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*Research Article*

# Boundary Value Problems Arising in Kalman Filtering

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The classic Kalman filtering equations for independent and correlated white noises are ordinary differential equations (deterministic or stochastic) with the respective initial conditions. Changing the noise processes by taking them to be more realistic wide band noises or delayed white noises creates challenging partial differential equations with initial and boundary conditions. In this paper, we are aimed to give a survey of this connection between Kalman filtering and boundary value problems, bringing them into the attention of mathematicians as well as engineers dealing with Kalman filtering and boundary value problems.



## 1. Introduction

In 1960-1961 Kalman [1] and Kalman and Bucy [2] proposed a method of estimation, called Kalman filtering, for linear dynamical systems corrupted by white noise processes. Briefly, Kalman filtering provides equations for the best estimate $\hat{x}_t$ of $x_t$ based on $z_s$, $0 \le s \le t$, where $x$ is treated as an unobservable signal process, satisfying

$$\begin{gathered} x_t' = Ax_t + Bw_t, \quad t > 0, \\ x_0 \text{ is given,} \end{gathered} \tag{1.1}$$

and $z$ as an observation process, depending on the signal in the linear form

$$\begin{gathered} z_t' = Cx_t + w_t, \quad t > 0, \\ z_0 = 0. \end{gathered} \tag{1.2}$$

However, (1.1)-(1.2) form a starting point for Kalman filtering problem, where $A$, $B$ and $C$ are matrices (resp., $x$ and $z$ are vector-valued) and $w$ is the so called vector-valued Gaussian white noise process with zero mean and covariance to be an identity matrix, all them of respective dimensions. It is assumed that $x_0$ is a Gaussian random vector with zero mean and known covariance $\operatorname{cov} x_0$ and independent on $w$.

The essence of Kalman filtering is that it presents $\hat{x}$ as a dynamical process to be a solution of the linear equation

$$\begin{gathered} \hat{x}_t' = A\hat{x}_t + (P_t C^* + B)(z_t' - C\hat{x}_t), \quad t > 0, \\ \hat{x}_0 = 0, \end{gathered} \tag{1.3}$$

where $C^*$ is the transpose of $C$ and $P$ is a solution of the matrix Riccati equation

$$\begin{gathered} P_t' = AP_t + P_t A^* + BB^* - (P_t C^* + B)(CP_t + B^*), \quad t > 0, \\ P_0 = \operatorname{cov} x_0. \end{gathered} \tag{1.4}$$

Here (1.4) can be solved a priori and the values of $P$ stored in a memory. Then (1.3) provides a linear transformation of the observation data $z_s$, $0 \le s \le t$, into the best estimate $\hat{x}_t$ for every $t > 0$. This transformation is called a Kalman filter. In applications the Kalman filter allows the replacement of the unknown signal $x_t$, which is very roughly expressed as a solution of (1.1), by its best possible estimate (in the mean square sense), which can be drawn from the available observations.

This result found wide applications in many applied areas, especially in space engineering. For the mathematical and engineering aspects of Kalman filtering we refer to Davis [3], Fleming and Rishel [4], Bensoussan [5], Liptser and Shiryayev [6], Curtain and Pritchard [7], Bucy and Joseph [8], Crassidis and Junkins [9].

In this paper, we give a survey of new results on Kalman filtering leading to boundary value problems. Such a connection between Kalman filtering and boundary value problems arise in cases when the noises involved to the Kalman filtering problem are delayed in time.

A delay of noises is not only a mathematical generalization of the basic Kalman filtering equations (1.3)-(1.4), but has a practical significance as well. It is well known that a white noise is an ideal process, approximating the noises in reality with more or less adequacy. In this regard, the remark in [4, page 126] by Fleming and Rishel is spectacular, where the authors describe wide band noises as a most adequate mathematical model of real noises. The issue on wide band noise was handled in Bashirov [10], where a wide band noise was represented in the form of distributed delay of a white noise, and on the base of this representation the Kalman filtering equations for the wide band noise model were derived. Now (1.3)-(1.4) of Kalman filtering change their form becoming two systems of equations combining as ordinary as well as partial differential equations with respective initial and boundary conditions.

