Euler Characteristic of Groups – C.T.C. Wall's Approach

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ABSTRACT

This work concentrates on the Euler characteristic of groups and points out that, this number can also be a rational one.

We give some background material on the Euler theorem and the relevant topics. We also review the Fundamental Group and the Covering Space theories. We give C.T.C. Wall's definition for the Euler characteristic of a group. We fill in the gaps in the proofs of certain formula given under the notions of $\mathcal{L}$-class and $\mathcal{M}$-class.

Finally, we describe the exact structure of subgroups of free products of groups via some examples.

Keywords: Euler characteristic, Euler theorem, Fundamental Group Theory, Covering Space Theory, C.T.C. Wall's definition, $\mathcal{L}$-class, $\mathcal{M}$-class, free products of groups.

ÖZ

Bu çalışmada grupların Euler karakteristiği baz alınarak, C.T.C. Wall'un bu sayının rasyonel olabileceğini gösterdiği çalışması irdelenmiştir. Euler Teorem'in temel özellikleri ve ilgili konuları verildi. Bunun yanında, Temel Gurup ve Örtü Uzayları Teorileri özetlendi. Gurupların Euler karakteristikleri için C.T.C. Wall'un tanımı verildi ve hemen ardından $\mathcal{L}$ ve $\mathcal{M}$ sınıfları için formüller ispatlandı. En son olarakta, serbest grupların alt gruplarının yapısı örneklerle incelendi.

Anahtar Kelimeler: Euler karakteristik, Euler teorem, Temel Gurup Teorisi, Örtü Uzayları Teorileri, $\mathcal{L}$ ve $\mathcal{M}$ sınıfları.

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INTRODUCTION

In this thesis, some topics are collected with the relevant definitions, theorems, lemmas, propositions and examples. The aim of this thesis is to fill in the gaps in the C.T.C Wall's paper [1961] where he gives a geometric definition for the Euler characteristic of a group and shows that this number can also be a rational one.

This thesis mainly consists of four chapters.

Before giving the contents of each section, we will make clear the description of the Euler characteristic by C.T.C. Wall and Ken Brown. Under certain topological conditions the two definitions coincide; however Kenneth Brown's definition;

$$\chi(G) = \sum_i (-1)^i rk_{\mathbb{Z}}(H_i(G))$$

involving the alternating sum of the ranks of the homology groups of G, is much more technical. Therefore, in this master thesis we'll study the more geometric approach introduced by C.T.C. Wall back in 1963.

In Chapter 2, we have first mentioned how the Euler theorem works on shapes. Later, we give the definition of topological equivalence and next, we write the quite significant theorem related with this notion. Moreover, we shortly explain the construction of surfaces. Lastly, before jumping to the following chapter, we introduce the loops which are relevant with Fundamental Group.

subsection under the Covering Spaces, we give the explanation of an $K(G,1)$ space and we show that S^1 and $S^1 \times S^1$ are $K(\mathbb{Z},1)$ and $K(\mathbb{Z}\times\mathbb{Z},1)$, respectively.

In the last chapter, we first describe the $\mathcal{L}$-class and according to C.T.C Wall, the Euler characteristic on this class is defined as;

$$\chi(G) = \chi(X)$$

where X is the classfying space.

Next, we state and prove the following three formula;

$$\chi(G \times H) = \chi(G)\chi(H) \tag{1}$$

$$\chi(G * H) = \chi(G) + \chi(H) - 1 \tag{2}$$

$$\chi(K) = r\chi(G) \tag{3}$$

Then, we introduce the $\mathcal{M}$-class with the definition

$$\chi(G) = \frac{1}{r}\chi(K)$$

where K is a subgroup of finite index r belonging to $\mathcal{L}$. In the following step, we state and prove a lemma which says that the definition on $\mathcal{M}$-class doesn't depend on to choice of the subgroup K. Later on, we prove that the same three formula, introduced above for $\mathcal{L}$-class, also hold for $\mathcal{M}$-class. Although formula (3) is clear, we use some properties for the rest of them.

Later on, by using $\chi(K) = r\chi(G)$, which belongs to $\mathcal{L}$-class, we are able to reach to the formula (1) for $\mathcal{M}$-class. Next, by using the homomorphism property between $G * H$ and $G/K \times H/L$, we apply the first isomorphism theorem to $G * H$ and we get the number of cosets of M in $G * H$.

In the folowing step, rs-fold cover helps us to complete the proof. Furthermore, we try to give relevant examples which are explained step by step according to the definitions of $\mathcal{L}$-class and $\mathcal{M}$-class.

In the rest of the chapter, a proposition is verified by using the Kurush's subgroup theorem and in the same way, we deduce the Diophantine equation that provides to find the exact structure of subgroups of free products of certain groups.

Chapter 2

EULER'S THEOREM

Before start, we ought to give some background material on Euler characteristic which shall form a base for the continuing chapters.

Euler characteristic is an extremely valuable topological property which was first mentioned by Leonhard Euler in a paper [3] in 1736 under “The Seven Bridges of Köningsberg” problem.

Let's consider an island which has a couple of rivers flowing with seven branches on the two sides. The aim of the paper is to walk around the region by using each branch not more than once. Unfortunately, traveling around the region is impossible and nobody had achieved any method to overcome this problem before Euler.

Actually, the method of Euler was quite basic. He had managed a solution and he explained his idea on the graph, that is, he pointed out the land regions as vertices and in the same way, edges were represented by bridges. Finally, we obtain the following shape;

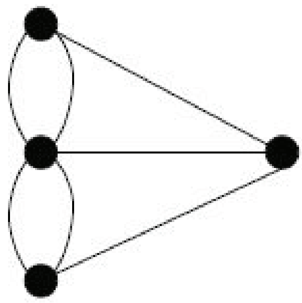

Figure 2.1: The Seven Bridges Graph

Then, Euler verified that, to walk surrounding the Fig. 2.1 [3] above by using of each edge exactly once, at most two vertices ought to have an odd number of edges touching them. By the way, we shall use the Euler method to solve the bridge problem.

In that point, we can talk about the “planar” and “nonplanar” graphs. If a graph can be drawn with no edge intersection by another, it is called **planar** . In contrast, it is defined by **nonplanar**.

By using the graph theory of Euler, the number of vertices minus the number of edges plus the number of faces always returns 2, for certain polyhedra. More precisely, we will learn in the continuing chapter that, if two shapes are homeomorphic to one-another then, the Euler characteristic of them are the same.

2.1 Euler’s theorem

Every polyhedra has a different aspect and each of them has Euler characteristic. The number of vertices subtract the number of edges, plus the number of faces always gives 2, if the Polyhedra satisfies the Euler’s theorem.

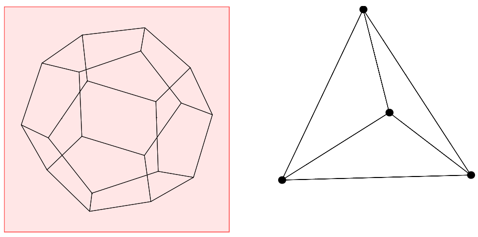

Figure 2.2: Dodecahedron, Tetrahedron

As we consider Figure 2.2 [2] (the tetrahedron), by the Euler's therem, $v - e + f = 4 - 6 + 4 = 2$. However, for certain polyhedra, the result may not be equal to the 2. If we consider the Fig. 2.3 [2], we obtain $v - e + f = 16 - 24 + 12 = 4$ and $v - e + f = 20 - 40 + 20 = 0$ from the first shape and second shape, respectively.

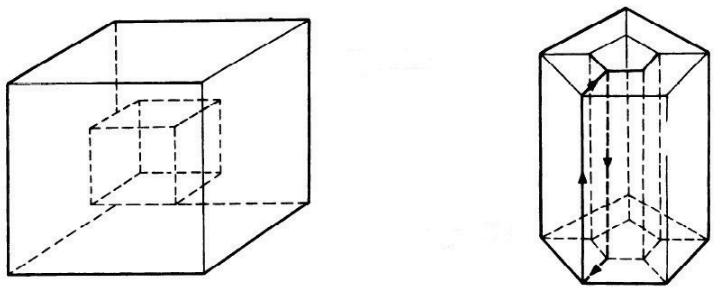

Figure 2.3: A cube with a smaller one removed from inside, A prism with a hole

Theorem 2.1.1 *(Euler's theorem) [2] Let P be a polyhedron which satisfies:*

a) Any two vertices of P can be connected by a chain of edges.

b) Any loop on P which is made up of straight line segments (not necessarily edges) separates P into two pieces.

Then

$$v - e + f = 2.$$

for P.

So as to find how Euler theorem works, we apply it to the Fig. 2.3. Euler's theorem fails for the first picture because there is no edge between all the vertices. Also, as we cut any loop via scissor from the second one, P will not be seperated into two pieces. It is still only one piece at the end of the process.

Proof. There is more than one proof for Euler's theorem. First of all, we should give the definition of a graph. If P has a connected set of vertices and edges, it is defined by a **graph**. In another words, a chain of edges connects any two vertices in the graph. If a graph does not contain any loop, it's is called a **tree**. If a tree is defined by T, we say

$$v(T) - e(T) = 1.$$

Moreover, if we wish, we can find any subgraph of a tree. In fact, subgraph contains all vertices of the tree (but you do not need to use all of the edges). Now, we are going to make a dual for T. Note that, the dual will come from a graph ψ. We have a polyhedra (P) and we try to find a tree on the polyhedra. Aim of the tree is to reach all vertices via edges. But any loop might not as well be on the tree. We have constructed a tree on P and for each face E of P we put a vertex $\hat{E}$ to the ψ. Two vertices of ψ, $\hat{E}$ and $\hat{U}$ are joined by an edge (it does not belong to the tree) if and only if the corresponding faces E and U are adjacent with intersection an edge that doesn't belong to T. We can easily say that, the number of edges which belong to T is equal to the number of edges of ψ.

How can we conclude that ψ is a tree? We turn to Euler's theorem to explain this. Hypothesis (b) states that any loop on P which is made up of straight line segments (not necesserily edges) seperates P into two pieces. That's why, if there is a loop in ψ, it might as well separate P into two distinct pieces and at least one vertex must belong to T . Any attempt to connect two vertices of T which lie in different pieces by a chain of edges results in a chain which meets the separating loop, and therefore in a chain which cannot lie entirely in T. This contradicts the fact that is T is connected. Therefore, ψ is a tree. Since the T and ψ are trees, we have

$$v(T) - e(T) = 1$$

and

$$v(\psi) - e(\psi) = 1.$$

So,

$$v(T) - [e(T) + e(\psi)] + v(\psi) = 2$$

$$e(T) + e(\psi) = e \quad \text{and} \quad v(\psi) = f$$

Hence

$$v - e + f = 2.$$

■

Note that, there are some properties of trees on n vertices:

i) Cycle free (has no cycle).

ii) $(n-1)$ edges.

iii) There is a unique path between every pair of vertices in the tree.

iv) If an edge is added to the tree, a cycle is created.

v) If an edge is left out from the tree, then the graph is not connected.

2.2 Topological Equivalence

Topological equivalence is also known as *homeomorphism*. It has both an algebraic and a geometric definition. In topology, we are more interested in the former one.

$f : X \to Y$ is continuous if for each point x of X and each neighbourhood V of $f(x)$ in

Y, the set $f^{-1}(V)$ is a neighbourhood of x in X. A function $g: A \rightarrow B$ is called a **homeomorphism** if it is one to one, onto, continuous, and has a continuous inverse. When such a function exists, both X and Y are called **homeomorphic** (or **topologically equivalent**) spaces.

Definition 2.2.1 *(Topological equivalence: Geometric approach) [2] Two objects are topologically equivalent if one object can continuously be deformed to the other one. To continuously deform a surface means to stretch it, expand it, bend it, shrink it, crumple it, etc- anything that we can do without actually tearing the surface or gluing parts of it together.*

For example, as we look at the Fig. 2.5 [2], the given polyhedron and the sphere are topological equivalent to each other. In order to see this clearly, first of all, we will explain how the sphere is constructed. In the beginning, we take a line and identify the end points obtaining a 'ring'. Then, two discs, the northern and southern hemispheres are connected along the ring. Hence, a sphere consists of two discs (see Fig. 2.4 [2]).

