The Block-Hexagonal Grid Method for Laplace's Equation with Singularities

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ABSTRACT

A fourth order accurate matching operator is constructed on a hexagonal grid, for the interpolation of the mixed boundary value problem of Laplace's equation, by using the harmonic properties of the solution. With the application of this matching operator for the connection of the subsystems, the Block-Grid method (BGM), which is a difference-analytical method, has been analysed on a hexagonal grid, for the solution of both the Dirichlet and mixed boundary value problems of Laplace's equation with singularities. First of all, BGM is considered on staircase polygons and it is justified that when the boundary functions outside the finite neighbourhood of the singular points are from the Hölder classes $C^{6,\lambda}$, $0 < \lambda < 1$, the error of approximation has an accuracy of $O\left(h^4\right)$, where h is the mesh size. The analysis of this method is extended to special polygons whose interior angles are $\alpha_j\pi$, $\alpha_j \in \left\{\frac{1}{3}, \frac{2}{3}, 1, 2\right\}$, and for the Dirichlet problem of Laplace's equation it is proved that, with the application of BGM, it is possible to lower the smoothness requirement on the boundary functions to $C^{4,\lambda}$, $0 < \lambda < 1$, outside the finite neighbourhood of the singular points, in order to obtain an accuracy of $O\left(h^4\right)$. For the demonstration of the theoretical results on staircase polygons, BGM has been applied on an L-shaped domain for two examples, which has a singularity at the vertex with an interior angle of $\frac{3\pi}{2}$, where Dirichlet and mixed boundary conditions are assumed respectively. The slit problem, which has the strongest singularity due to the interior angle of 2π at the vertex of the slit, has been considered on a parallelogram with a slit, in order to illustrate the results obtained on polygons with interior angles of $\alpha_j\pi$, $\alpha_j \in \left\{\frac{1}{3}, \frac{2}{3}, 1, 2\right\}$. The second example on a parallelogram demonstrates the application of BGM on a domain with two singularities as it is assumed that the vertices with interior angles of $\frac{2\pi}{3}$ are singular points. Solutions of the numerical examples are consistent with the theoretical results obtained.

Keywords: Hexagonal grids, Laplace's equation, singularity problem, block-grid method.

ÖZ

Laplace denklemi sınır problemleri için, dördüncü derece hata payı olan birleştirme (matching) operatörü petek düğümleri üzerinde kurulmuştur. Bu enterpolasyon operatörünün kurulumu için çözümün harmonik özellikleri kullanılmıştır. Alt sistemlerin birleştirilmesinde uygulanan matching operatörü ile Block-Grid metodu (BGM), petek ağlar üzerinde analiz edilmiştir. Bu metod, tekilliği olan Laplace denkleminin Dirichlet ve karışık (mixed) sınır problemlerine uygulanmıştır.

İlk önce BGM, iç açıları $\alpha_j\pi$, $\alpha_j \in \left\{\frac{1}{2}, 1, \frac{3}{2}, 2\right\}$ olan çokgenler üzerinde incelenmiştir. Tekil noktalardan belli bir uzaklıkta olan sınır üzerindeki fonksiyonlar $C^{6,\lambda}$, $0 < \lambda < 1$, Hölder gruplarından olduğu zaman yakınsaklık hatasının $O\left(h^4\right)$ olduğu kanıtlanmıştır (h ağ aralığıdır).

İlaveten, BGM'nin analizi özel çokgenler üzerine genişletilmiştir. Bu özel çokgenlerin iç açıları $\alpha_j\pi$, $\alpha_j \in \left\{\frac{1}{3}, \frac{2}{3}, 1, 2\right\}$, olarak verilmiştir. Laplace'ın Dirichlet probleminin yaklaşık çözümü için, bu çokgenler üzerinde, tekil noktalardan belli bir uzaklıkta olan sınır fonksiyonlarının $C^{4,\lambda}$, $0 < \lambda < 1$, Hölder grubundan olması ve BGM metodunun uygulanması ile hata payının yine $O\left(h^4\right)$ olduğu kanıtlanmıştır.

Teorik sonuçların nümerik çözümlemesi için BGM, iç açılarından biri $\frac{3\pi}{2}$ olan L-şekilli (L-shaped) çokgende uygulanmıştır. Açıları $\alpha_j\pi$, $\alpha_j \in \left\{\frac{1}{3}, \frac{2}{3}, 1, 2\right\}$, olan çokgenler üzerinde BGM'nin uygulanmasını göstermek üzere, iç açısı 2π olduğundan dolayı en güçlü tekilliğe sahip olan kesik problemi (slit problem), paralelkenar üzerinde çözülmüştür. Yine paralelkenar üzerinde, $\frac{2\pi}{3}$ iç açılı kenarların ikisinde de tekillik olduğu varsayılarak BGM ile Laplace sınır problemi çözümlenmiştir. Elde edilen sayısal çözümlerin teorik sonuçlarla uyumlu olduğu sergilenmiştir.

Anahtar Kelimeler: Laplace denklemi, tekil problemi, Block-Grid metodu, petek ağlar.

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INTRODUCTION

Elliptic equations are widely used in many applied sciences to represent equilibrium or steady-state problems. Among these Laplace's equation, which is one of the most encountered elliptic equations, has been used to model many real-life situations such as the steady flow of heat or electricity in homogeneous conductors, the irrotational flow of incompressible fluid, problems arising in magnetism, and so on. However, obtaining the approximate solution of elliptic equations is not straight-forward, as generally singularities are experienced in the domain of definition.

These singularities can be categorised into three different types: angular singularities, interface singularities and infinity when the domain is unbounded (see [4] and references therein). Angular singularities, in particular, arise as a result of reentrant angles in the domain, discontinuity in the boundary functions or having mixed boundary conditions. This leads to a reduction in the order of approximation if the classical finite-difference or finite-element methods are applied, as the low-order derivatives of the exact solution become unbounded at the singular points.

The angular singularity is easily demostrated in the example of Laplace's equation with Neumann-Dirichlet boundary conditions. Let $\overline{D} = D \cup \gamma$ be a closed polygonal domain, γ denotes the sides of the polygon, and consider the following boundary-value problem:

$$
\begin{aligned}
\Delta u &= 0 \text{ on } D, \\
\frac{\partial u}{\partial \upsilon} &= -\frac{1}{r}\frac{\partial u}{\partial \theta} = A \text{ when } \theta = 0, \\
u &= B \text{ when } \theta = \Theta,
\end{aligned}
$$

where $\Delta = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$, A and B are constants. The exact solution of this problem is:

i) $u = B - Ar\sin\theta + \frac{A\sin\theta}{\cos\Theta} r\cos\theta + \sum_{k=0}^{\infty} a_k r^{\alpha_k} \cos\alpha_k\theta$, where $\alpha_k = \frac{\pi}{\Theta}\left(k+\frac{1}{2}\right)$, $\Theta \neq \frac{\pi}{2}, \frac{3\pi}{2}$,

ii) $u = B - \frac{Ar}{\Theta}(\ln r\cos\theta - \theta\sin\theta) - Ar\sin\theta + \sum_{k=0}^{\infty} a_k r^{\alpha_k}\cos\alpha_k\theta$, where $\alpha_k = 2k+1$ when $\Theta = \pi/2$, $\alpha_k = (2k+1)/3$ when $\Theta = 3\pi/2$.

As can be seen from the exact solution, the strength of the singularity can be analysed by looking at the different values angle Θ takes. For instance, the solution is only analytic when $\Theta = \pi/2$ and $A = 0$. In the case when $\Theta < \pi/2$, it is easy to show that $u \in C^1$. However, when $\pi/2 < \Theta < 2\pi$, we obtain $1/4 < \alpha_1 < 1$. Since

$$\frac{\partial u}{\partial r} = O\left(r^{\alpha_1 - 1}\right),$$

the first derivative becomes unbounded as r tends to zero and $u \notin C^1$. Furthermore, when $\Theta = 2\pi$, $\alpha_1 = 1/4$ and hence

$$u = O\left(r^{1/4}\right),$$

which is the strongest singularity. Similar results are also obtained when we consider Laplace's equation with Dirichlet-Dirichlet, Dirichlet-Neumann or Neumann-Neumann boundary conditions.

E.A. Volkov justified in [2] that the smoothness requirement on the boundary functions can be lowered in order to obtain a second-order approximation using the $5-point$ scheme in square grids, on a bounded domain. It was shown that if the boundary functions belong to $C^{2,\lambda}$, $0<\lambda<1$, it is still possible to obtain the same order of accuracy everywhere in the closed domain. Furthermore, A.A. Dosiyev proved in [3] that when the $9-point$ scheme is considered in square grids, on a rectangular domain, in order to acquire an accuracy of $O\left(h^k\right)$, where h is the step size, $k=4,6$, the requirement of smoothness of the boundary functions can be reduced, and with the boundary functions belonging to the Hölder classes $C^{k,\lambda}$, $0<\lambda<1$, $k=4,6$ this order of accuracy can be obtained.

Clearly, the harmonic functions $u(x,y)=r^{1/\alpha}\cos\frac{\theta}{\alpha}$ and $v(x,y)=r^{1/\alpha}\sin\frac{\theta}{\alpha}$, when considered in a domain with an interior angle of $\alpha\pi$, $1/2<\alpha\leq 2$, do not belong to $C^{2,\lambda}$, $0<\lambda<1$. Even in the presence of singularities, E.A. Volkov has proved in [40] that it is possible to obtain an order of approximation around the singular points, depending on the interior angles of the polygon. It was justified that when the $5-point$ scheme is applied in square grids, for the numerical solution of Laplace's equation with Dirichlet boundary conditions, on a bounded domain with an interior angle of $\alpha\pi$, $1/2<\alpha\leq 2\pi$, $\alpha\neq 1$, the order of approximation obtained is $O\left(h^{1/\alpha}\right)$. Similarly, for the mixed boundary-value problem, $O\left(h^{1/2\alpha}\right)$ is obtained. Hence the approximation is considerably worse than $O\left(h^2\right)$.

Throughout the last century, many methods have been constructed for highly accurate approximations around singular points (for example [4]-[12] and references therein). These methods are generally based on four main ideas, the first of these being classified as Conformal Transformation Methods (CTM).

CTM is based on the idea that "If a domain Ω can be transformed to a simple domain Ω^* such that the Laplacian solutions are explicity obtained, then the harmonic functions on Ω can also be explicitly obtained", see [9], [13]. Hence the Schwarz-Christoffel transformation is applied to polygons with angular singularities, mapping them onto rectangular domains.

Another set of methods is based on the idea of local refinement, where the domain is separated into two as the "singular" part and the "nonsingular" part. The "nonsingular" part is approximated using the finite-difference or finite-element methods, with step size h. In order to balance the errors in the "singular" part, however, h is taken as a much smaller value, and the same method is applied as in the "nonsingular" part with the new value of h (see [10], [11], [14], [35]-[38]).

The singular functions method also provides a basis for the derivation of methods approximating around singular points. We let

$$u(r,\theta) = \sum_{i=1}^{\infty} D_i r^{\alpha_i} \sin \alpha_i \theta,$$

be the solution near the singular point O, where

$$D_i = \frac{2}{\Theta} r_0^{-\alpha_i} \int_0^{\Theta} u(r_0,\theta) \sin \alpha_i \theta d\theta$$

is the exact solution of the coefficients D_i, $i = 1,2,\ldots$, where r_0 denotes the radius of the sector separating the singular point. Hence, approximating the coefficients D_i, and applying a transformation of the form

$$w = u - \sum_{i=1}^{L} \widehat{D}_i r^{\alpha_i} \sin \alpha_i \theta,$$

where $\widehat{D}_i$ is the approximation of D_i, the singularity can be removed. Usually, the approximation of one or two coefficients is enough to remove the singularity of the series $u(r,\theta)$ (see [8], [16]).

Finally, Combined Methods are also widely applied for the approximation of elliptic equations in domains containing singular points. Similar to Local Refinement, the domain is partitioned as the "singular" part and the "nonsingular" part. However, different methods are applied in the separated parts of the domain, providing the advantage of using the most suitable method for the subdomain. Nevertheless, special care must be taken for the connection of subsystems. Some of these methods are given in [15], [20], [35], [17], [31].

It was commented in [4] by Z.C. Li that " The ideal numerical methods of the 21*st* century should be like the combined methods, where all methods can be employed together, and integrated in a very harmonious way such that to utilize fully their merits and also to avoid their shortcomings". Thus, drawing attention to the significance of exploring the combination of existing methods, in the improvement of numerical methods.

Among many combined methods, the Block-Grid Method (BGM) introduced in [15] by A.A. Dosiyev, for the solution of Laplace's boundary-value problem, is considered as one of the more highly accurate methods, not only for the approximation of the solution, but also for the approximation of its derivatives around singular points. BGM, a difference-analytical method, is the combination of two methods: the finite-difference method, which is regarded as one of the simplest methods in realization and is highly accurate, is applied in the "nonsingular" part of the domain, and the Block Method (BM), is applied in the "singular" part.

BM was first introduced by E.A. Volkov in [1], and is an extremely accurate method, which can be used for the numerical solution of Laplace's boundary-value problem. The method is based on the approximation of the integral representation of harmonic functions using rectangular quadrature nodes, inside the finite number of sectors of disks, half-disks and disks covering the domain. The approximate solution and its derivatives converge exponentially in proportion to the number of quadrature nodes, and the method can be successfully applied when the boundary functions are algebraic polynomials or analytic functions (see [18], [19]).

Therefore, the application of this method only on the "singular" parts of the domain removes the restriction on the boundary functions to be analytic or algebraic polynomials in the "nonsingular" part, making the BGM more fitting to a wider number of boundary value problems. In [20], the BGM is applied for the approximation of the mixed boundary-value problem of Laplace's equation on staircase polygons. The "singular" parts of the domain are covered by blocks and are separated from the rest of the polygon with the use of artificial boundaries, and the remaining parts of the polygon are covered by overlapping rectangles, which are approximated with the use of the $9-point$ scheme on square grids with step size h. A sixth-order interpolation operator, called the matching operator, is constructed for connecting all the subsystems, and thus it is justified that it is possible to obtain sixth order accuracy everywhere in the polygon, including the "singular" parts. Despite the high accuracy obtained by BGM, the application of the method was restricted to having square grids in the "nonsingular" part of the domain and using a staircase polygon.

Hexagonal grids are favored in many applied problems such as dynamical meteorology and oceanography (see [24]-[26]), due to its wavelike structure. Another advantage of using hexagonal grids is that eventhough the $7-point$ scheme on a hexagonal grid and the $9-point$ scheme on a rectangular grid both give fourth order accuracy when the boundary functions are from the Hölder classes $C^{6,\lambda}$, $0<\lambda<1$, the $7-point$ scheme on a hexagonal grid has the computational advantages of having easier algorithms to implement and requiring less memory space, due to having a 7-diagonal matrix rather than 9-diagonal.

However, they have not been widely applied in the approximation of the singularity problem using combined methods, as an interpolation function for connecting the subsystems, with the required order of accuracy, did not exist. Moreover, when hexagonal grids are considered on a rectangular domain, applying the $7-point$ scheme for the approximation of near-boundary nodes resulted in some nodes of evaluation emerging through the side of the domain. Thus, making the use of hexagonal grids difficult on staircase polygons. Moreover in [22], it was justified by A.A. Dosiyev and S.C. Buranay that when square grids are used in the "nonsingular" part of the staircase polygon, and the boundary functions in this part of the domain are from the Hölder classes $C^{4,\lambda}$, $0 < \lambda < 1$, the application of the BGM still gives fourth order accuracy. Hence, giving the same order of accuracy as the hexagonal grid, but with less requirement of smoothness on the boundary functions.

In this thesis, the use of hexagonal grids have been investigated for the solution of Laplace's equation with singularities, with the application of BGM, and it is justified that it is possible to approximate Laplace's equation by retaining the advantages provided by hexagonal grids. Moreover, it is justified that in certain type of polygons it is more advantageous to use the $7-point$ scheme on a hexagonal grid, rather than the $9-point$ scheme on a square grid.

In Chapter 2, we derive the hexagonal grid version of the BGM on staircase polygons. Section 2.2 is devoted to the analysis of the $7-point$ scheme on a rectangular domain, and in Section 2.3 an interpolation operator, called the matching operator, is constructed on hexagonal grids with fourth order accuracy, for the connection of the subsystems within the polygon. With the aid of this matching operator, the hexagonal grid version of BGM is applied for the Dirichlet problem of Laplace's equation. It is justified that it is possible to obtain fourth-order accuracy everywhere in the polygon, when the boundary functions in the "nonsingular" part are from $C^{6,\lambda}$, $0 < \lambda < 1$. The solution in the "singular" part of the domain is defined as a harmonic function, and the derivatives of the solution are also approximated in these parts of the domain by a simple differentiation of this function. It is proved that the errors of the derivatives of order p, $p = 1, 2, ...$, are $O\left(h^4/r_j^{p-\lambda_j}\right)$, where $\lambda_j = \frac{1}{\alpha_j}$, and $\alpha_j \pi$ is the interior angle at the vertices of the polygon, $\alpha_j = \left\{\frac{1}{2}, 1, \frac{3}{2}, 2\right\}$.

In Chapter 3, the hexagonal grid version of BGM is applied for the numerical solution of Laplace's equation with mixed boundary conditions, again on a staircase polygon. For the approximation in the rectangles covering the "nonsingular" part of the domain, interpolation formulae are constructed for near-boundary nodes and nodes lying on the boundary of the sides with Neumann conditions, by using the harmonic properties of the solution. Furthermore, the construction of the matching operator is extended for the interpolation of the points near sides with Neumann boundary conditions. Again it is justified that when the boundary functions in the "nonsingular" part are from $C^{6,\lambda}$, $0 < \lambda < 1$, fourth-order accuracy is obtained everywhere in the polygon.

In Chapter 4, it is proved that the hexagonal grid version of BGM can be extended to the approximation of Laplace's equation with Dirichlet boundary conditions on polygons with interior angles of $\alpha_j \pi$, $\alpha_j \in \left\{\frac{1}{3}, \frac{2}{3}, 1, 2\right\}$. Moreover, it is justified that in order to obtain fourth-order accuracy everywhere in this domain, the requirement for the smoothness of the boundary functions can be lowered so that when the boundary functions outside the "singular" parts of the domain are from the Hölder classes $C^{4,\lambda}$, $0 < \lambda < 1$, an accuracy of $O\left(h^4\right)$ is obtained, where h is the step size.

Chapter 5 demonstrates the numerical realization of the theoretical results obtained in Chapters 2, 3 and 4.

The results of this thesis are presented in [31] and [41]-[44].

Chapter 2

HEXAGONAL GRID VERSION OF THE BLOCK-GRID METHOD FOR THE DIRICHLET PROBLEM ON STAIRCASE POLYGONS

2.1 Description of the Block-Grid Method (BGM)

We define by G a simply connected polygon and denote the sides of this polygon by γ_j, $j = 1,2,...,N$, $(\gamma_0 = \gamma_N)$, numbered in the positive (counterclockwise) direction, with $\gamma = \cup_{j=1}^{N} \gamma_j$, and the vertices of this polygon are represented by $\dot{\gamma}_j = \gamma_{j-1} \cap \gamma_j$. These vertices have an interior angle of $\alpha_j \pi$, where $\alpha_j \in \left\{\frac{1}{2}, 1, \frac{3}{2}, 2\right\}$, i.e. G is a staircase polygon. Moreover, s is used to define the arclength measured along the sides of this polygon in the positive direction, where s_j is the value of s at $\dot{\gamma}_j$, and r_j, θ_j represent the polar system of coordinates, measured in the positive direction from γ_j, with pole at $\dot{\gamma}_j$.