Representation of wide band noises as a distributed delay of white noises became fruitful in order to derive Kalman filtering equations for pointwise delayed white noises as well. Such noises arise in real cases when a communication between the observer and the object takes considerable time. For example, in [11, 12] the case when the signal is corrupted by pointwise delayed white noise is suggested for the improvement of the preciseness of the Global Positioning Systems. A basic tool for derivation of Kalman filtering equations for pointwise delayed white noises, used in [11, 13], is an approximation of a white noise by wide band noises.

Our aim in this paper is to bring all these boundary value problems to the attention of the community of scientists dealing with boundary value problems and suggest the investigation of numerical methods for them.

## 2. The signal corrupted by wide band noise

The wide band noise Kalman filtering equations (8.60)–(8.66) from Bashirov [10] are too heavy since they are derived in Hilbert space case compressing two essentially different cases: wide band noise corrupting the signal and observations simultaneously. Here we delineate these cases, which lead to distinct patterns of boundary value problems and, respectively, require different numerical approaches. This essentially reduces the complication of these equations from [10], making a proper concentration on numerical methods.

Assume that the system (1.1) is disturbed by the wide band noise $\varphi$, represented as a distributed delay of the white noise $w$ in the form

$$\varphi_t = \int_{\min(0,t-\varepsilon)}^{t} \Phi_{\theta-t} w_\theta d\theta, \tag{2.1}$$

where $\Phi$ is a differentiable function on $[-\varepsilon, 0]$, satisfying $\Phi_{-\varepsilon} = 0$, and $\varepsilon > 0$ is a constant:

$$\begin{gathered} x_t' = Ax_t + \varphi_t, \quad t > 0, \\ x_0 \text{ is given.} \end{gathered} \tag{2.2}$$

Then the Kalman filtering equations for the systems (2.2) and (1.2) are

$$\begin{gathered} \hat{x}_t' = A\hat{x}_t + \varphi_{t,0} + P_t C^* \left(z_t' - C\hat{x}_t\right), \quad t > 0, \\ \hat{x}_0 = 0, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)\varphi_{t,\theta} = \left(Q_{t,\theta}^* C^* + \Phi_\theta\right)\left(z_t' - C\hat{x}_t\right), \quad -\varepsilon < \theta \le 0,\ t > 0, \\ \varphi_{0,\theta} = \varphi_{t,-\varepsilon} = 0, \quad -\varepsilon \le \theta \le 0,\ t > 0, \end{gathered} \tag{2.3}$$

$$\begin{gathered} P_t' = AP_t + P_t A^* + Q_{t,0} + Q_{t,0}^* - P_t C^* C P_t, \quad t > 0, \\ P_0 = \operatorname{cov} x_0, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right) Q_{t,\theta} = AQ_{t,\theta} + R_{t,0,\theta} - P_t C^* \left(CQ_{t,\theta} + \Phi_\theta^*\right), \quad -\varepsilon < \theta \le 0,\ t > 0, \\ Q_{0,\theta} = Q_{t,-\varepsilon} = 0, \quad -\varepsilon \le \theta \le 0,\ t > 0, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \tau}\right) R_{t,\theta,\tau} = \Phi_\theta \Phi_\tau^* - \left(Q_{t,\theta}^* C^* + \Phi_\theta\right)\left(CQ_{t,\tau} + \Phi_\tau^*\right), \quad -\varepsilon < \tau \le \theta \le 0,\ t > 0, \\ R_{0,\theta,\tau} = R_{t,\theta,-\varepsilon} = 0, \quad -\varepsilon \le \tau \le \theta \le 0,\ t > 0. \end{gathered} \tag{2.4}$$

Thus a distributed delay of white noise splits the stochastic ordinary differential equation (1.3) into two equations, given in (2.3), the first one being again a stochastic ordinary differential equation, and the second one a stochastic partial differential equation. Respectively, the Riccati equation (1.4) is split into three equations, given in (2.4), the first one being again a deterministic ordinary differential equation, and the second and third ones a deterministic partial differential equation. These partial differential equations serve for transformation of the zero initial and boundary values of $\psi$ and $Q$ along the boundary lines $t = 0$ and $\theta = -\varepsilon$ into their values along the other boundary line $\theta = 0$.