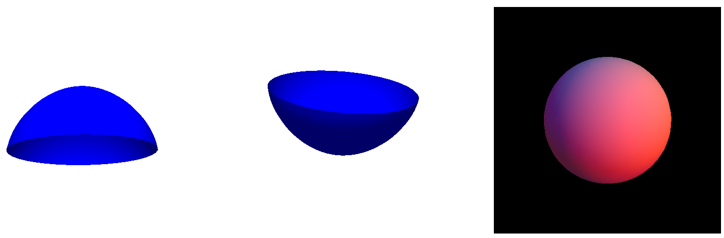

Figure 2.4: Constructing Sphere

If polygon is stretched and bended around the sphere, Fig. 2.5 is obtained.

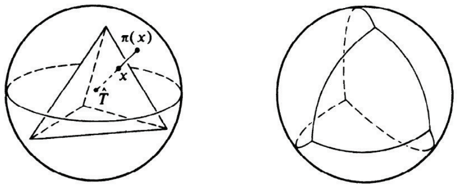

Figure 2.5: Radial Projection on a Sphere

Let's now return to the Euler's theorem. If P is homeomorphic to the sphere then both hypothesis (a) and (b) are satisfied by P and therefore The Euler's theorem holds. I mean that, P and sphere have the same Euler number, $v - e + f = 1 - 1 + 2 = 2$.

The three spaces given below (Fig.2.6) are homeomorphic to one another.

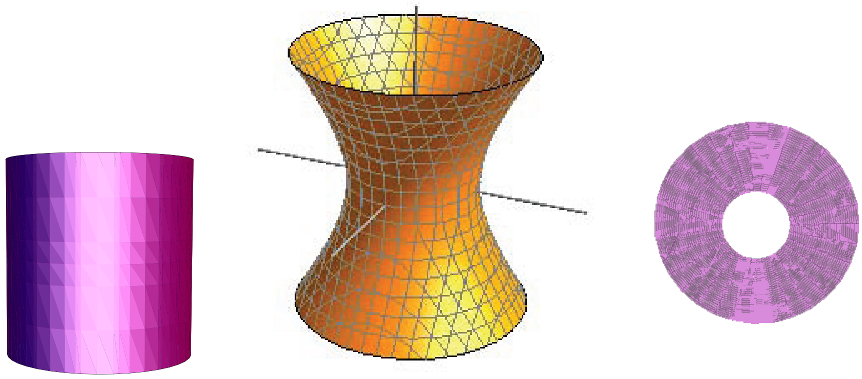

Figure 2.6: Cylinder, Hyperboloid, Annulus

1. The surface of a cylinder of finite height;

2. The one-sheated hyperboloid

3. The open annulus in the complex plane.

Let's take a rectangular paper and identify the opposite edges. We obtain a torus which is empty inside. As we calculate its Euler number, we get $(v - e + f = 1 - 2 + 1 = 0)$.

Theorem 2.2.2 *[2] Topologically equivalent polyhedra have the same Euler number.*

2.3 Surfaces

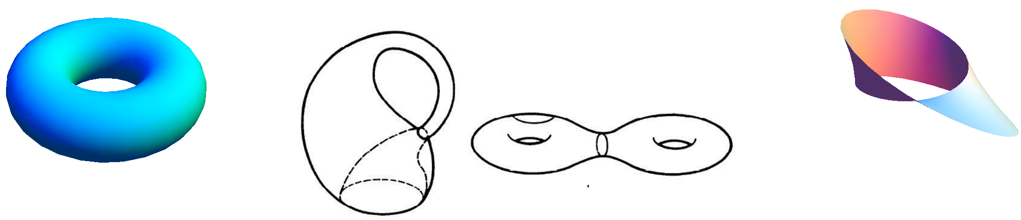

Figure 2.7: Different Surfaces

We have different surfaces in topology as above. Some of them (like Klein Bottle) are more difficult to visualize than others.

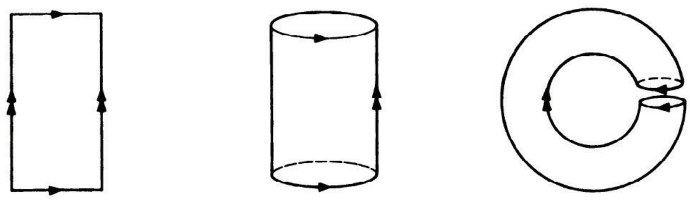

Figure 2.8: Constructing Torus

Fig. 2.8 [2] shows how we can construct the Torus. Firstly, take a rectangular paper and identify one pair of opposite edges in the same direction, getting a cylinder. Finally, to obtain the torus, identify the remaining pair of edges again in the same direction.

To build a Klein Bottle, again a rectangular paper is taken and cylinder is obtained. Once obtaining a cylinder, the ends of the latter is identified in the opposite direction as in Fig. 2.9 [2]. Later, the cylinder is bended around itself and pushed through the side.

We have a different shape for Möbius strip in Fig. 2.10 [2].

To obtain the Möbius strip, one begins with a rectangular piece of paper and identifies a pair of opposite edges with a half twist as in Fig. 2.11 [2]. It is enough to see Fig. 2.10

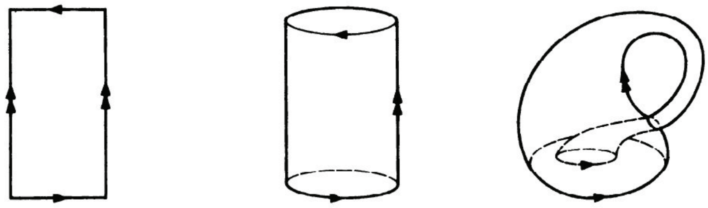

Figure 2.9: Constructing Klein Bottle

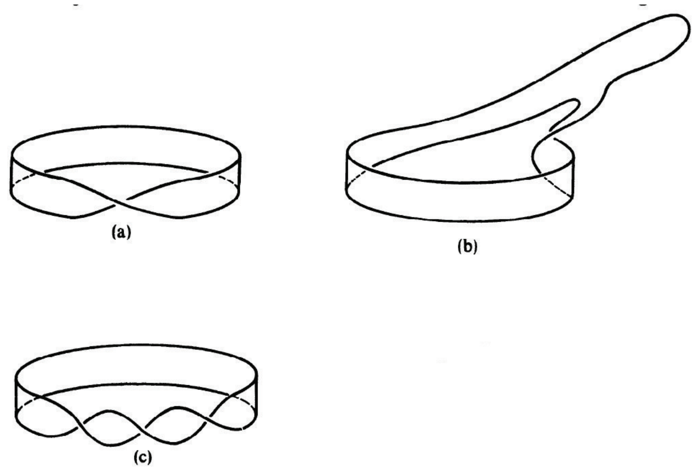

Figure 2.10: Different Möbius Strips

(b) pushed from inside to out. Therefore, Fig. 2.10 (a) and (b) are homeomorphic. Yet, Fig. 2.10 (a) and (c) are not topologically equivalent because they have different Euler numbers; 0 and 2, respectively.

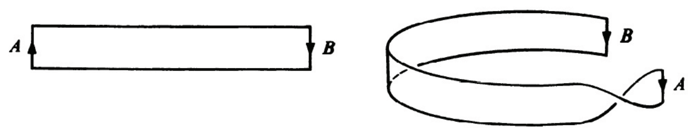

Figure 2.11: Constructing Möbius Strip

Definition 2.3.1 *[2] A surface is a topological space in which each point has a neighbourhood homeomorphic to the plane and where any two distinct points have disjoint neighbourhoods.*

2.4 A classification theorem

We might as well consider closed surfaces which have no boundary and closed up on themselves to explain classification theorem. The sphere, the torus, and the Klein bottle are examples for surfaces that we have in mind. However, since the Cylinder and the Möbius strip have edges, they are omitted.

The interesting property which comes out is that if we work with closed surfaces, which were mentioned before, then we can classify them and say exactly how many there are.

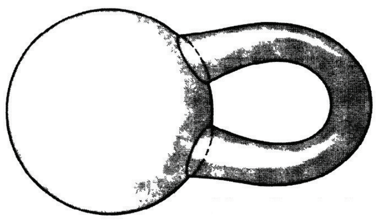

Figure 2.12: Sphere with one handle

One begins with any sphere, removes two disjoint discs and then adds on a cylinder which has two boundary circles, to the holes in the sphere as in Fig. 2.12 [2]. This progression is defined by **adding a handle** to the sphere. Also, we get a sphere with two, three or any finite number of handles. The sphere with one handle is essentially same as the torus. In other words, sphere with one handle is homeomorphic to the torus.

Theorem 2.4.1 *(Classification theorem) [2] Any closed surface is homeomorphic either to the sphere or to the sphere with a finite number of handles added, or to the sphere with a finite number of discs removed and replaced by Möbius strips. No two of these surfaces are homeomorphic.*

An *orientable surface* of genus n is a sphere with n handles added. As examples, the torus and the Möbius strip are given in Fig. 2.13 [2]. If we follow closed curve around the boundary by our finger, direction does not change on the way. However, the Möbius strip is non-orientable.

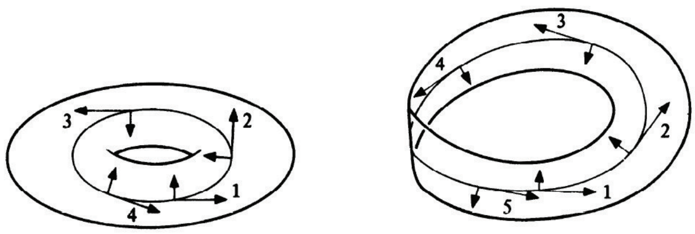

Figure 2.13: Torus and Möbius Strip

2.5 Topological invariants

Poincare also helped us to understand what topological invariance is. The idea is to assign a group to each topological space in such a way that homeomorphic spaces have isomorphic groups. If we want to distinguish between two spaces, we can try to solve the problem algebraically by first computing their groups and then looking to see whether or not the groups are isomorphic. If the groups are not isomorphic then the spaces are different (not homeomorphic).

As we look at the Fig. 2.14 [2], we have a disc and an annulus.

If we take any loop inside from the first shape, we can shrink it continuously to a point. However, loops can not be shrunk continuously in the annulus. Therefore, these two figures are not homeomorphic.

Geometrically, a loop in a topological space Y is nothing more than a continuous function γ : $C \to Y$, where C is denoted by the unit circle in the complex plane, and we conclude that

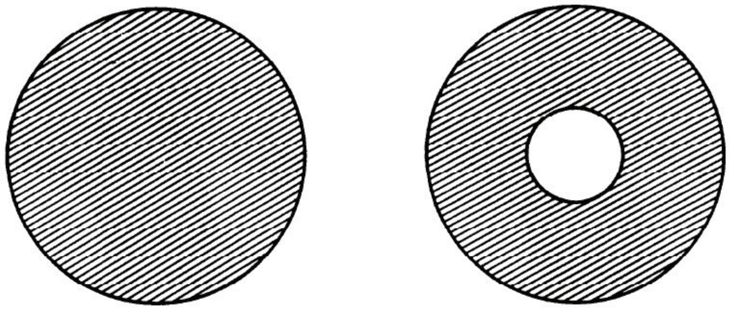

Figure 2.14: Disc and Annulus

if $\gamma(0) = \gamma(1) = a$, where a is the point on Y, the beginning and end points are same. The round shapes denote on the loops in our sketches, where we parametrize C by using $\{e^{i\theta}|$ $0 \leq \theta \leq 2\pi\}$.Changing the direction of the arrow creates a different loop and is equivalent to taking the element in question in the fundamental group. The fundamental group of the disc is the trivial group since any loop can continuously be shrunk to a point. Moreover, the infinite cylic group of integers is obtained for the annulus. Loops representing 0, -1, and $+2$ are shown in Fig. 2.15 [2].

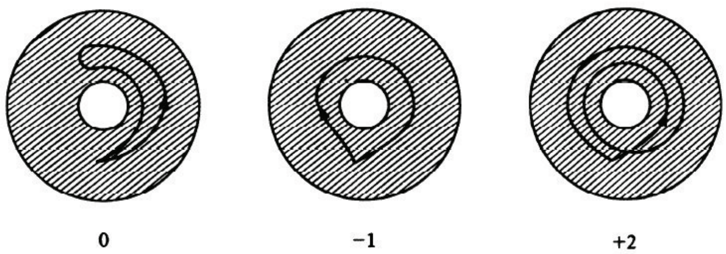

Figure 2.15: Loops on Annulus

It is not hard to imagine that homeomorphic spaces will have isomorphic fundamental groups. More precisely, if $\gamma : C \to Y$ is a loop in Y and $f : Y \to Z$ is a homeomorphism, then $f\gamma : C \to Z$ produces a loop in Z. Continuous deformations are also carried over in the same way. Eventually, the disc and the annulus are not topologically equivalent.