We consider the boundary value problem

$$\Delta u = 0 \text{ on } G, \tag{2.1.1}$$

$$u = \varphi_j \text{ on } \gamma_j, \; j = 1,2,...,N, \tag{2.1.2}$$

where φ_j are given functions, and

$$\varphi_j \in C^{6,\lambda}(\gamma_j), \; 0 < \lambda < 1, \; 1 \le j \le N. \tag{2.1.3}$$

In addition, when the interior angle at the vertex $\dot{\gamma}_j$ is $\pi/2$, the following conjugation conditions are assumed to be satisfied:

$$\varphi_j^{(2q)}(s_j) = (-1)^q \varphi_{j-1}^{(2q)}(s_j),\ q = 0,1,2,3. \tag{2.1.4}$$

At the vertices $\dot{\gamma}_j$, for $\alpha_j \neq \frac{1}{2}$, conditions (2.1.4) are not required to be satisfied, more precisely, the values of φ_{j-1} and φ_j at these vertices might not be the same. However, the condition imposed on the boundary functions on γ_{j-1} and γ_j, when $\alpha_j \neq 1/2$, is that the boundary functions should be given as algebraic polynomials of arclength s measured along γ represented as

$$\sum_{k=0}^{\tau_{j-1}} a_{jk} r_j^k \text{ and } \sum_{k=0}^{\tau_j} b_{jk} r_j^k, \tag{2.1.5}$$

respectively, where a_{jk} and b_{jk} are numerical coefficients, and τ_{j-1} and τ_j are the degrees of these polynomials.

Let $E = \{j : \alpha_j \neq 1/2,\ j = 1,2,...,N\}$ denote the set of vertices of G, called the "singular" vertices. We construct two fixed block sectors in the neighborhood of $\dot{\gamma}_j$, $j \in E$, denoted by $T_j^i = T_j(r_{ji}) \subset G$, $i = 1,2$, where $0 < r_{j2} < r_{j1} <\min\{s_{j+1} - s_j, s_j - s_{j-1}\}$, and $T_j(r) = \{(r_j, \theta_j) : 0 < r_j < r, 0 < \theta_j < \alpha_j \pi\}$. The function $Q_j(r_j, \theta_j)$ is constructed on the closed sector $\overline{T}_j^1, j \in E$. It is required that:

i) $Q_j(r_j, \theta_j)$ is harmonic and bounded on the open sector T_j^1,

ii) continuous everywhere on $\overline{T}_j^1$ apart from the point $\dot{\gamma}_j$, $j \in E$, when $\varphi_{j-1} \neq \varphi_j$, *iii*) continuously differentiable on $\overline{T}_j^1 \backslash \dot{\gamma}_j$,

iv) satisfies the given boundary conditions on $\gamma_{j-1} \cap \overline{T}_j^1$ and $\gamma_j \cap \overline{T}_j^1$, $j \in E$.

The function $Q_j(r_j, \theta_j)$ with the properties $i) - iv)$ is given in [1] in the form

$$Q_j(r_j,\theta_j) = b_{j0} + \frac{a_{j0} - b_{j0}}{\alpha_j \pi}\theta_j + \sum_{k=0}^{\tau_{j-1}} a_{jk} r_j^k \zeta_{jk}\left(r_j, \theta_j\right) + \sum_{k=0}^{\tau_j} b_{jk} r_j^k \zeta_{jk}\left(r_j, \alpha_j \pi - \theta_j\right), \tag{2.1.6}$$

where

$$\zeta_{jk}\left(r_j, \theta_j\right) = \begin{cases} r_j^k \frac{\theta_j \cos k\theta_j + \ln r_j \sin k\theta_j}{\alpha_j \pi \cos k\alpha_j \pi}, & \sin k\alpha_j \pi = 0, \\ r_j^k \frac{\sin k\theta_j}{\sin k\alpha_j \pi}, & \sin k\alpha_j \pi \neq 0. \end{cases} \tag{2.1.7}$$

Let

$$R_j(r_j, \theta_j, \eta) = \frac{1}{\alpha_j} \sum_{k=0}^{1} (-1)^k R\left(\left(\frac{r}{r_{j2}}\right)^{1/\alpha_j}, \frac{\theta}{\alpha_j}, (-1)^k \frac{\eta}{\alpha_j}\right), j \in E, \tag{2.1.8}$$

where

$$R(r, \theta, \eta) = \frac{1 - r^2}{2\pi(1 - 2r\cos(\theta - \eta) + r^2)} \tag{2.1.9}$$

is the kernel of the Poisson integral for a unit circle. It can be easily verified that

$$R_j(r_j, \theta_j, \eta) > 0,\ 0 < \theta,\ \eta < \alpha_j \pi,\ j \in E. \tag{2.1.10}$$

Discretization of the integral representation given in the following Lemma, using rectangular quadrature nodes, is used for the approximation of problem (2.1.1), (2.1.2) around the "singular" vertices $\dot{\gamma}_j$, $j \in E$.

Lemma 2.1.1 *The solution u of problem (2.1.1), (2.1.2) can be represented on $\overline{T}_j^2 \backslash V_j$, $j \in E$, in the form*

$$u(r_j,\theta_j) = Q_j(r_j,\theta_j) + \int_0^{\alpha_j\pi} (u(r_{j2},\eta) - Q_j(r_{j2},\eta))R_j(r_j,\theta_j,\eta)d\eta, \qquad (2.1.11)$$

where V_j is the curvilinear part of the boundary of sector T_j^2.

Proof. The proof follows from Theorems 3.1 and 5.1 in [1]. ■

We define the approximate solution in the polygon G by applying a version of the BGM introduced in [15] (see also [20]).

In order to apply the BGM, two more sectors, T_j^3 and T_j^4, are added to the sectors T_j^1, T_j^2, with $0 < r_{j4} < r_{j3} < r_{j2}$, $r_{j3} = (r_{j2} + r_{j4})/2$ and $T_k^3 \cap T_l^3 = \emptyset$, $k \neq l$, where $k,l \in E$. Also, we define $G_T = G \backslash (\cup_{j \in E} T_j^4)$. Below we give an explanation of how the method is applied on the polygon G.

i) Double sectors $T_j^i = T_j(r_{ji})$, $i = 2,3$, are used to block the vertices $\dot{\gamma}_j$, $j \in E$. Overlapping rectangles $\Pi_k, k = 1,2,...,M$, cover the rest of the polygon such that the distance from Π_k to a singular point $\dot{\gamma}_j$ is greater than r_{j4} for all $k = 1,2,...,M$, and $\cup_{k=1}^M \Pi_k$ is called the "nonsingular" part of the domain. $G \backslash \cup_{k=1}^M \Pi_k$ is called the "singular" part of the domain and sectors $T_j^3, j \in E$, cover the "singular" parts, $j \in E$.

ii) On each rectangle Π_k, the seven point difference scheme for the approximation of Laplace's equation on a hexagonal grid is used, with step size $h_k \leq h$, h a parameter, and for the approximate solution on $\overline{T}_j^3$, $j \in E$, a quadrature formula of the harmonic function (2.1.11) is used.

iii) The subsystems are connected by the matching operator S^4 formed in Section 2.3

iv) Schwarz's alternating procedure is used for solving the finite difference system formed for Laplace's equation on the rectangles covering D_T

The application of this method is demonstrated in Figure 2.1, on a staircase polygon with one singular vertex, where the "nonsingular" part of the domain is covered by four overlapping rectangles.

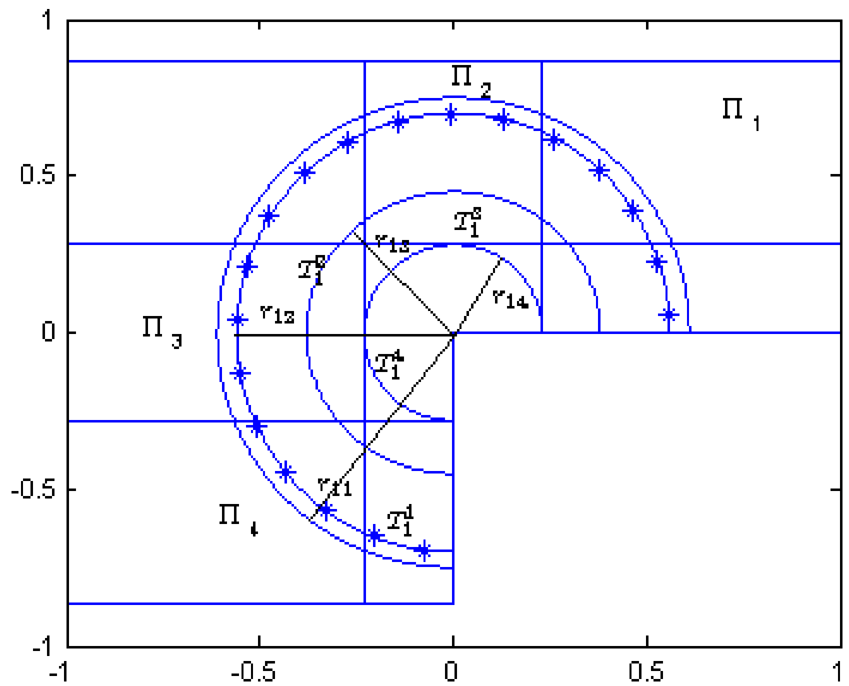

Figure 2.1. Application of the Block-Hexagonal Grid Method on a staircase polygon

In order to approximate problem (2.1.1), (2.1.2), the following steps are taken: We denote by $\Pi_k \subset G_T$, $k = 1, 2, ..., M$, fixed open rectangles, whose sides a_{1k} and a_{2k}

are parallel to to the sides of G, and $G \subset \left(\cup_{k=1}^{M} \Pi_k\right) \cup \left(\cup_{j\in E} T_j^3\right) \subset G$. The sides of Π_k are denoted by η_k, V_j is the curvilinear part of the boundary of the sector T_j^2 and $t_j = \left(\cup_{k=1}^{M} \eta_k\right) \cap \overline{T}_j^3$.

For the arrangement of the rectangles Π_k, $k = 1, 2, ..., M$, it is required that any point P lying on $\eta_k \cap G_T$, $1 \le k \le M$, or located on $V_j \cap G$, $j \in E$, lies inside at least one of the rectangles, i.e. $\Pi_{k(P),} 1 \le k(P) \le M$, and that the distance from P to $G_T \cap \eta_{k(P)}$ is not less than some constant $\varkappa_0$ independent of P. The quantity $\varkappa_0$ is called the gluing depth of the rectangles $\Pi_k, k = 1, 2, ..., M$.

We introduce the parameter $h \in (0, \varkappa_0/4]$ and consider a hexagonal grid on $\Pi_k, k = 1, 2, ..., M$, with maximal possible step $h_k \le \min\{h, \min\{a_{1k}, a_{2k}\}/4\}$. Let Π_k^h be the set of nodes on Π_k, η_k^h be the set of nodes on η_k, and let $\overline{\Pi}_k^h = \Pi_k^h \cap \eta_k^h$. We denote the set of nodes on the closure of $\eta_k \cap G_T$ by η_{k0}^h, and the set of nodes on Π_k^h whose distance from the boundary $\eta_k \cap G_T$ of Π_k is $\frac{h}{2}$ by η_{k0}^{*h}. We also have Π_k^{*h} denoting the set of nodes whose distance from the boundary η_{k1} of Π_k is $\frac{h}{2}$ and $\Pi_{k,}^{0h} = \Pi_k^h \setminus \left(\Pi_k^{*h} \cup \eta_{k0}^{*h}\right)$. Let t_j^h be the set of nodes on t_j, and let η_{k1}^h be the set of remaining nodes on η_k. We also specify a natural number $n \ge \left[\ln^{1+\varkappa} h^{-1}\right] + 1$, where $\varkappa > 0$ is a fixed number and the quantities $n(j) = \max\left\{4, \left[\alpha_j n\right]\right\}, \beta_j = \alpha_j \pi / n(j)$ and $\theta_j^m = (m - 1/2)\beta_j$, $j \in E$, $1 \le m \le n(j)$. On the arc V_j we choose the points $\left(r_{j2}, \theta_j^m\right)$ $, 1 \le m \le n(j)$ and denote the set of these points by V_j^n. Finally, let

$$\omega^{h,n} = \left(\cup_{k=1}^{M} \eta_{k0}^h\right) \cup \left(\cup_{k=1}^{M} \eta_{k0}^{*h}\right) \cup \left(\cup_{j\in E} V_j^n\right), \ \overline{G}_*^{h,n} = \omega^{h,n} \cup \left(\cup_{k=1}^{M} \overline{\Pi}_k^h\right).$$

Consider the system of equations

$$
\begin{aligned}
u_h &= Su_h \text{ on } \Pi_k^{0h}, && (2.1.12)\\
u_h &= S_m^* u_h + E_{mh}^*(\varphi_m) \text{ on } \Pi_k^{*h},\ \eta_{k1}^h \cap \gamma_m \neq \oslash, && (2.1.13)\\
u_h &= \varphi_m \text{ on } \eta_{k1}^h \cap \gamma_m, && (2.1.14)\\
u_h(r_j,\theta_j) &= Q_j(r_j,\theta_j)+\\
&\quad +\beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q)\left(u_h(r_{j2},\theta_j^q)-Q_j(r_{j2},\theta_j^q)\right) \text{ on } t_j^h, && (2.1.15)\\
u_h &= S^4(u_h,\varphi) \text{ on } \omega^{h,n}, && (2.1.16)
\end{aligned}
$$

where $1 \leq k \leq M$, $1 \leq m \leq N$, $j \in E$, $\varphi = \{\varphi_j\}_{j=1}^N$ and

$$
\begin{aligned}
Su(x,y) &= \frac{1}{6}\left(u(x+h,y)+u\left(x+\frac{h}{2},y+\frac{\sqrt{3}}{2}h\right)+u\left(x-\frac{h}{2},y+\frac{\sqrt{3}}{2}h\right)\right.\\
&\quad +u(x-h,y)+u\left(x-\frac{h}{2},y-\frac{\sqrt{3}}{2}h\right)+\\
&\quad \left.+u\left(x+\frac{h}{2},y-\frac{\sqrt{3}}{2}h\right)\right) && (2.1.17)
\end{aligned}
$$

$$
\begin{aligned}
S_j^* u(x,y) &= \frac{1}{7}\left(u\left(x+\frac{h}{2},y-\frac{\sqrt{3}h}{2}\right)+u(x+h,y)+\right.\\
&\quad u\left(x+\frac{h}{2},y+\frac{\sqrt{3}h}{2}\right), && (2.1.18)
\end{aligned}
$$

$$E^*_{jh}(\varphi_j) = \frac{1}{21}\left(2\varphi_j\left(y+\frac{\sqrt{3}h}{2}\right)+8\varphi_j(y)+2\varphi_j\left(y-\frac{\sqrt{3}h}{2}\right)\right). \qquad (2.1.19)$$

The operator (2.1.18) and the corresponding right-hand side (2.1.19) are constructed in the right coordinate system with the axis x_j directed along γ_{j+1} and the axis y_j directed along γ_j.

The solution of the system of equations (2.1.12)-(2.1.16) is an approximation of problem (2.1.1), (2.1.2) on $\overline{G}^{h,n}_*$.

Theorem 2.1.2 *There is a natural number n_0 such that for all $n \geq n_0$ and $h \in (0, \frac{\varkappa_0}{4}]$, where $\varkappa_0$ is the gluing depth, the system of equations $(2.1.12)-(2.1.16)$ has a unique solution.*

Proof. Let v_h be a solution of the system of equations

$$
\begin{aligned}
u_h &= Su_h \text{ on } \Pi^{0h}_k, &(2.1.20)\\
u_h &= S^*_m u_h \text{ on } \Pi^{*h}_k,\ \eta^h_{k1}\cap\gamma_m \neq \oslash, &(2.1.21)\\
u_h &= 0 \text{ on } \eta^h_{k1}\cap\gamma_m, &(2.1.22)\\
u_h(r_j,\theta_j) &= \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta^q_j)u_h(r_{j2},\theta^q_j) \text{ on } t^h_j, &(2.1.23)\\
u_h &= S^4 u_h \text{ on } \omega^{h,n}, &(2.1.24)
\end{aligned}
$$

where $1 \leq k \leq M,\ 1 \leq m \leq N,\ j \in E$. To prove the given theorem, it is necessary and sufficient to show that $\max_{\overline{G}^{h,n}_*}|v_h| = 0$. Since the operators S, S^*_j and S^4 have non-negative coefficients and their sum is less than or equal to one, by the maximum principle (see Chapter 4 in [21]) follows that the nonzero maximum value of the function v_h can be at the points on $\cup_{j\in E} t_j^h$. From the estimation (2.29) in [33] follows the existence of the positive constants n_0 and $\sigma > 0$ such that for $n \geq n_0$,

$$\max_{(r_j,\theta_j)\in\overline{T}_j^3} \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q) \leq \sigma < 1. \quad (2.1.25)$$

Taking (2.1.25) into account in (2.1.23) follows that the nonzero maximum value can not be at the points on $\cup_{j\in E} t_j^h$ either. Since the set $\overline{G}_*^{h,n}$ is connected, from (2.1.22) follows that $\max_{\overline{G}_*^{h,n}} |v_h| = 0$. ■

Let u_h be the solution of the system of equations (2.1.12)-(2.1.16). The function

$$U_h(r_j,\theta_j) = Q_j(r_j,\theta_j) + \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q)\left(u_h(r_{j2},\theta_j^q) - Q_j(r_{j2},\theta_j^q)\right) \quad (2.1.26)$$

is the discretization of the integral representation (2.1.11) with the use of the composite mid-point rule. The solution u of problem (2.1.1), (2.1.2), in the "singular" parts of the polygon G, is approximated with the use of the function $U_h(r_j,\theta_j)$ on the closed blocks $\overline{T}_j^3, j \in E$.

2.2 Approximation on a rectangular domain using the seven-point scheme in a hexagonal grid

Let $\Pi = \{(x,y) : 0 < x < a, 0 < y < b\}$ be an open rectangle, $\gamma_j, j = 1,2,3,4$, be its sides, including the ends, numbered in the positive direction starting from the left-hand side, $(\gamma_0 \equiv \gamma_4, \gamma_1 \equiv \gamma_5)$, $\gamma = \cup_{j=1}^4 \gamma_j$ stands for the boundary of Π and $\dot{\gamma}_j = \gamma_{j-1} \cap \gamma_j$

is the jth vertex. We consider the boundary value problem

$$\Delta u = 0 \text{ on } \Pi, \tag{2.2.1}$$

$$u = \varphi_j \text{ on } \gamma_j, j = 1,2,3,4, \tag{2.2.2}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, φ_j is a given function of arclength s taken along γ, and

$$\varphi_j \in C^{6,\lambda}(\gamma_j),\ 0 < \lambda < 1,\ j = 1,2,3,4. \tag{2.2.3}$$

At the vertices $s = s_j$, the conjugation conditions

$$\varphi_j^{(2q)}(s_j) = (-1)^q \varphi_{j-1}^{(2q)}(s_j),\ q = 0,1,2,3, \tag{2.2.4}$$

are satisfied.

Let $h > 0$, with $a/h \geq 2$, $b/\sqrt{3}h \geq 2$ integers. We assign Π^h a hexagonal grid on Π, with step size h, defined as the set of nodes

$$\Pi^h = \left\{ (x,y) \in \Pi : x = \frac{k-l}{2}h,\ y = \frac{\sqrt{3}(k+l)}{2}h,\ k = 1,2,...;l = 0,\pm1,\pm2,... \right\}. \tag{2.2.5}$$

Let γ_j^h stand for the set of nodes lying on γ_j and let $\dot{\gamma}_j^h = \gamma_j \cap \gamma_{j+1}$, $\gamma^h = \cup(\gamma_j^h \cup \dot{\gamma}_j^h)$, $\overline{\Pi}^h = \Pi^h \cup \gamma^h$. Also let Π^{*h} denote the set of nodes whose distance from the boundary γ of $\overline{\Pi}$ is $\frac{h}{2}$ and $\Pi^{0h} = \Pi^h \backslash \Pi^{*h}$.