## 3. The observations corrupted by wide band noise

Now disturb the observation system (1.2) by the sum of white and wide band noises $w$ and $\varphi$, respectively:

$$\begin{gathered} z_t' = Cx_t + w_t + \varphi_t, \quad t > 0, \\ z_0 = 0, \end{gathered} \tag{3.1}$$

where again $\varepsilon > 0$ is fixed and $\varphi$ is defined by (2.1), satisfying the same conditions as in Section 2, but the dimensions of the matrix $\Phi_\theta$ is consistent with the dimension of $z(t)$. Here the presence of non-degenerate white noise in observations is a restriction coming from the nature of Kalman filtering.

The Kalman filtering equations for the systems (1.1) and (3.1) have been derived in the form

$$\begin{gathered} \widehat{x}_t' = A\widehat{x}_t + \left(P_tC^* + Q_{t,0} + B\right)\left(z_t' - C\widehat{x}_t - \psi_{t,0}\right), \quad t > 0, \\ \widehat{x}_0 = 0, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)\psi_{t,\theta} = \left(Q_{t,\theta}^*C^* + R_{t,0,\theta}^* + \Phi_\theta\right)\left(z_t' - C\widehat{x}_t - \psi_{t,0}\right), \quad -\varepsilon < \theta \le 0,\ t > 0, \\ \psi_{0,\theta} = \psi_{t,-\varepsilon} = 0, \quad -\varepsilon \le \theta \le 0,\ t > 0, \end{gathered} \tag{3.2}$$

where

$$\begin{gathered} P_t' = AP_t + P_tA^* + BB^* - \left(P_tC^* + Q_{t,0} + B\right)\left(CP_t + Q_{t,0}^* + B^*\right), \quad t > 0, \\ P_0 = \operatorname{cov} x_0, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)Q_{t,\theta} = AQ_{t,\theta} + B\Phi_\theta^* - \left(P_tC^* + Q_{t,0} + B\right)\left(CQ_{t,\theta} + R_{t,0,\theta} + \Phi_\theta^*\right), \quad -\varepsilon < \theta \le 0,\ t > 0, \\ Q_{0,\theta} = Q_{t,-\varepsilon} = 0, \quad -\varepsilon \le \theta \le 0,\ t > 0, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \tau}\right)R_{t,\theta,\tau} = \Phi_\theta\Phi_\tau^* - \left(Q_{t,\theta}^*C^* + R_{t,0,\theta}^* + \Phi_\theta\right)\left(CQ_{t,\tau} + R_{t,0,\tau} + \Phi_\tau^*\right), \\ -\varepsilon < \tau \le \theta \le 0, \quad t > 0, \\ R_{0,\theta,\tau} = R_{t,\theta,-\varepsilon} = 0, \quad -\varepsilon \le \tau \le \theta \le 0,\ t > 0. \end{gathered} \tag{3.3}$$

Again, (1.3) and (1.4) are split into two and three equations containing partial differential equations, but now they are different from (2.3)-(2.4).

## 4. The signal corrupted by pointwise delayed white noise

Originally, the equations of Kalman filtering for pointwise delayed white noises were conjectured in [10] and then they were proved in [11, 13] with some corrections in boundary conditions. But the equations from [11] still contain a misprint which is corrected in [12].

The Kalman filtering equations from Sections 2 and 3 include zero boundary conditions. In cases when the delay of noises is pointwise some terms fall from the partial differential equations to boundary conditions, creating challenging patterns of boundary conditions.