Classification of Surfaces. [2] No two surfaces on the list given in Theorem 2.0.5 have isomorphic fundamental groups, so these surfaces are all topologically distinct.

Jardon seperation theorem. [2] Any simple closed curve in the plane divides the plane into pieces.

Brouwer fixed-point theorem. [2] Any continuous function from a disc to itself leaves at least one point fixed.

Nielsen-Schreier theorem. [2] A subgroup of a free group is always free.

Chapter 3

THE FUNDAMENTAL GROUP

3.1 Homotopic Maps

Homotopic maps are obtained by loops which lie on a surface. The quite significant point for loops is that, they complete themselves in one second. Moreover, the beginning and end points are always same. During this time, loops follow each other continuously.

Assume that X is a space and α is any loop on it. A loop in a space X *is a map* α $: I \to X$ such that $\alpha(0) = \alpha(1)$, where it's based at the point $\alpha(0)$. In addition, if loops, namely α and β, are based at the same point on X, then the product loop $\alpha \cdot \beta$ is defined as follows;

$$\alpha \cdot \beta(s) = \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

In here, since the images of α and β in X are continuous on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, $\alpha \cdot \beta$ is also continuous.

Above, we've given the definition for the product of two loops. However, this multiplication does not satisfy the group properties; as an example, even associativity fails. To remove this problem, identify two loops if one is continuously deformed into the other, keeping the base point fixed throughout the deformation.

Before giving the definition of fundamental group of a space X, we provide the definition of homotopy. If $f, g : X \to Y$ are maps, we shall explain the meaning of deforming f into g. Continuous deformation of f into g is called a **homotopy** and is denoted by F. In detail, we would like to define a family $\{f_g\}$ of maps from X to Y, one for each point t of $[0, 1]$ with $f_0 = f$, $f_1 = g$ and the with property that f_t changes continuously as t varies between 0 and 1. To investigate this continuous change we make use of the product space $X \times I$, noting that a map $F : X \times I \to Y$ gives rise to a family $\{f_t\}$ if we set $f_t(x) = F(x, t)$.

Definition 3.1.1 *[6] If X and Y are topological spaces, two continuous maps $f, g : X \to Y$ are said to be **homotopic** if there is a continuous map*

$$F : X \times I \to Y,$$

such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. If f and g are homotopic, we write $f \simeq g$.

If $f \simeq g$ and g is a constant map, f is said to be **nullhomotopic**.

Moreover, if f and g take the same values on a subset A of X, we may want to deform f to g without changing the values of f on A. In this case we would like to construct a homotopy F from f to g with the extra property that

$$F(a, t) = f(a) \quad \text{for all } a \in A, \text{ for all } t \in I \text{ where } t \text{ is time.}$$

If such a homotopy exists we say that f is homotopic to g relative to A and write $f \underset{F}{\simeq} g$ rel A.

Definition 3.1.2 *[7] Two paths f and g, mapping the interval $I = [0, 1]$ into X, are said to be **path homotopic** if they have the same starting point x_0 and the same end point x_1, and if there is a continuous map $F : I \times I$ such that,*

$$F(s, 0) = f(s) \quad \textit{and} \quad F(s, 1) = g(s)$$

$$F(0,t) = x_0 \quad and \quad F(1,t) = x_1$$

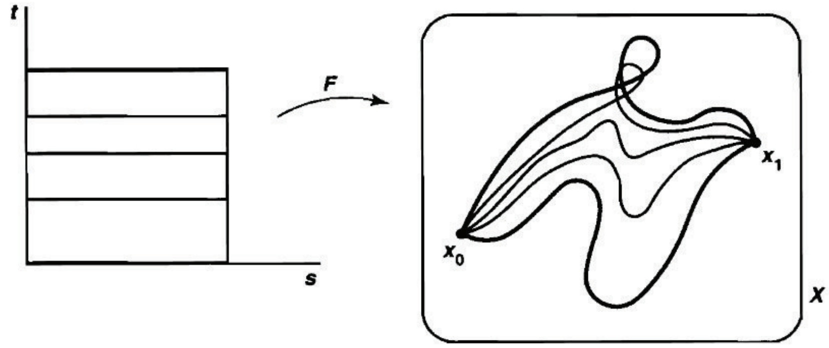

Figure 3.1: Homotopy between two maps

In the Fig. 3.1 [7], the first one says that, we have homotopy between f and g represented by F. In the second one, we have a family of paths namely f_x. It is clear that, $f_t(s) = F(s,t)$ is a path from x_0 to x_1. The difference of two conditions above is very important. I try to say that, in the first one, F provides a continuous deformation from path f to path g, and in the second one, initial and end points are fixed during the deformation. In the same way, we have two loops, $\alpha, \beta : I \to X$ and p is the based point of X.

Example 3.1.3 *Let A be a convex subset of euclidean space and $f, g : X \to A$ be defined as maps, where X is any topological space. For each point x of X, straight line starting at $f(x)$ and running to $g(x)$ lies in A. Define $F : X \times I \to A$ by $F(x,t) = (1-t)f(x) + tg(x)$. It is called a **straight-line homotopy**.*

It is clear that, this homotopy is achieved by sliding f along these straight lines. Note that, the general line is represented by $y = mt + c$, where t is time interval on $[0,1]$.

$m = g(x) - f(x) \implies y = [g(x) - f(x)]t + c \implies$ *if* $t = 0 \implies f(x) = c \implies$ $y = g(x)t - f(x)t + f(x) \implies F(x,t) = f(x)[1-t] + g(x)t.$

Example 3.1.4 *Let $f, g : X \longrightarrow S^n$ be maps defined in such a way that $f(x)$ and $g(x)$ are not antipodal (never appear at opposite ends of diameter).*

If we take two antipodal points at the opposite ends of a diameter, we get a line. However, this line is not enough to write homotopy. That's why, so as to express these maps on the surface of S^n, note that homotopy is divided by the norm of the line joining $f(x)$ to $g(x)$.

In conclusion, definition of $F : X \times I \longrightarrow S^n$ is given by

$$F(x,t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

Example 3.1.5 *S^1 is defined by the unit circle in the complex plane and two loops namely α, β are considered in S^1 (both based at the point 1).*

Let's

$$\alpha(s) = \begin{cases} \exp 4\pi i & 0 \leq s \leq \frac{1}{2} \\ \exp 4\pi i(2s-1) & \frac{1}{2} \leq s \leq \frac{3}{4} \\ \exp 8\pi i(1-s) & \frac{3}{4} \leq s \leq 1 \end{cases}$$

$$\beta(s) = \exp\ 2\pi i s \quad 0 \leq s \leq 1$$

When we consider the shape (as circle) of each $\alpha(s)$ in their segments $\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right]$ respectively, we see that, the first two circles run in anticlockwise and the third one in clockwise direction. Moreover, the loop β has anticlockwise orientation around the circle on the time interval $[0, 1]$ (see Fig. 3.2 [2]). Briefly, in here, α completes itself in one second. To do this, in the first time interval $[0, \frac{3}{4}]$, it rotates in anticlockwise direction and it follows the same direction for the next segment $[\frac{3}{4}, \frac{7}{8}]$. However, the second interval is travelled in shorter time compared to the previous one. Lastly, the last interval $[\frac{7}{8}, 1]$ keeps the opposite

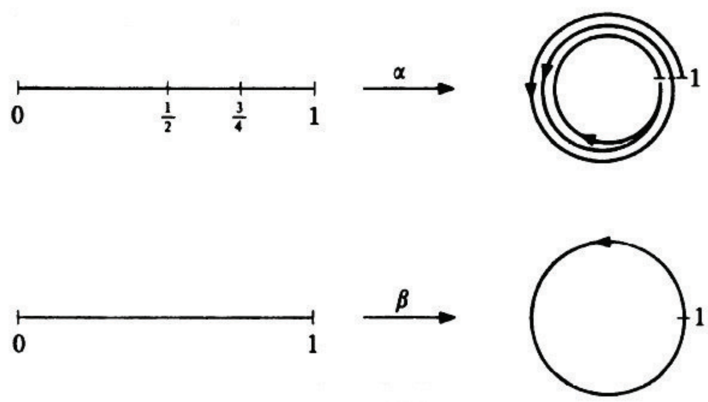

Figure 3.2: Loops on Circle

way to others.To conclude, we see that, α falls down to a point to complete the loop in one second.

The homotopy F from α to β relative to $\{0, 1\}$ is defined as follows;

$$F(s,t) = \begin{cases} \exp \frac{4\pi i s}{t+1} & 0 \leq s \leq \frac{t+1}{2} \\ \exp 4\pi i(2s - 1 - t) & \frac{t+1}{2} \leq s \leq \frac{t+3}{4} \\ \exp 8\pi i(1 - r) & \frac{t+3}{4} \leq s \leq 1 \end{cases}$$

As a picture of the homotopy F, we have (see Fig. 3.3 [2])

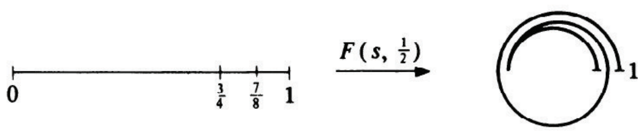

Figure 3.3: Middle Stage of Homotopy

Lemma 3.1.6 *[2] The relation of 'homotopy' is an equivalence relation on the set of all maps from X to Y.*

Proof. To get equivalence relation, we should have reflexivity, symmetry, and transitivity. Let's take maps f, g, h which run from X to Y. Taking any f, $f \underset{F}{\simeq} f$, where $F(x,t) = f(x)$; hence the relation is reflexive. If $f \underset{F}{\simeq} g$, then $g \underset{G}{\simeq} f$ where $G(x,t) = F(x, 1-t)$. So, we've symmetry. Lastly, the composition of two homotopies namely $f \underset{F}{\simeq} g$ and $g \underset{G}{\simeq} h$ gives $f \underset{H}{\simeq} h$ where H is defined as follows;

$$H(x,t) = \begin{cases} F(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ G(x, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

By H, we also have transitivity. ■

Lemma 3.1.7 *[2] The relation of 'homotopy relative to a subset A of X's is an equivalence relation on the set of all maps from X to Y which agree with some given map on A.*

Lemma 3.1.8 *[2] Homotopy behaves well with respect to composition of maps.*

Proof. Let's take maps namely f, g, h as

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\circlearrowleft"] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

As maps from X to Z, if $f \underset{F}{\simeq} g \ rel\ A$, then $hf \underset{hF}{\simeq} hg \ rel\ A$. Moreover, given

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\circlearrowleft"] & Z \end{tikzcd}$$

with $g \underset{G}{\simeq} h \ rel\ B$ where B is a subset of Y, $gf \underset{F}{\simeq} hf \ rel\ f^{-1}B$, by the homotopy $F(x,t) = G(f(x), t)$. ■

3.2 Construction of the Fundamental Group

Let X be a topological space and p is chosen as a base point of X. Also, consider the set of all loops in X based at p. By lemma (5.2), the relation of homotopy is an equivalence relation on the set of all maps from X to Y. By the way, we will call the equivalence classes homotopy classess and we'll represent the homotopy class of a loop α by $< \alpha >$.

We define the multiplication of loops by homotopy classes as follows;

$$< \alpha > \cdot < \beta > = < \alpha \cdot \beta > .$$

First of all, we have to show that, the multiplication above is well-defined. If $\alpha^{-1} \underset{F}{\simeq} \alpha$ $rel\ \{0,1\}$ and $\beta^{-1} \underset{G}{\simeq} \beta\ rel\ \{0,1\}$ then $\alpha' \cdot \beta' \underset{G}{\simeq} \alpha \cdot \beta\ rel\ \{0,1\}$

where

$$H(s,t) = \begin{cases} F(2s,t) & 0 \leq s \leq \frac{1}{2} \\ G(2s-1,t) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

The glueing lemma. [2] Let us take X and Y which are subsets of a topological space. Also X, Y, and $X \cup Y$ belong to the induced topology. If the given functions $f : X \to Z$ and $g : Y \to Z$ belong to the intersection of X and Y, then describe $f \cup g$ as follows;

$$f \cup g : X \cup Y \to Z.$$

Here $f \cup g(x) = f(x)$ for $x \in X$, and $f \cup g(y) = g(y)$ for $y \in Y$. We declare that, as a result of 'glueing together' functions f and g, $f \cup g$ is stated. The next lemma helps us to prove $f \cup g$ is continuous.