We consider the system of finite difference equations

$$
\begin{aligned}
u_h &= Su_h \text{ on } \Pi^{0h}, && (2.2.6)\\
u_h &= S_j^* u_h + E_{jh}^*(\varphi_j) \text{ on } \Pi^{*h}, && (2.2.7)\\
u_h &= \varphi_j \text{ on } \gamma_j^h,\ j=1,2,3,4, && (2.2.8)
\end{aligned}
$$

where

$$
\begin{aligned}
Su(x,y) = \frac{1}{6}\Bigg(& u(x+h,y) + u\left(x+\frac{h}{2}, y+\frac{\sqrt{3}}{2}h\right) + u\left(x-\frac{h}{2}, y+\frac{\sqrt{3}}{2}h\right) \\
& + u(x-h,y) + u\left(x-\frac{h}{2}, y-\frac{\sqrt{3}}{2}h\right) + u\left(x+\frac{h}{2}, y-\frac{\sqrt{3}}{2}h\right)\Bigg) \qquad (2.2.9)
\end{aligned}
$$

$$
\begin{aligned}
S_j^* u(x,y) = \frac{1}{7}\Bigg(& u\left(x+\frac{h}{2}, y-\frac{\sqrt{3}h}{2}\right) + u(x+h,y) + \\
& u\left(x+\frac{h}{2}, y+\frac{\sqrt{3}h}{2}\right)\Bigg), \qquad (2.2.10)
\end{aligned}
$$

$$
E_{jh}^*(\varphi_j) = \frac{1}{21}\left(2\varphi_j\left(y+\frac{\sqrt{3}h}{2}\right) + 8\varphi_j(y) + 2\varphi_j\left(y-\frac{\sqrt{3}h}{2}\right)\right). \qquad (2.2.11)
$$

From formulae (2.2.9) and (2.2.10) follows that the coefficients of the operators $Su(x,y)$ and $S_j^* u(x,y)$ are non-negative, and their sums do not exceed one. Hence, on the basis of maximum principle the solution of system (2.2.6)-(2.2.8) exists and is unique (see [21]).

We use $c, c_0, c_1, \ldots$, to stand for constants in the expressions below, which are independent of h.

Lemma 2.2.1 *Let*

$$\begin{aligned} v_1 &= Sv_1 + f_h \text{ on } \Pi^{0h}, \\ v_1 &= S_j^* v_1 \text{ on } \Pi^{*h}, \\ v_1 &= 0 \text{ on } \gamma_h, \end{aligned}$$

and

$$\begin{aligned} v_2 &= Sv_2 + \overline{f}_h \text{ on } \Pi^{0h}, \\ v_2 &= S_j^* v_2 + \overline{f}_h^* \text{ on } \Pi^{*h}, \\ v_2 &= \overline{\eta}_h \text{ on } \gamma_h, \end{aligned}$$

where $f_h, \overline{f}_h, \overline{f}_h^*$ and $\bar{\eta}_h$ are arbitrary grid functions. Assume the following inequalities hold:

$$\overline{f}_h^* \geq 0, \ |f_h| \leq \bar{f}_h \text{ and } \bar{\eta}_h \geq 0.$$

Then

$$|v_1| \leq v_2.$$

Proof. The proof of this lemma follows by analogy to the proof of the comparison theorem (see Chapter 4 in [21]). ■

Theorem 2.2.2 *Let u be the solution of problem* (2.2.1), (2.2.2) *and* u_h *be the solution of system* (2.2.6) – (2.2.8). *Then*

$$\max_{\overline{\Pi}^h} | u_h - u | \leq ch^4. \tag{2.2.12}$$

Proof. *Let*

$$\varepsilon_h = u_h - u, \tag{2.2.13}$$

where u is the trace of the solution of problem (2.2.1), (2.2.2) *on* $\overline{\Pi}^h$, *and* u_h *is the solution of system* (2.2.6) – (2.2.8). *Then, the error function* ε_h *satisfies the following system:*

$$\varepsilon_h = S\varepsilon_h + \Psi_h \text{ on } \Pi^{0h}, \tag{2.2.14}$$

$$\varepsilon_h = S_j^*\varepsilon_h + \Psi_h^* \text{ on } \Pi^{*h}, \tag{2.2.15}$$

$$\varepsilon_h = 0 \text{ on } \gamma^h, \tag{2.2.16}$$

where

$$\Psi_h = Su - u, \tag{2.2.17}$$

$$\Psi_h^* = S_j^*u - u + E_{jh}^*(\varphi_j) \tag{2.2.18}$$

are the truncation errors of equations (2.2.6) *and* (2.2.7), *respectively.*

On the basis of conditions (2.2.3) *and* (2.2.4), *and from Theorem 3.1 in [27] follows* *that* $u \in C^{6,\lambda}(\overline{\Pi})$, $0 < \lambda < 1$. *Then, by Taylor's formula, we obtain (see [28])*

$$\max_{(x,y)\in\,\overline{\Pi}} |\Psi_h(x,y)| \leq c_1 h^4 M_6, \tag{2.2.19}$$

where

$$M_q = \sup_{(x,y)\in\overline{\Pi}} \left\{ \left| \frac{\partial^q u(x,y)}{\partial x^p \partial y^{q-p}} \right|, p = 0,1,...,q \right\}. \tag{2.2.20}$$

We represent the solution of (2.2.14)-(2.2.16) as

$$\varepsilon_h = \varepsilon_h^1 + \varepsilon_h^2, \tag{2.2.21}$$

where

$$\varepsilon_h^1 = S\varepsilon_h^1 + \Psi_h \text{ on } \Pi^{0h}, \tag{2.2.22}$$

$$\varepsilon_h^1 = S_j^* \varepsilon_h^1 \text{ on } \Pi^{*h}, \tag{2.2.23}$$

$$\varepsilon_h^1 = 0 \text{ on } \gamma^h, \tag{2.2.24}$$

and

$$\varepsilon_h^2 = S\varepsilon_h^2 \text{ on } \Pi^{0h}, \tag{2.2.25}$$

$$\varepsilon_h^2 = S_j^* \varepsilon_h^2 + \Psi_h^* \text{ on } \Pi^{*h}, \tag{2.2.26}$$

$$\varepsilon_h^2 = 0 \text{ on } \gamma^h. \tag{2.2.27}$$

To estimate ε_h^1 *we use Gerschgorin's Majorant method (see [29], Chapter 5) by taking* *the function*

$$Y(x,y) = h^4 c_1 M_6 (a^2 + b^2 - x^2 - y^2). \tag{2.2.28}$$

For $Y(x,y)$, we have

$$Y = SY + h^6 c_1 M_6 \text{ on } \Pi^{0h}, \tag{2.2.29}$$

$$Y = S_j^* Y + \mu_h \text{ on } \Pi^{*h}, \tag{2.2.30}$$

$$Y = h^4 c_1 M_6 (a^2 + b^2 - x^2 - y^2) \text{ on } \gamma^h, \tag{2.2.31}$$

where $\mu_h = \frac{c_1 M_6 h^4}{7}(4a^2 + 4b^2 + 3h^2 + 4hx - 4x^2 - 4y^2) \geq 0$. On the basis of (2.2.22)-(2.2.24), (2.2.29)-(2.2.31) and Lemma 2.2.1, we obtain

$$\left|\varepsilon_h^1\right| \leq Y_h. \tag{2.2.32}$$

Hence,

$$\max_{(x,y)\in\Pi^h} \left|\varepsilon_h^1\right| \leq \max_{(x,y)\in\overline{\Pi}} |Y| \leq c_2 h^4 M_6. \tag{2.2.33}$$

Now the estimation of equations (2.2.25)-(2.2.27) is considered. By Taylor's formula about each of the points $(\frac{h}{2}, y) \in \Pi^{*h}$ and from (2.2.18), we have

$$\max_{(x,y)\in\Pi^{*h}} |\Psi^*| \leq c_3 M_4 h^4. \tag{2.2.34}$$

On the basis of maximum principle, we obtain

$$\max_{(x,y)\in\Pi^h} \left|\varepsilon_h^2\right| \leq \frac{7}{4} \max_{(x,y)\in\Pi^{*h}} |\Psi_h^*| \leq c_4 M_4 h^4. \tag{2.2.35}$$

From (2.2.16), (2.2.33) and (2.2.35) it follows that

$$\max_{(x,y)\in\Pi^h} |\varepsilon_h| \le ch^4. \tag{2.2.36}$$

■

2.3 Construction of the fourth order matching operator in a hexagonal grid

Let $z = x + iy$ be a complex variable and let $\Omega = \{z : |z| < 1\}$ be a unit circle. Using Taylor's formula, any harmonic function u on Ω with $u \in C^{4,0}(\overline{\Omega})$ can be expressed in the form:

$$u(x,y) = \sum_{k=0}^{3} a_k \mathrm{Re}\ z^k + \sum_{k=1}^{3} b_k \mathrm{Im}\ z^k + O(r^4), \tag{2.3.1}$$

where $(x,y) \in \Omega$ and $r = \sqrt{x^2+y^2}$,

$$a_0 = u(0,0), a_1 = \frac{\partial u(0,0)}{\partial x}, a_2 = \frac{1}{2}\frac{\partial^2 u(0,0)}{\partial x^2}, a_3 = \frac{1}{3!}\frac{\partial^3 u(0,0)}{\partial x^3}, \tag{2.3.2}$$

$$b_1 = \frac{\partial u(0,0)}{\partial y}, b_2 = \frac{1}{2}\frac{\partial^2 u(0,0)}{\partial x \partial y}, b_3 = \frac{1}{3!}\frac{\partial^3 u(0,0)}{\partial x^2 \partial y}. \tag{2.3.3}$$

In accordance with the solutions obtained in [15], the fourth order matching operator is constructed in a hexagonal grid, by assuming that the expression:

$$S^4 u = \sum \xi_k u_k, \tag{2.3.4}$$

where $u_k = u(P_k)$, P_k is a node of the hexagonal grid Π^h, gives the exact value of any harmonic polynomial of the form

$$F_3(x,y) = \sum_{k=0}^{3} a_k \operatorname{Re} z^k + \sum_{k=1}^{3} b_k \operatorname{Im} z^k,$$

at each point $P \in \Pi$, and

$$\xi_k \geq 0, \ \sum \xi_k \leq 1. \tag{2.3.5}$$

We use Π_0 to denote the set of points $P \in \Pi$ such that all the nodes P_k to evaluate $S^4 u$ by using the expression (2.3.4) lie in $\overline{\Pi}^h$, and Π_{01} contains the points P, where some of the nodes P_k emerge through the side γ_j, $j = 1,2,3,4$. Furthermore, "grid line" is used to mean the line connecting two neighbouring grid nodes.

Position 1. The point $P \in \Pi_0$ lies on a grid line. We place the origin of the rectangular system of coordinates on the node P_0 and direct the positive axis of x along the grid line, so that $P = P(\delta h, 0)$, $0 < \delta \leq 1/2$, and take the nodes (see Figure 2.2):

$$P_0(0,0),\ P_1(h,0),\ P_2(\frac{h}{2},\frac{\sqrt{3}h}{2}),\ P_3(-\frac{h}{2},\frac{\sqrt{3}h}{2}),$$
$$P_4(\frac{h}{2},-\frac{\sqrt{3}h}{2}),\ P_5\left(-\frac{h}{2},-\frac{\sqrt{3}h}{2}\right).$$

First, the coefficients λ_j', $j = 0,1,2,3$, satisfying the equation

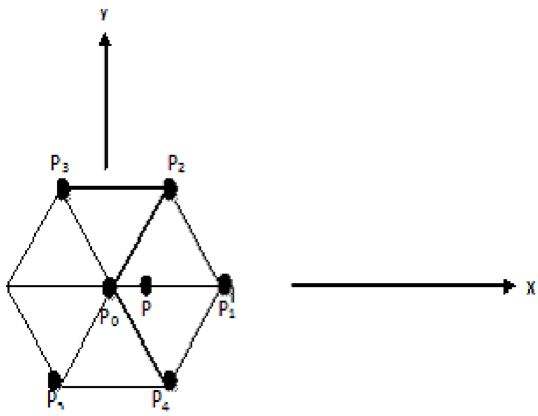

Figure 2.2. Nodes used on the hexagon

$$u_0 = \lambda_0' u + \lambda_1' u_1 + \lambda_2' u_2 + \lambda_3' u_3 \tag{2.3.6}$$

are obtained for the harmonic polynomials $\mathrm{Re}\, z^n$, $n = 0, 1, 2, 3$, where $u = u(P)$, $u_k = u(P_k)$, $k = 0, 1, 2, 3$, $z = x + iy$. Hence we attain the system

$$\begin{aligned}
\lambda_0' + \lambda_1' + \lambda_2' + \lambda_3' &= 1, \\
\delta\lambda_0' + \lambda_1' + \frac{1}{2}\lambda_2' - \frac{1}{2}\lambda_3' &= 0, \\
\delta^2\lambda_0' + \lambda_1' - \frac{1}{2}\lambda_2' - \frac{1}{2}\lambda_3' &= 0, \\
\delta^3\lambda_0' + \lambda_1' - \lambda_2' + \lambda_3' &= 0.
\end{aligned} \tag{2.3.7}$$

Solving system (2.3.7) we get

$$
\begin{aligned}
\lambda_0' &= \frac{-\mu_0}{-1+\delta}, \\
\lambda_1' &= \frac{\left(2\delta+\delta^3\right)\mu_0}{3(-1+\delta)}, \\
\lambda_2' &= -\delta\mu_0, \\
\lambda_3' &= \frac{1}{3}\left(-\delta+2\delta^2\right)\mu_0,
\end{aligned}
$$

where $\mu_0 = 1/(1-\delta+\delta^2)$. We rearrange (2.3.6) for u, thus obtaining

$$
u = \frac{u_0}{\lambda_0'} - \frac{\lambda_1'}{\lambda_0'}u_1 - \frac{\lambda_2'}{\lambda_0'}u_2 - \frac{\lambda_3'}{\lambda_0'}u_3. \tag{2.3.8}
$$

Next we consider the nodes $P_4(\frac{h}{2}, -\frac{\sqrt{3}h}{2})$ and $P_5\left(-\frac{h}{2}, -\frac{\sqrt{3}h}{2}\right)$ which are symmetric to the points P_2 and P_3, respectively, with respect to the x-axis. Since $\operatorname{Im} z^k = 0, k = 1,2,3$ for $y = 0$, and odd with respect to y, and $\operatorname{Re} z^k$, $k = 0,1,2,3$, is even with respect to y, from (2.3.8) we have

$$
u = \frac{u_0}{\lambda_0'} - \frac{\lambda_1'}{\lambda_0'}u_1 - \frac{\lambda_2'}{2\lambda_0'}u_2 - \frac{\lambda_3'}{2\lambda_0'}u_3 - \frac{\lambda_2'}{2\lambda_0'}u_4 - \frac{\lambda_3'}{2\lambda_0'}u_5 \tag{2.3.9}
$$

Hence the fourth order matching operator S^4 can be expressed as:

$$
S^4 u = \sum_{k=0}^{5} \lambda_k u_k, \tag{2.3.10}
$$

which gives the exact value of the harmonic polynomial $F_3(x,y)$ at the point P, with the coefficients

$$\lambda_0 = -(-1+\delta)\left(1-\delta+\delta^2\right), \tag{2.3.11}$$

$$\lambda_1 = \frac{2\delta+\delta^3}{3}, \tag{2.3.12}$$

$$\lambda_2 = \lambda_4 = \frac{-(-1+\delta)\,\delta}{2}, \tag{2.3.13}$$

$$\lambda_3 = \lambda_5 = \frac{(-1+\delta)\left(-\delta+2\delta^2\right)}{6}. \tag{2.3.14}$$

It can be easily verified that

$$\lambda_0 > 0, \lambda_j \geq 0,\ j = 1,2,3,\ \text{for } 0 < \delta \leq 1/2, \tag{2.3.15}$$

and

$$\sum_{k=0}^{5} \lambda_k = 1. \tag{2.3.16}$$

Remark 2.3.1 *When* $1/2 < \delta < 1$, *the node* P_1, *which is the nearest node to* P, *is taken as the origin.*

Position 2. The point $P \in \Pi_0$ lies inside a grid cell of the hexagonal grid.

Again, we place the origin of the rectangular system of coordinates at the node P_0 and direct the positive axis of x along the grid line, so that P has the coordinates $P\left(\delta h, \frac{\sqrt{3}h\kappa}{2}\right)$, where $0 < \delta, \kappa \leq 1/2$. A fictitious grid is formed from the arrangement of the following points:

$$P_0'\left(\frac{\kappa h}{2},\frac{\sqrt{3}h\kappa}{2}\right),P_1'\left(h+\frac{\kappa h}{2},\frac{\sqrt{3}h\kappa}{2}\right),P_2'\left(\frac{h}{2}+\frac{\kappa h}{2},\frac{\sqrt{3}h}{2}+\frac{\sqrt{3}h\kappa}{2}\right),$$
$$P_3'\left(-\frac{h}{2}+\frac{\kappa h}{2},\frac{\sqrt{3}h}{2}+\frac{\sqrt{3}h\kappa}{2}\right),P_4'\left(\frac{h}{2}+\frac{\kappa h}{2},-\frac{\sqrt{3}h}{2}+\frac{\sqrt{3}h\kappa}{2}\right),$$
$$P_5'\left(-\frac{h}{2}+\frac{\kappa h}{2},-\frac{\sqrt{3}h}{2}+\frac{\sqrt{3}h\kappa}{2}\right).$$

Each of the nodes P_k', $k=0,1,...,5$ of the fictitious grid falls on a grid line and for the approximation of P the expression

$$S^4u=\sum_{k=0}^{5}\lambda_k u(P_k') \tag{2.3.17}$$

is used. As P_k', $k=0,1,...,5$, all lie on grid lines, each of these points need to be approximated using the matching operator as follows:

$$S^4u=\sum_{k=0}^{5}\lambda_k S^4u(P_k'). \tag{2.3.18}$$

It is demonstrated by Figure 2.4 that only 17 nodes are needed for the evaluation of (2.3.18).

Hence, we form the matching operator as

$$S^4u=\sum_{k=0}^{16}\xi_k u(P_k), \tag{2.3.19}$$

where ξ_k, $k=0,...,16$, are defined by the coefficients obtained earlier and

$$\xi_k\geq 0,\ \sum_{k=0}^{16}\xi_k=1. \tag{2.3.20}$$

The structure of the hexagonal grid also plays an important role in the approximation of the solution using the matching operator. We consider the two types of triangles in each hexagon, *Type A* and *Type B* as shown in Figure. 2.3.

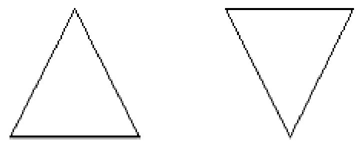

Figure 2.3. Shapes of triangles in a hexagon

It is obvious that when $\delta > \frac{\kappa}{2}$, the point $\left(\delta h, \frac{\sqrt{3}h}{2}\kappa\right)$ is in a triangle of *Type A* and when $\delta < \frac{\kappa}{2}$ it is in a triangle of *Type B*. In the case when $\delta = \frac{\kappa}{2}$, P is lying on a grid line.

We start by examining triangles of *TypeA*, with $0 < \delta, \kappa \leq 1/2$. The nodes needed in the evaluation of S^4u are shown in Figure. 2.4.

The case $1/2 < \delta < 1$, $0 < \kappa \leq 1/2$ has a similar layout, where the 17 nodes used have the same layout as the reflection of the nodes in Figure 2.4 about the line $x = 0$. The figure for the case $0 < \delta \leq 1/2$, $1/2 < \kappa < 1$ is also given below (see Figure. 2.5).