Change the signal system (1.1) by replacing $w_t$ by its delay $w_{t-\varepsilon}$, where $\varepsilon > 0$ is a constant:

$$\begin{gathered} x_t' = Ax_t + \Phi w_{t-\varepsilon}, \quad t > 0, \\ x_0 \text{ is given.} \end{gathered} \tag{4.1}$$

Then the Kalman filtering equations for the systems (4.1) and (1.2) are

$$\begin{gathered} \widehat{x}_t' = A\widehat{x}_t + \varphi_{t,0} + P_tC^*\left(z_t' - C\widehat{x}_t\right), \quad t > 0, \\ \widehat{x}_0 = 0, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)\varphi_{t,\theta} = Q_{t,\theta}^*C^*\left(z_t' - C\widehat{x}_t\right), \quad -\varepsilon < \theta \le 0,\ t > \theta + \varepsilon, \\ \varphi_{t,\theta} = 0, \quad -\varepsilon \le \theta \le 0,\ 0 \le t \le \theta + \varepsilon, \\ \varphi_{t,-\varepsilon} = \Phi\left(z_t' - C\widehat{x}_t\right), \quad t > 0, \end{gathered} \tag{4.2}$$

$$\begin{gathered} P_t' = AP_t + P_tA^* + Q_{t,0} + Q_{t,0}^* + \Phi\Phi^* I_{(0,\varepsilon]}(t) - P_tC^*CP_t, \quad t > 0, \\ P_0 = \operatorname{cov} x_0, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)Q_{t,\theta} = AQ_{t,\theta} + R_{t,0,\theta} - P_tC^*CQ_{t,\theta}, \quad -\varepsilon < \theta \le 0,\ t > \theta + \varepsilon, \\ Q_{t,\theta} = 0, \quad -\varepsilon \le \theta \le 0,\ 0 \le t \le \theta + \varepsilon, \\ Q_{t,-\varepsilon} = -P_tC^*\Phi^*, \quad t > 0, \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \tau}\right)R_{t,\theta,\tau} = -Q_{t,\theta}^*C^*CQ_{t,\tau}, \quad -\varepsilon < \tau \le \theta \le 0,\ t > \tau + \varepsilon, \\ R_{t,\theta,\tau} = 0, \quad -\varepsilon \le \tau \le \theta \le 0,\ 0 \le t \le \tau + \varepsilon, \\ R_{t,\theta,-\varepsilon} = -\Phi CQ_{t,-\varepsilon} - Q_{t,\theta}^*C^*\Phi^*, \quad -\varepsilon < \theta \le 0,\ t > 0, \end{gathered} \tag{4.3}$$

where $I_{(0,\varepsilon]}$ is the indicator function of the interval $(0, \varepsilon]$.

## 5. The observations corrupted by pointwise delayed white noise

Finally, we consider the case when the observations are corrupted by delayed white noise. Replace the system (1.2) by

$$z'_t = Cx_t + w_t + \Phi w_{t-\varepsilon} I_{(\varepsilon,\infty)}(t), \quad t > 0,$$
$$z_0 = 0, \tag{5.1}$$

where the delayed white noise effects to the observations starting the instant $\varepsilon > 0$. Then the Kalman filtering equations for the systems (1.1) and (5.1) are

$$\begin{gathered}
\widehat{x}'_t = A\widehat{x}_t + \left(P_tC^* + Q_{t,0} + B\right)\left(z'_t - C\widehat{x}_t - \varphi_{t,0}\right), \quad t > 0,\\
\widehat{x}_0 = 0,\\
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)\varphi_{t,\theta} = \left(Q^*_{t,\theta}C^* + R^*_{t,0,\theta}\right)\left(z'_t - C\widehat{x}_t - \varphi_{t,0}\right), \quad -\varepsilon < \theta \le 0,\ t > \theta + \varepsilon,\\
\varphi_{t,\theta} = 0, \quad -\varepsilon \le \theta \le 0,\ 0 \le t \le \theta + \varepsilon,\\
\varphi_{t,-\varepsilon} = \Phi\left(z'_t - C\widehat{x}_t - \varphi_{t,0}\right), \quad t > 0,
\end{gathered} \tag{5.2}$$