Lemma 3.2.1 *(Glueing Lemma) [2] If X and Y are closed in $X \cup Y$, and if both f and g are continuous, then $f \cup g$ is continuous.*

Proof. [13] Let's us take C; a closed subset of Z. Since f is continuous, $f^{-1}(C)$ is closed in X. Moreover, $f^{-1}(C)$ is also closed in $X \cup Y$ because X is closed in $X \cup Y$. In the same way, since g is continuous, the inverse image $g^{-1}(C)$ of C is closed in Y such that it is also closed in $X \cup Y$. Since the union of two closed subsets are closed, we have the following equation;

$$(f \cup g)^{-1}(C) = f^{-1}(C) \cup g^{-1}(C).$$

It says, $(f \cup g)^{-1}(C)$ is closed in $X \cup Y$ automatically.

Hence, we prove that, $f \cup g$ is continuous. ■

By the Glueing Lemma, H is continuous. As a result,

$$\langle \alpha' \rangle \cdot \langle \beta' \rangle = \langle \alpha \rangle \cdot \langle \beta \rangle.$$

Theorem 3.2.2 *[2] The set of homotopy classes of loops in X based at p forms a group under the multiplication*

$$< \alpha > \cdot < \beta > = < \alpha \cdot \beta > .$$

Proof. To show this, we should check the four conditions for a group. First of all, let us check whether multiplication is associative. We take any 3 loops α,β, and γ based at p and show $< \alpha \cdot \beta > \cdot < \gamma > = < \alpha > \cdot < \beta \cdot \gamma >$. To do this, we must show that $(\alpha \cdot \beta) \cdot \gamma$ is homotopic to $\alpha \cdot (\beta \cdot \gamma)$ relative to $\{0, 1\}$. We shall remember the lemma, stating that homotopy behaves well with respect to composition of maps. First, check that $(\alpha \cdot \beta) \cdot \gamma$ is equal to $(\alpha \cdot (\beta \cdot \gamma)) \circ f$ where f is a map from I to I defined by

$$f(s) = \begin{cases} 2s & 0 \leq s \leq \frac{1}{4} \\ s + \frac{1}{4} & \frac{1}{4} \leq s \leq \frac{1}{2} \\ \frac{s+1}{2} & \frac{1}{2} \leq s \leq 1 \end{cases}$$

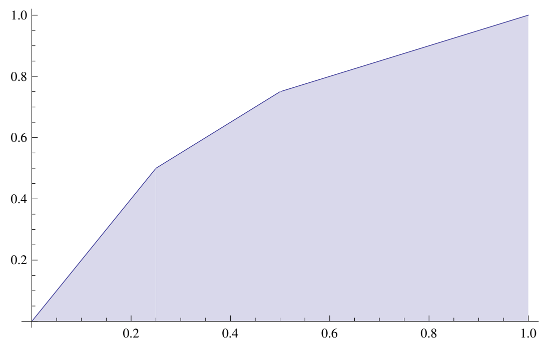

Figure 3.4: Construction of Fundamental Group (1)

I mean, let's check,

$$(\alpha \cdot \beta).\gamma(s) = (\alpha \cdot (\beta \cdot \gamma)) \circ f(s).$$

Before doing this remember that in the beginning of the chapter we've defined the product loop $\alpha \cdot \beta$ as follows;

$$\alpha \cdot \beta(s) = \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Let's look at the left hand side. In here, $(\alpha \cdot \beta)$ should be taken as α, and γ like β. If we apply product $\alpha \cdot \beta$ to them, we have

$$\begin{cases} \alpha \cdot \beta(2s) & 0 \leq s \leq \frac{1}{2} \\ \gamma(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Since $\alpha \cdot \beta$ is denoted by α, it is derived to

$$\begin{cases} \alpha(4s) & 0 \leq s \leq \frac{1}{2} & 0 \leq s \leq \frac{1}{4} \\ \beta(4s-1) & \frac{1}{2} \leq 2s \leq 1 & \frac{1}{4} \leq s \leq \frac{1}{2} \\ \gamma(2s-1) & \frac{1}{2} \leq s \leq 1 & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Now, jump to the right hand side. That part is a little complicate. By the same logic, α and

$\beta \cdot \gamma$ are denoted by α and β, respectively.

$$\alpha \cdot (\beta \cdot \gamma)(2s) = \begin{cases} \alpha(4s) & 0 \leq 2s \leq \frac{1}{2} \quad 0 \leq s \leq \frac{1}{4} \\ \beta \cdot \gamma(4s-1) & \frac{1}{2} \leq 2s \leq 1 \quad \frac{1}{2} \leq 2s \leq 1 \end{cases}$$

We easily see that, although the first part is satisfied, the second part is not. Therefore, we are now interested in $f(s+\frac{1}{4})$ and also s ranges between $\frac{1}{4}$ and $\frac{1}{2}$. Again,

$$\alpha \cdot (\beta \cdot \gamma)(s+\frac{1}{4}) = \begin{cases} \alpha(2s+\frac{1}{2}) & 0 \leq s+\frac{1}{4} \leq \frac{1}{2} \quad \frac{-1}{4} \leq s \leq \frac{1}{4} \\ \beta \cdot \gamma(2s-\frac{1}{2}) & \frac{1}{2} \leq s+\frac{1}{4} \leq 1 \quad \frac{1}{4} \leq s \leq \frac{3}{4} \end{cases}$$

The first part is not taken because α has already been found above. $\beta \cdot \gamma$ should be extended again.

$$\beta \cdot \gamma(2s-\frac{1}{2}) = \begin{cases} \beta(4s-1) & 0 \leq 2s-\frac{1}{2} \leq \frac{1}{2} \quad \frac{1}{4} \leq s \leq \frac{1}{2} \\ \gamma(4s-2) & \frac{1}{2} \leq 2s-\frac{1}{2} \leq 1 \quad \frac{1}{2} \leq s \leq \frac{3}{4} \end{cases}$$

It is clear that, β has also been found above. Now, apply the last condition of f to find γ. We have

$$\alpha \cdot (\beta \cdot \gamma)(\underline{s+1}_{2}) = \begin{cases} \alpha(s+1) & 0 \leq \underline{s+1}_{2} \leq \frac{1}{2} \quad -1 \leq s \leq 0 \\ \beta \cdot \gamma(s) & \frac{1}{2} \leq \underline{s+1}_{2} \leq 1 \quad 0 \leq s \leq 1 \end{cases}$$

$$\beta \cdot \gamma(s) = \begin{cases} \beta(2s) & 0 \leq s \leq \frac{1}{2} \\ \gamma(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Hence, we proved that, it satisfies the associativity condition.

Secondly, we must show that, homotopy classes have an identity element. To show that, let's take any loop namely α based at p to show $< e > \cdot < \alpha > = < \alpha >$ and $< \alpha > \cdot < e > = < \alpha >$. We should show that, $e \cdot \alpha$ and $\alpha \cdot e$ are both homotopic to $\alpha \; rel \; \{0,1\}$.

Let's us start with the first of these. We need to contruct a homotopy from $e \cdot \alpha$ to $\alpha \; rel \; \{0,1\}$. Let $f : I \to I$ be defined by

$$f(s) = \begin{cases} 0 & 0 \leq s \leq \frac{1}{2} \\ 2s-1 & \frac{1}{2} \leq s \leq 1 \end{cases}$$

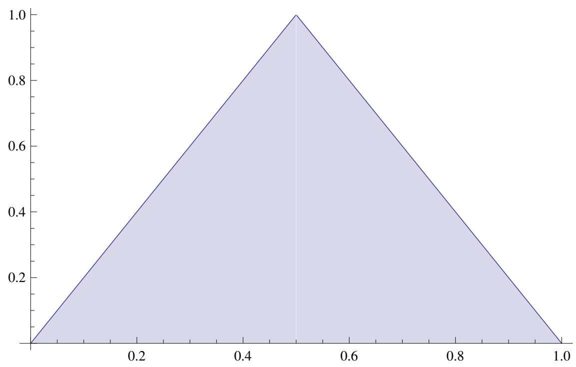

Figure 3.5: Construction of Fundamental Group (2)

Firstly, we will show that,

$$e \cdot \alpha(s) = \alpha \circ f(s)$$

For the LHS;

$$e \cdot \alpha(s) = \begin{cases} e(2s) & 0 \leq s \leq \frac{1}{2} \\ \alpha(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Similarly,

$$\alpha \circ f(s) = \begin{cases} \alpha(s) & 0 \leq s \leq \frac{1}{2} \\ \alpha(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

It is obvious that, the equation is satisfied by the product definition. I try to say that, $\alpha(2s-1)$ was found in the correct range. Also, since loop is based at p and $e(s) = p$ for $0 \leq s \leq 1$, $e(2s)$ and $\alpha(0)$ are equal to each other. Because, $\alpha(0)$ has the same value in the required range.

Secondly, we will show that,

$$\alpha \cdot e(s) = \alpha \circ f(s)$$

But, we should change f a little for that part. It could be taken as follows;

$$f(s) = \begin{cases} 2s & 0 \leq s \leq \frac{1}{2} \\ 0 & \frac{1}{2} \leq s \leq 1 \end{cases}$$

In the same way, for RHS;

$$\alpha \circ f(s) = \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1}{2} \\ \alpha(0) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

At the end, we see that,

$$e \cdot \alpha = \alpha \circ f \ \cong \alpha \circ 1_I \ rel\{0,1\}$$

$$= \alpha$$

and

$$\alpha \cdot e = \alpha \circ f \overset{\sim}{=} \alpha \circ I_I \; rel\{0,1\}$$

$$= \alpha$$

Lastly, the inverse of the homotopy class $< \alpha >$ is defined by $< \alpha^{-1} >$ where $\alpha^{-1}(s) = \alpha(1-s)$, $0 \leq s \leq 1$. It is clear that, α^{-1} travels in the opposite direction of α.

The inverse is well defined since if $\alpha \underset{F}{\simeq} \beta \; rel \; \{0,1\}$ then $\alpha^{-1} \underset{G}{\simeq} \beta^{-1}$ where $G(s,t) = F(1-s,t)$. If we are able to prove $\alpha \cdot \alpha^{-1} = \alpha \circ f(s)$, we will able to show that, $< \alpha > \cdot < \alpha^{-1} > = < e >$.

Let $f : I \to I$ be defined by

$$f(s) = \begin{cases} 2s & 0 \leq s \leq \frac{1}{2} \\ 2-2s & \frac{1}{2} \leq s \leq 1 \end{cases}$$

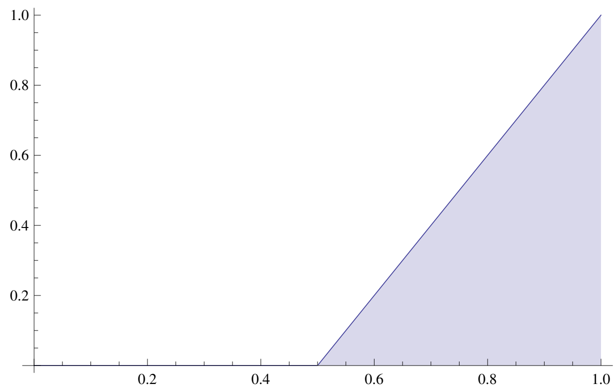

Figure 3.6: Construction of Fundamental Group (3)

Let's consider $\alpha \cdot \alpha^{-1} = \alpha \circ f(s)$ and let's consider the left hand side.

By product $\alpha \cdot \alpha^{-1}$, we have

$$\alpha \cdot \alpha^{-1} = \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1}{2} \\ \alpha^{-1}(2s-1) = \alpha(2-2s) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Notice that, $\alpha^{-1}(s) = \alpha(1-s)$, $0 \le s \le 1$.

For the RHS, we have

$$\alpha \circ f(s) = \begin{cases} \alpha(2s) & 0 \le s \le \frac{1}{2} \\ \alpha(2-2s) & \frac{1}{2} \le s \le 1 \end{cases}$$

We proved that,

$$\alpha \cdot \alpha^{-1}(s) = \alpha \circ f(s)$$

Since $f(0) = f(1) = 0$, we know that $f \simeq g \; rel \; \{0, 1\}$, where $g(s) = 0$, $0 \le s \le 1$.

Therefore,

$$\alpha \cdot \alpha^{-1} = \alpha \circ f \simeq \alpha \circ g \; rel\{0, 1\}$$

$$= e$$

In the same way, to show $< \alpha > \cdot < \alpha^{-1} >=< e >$, let's consider $\alpha^{-1} \cdot \alpha(s) = \alpha^{-1} \circ f(s)$.

