(|\arg(1+z)|\leq\pi-\varepsilon\quad(0<\varepsilon<\pi)\,;\quad\gamma,2\gamma-1\notin\mathbb{Z}_0^-\right),$$

in the last member of (5.8), we find that

$$\begin{aligned}\Delta_{\nu,N}(x,y,z,w)&=\sum_{n,k=0}^{\infty}\Omega(n+Nk)A_{n+Nk,k}\frac{(xw+2z)^n\left(yw^N\right)^k}{n!k!}\\&\quad\cdot{}_2F_1\left[-n,\nu;2\nu;\frac{4z}{xw+2z}\right]\\&=\sum_{n,k=0}^{\infty}\Omega(n+Nk)A_{n+Nk,k}\frac{(xw+2z)^n\left(yw^N\right)^k}{k!n!}\\&\quad\cdot\sum_{m=0}^{n}\frac{(-n)_m\,(\nu)_m}{(2\nu)_m\;m!}\left(\frac{4z}{xw+2z}\right)^m,\end{aligned}$$

which, upon rearranging the sums, yields

$$\begin{aligned}\Delta_{\nu,N}(x,y,z,w)&=\sum_{n,k=0}^{\infty}\sum_{m=0}^{n}\Omega(n+Nk)A_{n+Nk,k}\frac{(\nu)_m}{(2\nu)_m}\\&\quad\cdot\frac{(-4z)^m}{m!}\frac{(xw+2z)^{n-m}}{(n-m)!}\frac{\left(yw^N\right)^k}{k!}\\&=\sum_{m,n,k=0}^{\infty}\Omega(m+n+Nk)A_{m+n+Nk,k}\frac{(\nu)_m}{(2\nu)_m}\\&\quad\cdot\frac{(-4z)^m}{m!}\frac{(xw+2z)^n}{n!}\frac{\left(yw^N\right)^k}{k!}.\end{aligned}$$

Thus the proof is completed. □

In a similar manner, by applying the method used in proving Theorem 5.2.1 and following hypergeometric transformation:

$$ {}_2F_1\left(\alpha,\alpha+\frac{1}{2};\gamma;z\right)=(1-z)^{-\alpha}\,{}_2F_1\left(2\alpha,2\gamma-2\alpha-1;\gamma;\frac{\sqrt{1-z}-1}{2\sqrt{1-z}}\right) \tag{5.10}$$

$$\left(|\arg(1-z)|\leq\pi-\varepsilon\quad(0<\varepsilon<\pi)\,;\;\gamma\notin\mathbb{Z}_0^-\right),$$

we can easily prove the following Theorem 5.2.2.

Theorem 5.2.2. *Let $\{\Omega(n)\}_{n=0}^{\infty}$ be a bounded sequence of complex numbers. Then, we have*

$$\sum_{m,n=0}^{\infty} \frac{\Omega(2m+n)}{(\nu)_m} S_n^{2m,N}(x,y) \frac{z^m w^n}{m!} = \sum_{m,n,k=0}^{\infty} \Omega(m+n+Nk) A_{m+n+Nk,k}$$
$$\cdot \frac{(2\nu+n+m-1)_m}{(\nu)_m} \frac{\left\{\frac{1}{2}\left(wx - \sqrt{w^2x^2-4z}\right)\right\}^m}{m!}$$
$$\cdot \frac{(\sqrt{w^2x^2-4z})^n}{n!} \frac{\left(yw^N\right)^k}{k!} \qquad \left(\nu \notin \mathbb{Z}_0^-\right), \tag{5.11}$$

provided that each member of the series identity (5.11) exists.

By setting

$$w = \frac{-2z}{x}$$

in Theorem 5.2.1, we get Corollary 5.2.3 below.