The final case $1/2 < \delta, \kappa < 1$ again has the same distribution as the reflection of the nodes in Figure 2.5, about the line $x = 0$.

In the case when P falls into a triangle of *Type B*, we rotate the fictitious grids formed for *Type A* with an angle of 180°, for all four cases of δ and κ specified earlier.

Position 3. $P \in \Pi_{01}$, where $u = \varphi_j$ on the side γ_j, $j = 1,2,3,4$, and $\varphi_j \in C^{4,\lambda}\left(\gamma_j\right)$,

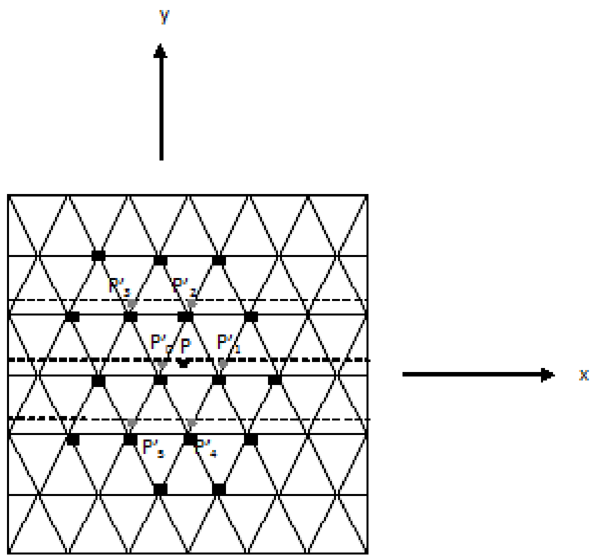

Figure 2.4. when P falls inside a grid cell and $0 < \delta, \kappa \leq 1/2$.

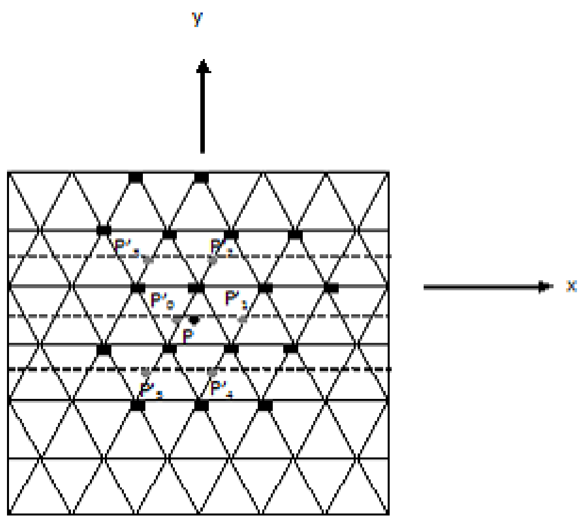

Figure 2.5. when P falls inside a grid cell and $0 < \delta \leq 1/2$, $1/2 < \kappa < 1$

$0 < \lambda < 1$.

We position the origin of the rectangular system of coordinates on γ_j so that the point P lies on the positive *y axis*, and the *x axis* is in the direction of the vertex $\dot{\gamma}_{j+1}$ along

γ_j. It is obvious that $\sum_{k=1}^{3} b_k \operatorname{Im} z^k = 0$ if $y = 0$, where $z = x + iy$. Hence, when the function $\varphi_j \in C^{4,\lambda}\left(\gamma_j\right)$, $0 < \lambda < 1$, is represented using Taylor's formula about the point $x = 0$ in the neighborhood $|z| \leq 4h$ of the origin, we define a_k, $k = 0, 1, 2, 3$, of (2.3.2) as

$$a_k = \frac{1}{k!} \frac{\partial^k \varphi_j(0)}{\partial x^k}.$$

We let

$$\widetilde{u}(x,y) = u(x,y) - \sum_{k=0}^{3} a_k \operatorname{Re} z^k = \sum_{k=1}^{3} b_k \operatorname{Im} z^k + O\left(h^4\right) \tag{2.3.21}$$

for $y > 0$, and keeping in mind that $\operatorname{Im} z^k$ is odd extendable, we complete the definition with $\widetilde{u}(x,y) = -\widetilde{u}(x,-y)$ for $y < 0$. Clearly, in the given neighborhood, $\widetilde{u}(x,y)$ is equal to the harmonic polynomial $\sum_{k=1}^{3} b_k \operatorname{Im} z^k$, with an accuracy of $O\left(h^4\right)$. To form an expression for the matching operator $S^4\widetilde{u}$ we use

$$S^4\widetilde{u} = \sum_{0 \leq j \leq 16} \mu_j \left(u - \sum_{k=0}^{3} a_k \operatorname{Re} z^k\right)(P_j),$$

or,

$$S^4\widetilde{u} = \sum_{0 \leq j \leq 5} \nu_j \left(u - \sum_{k=0}^{3} a_k \operatorname{Re} z^k\right)(P_j),$$

where

$$\mu_j \geq 0, \sum_{0\leq j\leq 16} \mu_j \leq 1; \; \nu_j \geq 0, \sum_{0\leq j\leq 5} \nu_j \leq 1. \tag{2.3.22}$$

Hence adding the term

$$\left(\sum_{k=0}^{3} a_k \operatorname{Re} z^k\right)(P),$$

to $S^4\widetilde{u}$, the approximation at any point $P \in \Pi_{01}$ can be obtained for the solution u of problem $(2.2.1),(2.2.2)$ as:

$$u = S^4\widetilde{u} + \left(\sum_{k=0}^{3} a_k \operatorname{Re} z^k\right)(P) + O(h^4). \tag{2.3.23}$$

Remark 2.3.2 *The expression (2.3.23) follows from the expressions (2.3.17) or (2.3.19) and contains less grid nodes P_l for the points on the boundary γ of Π.*

Let $\varphi = \left\{\varphi_j\right\}_{j=1}^{4}$. The matching operator S^4 is represented as:

$$S^4(u,\varphi) = \begin{cases} S^4 u \text{ on } \Pi_0 \\ S^4(u - \sum_{k=0}^{3} a_k \operatorname{Re} z^k) + \left(\sum_{k=0}^{3} a_k \operatorname{Re} z^k\right)(P), \text{ on } \Pi_{01} \cup \gamma \end{cases}. \tag{2.3.24}$$

Theorem 2.3.3 *Let the boundary functions φ_j, $j = 1,2,3,4$ in problem* $(2.2.1),(2.2.2)$ *satisfy the conditions*

$$\varphi_j \in C^{4,\lambda}(\gamma_j), 0 < \lambda < 1, \tag{2.3.25}$$

$$\varphi_j^{(2q)}(s_j) = (-1)^q \varphi_{j-1}^{(2q)}(s_j), \; q = 0,1,2. \tag{2.3.26}$$

Then

$$\max_{(x,y)\in\overline{\Pi}} \left|S^4(u,\varphi)-u\right| \le c_5h^4, \tag{2.3.27}$$

where u is the exact solution of problem (2.2.1), (2.2.2).

Proof. According to Theorem 3.1 in [27] from the conditions (2.3.25) and (2.3.26) follows that $u \in C^{4,\lambda}(\overline{\Pi})$. Then on the basis of (2.3.1), (2.3.10), (2.3.19), (2.3.23) and Remark 2.3.2, we obtain the inequality (2.3.27). ■

We define the function $\widehat{u}_h$ as follows

$$\widehat{u}_h = S^4(u_h,\varphi) \;\text{ on } \overline{\Pi}, \tag{2.3.28}$$

where u_h is the solution of the finite difference problem (2.2.6) – (2.2.8).

Theorem 2.3.4 *Let the conditions* (2.2.3) *and* (2.2.4) *be satisfied. Then the function* $\widehat{u}_h$ *is continuous on* $\overline{\Pi}$, *and*

$$\max_{(x,y)\in\overline{\Pi}} \mid \widehat{u}_h - u \mid \le c_6h^4, \tag{2.3.29}$$

where u is the solution of the problem (2.2.1), (2.2.2).

Proof. *From the construction of the expression* $S^4(u_h,\varphi)$ *it follows that* $\widehat{u}_h = u_h$ *on* Π^h, *and* $\widehat{u}_h = \varphi_j$ *on* γ_j^h, $j = 1,2,3,4$. *The continuity of* $\widehat{u}_h$ *on* Π *follows from the continuity* $S^4(u_h,\varphi)$ *on each closed triangle Type A and Type B, and from the equality* $\widehat{u}_h = u_h$ *on* Π^h. *By Remark 2.3.2 and from the condition* $\widehat{u}_h = \varphi_j$ *on* γ_j^h, $j = 1,2,3,4$, *follows the continuity of the function* $\widehat{u}_h$ *on the closed rectangle* $\overline{\Pi}$. *By virtue of (2.2.3) and (2.2.4)* *follows that* $u \in C^{6,\lambda}(\overline{\Pi})$, $0 < \lambda < 1$ *(see Theorem 3.1 in [27]). Then, on the basis of* (2.3.15), (2.3.16), (2.3.20), (2.3.23) *Theorem 2.2.2, Theorem 2.3.3 and (2.3.28), we obtain*

$$\begin{aligned}\max_{(x,y)\in\overline{\Pi}} \quad | \quad \widehat{u}_h - u \,| &\leq \max_{(x,y)\in\overline{\Pi}} |\, S^4(u,\varphi) - u \,| + \max_{(x,y)\in\overline{\Pi}} \left|S^4(u_h - u, 0)\right| \\ &\leq \quad c_5 h^4 + \sum_{k=0}^{16} \xi_k \max_{(x,y)\in\overline{\Pi}^h} |u_h - u| \leq c_6 h^4.\end{aligned}$$

■

2.4 Error analysis of the Block-Grid equations

Let

$$\varepsilon_h = u_h - u, \tag{2.4.1}$$

where u_h is the solution of the system (2.1.12)-(2.1.16) and u is the trace of the solution of (2.1.1), (2.1.2) on $\overline{G}_*^{h,n}$. On the basis of (2.1.1), (2.1.2), (2.1.12)-(2.1.16) and (2.4.1), ε_h satisfies the following difference equations:

$$\varepsilon_h \quad = \quad S\varepsilon_h + r_h^1 \;\text{ on } \Pi_k^{0h}, \tag{2.4.2}$$

$$\varepsilon_h \quad = \quad S_m^*\varepsilon_h + r_h^2 \;\text{ on } \Pi_k^{*h},\; \eta_{k1}^h \cap \gamma_m \neq \oslash,\;\; \varepsilon_h = 0 \text{ on } \eta_{k1}^h \cap \gamma_m, \tag{2.4.3}$$

$$\varepsilon_h(r_j,\theta_j) \quad = \quad \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q)\varepsilon_h(r_{j2},\theta_j^q) + r_{jh}^3,\; (r_j,\theta_j) \in t_j^h, \tag{2.4.4}$$

$$\varepsilon_h \quad = \quad S^4\varepsilon_h + r_h^4 \;\text{ on } \omega^{h,n}, \tag{2.4.5}$$

where $1 \le k \le M$, $1 \le m \le N$, $j \in E$ and

$$\begin{aligned} r_h^1 &= Su - u \text{ on } \cup_{k=1}^{M} \Pi_k^{0h},\ r_h^2 = S_m^* u + E_{mh}^*(\varphi_m) - u \text{ on } \cup_{1\le k\le M} \Pi_k^{*h}, && (2.4.6) \\ r_{jh}^3 &= \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q)\left(u(r_{j2},\theta_j^q) - Q_j(r_{j2},\theta_j^q)\right) && (2.4.7) \\ &\quad -(u\left(r_j,\theta_j\right) - Q_j(r_j,\theta_j)) \ \text{ on } \cup_{j\in E} t_j^h, \\ r_h^4 &= S^4(u,\varphi) - u \ \text{ on } \omega^{h,n}. && (2.4.8) \end{aligned}$$

Since all the rectangles Π_k, $k = 1,2,...,M$ are located away from the singular vertices $\dot{\gamma}_j$, $j \in E$ of the polygon G at a distance greater than $r_{j4} > 0$ independent of h, by virtue of the conditions (2.2.3) and (2.2.4), up to sixth order derivatives of the solution of problem (2.1.1),(2.1.2) are bounded on $\cup_{k=1}^{M}\Pi_k$. Then, by the Taylor formula, from (2.4.6), we obtain

$$\max_{\cup_{k=1}^{M}\Pi_k^{0h}} \left|r_h^1\right| \le c_1 h^6, \quad \max_{\cup_{k=1}^{M}\Pi_k^{*h}} \left|r_h^2\right| \le c_2 h^4. \tag{2.4.9}$$

Furthermore, as $\omega^{h,n} \subset \cup_{k=1}^{M}\Pi_k$ from (2.4.8) and Theorem 2.3.3, we have

$$\max_{\omega^{h,n}} \left|r_h^4\right| \le c_3 h^4. \tag{2.4.10}$$

By analogy to the proof of Lemma 6.2 in [20], it is shown that there exists a natural number n_0, such that for all $n \ge \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right]+1\right\}$, $\varkappa > 0$ being a fixed number,

$$\max_{j\in E} \left|r_{jh}^3\right| \le c_4 h^4. \tag{2.4.11}$$

Theorem 2.4.1 *There exists a natural number n_0 such that for all*

$n \geq \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right]\right\}$, $\varkappa > 0$ *being a fixed number,*

$$\max_{\overline{G}_*^{h,n}} |u_h - u| \leq ch^4. \tag{2.4.12}$$

Proof. Let $\Pi_{k^*}^h$ be one of the rectangles covering the domain G, with a hexagonal grid, and let $t_{k^*j}^h = \overline{\Pi}_{k^*}^h \cap t_j^h$. Furthermore, assume $t_{k^*j}^h \neq \emptyset$ and that v_h is a solution of the system (2.4.2)-(2.4.5) under the condition that r_{jh}^1, r_{jh}^2 and r_{jh}^3 are defined as in (2.4.6)-(2.4.8) in $\overline{\Pi}_{k^*}^h$, but are zero in $\overline{G}_*^{h,n} \backslash \overline{\Pi}_{k^*}^h$. It can be clearly seen that

$$W = \max_{\overline{G}_*^{h,n}} |v_h| = \max_{\overline{\Pi}_{k^*}^h} |v_h|. \tag{2.4.13}$$

We represent the function v_h on $\overline{G}_*^{h,n}$ as

$$v_h = \sum_{p=1}^{4} v_h^p, \tag{2.4.14}$$

where the functions $v_h^p, p = 2,3,4$, are defined on $\overline{\Pi}_{k^*}^h$ as a solution of the system of equations

$$\begin{aligned} v_h^2 &= \begin{cases} Sv_h^2 \text{ on } \Pi_{k^*}^{0h} \\ S_j^* v_h^2 \text{ on } \Pi_{k^*}^{*h} \end{cases}, v_h^2 = 0 \text{ on } \eta_{k^*1}^h, \\ v_h^2(r_j, \theta_j) &= r_{jh}^2, (r_j, \theta_j) \in t_{k^*j}^h, v_h^2 = 0 \text{ on } \omega^{h,n} \end{aligned} \tag{2.4.15}$$

$$\begin{aligned} v_h^3 &= \begin{cases} Sv_h^3 \text{ on } \Pi_{k^*}^{0h} \\ S_j^* v_h^3 \text{ on } \Pi_{k^*}^{*h} \end{cases}, v_h^3 = 0 \text{ on } \eta_{k^*1}^h, \\ v_h^3(r_j, \theta_j) &= 0, (r_j, \theta_j) \in t_{k^*j}^h, v_h^3 = r_{jh}^3 \text{ on } \omega^{h,n} \end{aligned} \tag{2.4.16}$$

$$
\begin{aligned}
v_h^4 &= \begin{cases} Sv_h^4 + r_{jh}^1 \text{ on } \Pi_{k^*}^{0h} \\ S_j^* v_h^4 + r_{jh}^1 \text{ on } \Pi_{k^*}^{*h} \end{cases}, v_h^4 = 0 \text{ on } \eta_{k^*1}^h, \qquad (2.4.17) \\
v_h^4(r_j, \theta_j) &= 0, (r_j, \theta_j) \in t_{k^*j}^h, v_h^4 = 0 \text{ on } \omega^{h,n}
\end{aligned}
$$

with

$$
v_h^p = 0, p = 2, 3, 4, \text{ on } \overline{G}_*^{h,n} \backslash \overline{\Pi}_{k^*}^h. \qquad (2.4.18)
$$

Moreover, keeping in mind equations (2.4.14)-(2.4.18), the function v_h^1 satisfies the system of equations

$$
\begin{aligned}
v_h^1 &= \begin{cases} Sv_h^1 \text{ on } \Pi_k^{0h} \\ S_j^* v_h^1 \text{ on } \Pi_k^{*h} \end{cases}, v_h^1 = 0 \text{ on } \eta_{k1}^h, \qquad (2.4.19) \\
v_h^1 &= \beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q) \sum_{p=1}^{4} v_h^p(r_{j2}, \theta_j^q), (r_j, \theta_j) \in t_j^h \\
v_h^1 &= S^4 \left(\sum_{p=1}^{4} v_h^p \right) \text{ on } \eta_{k0}^h, \ 1 \leq k \leq M, j \in E,
\end{aligned}
$$

where we presume that the functions $v_h^p, p = 2, 3, 4$, are known.

Taking into account (2.4.11), Remark 2.3.2 and Theorem 2.2.2, on the basis of (2.4.15)-(2.4.17) and the maximum principle, the following inequalities are obtained:

$$
\begin{aligned}
W_2 &= \max_{\overline{G}_*^{h,n}} \left| v_h^2 \right| \leq ch^4, \qquad (2.4.20) \\
W_3 &= \max_{\overline{G}_*^{h,n}} \left| v_h^3 \right| \leq ch^4, \qquad (2.4.21) \\
W_4 &= \max_{\overline{G}_*^{h,n}} \left| v_h^4 \right| \leq ch^4. \qquad (2.4.22)
\end{aligned}
$$

Next, the estimation of the function v_h^1 is considered. On the basis of (2.1.25), (2.3.15), (2.3.16), Remark 4.2.10 and the gluing condition of the rectangles $\Pi_k, k = 1,2,...,M$, by means of [30], for the estimation of the system (2.4.19), there exists a real number $\mu^*, 0 < \mu^* < 1$, independent of h, such that for all $h \leq \varkappa_0$ and $n \geq \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right] + 1\right\}$ we have

$$W_1 = \max_{\overline{G}_*^{h,n}} \left|v_h^1\right| \leq \mu^* W. \tag{2.4.23}$$

From (2.4.13), (2.4.14) and estimations (2.4.20)-(2.4.23), we obtain

$$W = \mu^* W + \sum_{i=2}^{4} W_i. \tag{2.4.24}$$

Hence,

$$W = \max_{\overline{G}_*^{h,n}} |v_h| \leq ch^4. \tag{2.4.25}$$

In the case when $t_{k^*j}^h = \emptyset$, (2.4.25) is proved similarly. As there is only a finite number of rectangles covering the domain G, inequality (2.4.12) follows. ■

Theorem 2.4.2 *We consider the approximation of the solution of problem (2.1.1), (2.1.2) on the sectors $\overline{T}_j^*$, $j \in E$, where $r_j^* = (r_{j2} + r_{j3})/r_{j2}$. Let u_h be the solution of the system of equations (2.1.12)-(2.1.16) and let an approximate solution of problem (2.1.1), (2.1.2) be found on blocks $\overline{T}_j^*$, $j \in E$, by (2.1.26). There is a natural number n_0 such that for all $n \geq \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right]\right\}$, $\varkappa > 0$ being a fixed number, the following* *estimations hold*

$$\left|U_h(r_j,\theta_j)-u(r_j,\theta_j)\right| \;\leq\; c_0h^4 \text{ on } \overline{T}_j^3, \tag{2.4.26}$$

$$\left|\frac{\partial^p}{\partial x^{p-q}\partial y^q}\left(U_h(r_j,\theta_j)-u(r_j,\theta_j)\right)\right| \;\leq\; c_ph^4/r_j^{p-1/\alpha_j} \text{ on } \overline{T}_j^3\backslash\dot{\gamma}_j, \tag{2.4.27}$$

where $j\in E,\, 0\leq q\leq p,\, p=1,2,\ldots$.