For the LHS, by product $\alpha \cdot \alpha^{-1}$ we have

$$\alpha^{-1} \cdot \alpha(s) = \begin{cases} \alpha^{-1}(2s) & 0 \le s \le \frac{1}{2} \\ \alpha(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

Since $\alpha^{-1}(s) = \alpha(1-s)$, it follows that,

$$\alpha^{-1} \cdot \alpha(s) = \begin{cases} \alpha(1-2s) & 0 \le s \le \frac{1}{2} \\ \alpha(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

Then, for the RHS,

$$\alpha^{-1} \circ f(s) = \begin{cases} \alpha^{-1}(2s) & 0 \le s \le \frac{1}{2} \\ \alpha^{-1}(2-2s) & \frac{1}{2} \le s \le 1 \end{cases}$$

Since $\alpha^{-1}(s) = \alpha(1-s), 0 \leq s \leq 1$, it follows that

$$\begin{cases} \alpha(1-2s) & 0 \leq s \leq \frac{1}{2} \\ \alpha(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Hence, we proved that, the set of homotopy classes space of loops in X based at p forms a group under the multiplication $< \alpha > \cdot < \beta > = < \alpha \cdot \beta >$. ■

3.3 Calculations

Space	**Fundamental group**
Convex subset of $\mathbb{R}^n$	Trivial
Circle	$\mathbb{Z}$
$S^n,\ n \geq 2$	Trivial
$P^n,\ n \geq 2$	$\mathbb{Z}_2$
Torus	$\mathbb{Z}$ x $\mathbb{Z}$
Klein bottle	$\{a, b \mid a^2 = b^2\}$
Lens space $L\{p, q\}$	$\mathbb{Z}_p$

In this section, fundamental groups of the spaces $\mathbb{R}^n$, Circle, S^n, P^n and Torus will be computed one by one.

Convex subset of $\mathbb{R}^n$. Since any loop can be continuously shrunk to a point through straight line homotopy, for any $\mathbb{R}^n$, $n \geq 2$ is trivial.

The circle. Introduce the unit circle in the complex plane by the map $\pi : \mathbb{R} \to S^1$ with $x \to e^{2\pi i x}$. In other words, $x \to cos2\pi x + sin2\pi x$. Then, $1 \in S^1$ is chosen as a base point. Moreover, for $n \in \mathbb{Z}$ let $\gamma_n(s) = ns$, $0 \leq s \leq 1$, be a path joining 0 to n in $\mathbb{R}$ space.

Since 1 is the base point of S^1, γ_n projects under π to a loop based at 1. The composition $\pi \circ \gamma_n$ rotates the circle n times in anticlockwise direction for n positive or clockwise for n negative.

Theorem 3.3.1 *[2] The function $\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, 1)$ defined by $\Phi(n) = \langle \pi \circ \gamma_n \rangle$ is an isomorphism.*

The n-sphere. By the above table, S^n has trivial fundamental group for $n \geq 2$. This is an immediate consequence of the following theorem;

Theorem 3.3.2 *[2] Let X be a space which can be written as the union of two simply connected open sets U, V in such a way that $U \cap V$ is path-connected. Then X is simply connected.*

Definition 3.3.3 *(Contractible)[1] A topological space X is contractible if there exists a homotopy equivalence between it and a one-point space, or equivalently, if there exists a homotopy $F : X \times I \to X$ that starts with the identity and ends with the constant map $c(x) = x_0$, namely 1_x is nullhomotopic. We call such a homotopy F a contraction.*

Example 3.3.4 *The unit ball is contractible, in contrast, the circle S^1 is not.*

Theorem 3.3.5 *[2]*

(a) A space is contractible if and only if it has the homotopy type of a point.

(b) A contractible space is simply connected.

(c) Any two maps into a contractible space are homotopic.

(d) If X is contractible, then 1_x is homotopic to the constant map at x for any $x \in X$.

3.4 Actions of group on spaces, Orbit spaces

Before giving the theorems and examples on group actions, we ought to know the following definition of orbit spaces.

Definition 3.4.1 *[2] A topological group G is said to act as a group of homeomorphisms on a space X if each group element induces a homeomorphism of the space in such a way that,*

(a) $hg(x) = h(g(x))$ for all $g, h \in G$, for all $x \in X$;

(b) $e(x) = x$ for all $x \in X$, where e is the identity element of G;

(c) the function $G \times X \to X$ defined by $(g, x) \longmapsto g(x)$ is continuous.

Theorem 3.4.2 *[2] Let G act on X and suppose that both G and X/G are connected, then X is connected.*

Theorem 3.4.3 *[2] If G acts as a group of homeomorphisms on a simply connected space X, and each point $x \in X$ has a neighbourhood U which satisfies $U \cap g(U)$ =Ø for all $g \in G - \{e\}$,then $\pi_1(X/G)$ is isomorphic to G.*

Example 3.4.4 *$\mathbb{Z}$ acts on $\mathbb{R}$ with orbit space the circle, giving $\pi_1(S^1) \cong \mathbb{Z}$.*

Let $X = \mathbb{R}$. Identify any point x on $\mathbb{R}$, with $x + n$ where $n \in \mathbb{Z}$. As a result of the equation, the real line $\mathbb{R}$ falls down to two points joined by an edge of length 1. Because these points are also identified, the resulting orbit space is the circle S^1.

$$\pi_1(\mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}.$$

Example 3.4.5 *$\mathbb{Z} \times \mathbb{Z}$ on $\mathbb{R}^2$ with orbit space the torus T, giving $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$.*

The action of $\mathbb{Z} \times \mathbb{Z}$ on the plane sends the point $(x, y) \in \mathbb{R}^2$ to $(x + m, y + n)$ where $(m, n) \in \mathbb{Z} \times \mathbb{Z}$. In this case, the orbit space is $S^1 \times S^1$ which is the torus. Geometrically, the plane is divided into unit squares by horizontal and vertical lines through points with integer coefficients. Under this action, all the unit squares are identified with each other and finally the opposite sides of the resulting one are identified to obtain the torus.

$$\mathbb{R}^2 \xrightarrow{\pi_1} \mathbb{R}^2/\mathbb{Z} \times \mathbb{Z} = T.$$

or

$$\pi_1(\mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}.$$

I mean, $\mathbb{Z} \times \mathbb{Z}$ acts on $\mathbb{R}^2$ and we obtain the Torus. Note that, $\mathbb{Z} \times \mathbb{Z}$ is the fundamental group of the torus.

Example 3.4.6 *$\mathbb{Z}_2$ acts on S^n with the orbit space P^n, giving $\pi_1(P^n) \cong \mathbb{Z}_2$.*

Before solving this problem, we know that, $\mathbb{Z}_2$ has two elements which are 0 and 1. It is clear that, 0 is the identity element.

Consider the S^2 the sphere. We have two hemispheres. Also, we know that, those two hemispheres have a common boundary in the middle. Take any pair of antipodal points on the sphere and identify them by the element $'1'$ of $\mathbb{Z}_2$. However, if we push the resulting hemisphere from up and down we get a disc. By the theorem 3.4.3, we have

$$\pi_1(S^2/\mathbb{Z}_2) \cong \pi_1(P^2) \cong \mathbb{Z}_2.$$

Theorem 3.4.7 *[2] If X and Y are path-connected spaces $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.*

Proof. Firstly, define two maps, $p_{1_\star} : \pi_1(X \times Y) \longrightarrow \pi_1(X)$, $p_{2_\star} : \pi_2(X \times Y) \longrightarrow \pi_2(Y)$

Then

$$\pi_1(X \times Y) \xrightarrow{\Psi} \pi_1(X) \times \pi_1(Y)$$

$$\langle \alpha \rangle \longmapsto (\langle p_1 \circ \alpha \rangle, \langle p_2 \circ \alpha \rangle)$$

α is defined by a loop in $X \times Y$, and if $p_1 \circ \alpha \underset{F}{\simeq} e_{x_0}$, $p_2 \circ \alpha \underset{G}{\simeq} e_{y_0}$, then $\alpha \underset{H}{\simeq} e(_{x_0, y_0})$ where $H(s,t) = (F(s,t), G(s,t))$. So, Ψ is one to one.

Secondly, we should show that, Ψ is onto. Contrary, β and γ are loops in X and Y respectivaly and $\alpha(s) = (\beta(s), \gamma(s))$ in $X \times Y$. Composition of p_1 and α gives β and in the same way, γ comes from $p_2 \circ \alpha$. At the end, $\Psi(\langle \alpha \rangle) = (\langle \beta \rangle, \langle \gamma \rangle)$ is found. That means, Ψ is onto.

■

Chapter 4

COVERING SPACE THEORY

To begin with, $\mathbb{R}^n$ has a trivial fundamental group as any loop can continuously be derived to a point. However, some surfaces don't have trivial fundamental groups. In these cases, we may use the notion of covering spaces to compute their fundamental group.

Definition 4.0.8 *[7] Let $p : E \to B$ be a continuous surjective map. The open set U of B is said to be* ***evenly covered*** *by p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into* ***slices.***

Let's take any open set U covered by p. In general, $p^{-1}(U)$ can be pictured as 'stack of pancakes' flying (in the air) above U. Consider an open subset W of U where U is evenly covered by p. It follows immediately that W is also evenly covered by p (see Fig. 4.1 [7]).

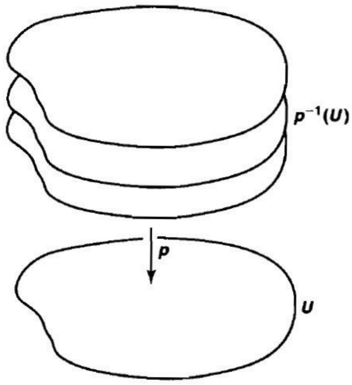

Figure 4.1: Stack of Pancakes

Definition 4.0.9 *[7] Let $p : E \to B$ be continuous and surjective. If every point b of B has a neighbourhood U that is evenly covered by p, then p is called a* ***covering map,*** *and E is said to be a* ***covering space*** *of B.*

There is a quite significant point in here. If we have the covering map $p : E \to B$, then for each $b \in B$ the subspace $p^{-1}(b)$ of E has the discrete topology. We know that, E includes slices of V_α which are open and the set $p^{-1}(b)$ intersects them in a single point in E. So, the intersection point is also open in $p^{-1}(b)$.

Moreover, covering maps $p : E \to B$ are open maps. Take an open set A from E and let $x \in p(A)$. U is chosen as a neighborhood of x which is evenly covered by p. Suppose that, the set $\{V_\alpha\}$ belongs to $p^{-1}(U)$. There is a point y of A such that $p(y) = x$. Let V_β be the slice which contains y. We know that, $V_\beta \cap A$ is open in E and it follows that, it is also open in V_β. Because p maps V_β homeomorphically onto U, the set $p(V_\beta \cap A)$ is open in U. Lastly, $p(V_\beta \cap A)$ is also open in B.

Example 4.0.10 *Let's consider any space Y and let us define a function $f : Y \to Y$as an identity map. Then, f is a covering map. More precisely, let E be a space $Y \times \{1, ..., n\}$*

based on n identical similarities of Y. The map $p : E \to Y$ given by $p(x, f) = x$ for all f is a covering map. Note that, we can image the whole space E as a stack of pancakes above Y.

Theorem 4.0.11 *[7] The map $p : \mathbb{R} \to S^1$ given by the equation*

$$p(x) = (\cos 2\pi x, \sin 2\pi x)$$

is a covering map.

Proof of the theorem will be omitted, but for an idea of a proof, we know that, the real line circles around the S^1 and $[n, n+1]$ is the interval mapped onto S^1 in that process.

Theorem 4.0.12 *[7] If $p : E \to B$ and $p^{'} : E^{'} \to B^{'}$ are covering maps.*

Then

$$p \times p' : E \times E^{'} \to B \times B^{'}$$

is a covering map.

Proof. Let's take $b \in B$ and $b^{'} \in B^{'}$, and let b and b' have neighborhoods N and N' evenly covered by p and $p^{'}$, respectively. Suppose that, the sets $\{V_\gamma\}$ and $\{V_\delta^{'}\}$ are partitions of $p^{-1}(N)$ and $(p^{'})^{-1}(N^{'})$ into slices, respectively.

The inverse image of the open set $U \times U^{'}$ can be represented as the union of disjoint open sets $V_\gamma \times V_\delta^{'}$ in $E \times E^{'}$ such that for each γ and δ, the restriction of $p \times p^{'}$ to $V_\gamma \times V_\delta^{'}$ is homeomorphism from $V_\gamma \times V_\delta^{'}$ to $N \times N^{'}$.