Corollary 5.2.3. *The following series identity holds true:*

$$\sum_{m,n=0}^{\infty} \frac{\Omega(2m+n)}{\left(\nu+\frac{1}{2}\right)_m} S_n^{2m,N}(x,y) \frac{z^{2m}}{m!} \left(\frac{-2z}{x}\right)^n$$
$$= \sum_{n,m=0}^{\infty} \Omega(m+Nn) A_{m+Nn,n}$$
$$\cdot \frac{(\nu)_m}{(2\nu)_m} \frac{(-4z)^m}{m!} \frac{\left(y\left(-2x^{-1}z\right)^N\right)^n}{n!}, \tag{5.12}$$

whenever each member (5.12) exists.

Upon setting

$$N = 1 \text{ and } A_{m,n} = (\alpha)_{m-n}(\beta)_n$$

in Corollary 5.2.3 and taking into account the following relationship:

$$S_n^{2m,1}(x,y) = (\alpha)_{2m}\, g_n^{(\alpha+2m,\beta)}(x,y)$$

where ([32], p. 441, Equation 8.5, 12)

$$g_n^{(\alpha,\beta)}(x,y) = \sum_{k=0}^{n} (\alpha)_{n-k}(\beta)_k \frac{x^{n-k}y^k}{(n-k)!k!} \tag{5.13}$$

are the familiar Lagrange polynomials in two variables, we obtain

$$\sum_{m,n=0}^{\infty} \frac{\Omega(2m+n)}{\left(\nu+\frac{1}{2}\right)_m} (\alpha)_{2m}\, g_n^{(\alpha+2m,\beta)}(x,y) \frac{z^{2m}}{m!} \left(\frac{-2z}{x}\right)^n$$
$$= \sum_{m,n=0}^{\infty} \Omega(m+n)\, (\alpha)_m\, (\beta)_n\, \frac{(\nu)_m}{(2\nu)_m} \frac{(-4z)^m}{m!} \frac{(-2x^{-1}yz)^n}{n!}. \tag{5.14}$$

On the other hand, by taking $\Omega(n) = 1$ $(n \in \mathbb{N}_0)$ in (5.14), we can derive the following generating function for Lagrange polynomials $g_n^{(\alpha,\beta)}(x,y)$ given explicitly by (5.13):

$$\sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m}}{\left(\nu+\frac{1}{2}\right)_m} g_n^{(\alpha+2m,\beta)}(x,y) \frac{z^{2m}}{m!} \left(\frac{-2z}{x}\right)^n$$
$$= \sum_{m,n=0}^{\infty} \frac{(\alpha)_m\, (\nu)_m}{(2\nu)_m} \frac{(-4z)^m}{m!} (\beta)_n \frac{(-2x^{-1}yz)^n}{n!}$$
$$= \left(1+2x^{-1}yz\right)^{-\beta} {}_2F_1\left(\alpha, \nu; 2\nu; -4z\right) \qquad \left(\left|\frac{yz}{x}\right| < \frac{1}{2}\right). \tag{5.15}$$

In view of the following relationship between the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ and the Lagrange polynomials $g_n^{(\alpha,\beta)}(x,y)$ ([32], p. 442, Equation 8.5, 17):

$$g_n^{(\alpha+2m,\beta)}(x,y) = (y-x)^n P_n^{(-\alpha-2m-n,-\beta-n)}\left(\frac{x+y}{x-y}\right),$$

we can rewrite (5.15) as a generating function for the Jacobi polynomials as follows :

$$\sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m}}{\left(\nu+\frac{1}{2}\right)_m} (y-x)^n P_n^{(-\alpha-2m-n,-\beta-n)}\left(\frac{x+y}{x-y}\right) \frac{z^{2m}}{m!} \left(\frac{-2z}{x}\right)^n$$
$$= \left(1+2x^{-1}yz\right)^{-\beta} {}_2F_1\left(\alpha, \nu; 2\nu; -4z\right), \qquad \left(\left|\frac{yz}{x}\right| < \frac{1}{2}\right). \tag{5.16}$$

Next, by setting

$$z = \left(\frac{wx}{2}\right)^2$$

in Theorem 5.2.2, we get Corollary 5.2.4 below.