Proof. On the basis of (2.1.17) we have, on the closed block $\overline{T}_j^*, j\in E$

$$\begin{aligned} U_h(r_j,\theta_j)-u(r_j,\theta_j) \;=\;& \beta_j\sum_{q=1}^{n(j)}R_j(r_j,\theta_j,\theta_j^q)\left(u(r_{j2},\theta_j^q)-Q_j(r_{j2},\theta_j^q)\right)\\ &-\int_0^{\alpha_j\pi}(u(r_{j2},\eta)-Q_j(r_{j2},\eta))R_j(r_j,\theta_j,\eta)d\eta\\ &+\beta_j\sum_{q=1}^{n(j)}R_j(r_j,\theta_j,\theta_j^q)\left(u_h(r_{j2},\theta_j^q)-u(r_{j2},\theta_j^q)\right) \end{aligned} \tag{2.4.28}$$

Since $r_j^*=(r_{j2}+r_{j3})/r_{j2}$, by (2.4.11),

$$\left|\beta_j\sum_{q=1}^{n(j)}R_j(r_j,\theta_j,\theta_j^q)\left(u(r_{j2},\theta_j^q)-Q_j(r_{j2},\theta_j^q)\right)\right. \tag{2.4.29}$$

$$\left.-\int_0^{\alpha_j\pi}(u(r_{j2},\eta)-Q_j(r_{j2},\eta))R_j(r_j,\theta_j,\eta)d\eta\right|\leq ch^4 \text{ on } \overline{T}_j^*, j\in E$$

On the basis of (2.1.17), Theorem 2.4.1 and using the boundedness of the kernel R_j we obtain

$$\left|\beta_j\sum_{q=1}^{n(j)}R_j(r_j,\theta_j,\theta_j^q)\left(u_h(r_{j2},\theta_j^q)-u(r_{j2},\theta_j^q)\right)\right|\leq ch^4 \text{ on } \overline{T}_j^*, j\in E \tag{2.4.30}$$

Combining (2.4.29) and (2.4.30), as $\overline{T}_j^3 \subset \overline{T}_j^*$ we obtain the inequality

$$\left|U_h(r_j,\theta_j) - u(r_j,\theta_j)\right| \leq c_0 h^4 \text{ on } \overline{T}_j^3, j \in E \tag{2.4.31}$$

Let

$$\varepsilon_h(r_j,\theta_j) = U_h(r_j,\theta_j) - u(r_j,\theta_j) \text{ on } \overline{T}_j^*, j \in E \tag{2.4.32}$$

From (2.1.17) and (2.4.31) follows that $\varepsilon_h(r_j,\theta_j)$ is continuous on $\overline{T}_j^*$, and is a solution of the boundary value problem (2.1.1), (2.1.2), where $0 \leq \theta_j \leq \alpha_j \pi$. As $\overline{T}_j^3 \subset \overline{T}_j^*, j \in E$, considering (2.4.30)-(2.4.32) and taking into account Lemma 6.12 in [1], inequality (2.4.27) follows. ■

2.5 The use of the Schwarz's alternating method for the solution of the system of block-grid equations

It is clear from Section 2.1 that for the approximate solution of problem (2.1.1), (2.1.2), it is first necessary to consider the solution in the domain $\overline{G}_*^{h,n}$. Hence, first of all, the solution of the system of equations (2.1.12)-(2.1.16) is taken into account. Then the solution itself and its derivatives of order p, $p = 1,2,...$, follows for any point of $\overline{T}_j^3$ and $\overline{T}_j^3 \backslash \dot{\gamma}_j$, with the use of formula (2.1.17). Therefore, it is only necessary to justify the method of finding a solution of the system of equations (2.1.12)-(2.1.16), as stated in [15].

In a similar manner to [15], we define classes $\Phi_\tau, \tau = 1,2,...,\tau^*$, of rectangles $\Pi_k, k = 1,2,...,M$. Class Φ_1 includes all rectangles whose intersection with the boundary γ of the polygon G contains a certain segment of positive length. Class Φ_2 contains all of the rectangles which are not in the class Φ_1, whose intersection with rectangles of Φ_1 contains a segment of finite length, and so on.

Let

$$
\begin{aligned}
\Phi^h_{\tau 0} &= \cup_{k:\Pi_k\in\Phi_\tau}\Pi^h_{k0}, \tau = 1,2,...,\tau^*, \\
G^h_{*0} &= \cup^{\tau^*}_{\tau=1}\Phi^h_{\tau 0}.
\end{aligned}
$$

For the solution of the system of equations (2.1.12)-(2.1.16), Schwarz's alternating method is carried out in the following form. We start with a zero approximation $u^{(0)}_h$ to the exact solution u_h of system (2.1.12)-(2.1.16). Finding $u^{(1)}_h$ for all $j \in E$ with (2.1.15) on t^h_j and with (2.1.16) on η_{k0}, we solve system (2.1.12)-(2.1.16) on the grids $\overline{\Pi}^h_k$ constructed on the rectangles belonging to the class Φ_1 and then to the class Φ_2 and so on. The next iteration follows in a similar manner. Consequently, we have the sequence $u^{(1)}_h, u^{(2)}_h, ...$ defined as follows:

$$
\begin{aligned}
u^{(m)}_h(r_j,\theta_j) &= Q_j(r_j,\theta_j)+ && (2.5.1)\\
&+\beta_j\sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta^q_j)\left[S^4(u^{(m-1)}_h(r_{j2},\theta^q_j),\varphi)-Q_j(r_{j2},\theta^q_j)\right] \text{ on } t^h_j, \\
u^{(m)}_h &= S^4u^{(m-1)}_h \text{ on } \omega^{h,n} && (2.5.2)
\end{aligned}
$$

$$u_h^{(m)} = Su_h^{(m)} \text{ on } \Pi_k^{0h}, \tag{2.5.3}$$

$$u_h^{(m)} = S_j^* u_h^{(m)} + E_{jh}^*(\varphi_j) \text{ on } \Pi_k^{*h}, \tag{2.5.4}$$

$$u_h^{(m)} = \varphi_j \text{ on } \eta_{k1}^h, \tag{2.5.5}$$

Theorem 2.5.1 *For any $h \leq \varkappa_0 \backslash 4$ and $n \geq \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right] + 1\right\}$, system (2.1.12)-(2.1.16) can be solved by Schwarz's alternating method with an accuracy of $\varepsilon > 0$, in a uniform metric with the number of iterations $O(\ln \varepsilon^{-1})$, independent of h and n, where $\varkappa_0$ is the gluing depth and $\varkappa$ is a constant independent of h.*

Proof. Theorem 2.5.1 is proved by analogy to Theorem 3 in [15], with the system under consideration being (2.5.1)-(2.5.5). ■

Chapter 3

HEXAGONAL GRID VERSION OF THE BLOCK-GRID METHOD FOR THE MIXED PROBLEM ON STAIRCASE POLYGONS

3.1 Approximation on a rectangular domain using a hexagonal grid with mixed boundary conditions

Let $\Pi = \{(x,y) : 0 < x < a,\ 0 < y < b\}$ be an open rectangle, $\gamma_j, j = 1,2,3,4$, be its sides, numbered in the positive direction starting from the left-hand side, $(\gamma_0 \equiv \gamma_4, \gamma_1 \equiv \gamma_5)$. Also let $\dot{\gamma}_j = \gamma_j \cap \gamma_{j+1}$ be the jth vertex, $\dot{\gamma} = \cup_{j=1}^{4}\left(\gamma_j \cap \gamma_{j+1}\right)$ be the set of all vertices of Π and $\gamma = \cup_{j=1}^{4}\gamma_j$ represent the whole boundary of Π. We consider the boundary value problem

$$\Delta u = 0 \text{ on } \Pi, \tag{3.1.1}$$

$$\nu_j u + \overline{\nu}_j u_n^{(1)} = \nu_j \varphi_j + \overline{\nu}_j \psi_j \text{ on } \gamma_j, j = 1,2,3,4, \tag{3.1.2}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, ν_j is a parameter taking the values 0 or 1, and $\overline{\nu}_j = 1 - \nu_j$. Furthermore, $u_n^{(1)}$ is the derivative along the inner normal, φ_j and ψ_j are given functions and

$$1 \leq \nu_1 + \nu_2 + \nu_3 + \nu_4 \leq 4, \tag{3.1.3}$$

$$\nu_j \varphi_j + \overline{\nu}_j \psi_j \in C^{6,\lambda}(\gamma_j),\ 0 < \lambda < 1,\ j = 1,2,3,4. \tag{3.1.4}$$

At the vertices $s = s_j$ (s is defined the same as in Section 2.1 and s_j is the beginning of γ_j), the conjugation conditions

$$\nu_j \varphi_j^{(2q+\delta_{\tau-2})} + \overline{\nu}_j \psi_j^{(2q+\delta_\tau)} = (-1)^{q+\delta_\tau+\delta_{\tau-1}} (\nu_{j-1} \varphi_{j-1}^{(2q+\delta_{\tau-1})} + \overline{\nu}_{j-1} \psi_{j-1}^{(2q+\delta_\tau)}) \tag{3.1.5}$$

are satisfied, where $\tau = \nu_{j-1} + 2\nu_j$, $\delta_\omega = 1$ for $\omega = 0$, $\delta_\omega = 0$ for $\omega \neq 0$, $q = 0, 1, ..., Q$, $Q = [(6 - \delta_{\tau-1} - \delta_{\tau-2})/2] - \delta_\tau$.

Let $h > 0$, with $a/h \geq 2$, $b/\sqrt{3}h \geq 2$ integers. We let Π^h stand for a hexagonal grid on Π, with step size h, where the set of these nodes are expressed as

$$\Pi^h = \left\{ (x,y) \in \Pi : x = \frac{k-l}{2} h,\ y = \frac{\sqrt{3}(k+l)}{2} h,\ k = 1,2,...; l = 0, \pm 1, \pm 2, ... \right\}.$$

Let γ_j^h be the set of nodes on the interior of γ_j, $\dot{\gamma}_j^h = \gamma_j \cap \gamma_{j+1}$ and $\gamma^h = \cup_{j=1}^4 \gamma_j^h$. In addition, let Π^{*h} stand for the set of nodes whose distance from the boundary γ of $\overline{\Pi}$ is $\frac{h}{2}$ and $\Pi^{0h} = \Pi^h / \Pi^{*h}$. Hence $\overline{\Pi}^h = \Pi^{0h} \cup \Pi^{*h} \cup \gamma^h$.

We consider the system of finite difference equations

$$u_h = S u_h \text{ on } \Pi^{0h}, \tag{3.1.6}$$

$$u_h = S_j^* u_h + E_{jh}^*(\varphi_j, \psi_j) \text{ on } \Pi^{*h}, \tag{3.1.7}$$

$$u_h = \overline{\nu}_j S_j u_h + E_{jh}(\varphi_j, \psi_j) \text{ on } \gamma_j^h, \tag{3.1.8}$$

$$u_h = \overline{\nu}_j \overline{\nu}_{j+1} \dot{S}_j u_h + \dot{E}_{jh}(\varphi_j, \varphi_{j+1}, \psi_j, \psi_{j+1}) \text{ on } \dot{\gamma}_j^h,\ j = 1,2,3,4, \tag{3.1.9}$$

where

$$
\begin{aligned}
Su(x,y) &= \frac{1}{6}\left(u\left(x+h,y\right)+u\left(x+\frac{h}{2},y+\frac{\sqrt{3}}{2}h\right)+u\left(x-\frac{h}{2},y+\frac{\sqrt{3}}{2}h\right)+\right.\\
&\quad +u\left(x-h,y\right)+u\left(x-\frac{h}{2},y-\frac{\sqrt{3}}{2}h\right)+\\
&\quad \left.+u\left(x+\frac{h}{2},y-\frac{\sqrt{3}}{2}h\right)\right), \qquad (3.1.10)
\end{aligned}
$$

the operators $S_j^*, E_{jh}^*, S_j, E_{jh}, \dot{S}_j$ and $\dot{E}_{jh}$ are constructed in the right coordinate system with the axis x_j directed along γ_{j+1} and the axis y_j directed along γ_j, and have the expressions:

$$
\begin{aligned}
S_j^*u(x,y) &= \frac{\nu_j}{7}\left(u\left(x+\frac{h}{2},y-\frac{\sqrt{3}h}{2}\right)+u(x+h,y)+u\left(x+\frac{h}{2},y+\frac{\sqrt{3}h}{2}\right)\right)+\\
&\quad \frac{\overline{\nu}_j}{5}\left(u\left(x-\frac{h}{2},y-\frac{\sqrt{3}h}{2}\right)+u\left(x+\frac{h}{2},y-\frac{\sqrt{3}h}{2}\right)+u(x+h,y)+\right.\\
&\quad \left.u\left(x+\frac{h}{2},y+\frac{\sqrt{3}h}{2}\right)+u\left(x-\frac{h}{2},y+\frac{\sqrt{3}h}{2}\right)\right), \qquad (3.1.11)
\end{aligned}
$$

$$
\begin{aligned}
E_{jh}^*(\varphi_j,\psi_j) &= \frac{\nu_j}{7}\left(\varphi_j\left(y+\frac{\sqrt{3}h}{2}\right)+\varphi_j\left(y-\frac{\sqrt{3}h}{2}\right)+2\varphi_j(y)-\right.\\
&\quad \left.\frac{h^2}{4}\varphi_j^{(2)}(y)+\frac{h^4}{4!8}\varphi_j^{(4)}(y)\right)+\\
&\quad \frac{\overline{\nu}_j}{5}\left(h\psi_j-\frac{h^3}{3!4}\psi_j^{(2)}+\frac{h^5}{5!16}\psi_j^{(4)}\right), \qquad (3.1.12)
\end{aligned}
$$

$$
S_ju(x,y)=\begin{cases}
\frac{1}{3}\left(u\left(x+\frac{h}{2},y-\frac{\sqrt{3}h}{2}\right)+u(x+h,y)+u\left(x+\frac{h}{2},y+\frac{\sqrt{3}h}{2}\right)\right) \text{ on } j=1,3,\\
\frac{1}{6}\left(u(x-h,y)+2u\left(x-\frac{h}{2},y+\frac{\sqrt{3}h}{2}\right)\right.\\
\left.+2u\left(x+\frac{h}{2},y+\frac{\sqrt{3}h}{2}\right)+u(x+h,y)\right) \text{ on } j=2,4,
\end{cases} \qquad (3.1.13)
$$

$$E_{jh}(\varphi_j,\psi_j)=\begin{cases}\overline{\nu}_j\left(\frac{2h}{3}\psi_j+\frac{h^3}{3!3}\psi_j^{(2)}+\frac{2h^5}{5!3}\psi_j^{(4)}\right)+\nu_j\varphi_j \text{ on } j=1,3,\\ \overline{\nu}_j\left(-\frac{\sqrt{3}h}{3}\psi_j+\frac{\sqrt{3}h^5}{5!3}\psi_j^{(4)}\right)+\nu_j\varphi_j \text{ on } j=2,4,\end{cases} \tag{3.1.14}$$

and

$$\dot{S}_j u(x,y)=\frac{1}{3}\left(u(x+h,y)+2u\left(x+\frac{h}{2},y+\frac{\sqrt{3}h}{2}\right)\right), \tag{3.1.15}$$

$$\begin{aligned}\dot{E}_{jh}(\varphi_j,\varphi_{j+1},\psi_j,\psi_{j+1}) &= \nu_j\varphi_j+\overline{\nu}_j\nu_{j+1}\varphi_{j+1}\\ &\quad-\overline{\nu}_j\overline{\nu}_{j+1}\left[\frac{1}{6}\left(2h\left(2\psi_j+\sqrt{3}\psi_{j+1}\right)+\sqrt{3}h^2\psi_{j+1}^{(1)}\right.\right.\\ &\quad+\frac{2h^3}{3!}\psi_j^{(2)}-\frac{2\sqrt{3}h^4}{4!}\left(\psi_j^{(3)}+\psi_{j+1}^{(3)}\right)+\\ &\quad\left.\left.+\frac{2h^5}{5!}\left(2\psi_j^{(4)}-\sqrt{3}\psi_{j+1}^{(4)}\right)\right)\right]\end{aligned} \tag{3.1.16}$$

Since the coefficients in the operators (3.1.10), (3.1.11), (3.1.13) and (3.1.15) are non-negative and their sum do not exceed one, taking the maximum principle into account, system (3.1.6)-(3.1.9) has a unique solution (see [21]).

Let $\gamma=\gamma^1\cup\gamma^2$, where γ^1 and γ^2 contain the prescribed values of u and the normal derivative $u_n^{(1)}$ respectively. Accordingly, the set of nodes on the interior of γ^1 and γ^2 are denoted by γ^{1h} and γ^{2h}, $\dot{\gamma}_m^{2h}=\gamma_m^2\cap\gamma_{m-1}^2$, $1\le m\le 4$, $\dot{\gamma}^{2h}=\cup_{1\le j\le 4}\dot{\gamma}_j^{2h}$ and $\dot{\gamma}^{1h}=\dot{\gamma}^h\backslash\dot{\gamma}^{2h}$. In addition, the set of nodes whose distance from the boundary γ^1 of $\overline{\Pi}$ is $h/2$ is denoted as Π_1^{*h}, and $\Pi_2^{*h}=\Pi^{*h}\backslash\Pi_1^{*h}$ denotes the set of nodes whose distance from γ^2 is $h/2$.

Lemma 3.1.1 *Let*

$$
\begin{aligned}
v_1 &= Sv_1 + f_h \ on \ \Pi^{0h}, \\
v_1 &= S_j^* v_1 \ on \ \Pi_1^{*h}, \\
v_1 &= S_j^* v_1 + f_h^* \ on \ \Pi_2^{*h}, \\
v_1 &= 0 \ on \ \gamma_j^{1h} \cup \dot{\gamma}_j^{1h}, \\
v_1 &= S_j v_1 + f_h' \ on \ \gamma_j^{2h}, \\
v_1 &= \dot{S}_j v_1 + \dot{f}_h \ on \ \dot{\gamma}_j^{2h}, j = 1,2,3,4,
\end{aligned}
$$

and

$$
\begin{aligned}
v_2 &= Sv_2 + \overline{f}_h \ on \ \Pi^{0h}, \\
v_2 &= S_j^* v_2 + \widetilde{f}_h \ on \ \Pi_1^{*h}, \\
v_2 &= S_j^* v_2 + \overline{f}_h^* \ on \ \Pi_2^{*h}, \\
v_2 &= \widehat{f}_h \ on \ \gamma_j^{1h} \cup \dot{\gamma}_j^{1h}, \\
v_2 &= S_j v_2 + \overline{f}_h' \ on \ \gamma_j^{2h}, \\
v_2 &= \dot{S}_j v_2 + \overline{\dot{f}}_h \ on \ \dot{\gamma}_j^{2h}, j = 1,2,3,4,
\end{aligned}
$$

where $f_h, f_h^*, f_h', \dot{f}_h$ *and* $\overline{f}_h, \widetilde{f}_h, \overline{f}_h^*, \widehat{f}_h, \overline{f}_h', \overline{\dot{f}}_h$ *are arbitrary grid functions. If the conditions*

$$
\begin{aligned}
\widetilde{f}_h, \widehat{f}_h &\geq 0, \\
|f_h| &\leq \overline{f}_h, \ |f_h^*| \leq \overline{f}_h^*, \ |f_h'| \leq \overline{f}_h' \ and \ \left|\dot{f}_h\right| \leq \overline{\dot{f}}_h
\end{aligned}
$$

are satisfied, then

$$|v_1| \le v_2.$$

Proof. The proof of this lemma follows by analogy to the proof of the comparison theorem (see [21]). ■

Everywhere below we will denote constants which are independent of h and of the cofactors on their right by $c, c_0, c_1, ...$, generally using the same notation for different constants for simplicity.