$$\begin{gathered}
P'_t = AP_t + P_tA^* + BB^* - \left(P_tC^* + Q_{t,0} + B\right)\left(CP_t + Q^*_{t,0} + B^*\right), \quad t > 0,\\
P_0 = \operatorname{cov} x_0,\\
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta}\right)Q_{t,\theta} = AQ_{t,\theta} - \left(P_tC^* + Q_{t,0} + B\right)\left(CQ_{t,\theta} + R_{t,0,\theta}\right), \quad -\varepsilon < \theta \le 0,\ t > \theta + \varepsilon,\\
Q_{t,\theta} = 0, \quad -\varepsilon \le \theta \le 0,\ 0 \le t \le \theta + \varepsilon,\\
Q_{t,-\varepsilon} = -\left(P_tC^* + Q_{t,0}\right)\Phi^*, \quad t > 0,\\
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \tau}\right)R_{t,\theta,\tau} = -\left(Q^*_{t,\theta}C^* + R^*_{t,0,\theta}\right)\left(CQ_{t,\tau} + R_{t,0,\tau}\right), \quad -\varepsilon < \tau \le \theta \le 0,\ t > \tau + \varepsilon,\\
R_{t,\theta,\tau} = 0, \quad -\varepsilon \le \tau \le \theta \le 0,\ 0 \le t \le \tau + \varepsilon,\\
R_{t,\theta,-\varepsilon} = -\Phi\left(CQ_{t,-\varepsilon} + R_{t,0,-\varepsilon}\right) - \left(Q^*_{t,\theta}C^* + R^*_{t,0,\theta}\right)\Phi^*, \quad -\varepsilon < \theta \le 0,\ t > 0.
\end{gathered} \tag{5.3}$$

## 6. Remarks on numerical solutions

Numerical solution of the Riccati systems of equations (2.4), (3.3), (4.3), and (5.3), which replace the Riccati equation (1.4) for delay cases, is very important for realization of the Kalman filters defined by systems (2.3), (3.2), (4.2), and (5.2), respectively. Note that the existence of the unique symmetric and positive solutions of these systems has been proved. This additionally makes these systems interesting in the light of increasing demand to investigations of positive solutions of boundary value problems (see, e.g., [14, 15]).

Each of the systems (2.4), (3.3), (4.3), and (5.3) consists of three equations; the first of them being a modification of the Riccati equation (1.4) and the other two for generation the

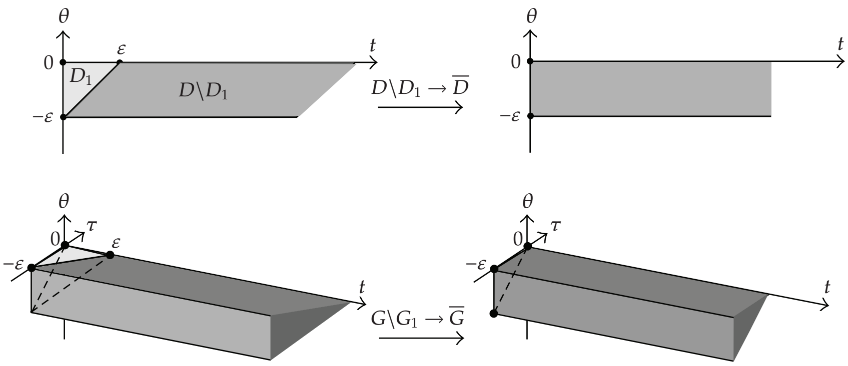


**Figure 1:** Transformation of $D$ onto $\overline{D}$ and $G$ onto $\overline{G}$.

values of $Q$. Let $D$ (a plane region) and $G$ (a solid) be the domains of the functions $Q$ and $R$. They are

$$D = \{(t, \theta) : -\varepsilon \le \theta \le 0,\ t \ge 0\},$$
$$G = \{(t, \theta, \tau) : -\varepsilon \le \tau \le \theta \le 0,\ t \ge 0\}, \tag{6.1}$$

and pictured in Figure 1 (two regions on the left), where both $D$ and $G$ are unbounded from the right hand side. In all the cases $Q$ and $R$ satisfy zero initial conditions on the line segment