Corollary 5.2.4. *The following series identity holds true:*

$$\sum_{m,n=0}^{\infty} \frac{\Omega(2m+n)}{(\nu)_m} S_n^{2m,N}(x,y) \frac{x^{2m}}{m!} \frac{w^{2m+n}}{2^m}$$
$$= \sum_{m,n=0}^{\infty} \Omega(m+Nn) A_{m+Nn,n} \frac{(2\nu+m-1)_m}{(\nu)_m\, m!} \left(\frac{wx}{2}\right)^m \frac{\left(yw^N\right)^n}{n!}, \tag{5.17}$$

whenever each member of (5.17) exists.

Choosing

$$N = 2 \text{ and } A_{m,n} = (\alpha)_{m-2n}(\beta)_n$$

in Corollary 5.2.4 and taking into account the following relationship:

$$S_n^{2m,2}(x,y) = (\alpha)_{2m} h_n^{(\alpha+2m,\beta)}(x,y),$$

where $h_n^{(\alpha,\beta)}(x,y)$ denotes the Lagrange-Hermite polynomials given explicitly by

$$h_n^{(\alpha,\beta)}(x,y) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (\alpha)_{n-2k}(\beta)_k \frac{x^{n-2k}y^k}{(n-2k)!k!}, \tag{5.18}$$

we obtain

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{\Omega(2m+n)}{(\nu)_m} (\alpha)_{2m} h_n^{(\alpha+2m,\beta)}(x,y) \frac{x^{2m}}{m!} \frac{w^{2m+n}}{2^m} \\
&= \sum_{m,n=0}^{\infty} \Omega(m+2n)(\alpha)_m(\beta)_n \frac{(2\nu+m-1)_m}{(\nu)_m \, m!} \\
&\cdot \left(\frac{wx}{2}\right)^m \frac{(yw^2)^n}{n!}.
\end{aligned} \tag{5.19}$$

In view of the elementary identity:

$$(2\nu+m-1)_m = \frac{2^{2m}\left(\frac{2\nu-1}{2}\right)_m (\nu)_m}{(2\nu-1)_m} \qquad (m \in \mathbb{N}_0),$$

by setting $\Omega(n) = 1 \quad (n \in \mathbb{N}_0)$ in (5.19), we can deduce the following generating function for the Lagrange-Hermite polynomials $h_n^{(\alpha,\beta)}(x,y)$ given explicitly by (5.18):

$$\begin{aligned}
&\sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m}}{(\nu)_m} h_n^{(\alpha+2m,\beta)}(x,y) \frac{x^{2m}}{m!} \frac{w^{2m+n}}{2^m} \\
&= \sum_{n,m=0}^{\infty} (\alpha)_m(\beta)_n \frac{\left(\frac{2\nu-1}{2}\right)_m}{(2\nu-1)_m \, m!} (2wx)^m \frac{(yw^2)^n}{n!} \\
&= \left(1-yw^2\right)^{-\beta} {}_2F_1\left(\alpha, \frac{2\nu-1}{2}; 2\nu-1; 2wx\right) \qquad \left(\left|yw^2\right| < 1\right).
\end{aligned} \tag{5.20}$$

Remark 5.2.5. *If we take $z = 0$ in Theorem 5.2.1 or in Theorem 5.2.2, we get the following generating function:*

$$\sum_{n=0}^{\infty} \Omega(n) S_n^{0,N}(x,y) w^n = \sum_{m,n=0}^{\infty} \Omega(n+Nm) A_{n+Nm,m} \frac{\left(yw^N\right)^m}{m!} \frac{(xw)^n}{n!},$$

which was derived earlier by Altın et al. [1].

5.3 Multilinear and Multilateral Generating Functions

The aim of this section is to obtain several families of bilinear, bilateral or mixed multilateral generating functions for the polynomials $S_n^{m,N}(x,y)$, given in (5.3), with the help of the method considered in such earlier works [20] and [23] (see also [[32], Chapter 8]) and very recently in [24].