Theorem 3.1.2

Let u be the trace of the solution of problem (3.1.1), (3.1.2) on $\overline{\Pi}^h$, and u_h be the solution of system (3.1.6)-(3.1.9). Then

$$\max_{\overline{\Pi}^h} |u_h - u| \le ch^4. \tag{3.1.17}$$

Proof. Let $\varepsilon_h = u_h - u$, where u is the trace of the solution of problem (3.1.1), (3.1.2) on $\overline{\Pi}^h$ and u_h is the solution of system (3.1.6)-(3.1.9). The error function ε_h satisfies the following system:

$$\begin{aligned}
\varepsilon_h &= S\varepsilon_h + \Psi_h^0 \text{ on } \Pi^{0h}, &(3.1.18)\\
\varepsilon_h &= S_j^*\varepsilon_h + \Psi_{jh}^* \text{ on } \Pi^{*h}, &(3.1.19)\\
\varepsilon_h &= \overline{\nu}_j S_j\varepsilon_h + \Psi_{jh} \text{ on } \gamma_j^h, &(3.1.20)\\
\varepsilon_h &= \overline{\nu}_j\overline{\nu}_{j+1}\dot{S}_j\varepsilon_h + \dot{\Psi}_{jh} \text{ on } \dot{\gamma}_j^h,\ j=1,2,3,4, &(3.1.21)
\end{aligned}$$

where

$$\begin{aligned}
\Psi_h^0 &= Su-u, &(3.1.22)\\
\Psi_{jh}^* &= S_j^*u-u+E_{jh}^*(\varphi_j,\psi_j), &(3.1.23)\\
\Psi_{jh} &= \overline{\nu}_j S_j u-u+E_{jh}(\varphi_j,\psi_j), &(3.1.24)\\
\dot{\Psi}_{jh} &= \overline{\nu}_j\overline{\nu}_{j+1}\dot{S}_j u-u+\dot{E}_{jh}(\varphi_j,\varphi_{j+1},\psi_j,\psi_{j+1}), &(3.1.25)
\end{aligned}$$

are the truncation errors of equations (3.1.6)-(3.1.9).

On the basis of conditions (3.1.3)-(3.1.5) and by Theorem 3.1 in [27], it follows that $u \in C^{6,\lambda}(\overline{\Pi})$, $0<\lambda<1$. Hence by Taylor's formula, we obtain (see [28]),

$$\max_{(x,y)\in\overline{\Pi}}\left|\Psi_h^0(x,y)\right| \le c_1h^6M_6. \qquad (3.1.26)$$

By using Taylor's formula we also obtain

$$\max_{(x,y)\in \overline{\Pi}} \left|\Psi_{jh}(x,y)\right| \le c_2 h^6 M_6, \tag{3.1.27}$$

and

$$\max_{(x,y)\in \overline{\Pi}} \left|\dot{\Psi}_{jh}(x,y)\right| \le c_3 h^6 M_6, \tag{3.1.28}$$

where

$$M_q = \sup_{(x,y)\in\overline{\Pi}} \left\{ \left| \frac{\partial^q u(x,y)}{\partial x^p \partial y^{q-p}} \right|, p = 0,1,...,q \right\}. \tag{3.1.29}$$

Finally, using Taylor's formula about the point $(h/2, y) \in \Pi^{*h}$ we obtain

$$\max_{(x,y)\in \overline{\Pi}} \left|\Psi^*_{jh}(x,y)\right| \le c_4 h^4 M_4 \tag{3.1.30}$$

when $\nu_j = 1$, and

$$\max_{(x,y)\in \overline{\Pi}} \left|\Psi^*_{jh}(x,y)\right| \le c_5 h^6 M_6 \tag{3.1.31}$$

when $\nu_j = 0$.

We represent the solution of (3.1.18)-(3.1.21) as

$$\varepsilon_h = \varepsilon_h^1 + \varepsilon_h^2, \tag{3.1.32}$$

where

$$\begin{aligned}
\varepsilon_h^1 &= S\varepsilon_h^1 + \Psi_h^0 \text{ on } \Pi^{0h}, && (3.1.33)\\
\varepsilon_h^1 &= S_j^*\varepsilon_h^1 \text{ on } \Pi_1^{*h}, && (3.1.34)\\
\varepsilon_h^1 &= S_j^*\varepsilon_h^1 + \Psi_{jh}^* \text{ on } \Pi_2^{*h}, && (3.1.35)\\
\varepsilon_h^1 &= 0 \text{ on } \gamma_j^{1h} \cup \dot{\gamma}_j^{1h}, && (3.1.36)\\
\varepsilon_h^1 &= S_j\varepsilon_h^1 + \Psi_{jh} \text{ on } \gamma_j^{2h}, && (3.1.37)\\
\varepsilon_h^1 &= \dot{S}_j\varepsilon_h^1 + \dot{\Psi}_{jh} \text{ on } \dot{\gamma}_j^{2h}, j = 1,2,3,4, && (3.1.38)
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon_h^2 &= S\varepsilon_h^2 \text{ on } \Pi^{0h}, && (3.1.39)\\
\varepsilon_h^2 &= S_j^*\varepsilon_h^2 + \Psi_{jh}^* \text{ on } \Pi_1^{*h}, && (3.1.40)\\
\varepsilon_h^2 &= S_j^*\varepsilon_h^2 \text{ on } \Pi_2^{*h}, && (3.1.41)\\
\varepsilon_h^2 &= \overline{\nu}_j S_j\varepsilon_h^2 \text{ on } \gamma_j^h, && (3.1.42)\\
\varepsilon_h^2 &= \overline{\nu}_j\overline{\nu}_{j+1}\dot{S}_j\varepsilon_h^2 \text{ on } \dot{\gamma}_j^h, j = 1,2,3,4. && (3.1.43)
\end{aligned}$$

To estimate ε_h^1, we take the function v_2 in Lemma 3.1.1 as

$$v_2(x,y) = h^4 c_6 M_6 (a^2 + b^2 - x^2 - y^2). \tag{3.1.44}$$

Hence,

$$\max_{\overline{\Pi}^h}\left|\varepsilon_h^1\right| \le \max_{(x,y)\in\overline{\Pi}}|v_2| \le c_7 h^4 M_6. \tag{3.1.45}$$

Now taking (3.1.30) and (3.1.39)-(3.1.43) into account, on the basis of maximum principle, we obtain

$$\max_{\overline{\Pi}^h}\left|\varepsilon_h^2\right| \le \frac{7}{4}\max_{\Pi_1^{*h}}\left|\Psi_{jh}^*\right| \le c_8 h^4 M_4. \tag{3.1.46}$$

From (3.1.18)-(3.1.21), (3.1.32), (3.1.45) and (3.1.46) it follows that

$$\max_{\overline{\Pi}^h}|\varepsilon_h| \le ch^4. \tag{3.1.47}$$

■

3.2 Construction of the matching operator for the mixed boundary value problem

The construction of the fourth order matching operator S^4u on a hexagonal grid, for approximating the solution of Laplace's equation with Dirichlet boundary conditions, is given in detail in [31] and Section 2.3. A summary of these results is provided here before extending the method to the construction with Neumann boundary conditions.

Let φ_j and ψ_j be the given functions defined in (3.1.2), and $\varphi = \left\{\varphi_j\right\}_{j=1}^4$, $\psi = \left\{\psi_j\right\}_{j=1}^4$. The estimation $P \in \overline{\Pi}$ by $S^4(u_h,\varphi,\psi)$ is given linearly by the values of the function u_h at the nodes of the hexagonal grid constructed in the rectangle $\overline{\Pi}_P^h$ and the assigned boundary values $\varphi^{(p)}$, $p = 0,1,2,3$, $\psi^{(q)}$, $q = 1,2,3$. The pattern of S^4 lies in a neighbourhood $O(h)$ of the point P, where

$$\begin{aligned} S^4(u_h,\varphi,\psi) &= \sum_{k=0}^{16} \lambda_k u_h(P_k), \\ \lambda_k &\geq 0, \; \sum_{k=0}^{16} \lambda_k = 1, \end{aligned} \tag{3.2.1}$$

and

$$u - S^4(u,\varphi,\psi) = O\left(h^4\right),$$

uniformly on $\overline{\Pi}$.

Let $\overset{\circ}{\Pi}$ denote the set of points $P \in \overline{\Pi}$ such that all the nodes P_k to determine expression (3.2.1) belong to $\overline{\Pi}^h$ and $\overset{\circ}{\Pi}_1$, $\overset{\circ}{\Pi}_2$ contain the points $P \in \overline{\Pi}$ where some of the nodes P_k emerge through the side $\gamma^h_{1,m}$, $\gamma^h_{2,m}$, $1 \leq m \leq 4$, respectively. The cases when the point P belongs to one of the sets $\overset{\circ}{\Pi}$, $\overset{\circ}{\Pi}_1$ is given in detail in [31] and Section 2.3. Hence, we consider the case when P lies inside the set $\overset{\circ}{\Pi}_2$.

Assume $P \in \overset{\circ}{\Pi}_2$, where $u = \psi_m$ on the side $\gamma_{2,m}$ and $\psi_m \in C^{4,\lambda}(\gamma_{2,m})$, $0 < \lambda < 1$, $1 \leq m \leq 4$.

Let $z = x + iy$ be a complex variable and let $\Omega = \{z : |z| < 1\}$ be a unit circle. For a harmonic function u on Ω with $u \in C^{4,0}(\overline{\Omega})$, by Taylor's formula, any point $(x,y) \in \Omega$ can be represented as

$$u(x,y) = \sum_{k=0}^{3} a_k \mathrm{Re}\; z^k + \sum_{k=1}^{3} b_k \mathrm{Im}\; z^k + O(r^4), \tag{3.2.2}$$

where $r = \sqrt{x^2 + y^2}$,

$$\begin{aligned} a_0 &= u(0,0), a_1 = \frac{\partial u(0,0)}{\partial x}, a_2 = \frac{1}{2}\frac{\partial^2 u(0,0)}{\partial x^2}, a_3 = \frac{1}{3!}\frac{\partial^3 u(0,0)}{\partial x^3}, \\ b_1 &= \frac{\partial u(0,0)}{\partial y}, b_2 = \frac{1}{2}\frac{\partial^2 u(0,0)}{\partial x \partial y}, b_3 = \frac{1}{3!}\frac{\partial^3 u(0,0)}{\partial x^2 \partial y}. \end{aligned} \tag{3.2.3}$$

The origin of the rectangular system of coordinates is placed on $\gamma_{2,m}$ so that the point P lies on the positive $y-axis$, and the $x-axis$ is in the direction of the vertex $\dot{\gamma}_{m+1}$ along $\gamma_{2,m}$. Since $\psi_m \in C^{4,\lambda}(\gamma_{2,m})$, the solution u of problem (3.1.1), (3.1.2) is

$$u \in C^{5,\lambda}(\overline{\Pi}).$$

Hence, in the neighbourhood $|z| \leq 4h$ of the origin, by Taylor's formula, we obtain

$$\begin{aligned} \left.\frac{\partial u(x,y)}{\partial y}\right|_{y=0} &= \frac{\partial u(0,0)}{\partial y} + x\frac{\partial^2 u(0,0)}{\partial x \partial y} \\ &+ \frac{x^2}{2!}\frac{\partial^3 u(0,0)}{\partial x^2 \partial y} + \frac{x^3}{3!}\frac{\partial^4 u(0,0)}{\partial x^3 \partial y} + O\left(h^4\right). \end{aligned} \tag{3.2.4}$$

Keeping in mind that $u_n^{(1)} = \psi_m$, we have

$$\begin{aligned} \left.\frac{\partial u(x,y)}{\partial y}\right|_{y=0} &= \psi_m(x) = \psi_m(0) + x\frac{\partial \psi_m(0)}{\partial x} \\ &+ \frac{x^2}{2!}\frac{\partial^2 \psi_m(0)}{\partial x^2} + \frac{x^3}{3!}\frac{\partial^3 \psi_m(0)}{\partial x^3} + O\left(h^4\right). \end{aligned} \tag{3.2.5}$$

Based on (3.2.4) and (3.2.5), we have the expressions

$$\begin{aligned} \frac{\partial u(0,0)}{\partial y} &= \psi_m(0),\ \frac{\partial^2 u(0,0)}{\partial x \partial y} = \frac{\partial \psi_m(0)}{\partial x}, \\ \frac{\partial^3 u(0,0)}{\partial x^2 \partial y} &= \frac{\partial^2 \psi_m(0)}{\partial x^2},\ \frac{\partial^4 u(0,0)}{\partial x^3 \partial y} = \frac{\partial^3 \psi_m(0)}{\partial x^3}. \end{aligned} \tag{3.2.6}$$

By (3.2.3) and (3.2.6), we can write the coefficients b_k, $k = 1,2,3$, as

$$b_k = \frac{1}{k!}\frac{\partial^{k-1}\psi_m(0)}{\partial x^{k-1}},\ k = 1,2,3.$$

Let

$$\widehat{u}(x,y) \equiv u(x,y) - \sum_{k=1}^{3} b_k \operatorname{Im} z^k = \sum_{k=0}^{3} a_k \operatorname{Re} z^k + O(h^4), \tag{3.2.7}$$

for $y > 0$, and for $y < 0$ we have $\widehat{u}(x,y) = \widehat{u}(x,-y)$. By (3.2.7), in the neighbourhood $|z| \le 4h$, $\widehat{u}(x,y)$ corresponds to $\sum_{k=0}^{3} a_k \operatorname{Re} z^k$ with an accuracy of $O(h^4)$, as this polynomial is even relative to the $x-axis$. Hence, using the expression for $S^4\widehat{u}$, $u(P)$ can be approximated by the equation

$$u(x,y) = S^4\widehat{u} + \left(\sum_{k=1}^{3} b_k \operatorname{Im} z^k\right)(P). \tag{3.2.8}$$

Combining the result obtained above with the expressions in [31], the matching operator can be expressed as:

$$S^4(u,\varphi,\psi) = \begin{cases} S^4 u \text{ on } \overset{\circ}{\Pi}, \\ S^4\left(u - \sum_{k=0}^{3} a_k \operatorname{Re} z^k\right) + \left(\sum_{k=0}^{3} a_k \operatorname{Re} z^k\right)(P) \text{ on } \overset{\circ}{\Pi}_1, \\ S^4\left(u - \sum_{k=1}^{3} b_k \operatorname{Im} z^k\right) + \left(\sum_{k=1}^{3} b_k \operatorname{Im} z^k\right)(P) \text{ on } \overset{\circ}{\Pi}_2. \end{cases} \tag{3.2.9}$$

Theorem 3.2.1 *Let the boundary functions φ_j, ψ_j, $j = 1,2,3,4$, in problem (3.1.1),* *(3.1.2) satisfy the conditions*

$$\begin{aligned} \nu_j\varphi_j+\overline{\nu}_j\psi_j &\in C^{4,\lambda}(\gamma_j),\ 0<\lambda<1, && (3.2.10)\\ \nu_j\varphi_j^{(2q+\delta_{\tau-2})}+\overline{\nu}_j\psi_j^{(2q+\delta_\tau)} &= (-1)^{q+\delta_\tau+\delta_{\tau-1}}\Big(\nu_{j-1}\varphi_{j-1}^{(2q+\delta_{\tau-1})}+\\ &\quad +\overline{\nu}_{j-1}\psi_{j-1}^{(2q+\delta_\tau)}\Big), && (3.2.11)\end{aligned}$$

with $q=0,1,2$. Then

$$\max_{(x,y)\in\overline{\Pi}}\left|S^4u-u\right|\le c_9h^4, \qquad (3.2.12)$$

where u is the exact solution of problem (3.1.1), (3.1.2).

Proof. According to Theorem 3.1 in [27], from the conditions (3.2.10) and (3.2.11) follows that $u\in C^{4,\lambda}(\overline{\Pi})$. Then, the inequality (3.2.12) follows from Theorem 3.4 in [31]. ■

3.3 Block-Grid equations with mixed boundary conditions

The BGM is applied for the approximation of Laplace's equation with mixed boundary conditions, with the employment of the following changes in the method described in Section 2.1.

We consider the approximation of the following problem, in the staircase polygon G defined in Section 2.1:

$$\begin{aligned} \Delta u &= 0 \text{ on } G, && (3.3.1)\\ \nu_j u + \overline{\nu}_j u_n^{(1)} &= \nu_j \varphi_j + \overline{\nu}_j \psi_j \text{ on } \gamma_j,\ j = 1,2,...,N, && (3.3.2) \end{aligned}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, ν_j is a parameter taking the values 0 or 1 and $\overline{\nu}_j = 1 - \nu_j$. Furthermore, $u_n^{(1)}$ is the derivative along the inner normal, φ_j and ψ_j are given functions and

$$\begin{aligned} 1 &\leq \sum_{k=1}^{N} \nu_k \leq N, && (3.3.3)\\ \nu_j \varphi_j + \overline{\nu}_j \psi_j &\in C^{6,\lambda}(\gamma_j),\ 0 < \lambda < 1,\ j = 1,2,...,N. && (3.3.4) \end{aligned}$$

We presume that conjugation conditions (3.1.5) are satisfied at the vertices $\dot{\gamma}_j$ whose interior angles are $\pi/2$. It is not required that the boundary functions at the vertices with an interior angle of $\alpha_j \pi \neq \pi/2$ are compatible, however, it is requested that the boundary functions on the adjacent sides of these vertices are algebraic polynomials of the form (2.1.5).

Let $E = \left\{ j : \alpha_j \neq 1/2,\ j = 1,2,...,N \right\}$. Two fixed block sectors are constructed in the same form as in Section 2.1, in the neighbourhood of $\dot{\gamma}_j$, $j \in E$, denoted by $T_j^i = T_j(r_{ji}) \subset G$, $i = 1,2$. The function $Q_j(r_j, \theta_j)$ will have one of the forms (3.2) – (3.9), defined in [1], depending on the nature of the boundary conditions specified on γ_{j-1} and γ_j.

We set, (see [1])

$$\begin{aligned} R(m,m,r,\theta,\eta) &= R(r,\theta,\eta)+(-1)^m R(r,\theta,-\eta), \\ R(1-m,m,r,\theta,\eta) &= R(m,m,r,\theta,\eta)-(-1)^m R(m,m,r,\theta,\pi-\eta), \end{aligned}$$

where

$$R(r,\theta,\eta)=\frac{1-r^2}{2\pi(1-2r\cos(\theta-\eta)+r^2)}$$

is the kernel of the Poisson integral for a unit circle. The kernel is specified as

$$R_j(r_j,\theta_j,\eta)=\lambda_j R\left(\nu_{j-1},\nu_j,\left(\frac{r_j}{r_{j2}}\right)^{\lambda_j},\lambda_j\theta_j,\lambda_j\eta\right),\ j\in E,$$

where

$$\lambda_j=\frac{1}{(2-\nu_{j-1}\nu_j-\overline{\nu}_{j-1}\overline{\nu}_j)\alpha_j}. \tag{3.3.5}$$

We outline the procedure for obtaining the algebraic system of equations, for the numerical solution of problem (3.3.1), (3.3.2).