$$\{(t, \theta) : -\varepsilon \le \theta \le 0,\ t = 0\} \tag{6.2}$$

and on the triangle

$$\{(t, \theta, \tau) : -\varepsilon \le \tau \le \theta \le 0,\ t = 0\}, \tag{6.3}$$

respectively. The essence of the second and third equations in (2.4), (3.3), (4.3), and (5.3) is that they transform the boundary conditions on the line

$$\{(t, \theta) : \theta = -\varepsilon,\ t > 0\} \tag{6.4}$$

and on the rectangle

$$\{(t, \theta, \tau) : -\varepsilon = \tau \le \theta \le 0,\ t > 0\} \tag{6.5}$$

onto the values of $Q$ interior of $D$ and on the other boundary line

$$\{(t, \theta) : \theta = 0,\ t > 0\} \tag{6.6}$$

of $D$.

One can observe that the systems (2.4), (3.3), (4.3), and (5.3) obey different kinds of boundary conditions. The boundary conditions of the systems (2.4) and (3.3) are constantly zero. Therefore, for numerical solution of them it suffices to use rectangular grids on $D$ and $G$.

Whereas the boundary conditions of (4.3) and (5.3) are complicated for numerical solution by rectangular grids; they require data which are not yet calculated. But this complication can be removed by use of continuity: if a step of the grid is too small, then the required data $P_{t_i}$, $Q_{t_i,\theta_j}$, and $R_{t_i,0,\theta_j}$ on grid points can be approximated by already calculated data $P_{t_{i-1}}$, $Q_{t_{i-1},\theta_j}$ and $R_{t_{i-1},0,\theta_j}$. This idea was used in Bashirov and Mazhar [12] for the system (4.3) in one dimensional case, where some significant conclusions were obtained. In particular, it was demonstrated that neglecting the delay in (4.3) causes a loss of information, which is not recovered as time increases.

But applied problems require a consideration of (4.3) and (5.3) in a multidimensional case and a development of fast computational methods for them. In this regard the following observation may be useful. One can see that on the interval $(0,\varepsilon]$ the values of $\widehat{x}$ and $P$ from (4.2)-(4.3) and (5.2)-(5.3) can be calculated without any contribution of $\psi$, $Q$ and $R$ because they are identically zero on the lightly colored subregions on the left hand side of $D$ and $G$; on the triangle

$$D_1 = \{(t,\theta) : -\varepsilon \le \theta \le 0,\ 0 \le t \le \theta + \varepsilon\} \tag{6.7}$$

and on the tetrahedron

$$G_1 = \{(t,\theta) : -\varepsilon \le \tau \le \theta \le 0,\ 0 \le t \le \tau + \varepsilon\}. \tag{6.8}$$

Therefore, a rhombic grid seems to be more natural for the systems (4.3) and (5.3). For this, it is suitable to consider $P$ from (4.3) and (5.3) on the interval $[0,\varepsilon]$ and transform the rest of its domain, that is, $(\varepsilon,\infty)$, onto $(0,\infty)$ by $t \to t-\varepsilon$. This suggests also a transformation of

$$D \setminus D_1, \qquad G \setminus G_1 \tag{6.9}$$

onto

$$\overline{D} = \{(t,\theta) : -\varepsilon \le \theta \le 0,\ t > 0\}, \qquad \overline{G} = \{(t,\theta,\tau) : -\varepsilon \le \tau \le \theta \le 0,\ t > 0\}, \tag{6.10}$$

respectively, by

$$(t,\theta) \longrightarrow (t-\theta-\varepsilon,\theta), \qquad (t,\theta,\tau) \longrightarrow (t-\tau-\varepsilon,\theta,\tau). \tag{6.11}$$