First of all, by making use of (5.15), we have the following result.

Theorem 5.3.1. *Corresponding to an identically nonvanishing function* $\Phi_{k,l}(\xi_1,\xi_2,...,\xi_s)$ *of s (real or complex) variables*

$$\xi_1,\xi_2,...,\xi_s \quad (s\in\mathbb{N}),$$

let

$$\Delta(\xi_1,\xi_2,...,\xi_s;\tau,\theta)=\sum_{k,l=0}^{\infty}a_{k,l}\Phi_{k,l}(\xi_1,\xi_2,...,\xi_s)\tau^k\theta^l \qquad (\,a_{k,l}\neq 0\,)\,. \tag{5.21}$$

Suppose also that

$$\begin{aligned}\Theta_{m,n}(x,y;\xi_1,\xi_2,...,\xi_s;\zeta,\omega)&=\sum_{k=0}^{\left[\frac{n}{q}\right]}\sum_{l=0}^{\left[\frac{m}{p}\right]}\frac{a_{k,l}}{(m-pl)!}\frac{(\alpha)_{2m-2pl}}{(\nu+\frac{1}{2})_{m-pl}}\\&.\Phi_{k,l}(\xi_1,\xi_2,...,\xi_s)\,g_{n-qk}^{(\alpha+2m-2pl,\beta+\lambda k+\rho l)}(x,y)\zeta^k\omega^l\\&(n,q,m,p\in\mathbb{N}\quad\lambda,\rho\in\mathbb{C}).\end{aligned} \tag{5.22}$$

Then

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Theta_{m,n}\left(x,y;\xi_1,\xi_2,...,\xi_s;\frac{\delta x^q}{(-2t)^q},\frac{\eta}{t^{2p}}\right)t^{2m}\left(\frac{-2t}{x}\right)^n\\&=\left(1+\frac{2yt}{x}\right)^{-\beta}{}_2F_1\left(\alpha,\nu;2\nu;-4t\right)\\&.\Delta\left(\xi_1,\xi_2,...,\xi_s;\frac{\delta}{(1+2ytx^{-1})^{\lambda}},\frac{\eta}{(1+2ytx^{-1})^{\rho}}\right)\\&\left(\left|\frac{yt}{x}\right|<\frac{1}{2}\right),\end{aligned} \tag{5.23}$$

provided that each member of (5.23) exists.

Proof. For convenience, let $\mathcal{M}(x,y,t)$ denote the first member of assertion (5.23) of Theorem 5.3.1. Then, upon substituting for

$$\Theta_{m,n}(x,y;\xi_1,\xi_2,...,\xi_s;\zeta,\omega)$$

from (5.22) into the left hand side of (5.23), we find that

$$\begin{aligned}
\mathcal{M}(x,y,t) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Theta_{m,n}\left(x,y;\xi_1,\xi_2,...,\xi_s;\frac{\delta x^q}{(-2t)^q},\frac{\eta}{t^{2p}}\right)t^{2m}\left(\frac{-2t}{x}\right)^n \\
&= \sum_{m,n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{q}\right]}\sum_{l=0}^{\left[\frac{m}{p}\right]}\frac{a_{k,l}}{(m-pl)!}\Phi_{k,l}(\xi_1,\xi_2,...,\xi_s)\frac{(\alpha)_{2m-2pl}}{(\nu+\frac{1}{2})_{m-pl}} \\
&\quad .g_{n-qk}^{(\alpha+2m-2pl,\beta+\lambda k+\rho l)}(x,y)\delta^k\eta^l t^{2m-2pl}\left(\frac{-2t}{x}\right)^{n-qk}.
\end{aligned} \tag{5.24}$$