Let $\Pi_k\subset G_T$, $k=1,2,...,M$, be certain fixed open rectangles with sides a_{1k} and a_{2k} parallel to the x and y axes, and $G\subset\left(\cup_{k=1}^M\Pi_k\right)\cup\left(\cup_{j\in E}T_j^3\right)\subset G$. We use η_k to represent the sides of the rectangle Π_k, V_j denotes the curvilinear part of the boundary of the sector T_j^2 and $t_j=\left(\cup_{k=1}^M\eta_k\right)\cap\overline{T}_j^3$. The overlapping condition is defined the same as in Section 2.1, and the gluing depth is denoted by $\varkappa_0$.

Let Π_k^h be the set of nodes on Π_k, η_k^h is the set of nodes on η_k and $\overline{\Pi}_k^h=\Pi_k^h\cap\eta_k^h$.

Also let η_{k0}^h stand for the set of nodes on $\eta_k \cap G_T$, t_j^h be the set of nodes on t_j and η_{k1}^h be the remaining nodes on η_k^h. Furthermore, we let $\widetilde{\Pi}_k^h$ denote the set of nodes whose distance from the boundary η_{k1}^h of Π_k is $\frac{h}{2}$, $\widetilde{\omega}_k^h$ stands for the set of nodes whose distance from η_{k0}^h or t_j^h is $\frac{h}{2}$ and $\widehat{\Pi}_k^h = \Pi_k^h \setminus \left(\widetilde{\Pi}_k^h \cup \widetilde{\omega}_k^h\right)$. The expressions n, $n(j)$, β_j and V_j^n are defined the same as in Section 2.1.

Hence, we have

$$\omega^{h,n} = \left(\cup_{k=1}^M \eta_{k0}^h\right) \cup \left(\cup_{j\in E} V_j^n\right) \cup \left(\cup_{k=1}^M \widetilde{\omega}_k^h\right), \; \overline{G}_*^{h,n} = \omega^{h,n} \cup \left(\cup_{k=1}^M \overline{\Pi}_k^h\right).$$

Let

$$R_j^{(q)}\left(r_j, \theta_j\right) = \frac{R_j(r_j, \theta_j, \theta_j^q)}{\max\left\{1, \beta_j \sum_{p=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^p)\right\}} \tag{3.3.6}$$

By (3.3.6), it is easy to demonstrate that

$$0 \le R_j^{(q)}\left(r_j, \theta_j\right) \le R_j(r_j, \theta_j, \theta_j^q), \; 0 < \theta_j, \; \theta_j^q < \alpha_j \pi, \; 1 \le q \le n(j), \; j \in E. \tag{3.3.7}$$

Furthermore, as it was stated in [33], there exists positive constants n_0 and $\sigma > 0$, such that for $n \ge n_0$ and $\nu_{j-1} + \nu_j \ge 1$,

$$\max_{\overline{T}_j^3} \beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q) \le \sigma < 1, \tag{3.3.8}$$

and on the basis of (3.3.6) and (3.3.7),

$$0 \le \beta_j \sum_{q=1}^{n(j)} R_j^{(q)}\left(r_j, \theta_j\right) \le 1, \; j \in E, \tag{3.3.9}$$

when $\nu_{j-1} = \nu_j = 0$.

Consider the system of difference equations

$$u_h = Su_h \text{ on } \widehat{\Pi}_k^h, \tag{3.3.10}$$

$$u_h = S_m^* u_h + E_{mh}^*(\varphi_m, \psi_m) \text{ on } \widetilde{\Pi}_k^h, \tag{3.3.11}$$

$$u_h = \overline{\nu}_m S_m u_h + E_{mh}(\varphi_m, \psi_m) \text{ on } \eta_{k1}^h \cap \gamma_m, \tag{3.3.12}$$

$$u_h = \overline{\nu}_m \overline{\nu}_{m+1} \dot{S}_m u_h + \dot{E}_{mh}(\varphi_m, \varphi_{m+1}, \psi_m, \psi_{m+1}) \text{ on } \eta_{k1}^h \cap \gamma_m \cap \gamma_{m+1}, \tag{3.3.13}$$

$$\begin{aligned} u_h(r_j, \theta_j) &= Q_j(r_j, \theta_j) + \\ &+ \beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q) \left[u_h(r_{j2}, \theta_j^q) - Q_j(r_{j2}, \theta_j^q) \right] \text{ on } t_j^h, \end{aligned} \tag{3.3.14}$$

$$u_h = S^4(u_h, \varphi, \psi) \text{ on } \omega^{h,n}, \tag{3.3.15}$$

where $1 \le m \le N$, $1 \le k \le M$, $j \in E$ and Su_h, $S_m^* u_h$, $E_{mh}^*(\varphi_m, \psi_m)$, $S_m u_h$, $E_{mh}(\varphi_m, \psi_m)$, $\dot{S}_m u_h$ and $\dot{E}_{mh}(\varphi_m, \varphi_{m+1}, \psi_m, \psi_{m+1})$ are defined as equations (3.1.10)-(3.1.16) in Section 3.1, respectively.

Theorem 3.3.1 *There is a natural number n_0 such that for all $n \ge n_0$ and $h \in (0, \varkappa_0]$, where $\varkappa_0$ is the gluing depth, system of equations (3.3.10)-(3.3.15) has a unique solu-*

tion.

Proof. Let v_h be a solution of the system of equations

$$v_h = Sv_h \text{ on } \widehat{\Pi}_k^h, \tag{3.3.16}$$

$$v_h = S_m^* v_h \text{ on } \widetilde{\Pi}_k^h, \tag{3.3.17}$$

$$v_h = \overline{\nu}_m S_m v_h \text{ on } \eta_{k1}^h \cap \gamma_m, \tag{3.3.18}$$

$$v_h = \overline{\nu}_m \overline{\nu}_{m+1} \dot{S}_m v_h \text{ on } \eta_{k1}^h \cap \gamma_m \cap \gamma_{m+1}, \tag{3.3.19}$$

$$v_h(r_j, \theta_j) = \beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q) v_h(r_{j2}, \theta_j^q) \text{ on } t_j^h, \tag{3.3.20}$$

$$v_h = S^4 v_h \text{ on } \omega^{h,n}, \tag{3.3.21}$$

where $1 \leq m \leq N$, $1 \leq k \leq M$ and $j \in E$. For the proof of this theorem, it is necessary and sufficient to show that $\max_{\overline{G}_*^{h,n}} |v_h| = 0$. Since the sum of the coefficients in the operators Sv_h, $S_m^* v_h$, $S_m v_h$, $\dot{S}_m v_h$ and $S^4 v_h$ are not more than one, and they are all nonnegative, on the basis of the maximum priciple (see Chapter 4 in [21]), v_h will not take its nonzero maximum value in $\widehat{\Pi}_k^h$, $\widetilde{\Pi}_k^h$, $\omega^{h,n}$, or in $\eta_{k1}^h \cap \gamma_m$, $\eta_{k1}^h \cap \gamma_m \cap \gamma_{m+1}$ when $\overline{\nu}_m = 1$. Hence we consider the nodes in $\cup_{j \in E} t_j^h$. Taking (3.3.7), (3.3.8) and (3.3.9) into account, again by the maximum principle, it is not possible to obtain the nonzero maximum value of v_h in $\cup_{j \in E} t_j^h$ either.

Therefore, the maximum value is attained at $\eta_{k1}^h \cap \gamma_m$ or $\eta_{k1}^h \cap \gamma_m \cap \gamma_{m+1}$ when $\overline{\nu}_m = 0$

and, by system (3.3.16)-(3.3.21), it follows that

$$\max_{\overline{G}_*^{h,n}} |v_h| = 0.$$

■

Let u_h be the solution of the system of equations (3.3.10)-(3.3.15). The function

$$U_h(r_j,\theta_j) = Q_j(r_j,\theta_j) + \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q)\left(u_h(r_{j2},\theta_j^q) - Q_j(r_{j2},\theta_j^q)\right) \quad (3.3.22)$$

is called an approximate solution of the problem (3.3.1),(3.3.2) on the closed block $\overline{T}_j^3$, $j \in E$.

3.4 Error analysis of the new system of Block-Grid Equations

Let

$$\varepsilon_h = u_h - u, \quad (3.4.1)$$

where u_h is the solution of the system (3.3.10)-(3.3.15) and u is the trace of the solution of (3.3.1), (3.3.2) on $\overline{G}_*^{h,n}$. On the basis of (3.3.1), (3.3.2), (3.3.10)-(3.3.15) and (3.4.1), ε_h satisfies the system of equations:

$$\varepsilon_h = S\varepsilon_h + r_h^1 \text{ on } \widehat{\Pi}_k^h, \tag{3.4.2}$$

$$\varepsilon_h = S_m^*\varepsilon_h + r_h^2 \text{ on } \widetilde{\Pi}_k^h, \tag{3.4.3}$$

$$\varepsilon_h = \overline{\nu}_m S_m \varepsilon_h + r_h^3 \text{ on } \eta_{k1}^h \cap \gamma_m, \tag{3.4.4}$$

$$\varepsilon_h = \overline{\nu}_m \overline{\nu}_{m+1} \dot{S}_m \varepsilon_h + r_h^4 \text{ on } \eta_{k1}^h \cap \gamma_m \cap \gamma_{m+1}, \tag{3.4.5}$$

$$\varepsilon_h(r_j, \theta_j) = \beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q) S^4 \varepsilon_h(r_{j2}, \theta_j^q) + r_{jh}^5 \text{ on } t_j^h, \tag{3.4.6}$$

$$\varepsilon_h = S^4\varepsilon_h + r_h^6 \text{ on } \omega^{h,n}, \tag{3.4.7}$$

where $1 \le m \le N$, $1 \le k \le M$, $j \in E$ and

$$r_h^1 = Su - u \text{ on } \cup_{k=1}^M \widehat{\Pi}_k^h, \tag{3.4.8}$$

$$r_h^2 = S_m^* u - u + E_{mh}^*(\varphi_m, \psi_m) \text{ on } \cup_{k=1}^M \widetilde{\Pi}_k^h, \tag{3.4.9}$$

$$r_h^3 = \overline{\nu}_m S_m - u + E_{mh}(\varphi_m, \psi_m) \text{ on } \left(\cup_{k=1}^M \eta_{k1}^h\right) \cap \gamma_m, \tag{3.4.10}$$

$$r_h^4 = \overline{\nu}_m \overline{\nu}_{m+1} \dot{S}_m u - u + \dot{E}_{mh}(\varphi_m, \varphi_{m+1}, \psi_m, \psi_{m+1}) \tag{3.4.11}$$

$$\text{on } \left(\cup_{k=1}^M \eta_{k1}^h\right) \cap \gamma_m \cap \gamma_{m+1},$$

$$r_{jh}^5 = \beta_j \sum_{q=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^q) \left(u_h(r_{j2}, \theta_j^q) - Q_j(r_{j2}, \theta_j^q)\right) -$$

$$- \left(u(r_j, \theta_j) - Q_j(r_j, \theta_j)\right) \text{ on } \cup_{j\in E} t_j^h, \tag{3.4.12}$$

$$r_h^6 = \begin{cases} S^4 u - u \text{ on } \omega_1^h, \\ S^4\left(u - \sum_{k=0}^3 a_k \operatorname{Re} z^k\right) + \left(\sum_{k=0}^3 a_k \operatorname{Re} z^k\right)(P) \text{ on } \omega_2^h, \\ S^4\left(u - \sum_{k=1}^3 b_k \operatorname{Im} z^k\right) + \left(\sum_{k=1}^3 b_k \operatorname{Im} z^k\right)(P) \text{ on } \omega_3^h. \end{cases} \tag{3.4.13}$$

where $P \in \omega_1^h$ if all nodes of evaluation P_k of the matching operator S^4 are in $\overline{G}_*^{h,n}$, $P \in \omega_2^h$ if the side where the nodes of evaluation P_k emerge through has Dirichlet boundary conditions, and $P \in \omega_3^h$ if the side where the nodes of evaluation P_k emerge through has Neumann boundary conditions, and thus $\omega^{h,n} = \omega_1^h \cup \omega_2^h \cup \omega_3^h$.

Analogous to the proof of Lemma 6.2 in [20], there exists a natural number n_0, such that for all $n \geq \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right] + 1\right\}$, $\varkappa > 0$ being a fixed number,

$$\max_{j \in E} \left| r_{jh}^5 \right| \leq c_1 h^4. \tag{3.4.14}$$

Furthermore, as the set $\omega^{h,n} \subset \cup_{k=1}^{M} \Pi_k$, by Theorem 3.2.1, we have

$$\max_{\omega^{h,n}} \left| r_h^6 \right| \leq c_2 h^4. \tag{3.4.15}$$

Theorem 3.4.1 *Assume that conditions (3.3.3), (3.3.4) are satisfied, and the conjugation conditions (3.1.5) hold at the vertices with interior angles of $\pi/2$. Then there exists a natural number n_0 such that for all $n \geq \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right]\right\}$, $\varkappa > 0$ being a fixed number,*

$$\max_{\overline{G}_*^{h,n}} |u_h - u| \leq c h^4. \tag{3.4.16}$$

Proof. Consider an arbitrary rectangle $\Pi_{k^*}^h$, which is one of the overlapping rectangles covering the "nonsingular" part of the domain G with a hexagonal grid, and let $t_{k^*j}^h = \overline{\Pi}_{k^*}^h \cap t_j^h$. Let $t_{k^*j}^h \neq \emptyset$ and assume v_h is a solution of system (3.4.2)-(3.4.7) in the case when r_h^1, r_h^2, r_h^3, r_h^4, r_{jh}^5 and r_h^6 are expressed the same as in (3.4.8)-(3.4.13) in $\overline{\Pi}_{k^*}^h$, but are zero in $\overline{G}_*^{h,n} \diagdown \overline{\Pi}_{k^*}^h$. It is clear that

$$W = \max_{\overline{G}_*^{h,n}} |v_h| = \max_{\overline{\Pi}_{k^*}^h} |v_h|\,. \tag{3.4.17}$$

We represent v_h on $\overline{G}_*^{h,n}$ in the form

$$v_h = \sum_{\chi=1}^{4} v_h^{\chi}, \tag{3.4.18}$$

where the functions $v_h^{\chi}, \chi = 2,3,4,$ are defined on $\overline{\Pi}_{k^*}^h$ as a solution of the system of equations

$$\begin{aligned}
v_h^{\chi} &= S v_h^{\chi} + r_1^{\chi}(h) \text{ on } \widehat{\Pi}_{k^*}^h, &(3.4.19)\\
v_h^{\chi} &= S_m^* v_h^{\chi} + r_1^{\chi}(h) \text{ on } \widetilde{\Pi}_{k^*}^h, &\\
v_h^{\chi} &= \overline{v}_m S_m v_h^{\chi} + r_1^{\chi}(h) \text{ on } \eta_{k^*1}^h \cap \gamma_m, &\\
v_h^{\chi} &= \overline{v}_m \overline{v}_{m+1} \dot{S}_m v_h^{\chi} + r_1^{\chi}(h) \text{ on } \eta_{k^*1}^h \cap \gamma_m \cap \gamma_{m+1}, &\\
v_h^{\chi}(r_j, \theta_j) &= r_2^{\chi}(h) \text{ on } t_{k^*j}^h, &\\
v_h^{\chi} &= r_3^{\chi}(h) \text{ on } \omega^{h,n}. &
\end{aligned}$$

with

$$v_h^{\chi} = 0,\ \chi = 2,3,4 \text{ on } \overline{G}_*^{h,n} \diagdown \overline{\Pi}_{k^*}^h, \tag{3.4.20}$$

$r_1^\chi(h) = 0$ when $\chi = 3,4$ and

$$r_1^\chi(h) = \begin{cases} r_h^1 \text{ on } \widehat{\Pi}_{k^*}^h, \\ r_h^2 \text{ on } \widetilde{\Pi}_{k^*}^h, \\ r_h^3 \text{ on } \eta_{k^*1}^h \cap \gamma_m, \\ r_h^4 \text{ on } \eta_{k^*1}^h \cap \gamma_m \cap \gamma_{m+1}, \end{cases}$$

when $\chi = 2$,

$$r_2^\chi(h) = \begin{cases} 0,\ \chi = 2,4, \\ r_{jh}^5,\ \chi = 3, \end{cases}$$

and

$$r_3^\chi(h) = \begin{cases} 0,\ \chi = 3,4, \\ r_h^6,\ \chi = 2. \end{cases}$$

Hence considering the systems of equations (3.4.18)-(3.4.20), we define a function v_h^1 satisfying

$$\begin{aligned} v_h^1 &= Sv_h^1 \text{ on } \widehat{\Pi}_k^h, \qquad (3.4.21) \\ v_h^1 &= S_m^* v_h^1 \text{ on } \widetilde{\Pi}_k^h, \\ v_h^1 &= \overline{\nu}_m S_m v_h^1 \text{ on } \eta_{k1}^h \cap \gamma_m, \\ v_h^1 &= \overline{\nu}_m \overline{\nu}_{m+1} \dot{S}_m v_h^1 \text{ on } \eta_{k1}^h \cap \gamma_m \cap \gamma_{m+1}, \\ v_h^1(r_j,\theta_j) &= \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q) \sum_{\chi=1}^{4} v_h^\chi(r_{j2},\theta_j^q) \text{ on } t_j^h, \\ v_h^1 &= S^4\left(\sum_{\chi=1}^{4} v_h^\chi\right) \text{ on } \omega^{h,n},\ 1 \le m \le N,\ 1 \le k \le M,\ j \in E, \end{aligned}$$

where the functions v_h^χ, $\chi = 2,3,4$ are presumed to be known.

Taking into account Theorem 2.1.2, (3.4.14), (3.4.15), (3.4.20) and since the rectangles Π_k, $k = 1, 2, ..., M$, are located away from the singular vertices at a distance more than r_{4j}, on the basis of the maximum principle,

$$W_2 = \max_{\overline{G}_*^{h,n}} \left|v_h^2\right| \leq ch^4, \tag{3.4.22}$$

$$W_3 = \max_{\overline{G}_*^{h,n}} \left|v_h^3\right| \leq ch^4, \tag{3.4.23}$$

$$W_4 = \max_{\overline{G}_*^{h,n}} \left|v_h^4\right| \leq ch^4. \tag{3.4.24}$$

The rest of the proof follows analogously to the proof of Theorem 6.3 in [20], by taking into account systems (3.4.19), (3.4.21) and inequalities (3.4.22)-(3.4.24). ■

Theorem 3.4.2 *There exists a natural number n_0 such that for all $n \geq \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right]\right\}$, $\varkappa > 0$ being a fixed number, the approximation of problem (3.3.1), (3.3.2) on the blocks $\overline{T}_j^3, j \in E$, by the function (3.3.22) satisfies the following inequalities:*

$$\left|U_h(r_j, \theta_j) - u(r_j, \theta_j)\right| \leq c_0 h^4 \text{ on } \overline{T}_j^3,\ j \in E, \tag{3.4.25}$$

We also have, for the case when λ_j is an integer, ν_{j-1} and ν_j are 0 or 1, and $p \geq \lambda_j$, or the case $\nu_{j-1} = \nu_j = 0$ when $p = 0$ and λ_j takes any value, the inequality

$$\left|\frac{\partial^p}{\partial x^{p-q} \partial y^q}\left(U_h(r_j, \theta_j) - u(r_j, \theta_j)\right)\right| \leq c_p h^4 \text{ on } \overline{T}_j^3. \tag{3.4.26}$$

Furthermore, for any λ_j, if $\nu_{j-1} + \nu_j \geq 1$, $0 \leq p < \lambda_j$ or $\nu_{j-1} = \nu_j = 0$, $1 \leq p < \lambda_j$, we have

$$\left|\frac{\partial^p}{\partial x^{p-q} \partial y^q}\left(U_h(r_j, \theta_j) - u(r_j, \theta_j)\right)\right| \leq c_p h^4 / r_j^{p-\lambda_j} \text{ on } \overline{T}_j^3. \tag{3.4.27}$$

Finally, for noninteger λ_j, any ν_{j-1}, ν_j, and $p > \lambda_j$, we have

$$\left|\frac{\partial^p}{\partial x^{p-q}\partial y^q}\left(U_h(r_j,\theta_j)-u(r_j,\theta_j)\right)\right| \le c_p h^4/r_j^{p-\lambda_j} \text{ on } \overline{T}_j^3 \setminus \dot{\gamma}_j, \tag{3.4.28}$$

where $0 \le q \le p$, λ_j is defined the same as (3.3.5) and u is the solution of the problem (3.3.1), (3.3.2).