Letting $\overline{P}_t = P_{t+\varepsilon}$, $\overline{Q}_{t,\theta} = Q_{t+\theta+\varepsilon,\theta}$ and $\overline{R}_{t,\theta,\tau} = R_{t+\tau+\varepsilon,\theta,\tau}$, we can write (4.3) in terms of new functions $\overline{P}$, $\overline{Q}$, and $\overline{R}$ in the form

$$
\begin{gathered}
P_t' = AP_t + P_tA^* + \Phi\Phi^* - P_tC^*CP_t, \quad 0 < t \le \varepsilon, \\
P_0 = \operatorname{cov} x_0, \\
\overline{P}_t' = A\overline{P}_t + \overline{P}_tA^* + \overline{Q}_{t,0} + \overline{Q}_{t,0}^* - \overline{P}_tC^*C\overline{P}_t, \quad t > 0, \\
\overline{P}_t = P_{t+\varepsilon}, \quad -\varepsilon \le t \le 0, \\
\frac{\partial}{\partial\theta}\overline{Q}_{t,\theta} = A\overline{Q}_{t,\theta} + \overline{R}_{t,0,\theta} - \overline{P}_{t+\theta}C^*C\overline{Q}_{t,\theta}, \quad -\varepsilon < \theta \le 0,\ t > 0, \\
\overline{Q}_{0,\theta} = 0, \quad -\varepsilon \le \theta \le 0, \\
\overline{Q}_{t,-\varepsilon} = -\overline{P}_{t-\varepsilon}C^*\Phi^*, \quad t > 0, \\
\left(\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\tau}\right)\overline{R}_{t,\theta,\tau} = -\overline{Q}_{t+\tau-\theta,\theta}^*C^*C\overline{Q}_{t,\tau}, \quad -\varepsilon < \tau \le \theta \le 0,\ t > 0, \\
\overline{R}_{0,\theta,\tau} = 0, \quad -\varepsilon \le \tau \le \theta \le 0, \\
\overline{R}_{t,\theta,-\varepsilon} = -\Phi C\overline{Q}_{t,-\varepsilon} - \overline{Q}_{t-\theta-\varepsilon,\theta}^*C^*\Phi^*, \quad -\varepsilon \le \theta \le 0,\ t > 0.
\end{gathered}
\tag{6.12}
$$

In a similar way, (5.3) can be written in the form

$$
\begin{gathered}
P_t' = AP_t + P_tA^* + BB^* - (P_tC^* + B)(CP_t + B^*), \quad 0 < t \le \varepsilon, \\
P_0 = \operatorname{cov} x_0, \\
\overline{P}_t' = A\overline{P}_t + \overline{P}_tA^* + BB^* - (\overline{P}_tC^* + \overline{Q}_{t,0} + B)(C\overline{P}_t + \overline{Q}_{t,0}^* + B^*), \quad t > 0, \\
\overline{P}_t = P_{t+\varepsilon}, \quad -\varepsilon \le t \le 0, \\
\frac{\partial}{\partial\theta}\overline{Q}_{t,\theta} = A\overline{Q}_{t,\theta} - (\overline{P}_{t+\theta}C^* + \overline{Q}_{t+\theta,0} + B)(C\overline{Q}_{t,\theta} + \overline{R}_{t,0,\theta}), \quad -\varepsilon < \theta \le 0,\ t > 0, \\
\overline{Q}_{0,\theta} = 0, \quad -\varepsilon \le \theta \le 0, \\
\overline{Q}_{t,-\varepsilon} = -(\overline{P}_{t-\varepsilon}C^* + \overline{Q}_{t-\varepsilon,0})\Phi^*, \quad t > 0, \\
\left(\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\tau}\right)\overline{R}_{t,\theta,\tau} = -(\overline{Q}_{t+\tau-\theta,\theta}^*C^* + \overline{R}_{t+\tau-\theta,0,\theta}^*)(C\overline{Q}_{t,\tau} + \overline{R}_{t,0,\tau}), \quad 0 < \tau \le \theta \le 0,\ t > 0, \\
\overline{R}_{0,\theta,\tau} = 0, \quad -\varepsilon \le \tau \le \theta \le 0, \\
\overline{R}_{t,\theta,-\varepsilon} = -\Phi(C\overline{Q}_{t,-\varepsilon} + \overline{R}_{t,0,-\varepsilon}) - (\overline{Q}_{t-\theta-\varepsilon,\theta}^*C^* + \overline{R}_{t-\theta-\varepsilon,0,\theta}^*)\Phi^*, \quad -\varepsilon \le \theta \le 0,\ t > 0.
\end{gathered}
\tag{6.13}
$$

A numerical solution of (4.3) and (5.3) by rhombic grid in fact means a numerical solution of (6.12) and (6.13) by rectangular grid, respectively.