■

Example 4.0.13 *Let's take the space $T = S^{'} \times S^{'}$; the **torus**. The product map*

$$p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$$

is a covering of the torus by the plane $\mathbb{R}^2$ *where p represents the covering map* $p(x) = (\cos 2\pi x, \sin 2\pi x)$. *Each of the unit squares* $[n, n+1] \times [m, m+1]$ *gets wrapped by* $p \times p$ *completely all over the torus.*

This map says, the torus, namely $S^1 \times S^1$, is covered by $\mathbb{R}^2$-plane. However, it is hard to picture getting the torus by $S^1 \times S^1$ which is in $\mathbb{R}^4$. That's why, let's take the circle C_1 around the xz-plane which has radius $\frac{1}{3}$ and centre at $(1, 0, 0)$ on the x-axis. Also, let's take the another circle C_2 which has radius 1 and centre at the origin on the xy-plane. Let $f : C_1 \times C_2 \to D$ and let $f(a, b)$ be the point a is carried to after rotating C_1 about the z-axis until its centre reaches the point b (see Fig.4.2 [7]). It can be shown that the map f is a homeomorphism between $C_1 \times C_2$ and D.

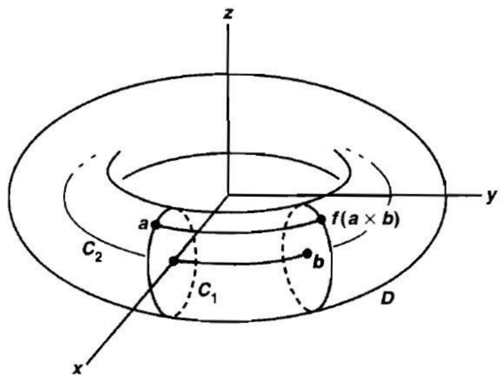

Figure 4.2: Torus

Example 4.0.14 *Consider the covering map* $p \times p$ *of the previous example. Let's take take* a_0 *to indicate the point* $p(0)$ *of* S^1. *Also, let* A_0 *imply the subspace*

$$A_0 = (S^1 \times a_0) \cup (a_0 \times S^1)$$

of $S' \times S'$. *Then,* A_0 *is the union of two circles tangent to each other at the point* a_0. *We usually call it the* ***figure-eight space***. *The space* $E_0 = p^{-1}(A_0)$ *is the 'infinite grid'*

$$E_0 = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$$

pictured in Fig. 4.3 [7];

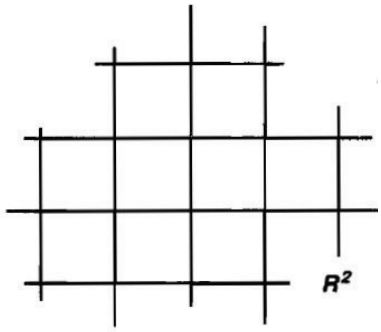

Figure 4.3: Infinite grid

The map $p_0 : E_0 \to A_0$ is obtained by restricting $p \times p$.

Let's consider $E_0 = (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$. If you bend the identified pairs of opposite edges which are in the same direction, you get two circles touching one another at a point. Even if you think you obtain the torus at the end, you get the figure-eight. Because not every point (x, y) belong to E_0.

Example 4.0.15 *Consider the covering map*

$$p \times i : \mathbb{R} \times \mathbb{R}_+ \to S^1 \times \mathbb{R}_+$$

i here is the identity map of $\mathbb{R}_+$ and p is the map $p(x) = (\cos 2\pi x, \sin 2\pi x)$. The map f : $S^1 \times \mathbb{R}_+ \to \mathbb{R}^2 - 0$ mapping (x, t) to tx is a homeomorphism, and the composition of f, with $p \times i$ above, gives us the covering map

$$\mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2 - 0$$

of the punctured plane by the open upper half-plane.

The first aim is to draw the figure of $\mathbb{R} \times \mathbb{R}_+$. Then, focus on $S^1 \times \mathbb{R}_+$ which is sent to $\mathbb{R}^2 - \{0\}$ via tx.

Now, consider the following homeomorphism

$$x \times t \to S^1 \times \mathbb{R}_+ \to \mathbb{R}^2 - 0$$

It is clear that, x represents a point on the circle and height is defined by t. Because of $\mathbb{R}_+$, the point 0 does not belongs to $\mathbb{R}^2$. Consequently, $S^1 \times \mathbb{R}_+$ and $\mathbb{R}^2 - 0$ are homeomorphic to each other. Hence, $\mathbb{R} \times \mathbb{R}_+$ is also covering space of $\mathbb{R}^2 - 0$ (see Fig. 4.4 [7]).

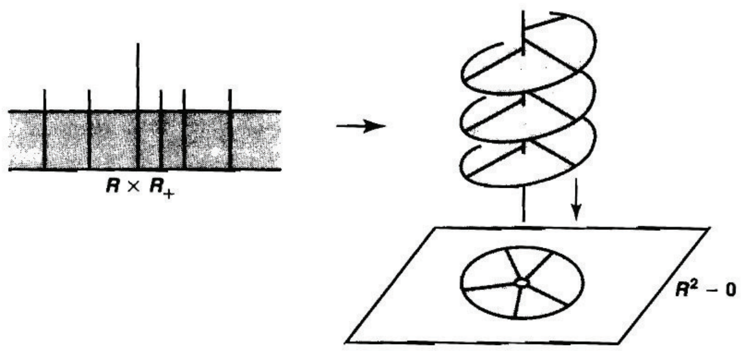

Figure 4.4: Covering of punctured plane by open upper half-plane

Definition 4.0.16 *(Simply connected) [1] A topological space X is said to be* ***simply connected*** *if it is path connected (0-connected) and for some base point $x_0 \in X$ the fundamental group $\pi_1(X, x_0)$ is trivial. Frequently, a simply connected space is also called* ***1-connected.***

Example 4.0.17 *Both a sphere and a disc have trivial fundamental groups since every loop can continuously be shrunk to a point. Therefore, both of them are simply connected. However, a circle is not simply connected because some loops can not be continuously contracted to a point.*

Definition 4.0.18 *(Universal covers) [7] Consider the covering map $p : E \to B$. If E is simply connected, then E is said to be a universal covering space of B. The name of* *universal cover comes from the following property: If the mapping $g : D \to X$ is a universal covering map of the space X and the mapping $p : C \to X$ is any other covering map of the space X, where the covering space C is connected, then there exists a covering map $f : D \to C$ such that $p \circ f = g$. This can be phrased as the universal cover of the space X covers all connected covers of the space X.*

Example 4.0.19 *Consider the covering map*

$$p : \mathbb{R} \to S^1$$

It is clear that, any loop on the real line can be contracted to a point. Moreover, since real line is a simply connected space, and the circle itself is connected, $\mathbb{R}$ is called a universal covering space of the circle.

Example 4.0.20 *The sphere is its own universal cover. Consider the following map*

$$p : S^2 \to S^2$$

When you take any loop on the sphere, it can continiusly fall down to a point in such a way that, the fundamental group of S^2 is trivial. Plus, S^2 is path connected. Therefore, S^2 is its own universal cover.

Example 4.0.21 *Consider the space $T = S^1 \times S^1$ which is the Torus. The following map;*

$$p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$$

is called the universal covering map of the torus. $\mathbb{R} \times \mathbb{R}$ is path connected and also when you take any loop on $\mathbb{R} \times \mathbb{R}$, it can continuously be shrunk to a point. So that, its fundamental group is trivial. Clearly, $\mathbb{R} \times \mathbb{R}$ is simply connected.

Example 4.0.22 *The universal cover of the connected topological space $S^1 \times \mathbb{R}_+$ is the simply connected space $\mathbb{R} \times \mathbb{R}_+$ with the covering map*

$$p \times i : \mathbb{R} \times \mathbb{R}_+ \to S^1 \times \mathbb{R}_+$$

Example 4.0.23 *The Cayley graph of the free group on two generators is the covering map of the figure-eight space which is connected. It is obvious that, the Cayley graph is a universal covering space of the figure-eight space. Since any path on the branch can be shrunk to a point, its fundamental group is trivial.*

Note that, every covering map is not a universal covering map. As an example, not all loops on $(\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$ can fall down to a point; so infinite grid is not simply connected.

Theorem 4.0.24 *[7] Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$.*

(a) The homomorphism $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is a monomorphism.

(b) Let $H = p_*(\pi_1(E, e_0))$. The lifting correspondence ϕ induces an injective map

$$\Phi : \pi_1(B, b_0)/H \to p^{-1}(b_0)$$

of the collection of right cosets of H into $p^{-1}(b_0)$, which is bijective if E is path connected.

(c) If f is a loop in B based at b_0, then $[f] \in H$ if and only if f lifts to a loop in E based at e_0.

4.1 $K(G, 1)$ Spaces

Definition 4.1.1 *[4] A path-connected space whose fundamental group is isomorphic to a given group G and which has a contractible universal covering space is called a $K(G, 1)$ space. The '1' here refers to π_1. All these spaces are called Eilenberg-Maclane spaces, though in the case $n = 1$.*

Example 4.1.2 S^1 *is a* $K(\mathbb{Z}, 1)$.

S^1 is path connected, $\pi_1(S^1) = \pi_1(B\mathbb{Z}) = \mathbb{Z}$ and the universal cover of S^1 is $\mathbb{R}$, which is contractible.

Example 4.1.3 $S^1 \times S^1$ *is a* $K(\mathbb{Z} \times \mathbb{Z}, 1)$.

Again here, $S^1 \times S^1$ is path connected, $\pi_1(S^1 \times S^1) = \pi_1(B(\mathbb{Z} \times \mathbb{Z})) = \mathbb{Z} \times \mathbb{Z}$ and the universal cover of $S^1 \times S^1$ is the contractible $\mathbb{R}^2$.

Example 4.1.4 S^2 *is not a* $K(\{e\}, 1)$.

Let's consider the covering map

$$p : S^2 \to S^2$$

S^2 is a path-connected space which has trivial fundamental group; isomorphic to $\{e\}$. S^2 is a universal cover of itself but it's not contractible.

Example 4.1.5 *Consider the cover* $(\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$ *of the figure-eight space which is path-connected. Since the fundamental group of* $(\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R})$ *is not trivial, it is not simply connected. So, it can not be a universal covering space. On the other hand, since the Cayley graph is contractible, it can be taken as the universal cover of the figure-eight space.*

Chapter 5

RATIONAL EULER CHARACTERISTICS

Assume that $\mathcal{L}$ is the class of groups that have finite classifying space (BG space) X. Euler characteristic on this class is defined as $\chi(G) = \chi(BG) = \chi(X)$. We'll show that in this class, the following formula [9] are satisfied;

$$\chi(G \times H) = \chi(G)\chi(H) \tag{1}$$

$$\chi(G * H) = \chi(G) + \chi(H) - 1 \tag{2}$$

$$\chi(K) = r\chi(G) \tag{3}$$

Here G, H are $\mathcal{L}$-groups, and K is a subgroup of index r in G.

5.1 Proofs of The Formula for $\mathcal{L}$-class

Before we prove formula (1), we need the following proposition. But first, let us define the term 'CW-Complex' which appears in the proof of the proposition below.

Definition 5.1.1 *(CW-Complex) [10], [11] Firstly, we may talk about the origin of the term 'CW-complex'. The capital letters 'C' and 'W' represent the 'closure-finite' and 'weak topology', respectively. CW-complexes are based on the n-skeleton of space X. To create a CW-complex, we need an initial point which is represented by X^0 (called zero-cell or D^0). In* *the same way, a closed line segment is referred to as X^1 (known as one-cell or D^1). Furthermore, by using the product of two closed line segments, we obtain a square named as two-cell or D^2, and so on for a new space which is in X^n. Intuitively, attaching infinitely many $(n-1)$ discs demonstrates X^{n-1}, in such a way that, we build X^n. It is clear that,*

$$S^{n-1} \to X^{n-1} \to X.$$

May be defined as a continuous map where S represents the boundary.

Proposition 5.1.2 *For general spaces K and L;*

$$\chi(K \times L) = \chi(K)\chi(L)$$

Proof. This formula is useful for finite sets via the cartesian product. However, proving this for general spaces is quite hard. Let's see how can we prove it within the simple way for CW-complexes.

The Euler characteristic of a CW-complex can be calculated by the alternating sum of its d-cells.