Now, by setting $n\to n+qk$ and $m\to m+pl$ in (5.24), we obtain

$$\begin{aligned}
\mathcal{M}(x,y,t) &= \sum_{n,m,k,l=0}^{\infty}a_{k,l}\Phi_{k,l}(\xi_1,\xi_2,...,\xi_s)\frac{(\alpha)_{2m}}{(\nu+\frac{1}{2})_m}g_n^{(\alpha+2m,\beta+\lambda k+\rho l)}(x,y) \\
&\quad .\delta^k\eta^l\frac{t^{2m}}{m!}\left(\frac{-2t}{x}\right)^n.
\end{aligned} \tag{5.25}$$

Finally, in view of (5.15) with $\beta\to\beta+\lambda k+\rho l$, we find from (5.25) that

$$\begin{aligned}
\mathcal{M}(x,y,t) &= \left(1+2x^{-1}yt\right)^{-\beta}\ {}_2F_1\left(\alpha,\nu;2\nu;-4t\right) \\
&\quad .\sum_{k,l=0}^{\infty}a_{k,l}\Phi_{k,l}(\xi_1,\xi_2,...,\xi_s)\frac{\delta^k}{(1+2x^{-1}yt)^{\lambda k}}\frac{\eta^l}{(1+2x^{-1}yt)^{\rho l}} \\
&= \left(1+2x^{-1}yt\right)^{-\beta}\ {}_2F_1\left(\alpha,\nu;2\nu;-4t\right) \\
&\quad .\Delta\left(\xi_1,\xi_2,...,\xi_s;\frac{\delta}{(1+2ytx^{-1})^{\lambda}},\frac{\eta}{(1+2ytx^{-1})^{\rho}}\right),
\end{aligned}$$

which proves the assertion (5.23) of Theorem 5.3.1. $\square$

In a similar way, by applying (5.20) instead of (5.15), we are led fairly easily to Theorem 5.3.2 below.

Theorem 5.3.2. *Corresponding to an identically nonvanishing function* $\Phi_{k,l}^{(1)}(\xi_1, \xi_2, ..., \xi_s)$ *of s (real or complex) variables*

$$\xi_1, \xi_2, ..., \xi_s \qquad (s \in \mathbb{N})$$

let

$$\Delta^{(1)}(\xi_1, \xi_2, ..., \xi_s; \tau, \theta) = \sum_{k,l=0}^{\infty} b_{k,l} \Phi_{k,l}^{(1)}(\xi_1, \xi_2, ..., \xi_s)\tau^k \theta^l \quad (b_{k,l} \neq 0)\,. \tag{5.26}$$

Suppose also that

$$\Theta_{n,m}^{(1)}(x, y; \xi_1, \xi_2, ..., \xi_s; \zeta, \omega) = \sum_{k=0}^{\left[\frac{n}{q}\right]} \sum_{l=0}^{\left[\frac{m}{p}\right]} \frac{b_{k,l}}{(m-pl)!} \frac{(\alpha)_{2m-2pl}}{(\nu)_{m-pl}}$$

$$.\Phi_{k,l}^{(1)}(\xi_1, \xi_2, ..., \xi_s) h_{n-qk}^{(\alpha+2m-2pl, \beta+\lambda k+\rho l)}(x, y)\zeta^k \omega^l \tag{5.27}$$

$$(n, q, m, p \in \mathbb{N} \quad \lambda, \rho \in \mathbb{C}).$$

Then

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \Theta_{n,m}^{(1)} \left(x, y; \xi_1, \xi_2, ..., \xi_s; \frac{\delta}{t^q}, \frac{2^p \eta}{(tx)^{2p}} \right) \frac{(tx)^{2m}}{2^m} t^n$$

$$= \left(1 - yt^2\right)^{-\beta} {}_2F_1 \left(\alpha, \frac{2\nu - 1}{2}; 2\nu - 1; 2tx \right)$$

$$.\Delta^{(1)} \left(\xi_1, \xi_2, ..., \xi_s; \frac{\delta}{(1 - yt^2)^{\lambda}}, \frac{\eta}{(1 - yt^2)^{\rho}} \right) \tag{5.28}$$

$$\left(\left|yt^2\right| < 1\right),$$

provided that each member of (5.28) exists.