## 7. Conclusion

The paper surveys new Kalman filtering results leading to boundary value problems. We consider simplest cases, stressing on partial differential equation nature of the Kalman filtering equations under delayed noises. Numerical solution of the Riccati equations is an integral part of Kalman filters. Its complexity increases very fast if the dimension of the signal and observation systems increases. In case of ordinary Riccati differential equation (1.4), efficient algorithms are already developed. But the Riccati systems in (2.4), (3.3), (4.3), and (5.3) are awaiting. A simple trial has been done in [12] for the system (4.3) in one-dimensional case. The paper is a call to the community of mathematicians and engineers, dealing with Kalman filtering and boundary value problems, to attract their attention to the new kinds of boundary value problems awaiting numerical solution methods.

## References


[1] R. E. Kalman, "A new approach to linear filtering and prediction problems," *Journal of Basic Engineering, Series D*, vol. 82, no. 1, pp. 35–45, 1960.

[2] R. E. Kalman and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory," *Journal of Basic Engineering, Series D*, vol. 83, pp. 95–108, 1961.

[3] M. H. A. Davis, *Linear Estimation and Stochastic Control*, Chapman & Hall Mathematics Series, Chapman & Hall, London, UK; John Wiley & Sons, New York, NY, USA, 1977.

[4] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*, vol. 1 of *Applications of Mathematics*, Springer, Berlin, Germany, 1975.

[5] A. Bensoussan, *Stochastic Control of Partially Observable Systems*, Cambridge University Press, Cambridge, UK, 1992.

[6] R. S. Liptser and A. N. Shiryayev, *Statistics of Random Processes. II. Applications*, Applications of Mathematics, Springer, New York, NY, USA, 1978.

[7] R. F. Curtain and A. J. Pritchard, *Infinite Dimensional Linear Systems Theory*, vol. 8 of *Lecture Notes in Control and Information Sciences*, Springer, Berlin, Germany, 1978.

[8] R. S. Bucy and P. D. Joseph, *Filtering for Stochastic Processes with Applications to Guidance*, Interscience Tracts in Pure and Applied Mathematics, no. 23, John Wiley & Sons, New York, NY, USA, 1968.

[9] J. L. Crassidis and J. L. Junkins, *Optimal Estimation of Dynamic Systems*, Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series, 2, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2004.

[10] A. E. Bashirov, *Partially Observable Linear Systems under Dependent Noises*, Systems and Control: Foundations and Applications, Birkhäuser, Basel, Switzerland, 2003.

[11] A. E. Bashirov, "Filtering for linear systems with shifted noises," *International Journal of Control*, vol. 78, no. 7, pp. 521–529, 2005.

[12] A. E. Bashirov and Z. Mazhar, "On asymptotical behavior of solution of Riccati equation arising in linear filtering with shifted noises," in *Mathematical Methods in Engineering*, K. Taş, J. A. T. Machado, and D. Baleanu, Eds., pp. 141–149, Springer, Dordrecht, The Netherlands, 2007.

[13] A. E. Bashirov, Z. Mazhar, and S. Ertürk, "Kalman type filter for systems with delaying observation noise," submitted to *Mathematics of Control, Signals and Systems*.

[14] C. Bai, "Existence of positive solutions for fourth-order three-point boundary value problems," *Boundary Value Problems*, vol. 2007, Article ID 68758, 10 pages, 2007.

[15] S. J. Yang, B. Shi, and D. C. Zhang, "Existence of positive solutions for boundary value problems of nonlinear functional difference equation with $p$-Laplacian operator," *Boundary Value Problems*, vol. 2007, Article ID 38230, 12 pages, 2007.