[12] Let's take two spaces K and L where both of them have a single edge and two vertices. It's clear that, the product $K \times L$ gives a "square" with 4 vertices, 4 edges, and an internal face. On the other hand, if we think about simplicial complexes, we have $2-$dimentional triangles. Therefore, we realize that, the calculation gets more complex. In conclusion, the products of CW-complexes are easier to visualise than products of simplicial complexes.

Let us now increase the level a little bit. In the previous explanation, we computed the Euler characteristic directly from an alternating sum. Now, instead of this, we will find it by evaluating a specially crafted polynomial. Indeed, this is very interesting.

Let's take two polynomials $P(K) = P(L) = 2+1.t$ where the degree of polynomial denotes the dimension, and the coefficient represents the number of cells of that dimention. We obtain the Euler characteristic by evaulating the polynomial at value $t = -1$.

Now, consider the polynomial $P(K \times L) = 4 + 4t + t^2$. Every k-dimentional cell in K and l-dimentional cell in L gives an $(k + l)$ dimentional cell in the product space.

Hence

$$P(K \times L) = P(K) \times P(L).$$

Therefore,

$$\chi(K \times L) = \chi(K)\chi(L).$$

■

Lemma 5.1.3 *(formula (1))*

$$\chi(G \times H) = \chi(G)\chi(H) \tag{1}$$

Proof. Remember that the Euler characteristic on class $\mathcal{L}$ is defined as;

$$\chi(G) = \chi(X)$$

where X is the classifying space.

So,

$$\chi(G \times H) = \chi(B(G \times H))$$

Is

$$B(G \times H) = BG \times BH \qquad ? \tag{$1'$}$$

Let's consider the right hand side of $(1')$. Since BG and BH are connected, $BG \times BH$ is also connected automatically.

Then, by the theorem, if BG and BH are path-connected spaces, then

$$\pi_1(BG \times BH) = \pi_1(BG) \times \pi_1(BH)$$

$$= G \times H.$$

In that point, we may need to remember the definition of $K(G,1)$ spaces. A $K(G,1)$ $\{BG\}$ space is a path-connected space with $\pi_1(BG) = G$ and with universal cover contractible.

So, we will show that the universal cover of $BG \times BH$ is contractible.

But we already know by the theorem that,

$$\mathcal{U}(BG \times BH) = \mathcal{U}(BG) \times \mathcal{U}(BH)$$

Since $\mathcal{U}(BG)$ and $\mathcal{U}(BH)$ are contractible, it will imply that $\mathcal{U}(BG \times BH)$ is also contractible. So, by the definition $BG \times BH$ is a $B(G \times H)$.

Hence

$$\chi(G \times H) = \chi(B(G \times H)) = \chi(BG \times BH) = \chi(BG) \times \chi(BH) = \chi(G)\chi(H).$$

■

Lemma 5.1.4 *(formula (2))*

$$\chi(G * H) = \chi(G) + \chi(H) - 1$$

The following definition will help us to prove the above lemma.

Definition 5.1.5 *If $X = BG$ and $Y = BH$, then*

$$X \vee Y = \frac{X \amalg Y}{x_0 = y_0} = B(G * H)$$

where X & Y are based spaces with base points x_0, y_0.

Proof. Since groups G and H belong to class $\mathcal{L}$, they have finite classifying spaces. For a model of $B(G * H)$, we take the disjoint union of BG and BH spaces and identify the base points of their fundamental groups. This is equivalent to taking the disjoint union of BG and BH spaces and joining them (through base points) with an edge. This will bring the Euler characteristic one down. ■

Lemma 5.1.6 *(formula (3))*

$$\chi(K) = r\chi(G) \tag{3}$$

Proof. If K is a subgroup of finite index r in G, $\exists\ \pi : BK \to BG$ covering map such that the induced map on fundamental groups $\pi_* : K \to G$ is injective. Since the map π_* is injective, $\pi_*(K)$, which is a subgroup of G, also has finite index r in G.

Thus, BK is an r-fold cover of BG.

Therefore,

$$\chi(BK) = r\chi(BG)$$

$$\chi(K) = r\chi(G).$$

■

Example 5.1.7 *Let $G = \mathbb{Z}$. Here S^1 can be taken as a model for $B\mathbb{Z}$; S^1 is connected, and $\pi_1(S^1) = \pi_1(B\mathbb{Z}) = \mathbb{Z}$.*

Since $\mathbb{R}$ is the universal cover of S^1 and $\mathbb{R}$, is contractible,

$$\chi(\mathbb{Z}) = \chi(B\mathbb{Z}) = \chi(S^1) = 0.$$

Example 5.1.8 *Let $G = \mathbb{Z} \times \mathbb{Z}$. We take $S^1 \times S^1 (= torus)$ as a model for the classifying space. $S^1 \times S^1$ is connected, $\pi_1(S^1 \times S^1) = \pi_1(B(\mathbb{Z} \times \mathbb{Z})) = \mathbb{Z} \times \mathbb{Z}$ and the universal cover* *of $S^1 \times S^1$ is $\mathbb{R} \times \mathbb{R}$, which is contractible.*

Hence,

$$\chi(\mathbb{Z} \times \mathbb{Z}) = \chi(B(\mathbb{Z} \times \mathbb{Z})) = \chi(S^1 \times S^1) = 0.$$

Note that, by the formula (1) of $\mathcal{L}$-class, we may also write

$$\chi(\mathbb{Z} \times \mathbb{Z}) = \chi(\mathbb{Z})\chi(\mathbb{Z})$$

which gives us again zero for the Euler characteristic of $\mathbb{Z} \times \mathbb{Z}$.

Example 5.1.9 *Let us take $G = \mathbb{Z} * \mathbb{Z}$. By the formula (2) we have*

$$\chi(G * H) = \chi(G) + \chi(H) - 1.$$

Since the Euler characteristic of $\mathbb{Z}$ is zero,

$$\chi(\mathbb{Z} * \mathbb{Z}) = -1.$$

Note that, let us take $G = \mathbb{Z} * \mathbb{Z}$ and let's consider figure-eight space, that is connected, as a classifying space for G. $\pi_1(figure{-}eight) = \pi_1(B(\mathbb{Z}*\mathbb{Z})) = \mathbb{Z}*\mathbb{Z}$ and the universal cover of $figure - eight$ is Cayley graph of the free group on two generators that is contractible.

Example 5.1.10 *Consider G as $4\mathbb{Z}$. Since $4\mathbb{Z}$ is a subgroup of $\mathbb{Z}$, $K = 4\mathbb{Z}$ and $G = \mathbb{Z}$ are taken. $B\mathbb{Z} = S^1$ so $\chi(\mathbb{Z}) = 0$.*

Since we have

$$\chi(4\mathbb{Z}) = r\chi(\mathbb{Z})$$

where $r = 4$ (r is the number of left cosets of $4\mathbb{Z}$ in $\mathbb{Z}$)

$$\chi(4\mathbb{Z}) = 0.$$

Define class $\mathcal{M}$ as the class of groups G which have a subgroup K of finite index r belonging to $\mathcal{L}$.

On $\mathcal{M}$, we will define

$$\chi(G) = \frac{1}{r}\chi(K).$$

5.2 Proofs of The Formula for $\mathcal{M}$-class

Lemma 5.2.1 *The definition above doesn't depend on the choice of K i.e. if $K' \leq G$ and $\left|G : K'\right| = r'$ and $K' \in \mathcal{L}$*

Then

$$\frac{1}{r}\chi(K) = \frac{1}{r'}\chi(K') = \chi(G).$$

To prove the lemma above, we use a couple of propositions given below.

Proposition 5.2.2 *Let H and K be subgroups of G. If they have finite indices in G, then so does their intersection.*

In fact,

$$|G : H \cap K| \leq |G : H|\,|G : K|\,.$$

Proof. To prove the above proposition, we will focus on the actions of G on the set of left cosets G/H and G/K.

In here, we will show that, if both $|G : H|$ and $|G : K|$ are finite indices, then $|G : H \cap K|$ is also finite. In other words, the index of $H \cap K$ is at most the index of H times the index of K. To show that, we consider the injection from the set of left cosets of $H \cap K$ to the set of pairs (A, B) where A and B are defined by the left cosets of H and K, respectively.

Let $g \in G$. Since $H \cap K \leq H$, $g(H \cap K)H = gH$. In the same way, since $H \cap K \leq K$, $g(H \cap K)K = gK$. To show that the given map is one to one, we use the properties of cosets. If $gH = g^{'}H$ and $gK = g^{'}K$ then $g^{-1}g^{'} \in H$ and $g^{-1}g^{'} \in K$, which follows that $g^{-1}g^{'} \in H \cap K$, in such a way that $g(H \cap K) = g^{'}(H \cap K)$. ■

Proposition 5.2.3 *Let K and $K^{'}$ be two subgroups of finite index in G. Let the indices of K and $K^{'}$ be r and $r^{'}$, respectively. Let also the intersection $K \cap K^{'}$ have index R in G.*

Then,

$$\left| K : K \cap K^{'} \right| = \frac{R}{r}.$$

Proof. By the above informations, the number of cosets of K in G is a finite number r. Since $K \cap K^{'} \leq K$, the number of cosets of $K \cap K^{'}$ in K is finite number x, in such a way that, x times r gives R.

So,

$$\left| K : K \cap K^{'} \right| = \frac{R}{r}.$$

■

Proof. (Lemma 5.2.1) Let $G \in \mathcal{M}$ and let G have subgroups K and $K^{'}$ of finite indices r and $r^{'}$, respectively.

Then by the definition, we have

$$\chi(G) = \frac{1}{r}\chi(K).$$

For the $\mathcal{L}$-class, the previous proposition helps us to write the following formula:

$$\chi(K \cap K^{'}) = \frac{R}{r}\chi(K) \tag{1}$$

and

$$\chi(K \cap K^{'}) = \frac{R}{r^{'}}\chi(K^{'}). \tag{$1^{'}$}$$

In here, since K and $K^{'}$ belong to $\mathcal{L}$-class, they have finite classifying spaces BK and $BK^{'}$, respectively. Also, it is obvious that, $K \cap K^{'} \leq K$. In the same way, $K \cap K^{'} \leq K^{'}$.

The formula (1) and (1$^{'}$) can be written as follows;

$$\chi(K) = \frac{r}{R}\chi(K \cap K^{'}) \tag{2}$$

and

$$\chi(K^{'}) = \frac{r^{'}}{R}\chi(K \cap K^{'}) \tag{2$'$}$$

If we combine (2) and (2$^{'}$), we get

$$\frac{1}{R}\chi(K \cap K^{'}) = \frac{\chi(K^{'})}{r^{'}} = \frac{\chi(K)}{r} = \chi(G)$$

where $\left|G : K \cap K^{'}\right| = R$.

It is clear that, the Euler characteristic $\chi(K \cap K^{'})$ does not depend on the choice of subgroups K and $K^{'}$. ■

During the next section, we will prove that the same formula (which were given at the beginning of Chapter 5) are satisfied for class $\mathcal{M}$.

Lemma 5.2.4 $\chi(K) = r\chi(G)$.

Proof. Let us come back to what we said before about the $\mathcal{M}$-class: $\mathcal{M}$-class is the class of groups having a subgroup K of finite index r which belongs to $\mathcal{L}$.

Since the formula (3) above is same as the definition of Euler characteristic, on class $\mathcal{M}$, it's clear. ■

Lemma 5.2.5 $\chi(G \times H) = \chi(G)\chi(H)$, *where* G *and* H *are* $\mathcal{M}$*-groups.*

Before the proof of this lemma, we state and prove the proposition below;

Proposition 5.2.6 *The map* $f : G/K \times H/L \to (G \times H)/(K \times L)$ *is a bijection.*

Proof. Let's take g and h which are belonging to G and H, respectively. Left cosets of K in G and H in L are written as gK and hL, respectively. Intuitively, a random element of left hand side is represented by (gK, hL).

On the other hand, the left cosets of right hand side are expressed as $(g, h)(K, L)$. Since $(g, h)(k, l) = (gk, hl)$ where $k \in K$ and $l \in L$, we get whole elements of $(g, h)(K, L)$. It's obvious that, by the bijection that (gK, hL) goes to $(g, h)(K, L)$. ■

Proof. (Lemma 5.2.5) Consider the groups G and H which belong to $\mathcal{M}$-class. By the definition of the latter, they have subgroups K and L of finite indices r and s, respectively that belong to $\mathcal{L}$-class.

As we pointed out in the previous lemma, we have

$$\chi(K) = r\chi(G) \tag{*}$$

where subgroup K has finite BK.