We conclude of our investigation by presenting the following illustrative example.

Example 5.3.3. *Upon setting*

$$s = 1, \quad \xi_1 = z, \quad \Phi_{k,l}(z) = L_{k,l}^{(\gamma_1, \gamma_2; \beta)}(z) \text{ and } a_{k,l} = \frac{1}{k!l!}$$

in Theorem 5.3.1, we obtain (see [32], p.858, Theorem 2.2)

$$\Delta(z; \tau, \theta) = \frac{1}{(1 - \tau)^{\gamma_1 + 1}(1 - \theta)^{\gamma_2 + 1}} \exp\left(\frac{\beta(\tau + \theta - \tau\theta) z}{(1 - \tau)(1 - \theta)} \right), \tag{5.29}$$

$L_{m,n}^{(\alpha_1,\alpha_2;\beta)}(z)$ being the case $r=2$ of the multiple Laguerre polynomials $L_{n_1,\ldots,n_r}^{(\alpha_1,\ldots,\alpha_r;\beta)}(z)$ studied recently by Lee ([32], p. 856, Equation 2). Making use of the generating function (5.29), it is not difficult to deduce the following interesting application of Theorem 5.3.1:

$$\sum_{m,n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{q}\right]}\sum_{l=0}^{\left[\frac{m}{p}\right]} L_{k,l}^{(\gamma_1,\gamma_2;\beta)}(z)\, g_{n-qk}^{(\alpha+2m-2pl,\beta+\lambda k+\rho l)}(x,y)$$
$$\cdot\frac{(\alpha)_{2m-2pl}}{(\nu+\frac{1}{2})_{m-pl}}\frac{\delta^k\eta^l}{k!l!}\frac{t^{2m-2pl}}{(m-pl)!}\left(\frac{-2t}{x}\right)^{n-qk}$$
$$=\left(1+\frac{2yt}{x}\right)^{-\beta}\left(1-\frac{\delta}{(1+2ytx^{-1})^{\lambda}}\right)^{-\gamma_1-1}\left(1-\frac{\eta}{(1+2ytx^{-1})^{\rho}}\right)^{-\gamma_2-1}$$
$$.\exp\left(\frac{\beta\left(\frac{\delta}{(1+2ytx^{-1})^{\lambda}}+\frac{\eta}{(1+2ytx^{-1})^{\rho}}-\frac{\delta}{(1+2ytx^{-1})^{\lambda}}\frac{\eta}{(1+2ytx^{-1})^{\rho}}\right)z}{\left(1-\frac{\delta}{(1+2ytx^{-1})^{\lambda}}\right)\left(1-\frac{\eta}{(1+2ytx^{-1})^{\rho}}\right)}\right)$$
$$\cdot{}_2F_1(\alpha,\nu;2\nu;-4t) \tag{5.30}$$
$$\left(\left|\frac{\delta}{(1+2ytx^{-1})^{\lambda}}\right|<1,\ \left|\frac{\eta}{(1+2ytx^{-1})^{\rho}}\right|<1,\ \left|\frac{yt}{x}\right|<\frac{1}{2}\right).$$

For each suitable choice of the following coefficients:

$$a_{k,l} \text{ and } b_{k,l} \quad (k,l\in\mathbb{N})$$

occuring in Theorems 5.3.1 and 5.3.2, the multivariable functions $\Phi_{k,l}$ and $\Phi_{k,l}^{(1)}$ can be expressed as a product of several simpler functions. In this way, each of Theorem 5.3.1 and 5.3.2 would yield various classes of bilinear, bilateral or mixed multilateral generating functions for the Lagrange polynomials $g_n^{(\alpha,\beta)}(x,y)$ and the Lagrange-Hermite polynomials $h_n^{(\alpha,\beta)}(x,y)$, respectively.

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