Let's start with the left-hand side of the above lemma.

We have

$$\chi(G \times H) = \frac{1}{rs}\chi(K \times L) \tag{**}$$

which comes from the $(*)$.

Note that, since K and L belong to $\mathcal{L}$-class, they have finite BK and BL, respectively.

Also

$$B(K \times L) = BK \times BL.$$

The product of two finite classifying spaces gives a finite classifying space as $B(K \times L)$. Hence, $B(K \times L)$ is also finite.

Since K and L belong to $\mathcal{L}$-class, by the formula (1) $(\textit{Lemma}\ 5.0.52)$ the right-hand side of equation $(**)$ may be expressed as the following

$$= \frac{1}{rs}\chi(K)\chi(L)$$

or

$$= \frac{1}{r}\chi(K)\frac{1}{s}\chi(L) \qquad (***)$$

As a result, by the $(**)$, we get

$$\chi(G \times H) = \chi(G)\chi(H).$$

Combining all this, we get

$$\chi(G \times H) = \frac{1}{rs}\chi(K \times L) = \frac{1}{rs}\chi(K)\chi(L) = \frac{1}{r}\chi(K)\frac{1}{s}\chi(L) = \chi(G)\chi(H)$$

■

Now, let us state the following theorem before we start to prove lemma below;

Theorem 5.2.7 *(Conjugate-intersection index theorem) [5] Let H be a subgroup of G of finite index r in G. Let $H_1, H_2, ..., H_s$ be distinct conjugates of H in G, so that $s \leq r$. Let $K = \bigcap_{i=1}^{s} H_i$. Then,*

(a) K is of finite index in G.

(b) The index of K in G is bounded from above by $r(r-1)...(r-s+1)$.

Lemma 5.2.8 $\chi(G * H) = \chi(G) + \chi(H) - 1.$

Proof. Let us take groups G and H, with $\mathcal{L}$ subgroups K and L. Let also K and L have indices r and s in G and H, respectively. The conjugate gKg^{-1} is a subgroup of K. So, $\bigcap(gKg^{-1})$ is also a subgroup in K. $|G : \bigcap(gKg^{-1})|$ is finite by the finite index theorem above [5]. Hence, $|K : \bigcap(gKg^{-1})|$ is also finite. Since K is an $\mathcal{L}$-group and $\bigcap(gKg^{-1})$ is a subgroup of finite index, $\bigcap(gKg^{-1})$ is also an $\mathcal{L}$ group. Note that, by this argument, we can assume that, K and L are normal subgroups.

The map $f : G * H \to G \times H$ is onto and so is the map $g : G \times H \to G \times H/K \times L$. As $G \times H/K \times L$ is isomorphic to $G/K \times H/L$, the composition map $g \circ f : G * H \to G/K \times H/L$ is also onto.

In other words,

$$G * H \twoheadrightarrow G \times H \twoheadrightarrow (G \times H)/(K \times L) \cong G/K \times H/L.$$

Next, if we apply the first isomorphism theorem, we have

$$(G * H)/M \cong Im \cong G/K \times H/L$$

where M is the Kernel.

Since the number of cosets of $K \times L$ in $G \times H$ is rs, the index of M in $G * H$ is equal to rs. Also, as M is a subgroup of $G * H$, the map between M and $G * H$ is one to one.

Next, we should show that M belongs to $\mathcal{L}$-class.

As the index of M in $G * H$ is rs, we try to find an rs-fold cover for $B(G * H)$. This is provided by the following diagram.

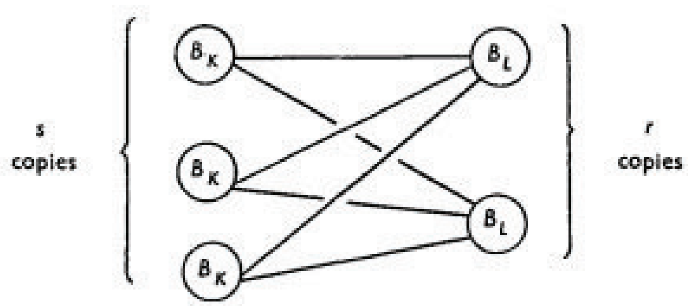

Figure 5.1: Rational Euler Characteristic diagram

Since K is a subgroup of index r in G, we have a finite BK. In the same way, since L is a subgroup of index s in H, we have another finite BL in such a way that BK and BL are the r-fold and s-fold covers of BG (may be infinite) and BH (may be infinite), respectively. By the figure above, we have s-times BK and r-times BL, that is, we have rs-fold cover for the free product. It is clear that, we obtain BM by the diagram.

By the above argument, we write

$$\chi(BM) = s\chi(BK) + r\chi(BL) - rs$$

or

$$\chi(M) = s\chi(K) + r\chi(L) - rs$$

Since M has finite index rs in $G * H$ and BM is finite, we have

$$\chi(M) = rs\chi(G * H)$$

or

$$\chi(G * H) = \frac{1}{rs}\chi(M)$$

$$\chi(G * H) = \frac{1}{r}\chi(K) + \frac{1}{s}\chi(L) - 1$$

$$\chi(G * H) = \chi(G) + \chi(H) - 1 \quad .$$

■

Example 5.2.9 *It's not very easy to consider a BG space for $\mathbb{Z}_2$. So, by formula (3),*

Let $G = \mathbb{Z}_2$ and $K = \{e\}$ where $r = 2$.

Hence

$$\chi(\mathbb{Z}_2) = \frac{1}{2}\chi(\{e\})$$

where $\chi(\{e\}) = 1$.

Note that, let us think a point for $B\{e\}$.

Since

$$\pi_1(\{e\}) = \pi_1(B\{e\}) = \{e\}$$

and the universal cover for a single point is itself, which is contractible,

$$\chi(\{e\}) = \chi(B(\{e\})) = 1$$

So, we get

$$\chi(\mathbb{Z}_2) = \frac{1}{2}.$$

In conclusion,

$$\chi(\mathbb{Z}_m) = \frac{1}{m}.$$

Example 5.2.10 *Let $G = \mathbb{Z}_2$ and $H = \mathbb{Z}_3$.*

Since

$$\chi(\mathbb{Z}_2) = \frac{1}{2} \quad \textit{and} \quad \chi(\mathbb{Z}_3) = \frac{1}{3}.$$

Also, by the formula (2) which says

$$\chi(G * H) = \chi(G) + \chi(H) - 1.$$

We have

$$\begin{aligned} \chi(\mathbb{Z}_2 * \mathbb{Z}_3) &= \chi(\mathbb{Z}_2) + \chi(\mathbb{Z}_3) - 1 \\ &= \frac{1}{2} + \frac{1}{3} - 1 = -\frac{1}{6}. \end{aligned}$$

Example 5.2.11 *Let $G = \mathbb{Z}_2$ and $H = \mathbb{Z}_3$.*

Since

$$\chi(\mathbb{Z}_2) = \frac{1}{2} \quad \textit{and} \quad \chi(\mathbb{Z}_3) = \frac{1}{3}$$

we have

$$\chi(\mathbb{Z}_2 \times \mathbb{Z}_3) = \chi(\mathbb{Z}_2)\chi(\mathbb{Z}_3)$$

$$= \frac{1}{2}.\frac{1}{3} = \frac{1}{6}.$$

Remark 5.2.12 *According to Kurosh's subgroup theorem, a subgroup of finite index of $\mathbb{Z}_2 * \mathbb{Z}_3$ is a free product of finite p, q, r copies of $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}$.*

Proposition 5.2.13 *By* (2) *above*

$$\chi(p\mathbb{Z}_2 * q\mathbb{Z}_3 * r\mathbb{Z}) = \frac{1}{2}p + \frac{1}{3}q - p - q - r + 1.$$

Proof. Since

$$\chi(G * H) = \chi(G) + \chi(H) - 1$$

we have

$\chi(p\mathbb{Z}_2 * q\mathbb{Z}_3 * r\mathbb{Z}) = \chi(\mathbb{Z}_2) + \chi[(\mathbb{Z}_2 * ... * \mathbb{Z}_2) * (\mathbb{Z}_3 * ... * \mathbb{Z}_3) * (\mathbb{Z} * ... * \mathbb{Z})] - 1$

$= \chi(\mathbb{Z}_2) + \chi(\mathbb{Z}_2) + \chi[(\mathbb{Z}_2 * ... * \mathbb{Z}_2) * (\mathbb{Z}_3 * ... * \mathbb{Z}_3) * (\mathbb{Z} * ... * \mathbb{Z})] - 1 - 1$

$= \chi(\mathbb{Z}_2) + ... + \chi(\mathbb{Z}_2) + \chi[(\mathbb{Z}_3 * ... * \mathbb{Z}_3) * (\mathbb{Z} * ... * \mathbb{Z})] - p$

$= \chi(\mathbb{Z}_2) + ... + \chi(\mathbb{Z}_2) + \chi(\mathbb{Z}_3) + \chi[(\mathbb{Z}_3 * ... * \mathbb{Z}_3) * (\mathbb{Z} * ... * \mathbb{Z})] - p - 1$

$= \chi(\mathbb{Z}_2) + ... + \chi(\mathbb{Z}_2) + \chi(\mathbb{Z}_3) + \chi(\mathbb{Z}_3) + \chi[(\mathbb{Z}_3 * ... * \mathbb{Z}_3) * (\mathbb{Z} * ... * \mathbb{Z})] - p - 1 - 1)$

$= \chi(\mathbb{Z}_2) + ... + \chi(\mathbb{Z}_2) + \chi(\mathbb{Z}_3) + ... + \chi(\mathbb{Z}_3) + \chi[(\mathbb{Z} * ... * \mathbb{Z})] - p - q$

$= \chi(\mathbb{Z}_2) + ... + \chi(\mathbb{Z}_2) + \chi(\mathbb{Z}_3) + ... + \chi(\mathbb{Z}_3) + \chi(\mathbb{Z}) + \chi[(\mathbb{Z} * ... * \mathbb{Z})] - p - q - 1$

$= \chi(\mathbb{Z}_2) + ... + \chi(\mathbb{Z}_2) + \chi(\mathbb{Z}_3) + ... + \chi(\mathbb{Z}_3) + \chi(\mathbb{Z}) + ... + \chi(\mathbb{Z}) + \chi[(\mathbb{Z} * \mathbb{Z})] - p - q - (r - 2)$

$= \chi(\mathbb{Z}_2) + ... + \chi(\mathbb{Z}_2) + \chi(\mathbb{Z}_3) + ... + \chi(\mathbb{Z}_3) + \chi(\mathbb{Z}) + ... + \chi(\mathbb{Z}) + \chi(\mathbb{Z}) + \chi(\mathbb{Z}) - p - q - (r - 2) - 1$

Since $\chi(\mathbb{Z}_m) = \frac{1}{m}$ where $m \in \mathbb{N}$,

we have

$$\frac{1}{2}p + \frac{1}{3}q - p - q - r + 1.$$

■

Suppose that $\mathbb{Z}_2 * \mathbb{Z}_4$ has a subgroup with finite index k. Then, $\chi(p\mathbb{Z}_2 * q\mathbb{Z}_3 * r\mathbb{Z}) = \frac{-1}{4}k$ by formula (3) such that,

$$\frac{-1}{4}k = \frac{1}{2}p + \frac{1}{4}q - p - q - r + 1.$$

Hence, we get

$$2p + 3q + 4r = k + 4.$$

This equation is called a *Diophantine equation*. It helps us to find the exact structure of subgroups $\mathbb{Z}_2 * \mathbb{Z}_4$.

Let's see this fact on an example;

Consider a subgroup which has index 3, that is q and r have 1 in such a way that the subgroup is isomorphic to $\mathbb{Z}_4 * \mathbb{Z}$. In the same way, if index is taken to be 6, $\mathbb{Z}_2 * \mathbb{Z}_4$ has a subgroup isomorphic to $\mathbb{Z}_4 *$ $\mathbb{Z}_4 * \mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z} * \mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_4 * \mathbb{Z}_4$.

Chapter 6

CONCLUSION

The most obvious point to emerge from this study is that, due to C.T.C. Wall the Euler characteristic of groups can be computed easily by using the geometric way. Another significant point in this study is that, this number may also be a rational one. In this dissertation, we were not only based on the groups which have finite classifying space. We could also evaluate the Euler characteristic of groups having a finite index subgroup with finite classifying space. The result of this thesis indicates that, the definition of Euler characteristic of groups by C.T.C. Wall is much more useful and easier when we compare it with K.S.Brown's one.

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