Proof. The proof is obtained on the basis of (3.3.22), Lemma 2.1.1, (3.3.8), (3.3.9) and Theorem 3.4.1, and follows by analogy to Theorem 6.4 in [20]. ■

3.5 The use of Schwarz's alternating method

As stated in Section 2.5 and [15], it is first required to justify the method of finding a solution of the system of equations (3.3.10)-(3.3.15). The classes $\Phi_\tau, \tau = 1,2,...,\tau^*$, of rectangles $\Pi_k, k = 1,2,...,M$, are defined the same as in [15].

For the solution of the system of equations (3.3.10)-(3.3.15), Schwarz's alternating method is carried out in the same form as in Section 2.5. Again, we start with a zero approximation $u_h^{(0)}$ to the exact solution u_h of the system (3.3.10)-(3.3.15), and we obtain the sequence $u_h^{(1)}, u_h^{(2)}, ...$ as follows:

$$
\begin{aligned}
u_h^{(m)}(r_j,\theta_j) &= Q_j(r_j,\theta_j)+ \\
&+\beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q)\left[S^4(u_h^{(m-1)}(r_{j2},\theta_j^q),\varphi)-Q_j(r_{j2},\theta_j^q)\right] \\
&\text{ on } t_j^h, \qquad (3.5.1)
\end{aligned}
$$

$$u_h^{(m)} = S^4 u_h^{(m-1)} \text{ on } \omega^{h,n} \qquad (3.5.2)$$

$$u_h^{(m)} = S u_h^{(m)} \text{ on } \widehat{\Pi}_k^h, \qquad (3.5.3)$$

$$u_h^{(m)} = S_p^* u_h^{(m)} + E_{ph}^*(\varphi_p,\psi_p) \text{ on } \widetilde{\Pi}_k^h, \qquad (3.5.4)$$

$$u_h^{(m)} = \overline{\nu}_p S_p u_h^{(m)} + E_{ph}(\varphi_p,\psi_p) \text{ on } \eta_{k1}^h \cap \gamma_p, \qquad (3.5.5)$$

$$
\begin{aligned}
u_h^{(m)} &= \overline{\nu}_p \overline{\nu}_{p+1} \dot{S}_p u_h^{(m)} + \dot{E}_{ph}(\varphi_p,\varphi_{p+1},\psi_p,\psi_{p+1}) \\
&\text{ on } \eta_{k1}^h \cap \gamma_p \cap \gamma_{p+1}. \qquad (3.5.6)
\end{aligned}
$$

where $1 \le k \le M$, $1 \le p \le N$, $j \in E$, $m = 1,2,\ldots$.

Remark 3.5.1 *Theorem 2.5.1 remains valid and is proved by analogy to Theorem 7.1 in [20], with the system under consideration being (3.5.1)-(3.5.6).*

Chapter 4

A FOURTH ORDER APPROXIMATION ON A SPECIAL TYPE OF POLYGON WHEN THE BOUNDARY FUNCTIONS ARE FROM $C^{4,\lambda}$

4.1 Boundary value problem on a special type of polygon

Let D be an open simply-connected polygon with interior angles $\alpha_j\pi$, $\alpha_j \in \left\{\frac{1}{3}, \frac{2}{3}, 1, 2\right\}$, let the sides of this polygon be denoted by γ_j, $j = 1,2,...,N$,enumerated counterclockwise, and $\dot{\gamma}_j = \gamma_j \cap \gamma_{j-1}$ be the vertices of D.

The following boundary value problem is taken into consideration in the domain D:

$$\Delta u = 0 \text{ on } D, \tag{4.1.1}$$

$$u = \varphi_j, \text{ on } \gamma_j, \ j = 1,2,...,N, \tag{4.1.2}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$, φ_j, $j = 1,2,...,N$, are given functions, and

$$\varphi_j \in C^{4,\lambda}\left(\gamma_j\right), \ 0 < \lambda < 1, \ 1 \leq j \leq N. \tag{4.1.3}$$

In addition, at the vertices $\dot{\gamma}_j$, for $\alpha_j = 1/3$, the following conjugation conditions are satisfied:

$$\varphi_{j-1}^{(3p)}\left(s_j\right) = \varphi_j^{(3p)}\left(s_j\right), \ p = 0,1. \tag{4.1.4}$$

It is not required that the boundary functions at the vertices with an interior angle of $\alpha_j\pi \neq \pi/3$ are compatible, however, same as before, it is requested that the boundary functions on the adjacent sides of these vertices are algebraic polynomials of the form (2.1.5).

We apply the BGM given in detail in [15] and Section 2.1 in the domain $\overline{D}$. The following is required for this application:

Let $E = \{j : \alpha_j \neq 1/3,\ j = 1,2,...,N\}$. Two fixed block sectors are constructed in the same form as in Section 2.1, in the neighbourhood of $\dot{\gamma}_j,\ j \in E$, denoted by $T_j^i = T_j(r_{ji}) \subset G$, $i = 1,2$, and we assume the function $Q_j(r_j,\theta_j)$ has the form (2.1.6), (2.1.7). Furthermore, we let $R_j(r_j,\theta_j,\eta)$ be defined as (2.1.8). The application of this method requires the construction of two more sectors T_j^3 and T_j^4, where $0 < r_{j4} < r_{j3} < r_{j2}$. Let $D_T = D \backslash \left(\cup_{j\in E}\overline{T}_j^4\right)$. The following steps are taken for the realization:

1) We blockade the singular corners $\dot{\gamma}_j,\ j \in E$, by the double sectors $T_j^i(r_{ji})$, $i = 2,3$, with $T_k^2 \cap T_l^3 = \emptyset$, $k \neq l$, $k,l \in E$, and cover the polygon D by overlapping parallelograms denoted by $D_l',\ l = 1,2,...,M$, and sectors $T_j^3,\ j \in E$, such that the distance from $\overline{D_l'}$ to $\dot{\gamma}_j$ is not less that r_{j4} for all $l = 1,2,...,M$.

2) On the parallelograms $\overline{D_l'}$, $l = 1,2,...,M$, we use the 7-point scheme for the hexagonal grid with the step size $h_l \leq h$, h a parameter, for the approximation of Laplace's equation, and the singular parts $\overline{T}_j^3,\ j \in E$, are approximated by using the harmonic function defined in Lemma 2.1.1.

The rest of the description follows by analogy to the description given in [15] and Section 2.1.

In order to obtain a numerical solution of problem (4.1.1), (4.1.2), an algebraic system of equations is formed using the following notation:

Let $D'_l \subset D_T$, $l = 1,2,...,M$, be open fixed parallelograms and $D \subset \left(\cup_{l=1}^{M} D'_l\right) \cup \left(\cup_{j \in E} T_j^3\right) \subset D$. We denote by η_l the boundary of D'_l, $l = 1,2,...M$, by V_j the curvilinear part of the boundary of the sector T_j^2, and let $t_j = \left(\cup_{l=1}^{M} \eta_l\right) \cap \overline{T}_j^3$. For the arrangement of the parallelograms D'_l, $l = 1,2,...,M$, it is required that any point P lying on $\eta_l \cap D_T$, $1 \leq l \leq M$, or lying on $V_j \cap D$, $j \in E$, falls inside at least on of the parallelograms $D'_{l(P)}$, $1 \leq l(P) \leq M$, depending on P, where the distance from P to $D_T \cap \eta_{l(P)}$ is not less than some constant κ_0 independent of P. κ_0 is called the gluing depth of the parallelograms D'_l, $l = 1,2,...,M$.

Let $h \in (0, \kappa_0/4]$ be a parameter, and define a hexagonal grid on D'_l, $1 \leq l \leq M$, with maximal positive step $h_l \leq h$, such that the boundary η_l lies entirely on the grid lines. Let D'_{lh} be the set of grid nodes on D'_l, η_l^h be the set of nodes on η_l, and let $\overline{D'_{lh}} = D'_{lh} \cup \eta_l^h$. Furthermore, η_{l0}^h denotes the set of nodes on $(\eta_l \cap D_T) \setminus t_j$, $\eta_{l1}^h = \eta_l^h \backslash \eta_{l0}^h$ and t_j^h denotes the set of nodes on t_j. Finally we have

$$\omega^{h,n} = \left(\cup_{l=1}^{M} \eta_{l0}^h\right) \cup \left(\cup_{j \in E} V_j^n\right), \; \overline{D}_*^{h,n} = \omega^{h,n} \cup \left(\cup_{l=1}^{M} \overline{D'}_{lh}\right).$$

Consider the system of difference equations

$$u_h = Su_h \text{ on } D'_{lh}, \tag{4.1.5}$$

$$u_h = \varphi \text{ on } \eta_{l1}^h, \tag{4.1.6}$$

$$
\begin{aligned}
u_h(r_j,\theta_j) &= Q_j(r_j,\theta_j)+ \\
&+\beta_j \sum_{k=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^k)\left[u_h(r_{j2},\theta_j^k)-Q_j(r_{j2},\theta_j^k)\right] \text{ on } t_j^h,
\end{aligned}
\tag{4.1.7}
$$

$$
u_h = S^4\left(u_h,\varphi\right) \text{ on } \omega^{h,n}, \tag{4.1.8}
$$

where $1 \le l \le M$, $j \in E$, and the operator S is defined as

$$
\begin{aligned}
Su(x,y) &= \frac{1}{6}\left(u\left(x+h,y\right)+u\left(x+\frac{h}{2},y+\frac{\sqrt{3}}{2}h\right)+u\left(x-\frac{h}{2},y+\frac{\sqrt{3}}{2}h\right)\right. \\
&\left.+u\left(x-h,y\right)+u\left(x-\frac{h}{2},y-\frac{\sqrt{3}}{2}h\right)+u\left(x+\frac{h}{2},y-\frac{\sqrt{3}}{2}h\right)\right).
\end{aligned}
$$

The solution of system (4.1.5)-(4.1.8) is the approximation of the solution of problem (4.1.1), (4.1.2) on $\overline{D}_*^{h,n}$.

Theorem 4.1.1 *There is a natural number n_0 such that for all $n \ge n_0$ and $h \le \kappa_0/4$, where κ_0 is the gluing depth, the system of equations (4.1.5)-(4.1.8) has a unique solution.*

Proof. Let w_h be a solution of the system of equations

$$
\begin{aligned}
w_h &= Sw_h \text{ on } D'_{lh}, \qquad (4.1.9) \\
w_h &= 0 \text{ on } \eta_{l1}^h, \qquad (4.1.10)
\end{aligned}
$$

$$
w_h(r_j,\theta_j) = \beta_j \sum_{q=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^q) w_h(r_{j2},\theta_j^q) \text{ on } t_j^h, \tag{4.1.11}
$$

$$
w_h = S^4\left(w_h,0\right) \text{ on } \omega^{h,n}, \tag{4.1.12}
$$

where $1 \le l \le M$ and $j \in E$. For the proof of this theorem, it is necessary and sufficient to show that $\max_{\overline{D}_*^{h,n}} |w_h| = 0$. Since the sum of the coefficients in the operators Sw_h, and $S^4 w_h$ do not exceed one, and they are all positive, by the maximum principle (see Chapter 4 in [21]), w_h will not take its nonzero maximum value on D'_{lh}, or $\omega^{h,n}$. Hence we consider the nodes in $\cup_{j\in E} t_j^h$. Taking (2.1.10) and (2.1.25) into account, again by the maximum principle, for all $n \ge n_0$ (n_0 is the given integer defined in Section 2.1), it is not possible to obtain the nonzero maximum value of w_h in $\cup_{j\in E} t_j^h$ either.

Therefore, the maximum value is attained at η_{l1}^h. As $\overline{D}_*^{h,n}$ is connected, by system (4.1.9)-(4.1.12) it follows that

$$\max_{\overline{D}_*^{h,n}} |w_h| = 0.$$

■

Next, the numerical solution in the "singular" parts of the domain is considered. For the approximation of problem (4.1.1), (4.1.2) on the closed block $\overline{T}_j^3$, $j \in E$, the function $U_h(r_j, \theta_j)$, which is defined as (2.1.26), is applied.

4.2 Error analysis of the 7-point approximation on the parallelogram D'

Let D' be one of the parallelograms covering the "nonsingular" part of the polygon D defined in Section 4.1. The boundaries of the parallelogram D' are denoted by γ'_j, enumerated counterclockwise starting from left, including the ends, $\dot{\gamma}'_j = \gamma'_{j-1} \cap \gamma'_j$, $j = 1,2,3,4$, denotes the vertices of D', $\gamma' = \cup_{j=1}^4 \gamma'_j$ and $\overline{D'} = D' \cup \gamma'$. Furthermore $\gamma \cap \gamma' \neq \emptyset$, but the vertices $\dot{\gamma}'_m$ with an interior angle of $\alpha_m \pi \neq \pi/3$ are located either inside of D, or on the interior of a side γ_m of D, $1 \le m \le N$. We define the open parallelogram D' as

$D' = \left\{(x,y) : 0 < y < a,\ d - y/\sqrt{3} < x < e - y/\sqrt{3}\right\}$, and the boundary value problem (4.1.1)-(4.1.4) is considered on D' :

$$\Delta v = 0 \text{ on } D', \tag{4.2.1}$$

$$v = \psi_j, \text{ on } \gamma'_j,\ j = 1,2,3,4, \tag{4.2.2}$$

where ψ_j are the values of the solution of problem (4.1.1)-(4.1.4) on γ'.

Let $h > 0$, where $(e-d)/h \ge 2$, $a/\sqrt{3}h \ge 2$ are integers. We assign to D' a hexagonal grid of the form

$D'_h = \left\{(x,y) \in D' : x = \frac{h}{2}(1-l) + kh,\ y = l\frac{\sqrt{3}h}{2},\ k,l = 0,\pm1,\pm2,\pm3,...\right\}$. Let γ'_{jh} be the set of nodes on the interior of γ'_j, and

$$\dot{\gamma}'_{jh} = \gamma'_{j-1} \cap \gamma'_j,\ \gamma'_h = \cup_{j=1}^{4}\gamma'_{jh},\ j = 1,2,3,4,$$
$$\overline{D'_h} = D'_h \cup \gamma'_h.$$

We consider the system of finite difference equations:

$$v_h = Sv_h \text{ on } D'_h \tag{4.2.3}$$

$$v_h = \psi_j \text{ on } \gamma'_{jh},\ j = 1,2,3,4, \tag{4.2.4}$$

where

$$
\begin{aligned}
Sv(x,y) &= \frac{1}{6}\left(v(x+h,y)+v\left(x+\frac{h}{2},y+\frac{\sqrt{3}}{2}h\right)+v\left(x-\frac{h}{2},y+\frac{\sqrt{3}}{2}h\right)\right. \\
&\left. +v(x-h,y)+v\left(x-\frac{h}{2},y-\frac{\sqrt{3}}{2}h\right)+v\left(x+\frac{h}{2},y-\frac{\sqrt{3}}{2}h\right)\right) \quad (4.2.5)
\end{aligned}
$$

Since expression (4.2.5) has nonnegative coefficients and their sum is equal to 1, the solution of system (4.2.3), (4.2.4) exists and is unique (see [21]).

Lemma 4.2.1 *Let*

$$\psi_j(s) \in C^{4,\lambda}(\gamma_j'),\ 0<\lambda<1, \tag{4.2.6}$$

and

$$\psi_{j-1}^{(3p)}(s_j) = \psi_j^{(3p)}(s_j),\ p=0,1, \tag{4.2.7}$$

be satisfied on the vertices whose interior angles are $\alpha_j\pi = \pi/3$, where $j = 1,2,3,4$. Then the solution of problem (4.2.1), (4.2.2)

$$v \in C^{4,\lambda}(\overline{D'}) \tag{4.2.8}$$

Proof. The closed parallelogram $\overline{D'}$ lies inside the polygon D defined in Section 4.1 and the vertices $\dot{\gamma}_m'$ with an interior angle of $\alpha_m\pi \neq \pi/3$ are located either inside of D or on the interior of a side γ_m of D, $1 \leq m \leq N$. Since the boundary functions $\varphi_j, j = 1,2,3,4$, in problem (4.1.1), (4.1.2) satisfy conditions (4.1.3) and (4.1.4), (4.2.8) follows from the results in [34]. ■

Let $D'_{h,k}$ be the set of nodes whose distance from the point $P \in D'_h$ to γ' is $\frac{\sqrt{3}}{2}kh$, $1 \leq k \leq a^*$, where $a^* = \left[\frac{d_t}{(\sqrt{3}h/2)}\right]$, $[c]$ denotes the integer part of c, and d_t is the minimum of the half-lengths of the sides of the parallelogram.

Lemma 4.2.2 *Let $w_h^k \neq const.$ be the solution of the system of equations*

$$\begin{aligned} w_h^k &= Sw_h^k + f_h^k \text{ on } D'_{h,k}, \\ w_h^k &= Sw_h^k \text{ on } D'_h \backslash D'_{h,k}, \\ w_h^k &= 0 \text{ on } \gamma'_h, \end{aligned}$$

and $z_h^k \neq const.$ be the solution of the system of equations

$$\begin{aligned} z_h^k &= Sz_h^k + g_h^k \text{ on } D'_{h,k}, \\ z_h^k &= Sz_h^k \text{ on } D'_h \backslash D'_{h,k}, \\ z_h^k &= 0 \text{ on } \gamma'_h, \end{aligned}$$

where $1 \leq k \leq a^*$. If $\left|f_h^k\right| \leq g_h^k$, then

$$\left|w_h^k\right| \leq z_h^k,\ 1 \leq k \leq a^*. \tag{4.2.9}$$

Proof. The proof follows analogously to the proof of the comparison theorem given in [21]. ■

Lemma 4.2.3 *Let v be the trace of the solution of problem (4.2.1), (4.2.2) on $\overline{D'_h}$, and v_h be the solution of system (4.2.3), (4.2.4). If*

$$\psi_j(s) \in C^{4,\lambda}(\gamma'_j),\ 0 < \lambda < 1,\ j = 1,2,3,4,$$

and

$$\psi_{j-1}^{(3p)}(s_j) = \psi_j^{(3p)}(s_j),\ p = 0,1,$$

on the vertices with an interior angle of $\alpha_j\pi = \pi/3,\ j = 1,2,3,4$, then

$$\max_{\overline{D'_h}} |v - v_h| \leq ch^4. \tag{4.2.10}$$

Proof. Let $\varepsilon_h = v_h - v$ on $\overline{D'_h}$. Clearly

$$\varepsilon_h = S\varepsilon_h + (Sv - v) \text{ on } D'_h, \tag{4.2.11}$$

$$\varepsilon_h = 0 \text{ on } \gamma'_h. \tag{4.2.12}$$

Let D'_{1h} contain the set of nodes whose distance from the boundary γ' is $\frac{\sqrt{3}h}{2}$, and hence for $(x,y) \in D'_{1h}$, $(x+sH, y+sK) \in \overline{D'}$ for $0 \leq s \leq 1$, $H = \pm\frac{h}{2}, \pm h$, $K = 0, \pm\frac{\sqrt{3}h}{2}$, $H^2 + K^2 > 0$, and $D'_{2h} = D'_h \backslash D'_{1h}$.

Moreover, let

$$\varepsilon_h = \varepsilon_h^1 + \varepsilon_h^2. \tag{4.2.13}$$

We rewrite problem (4.2.11), (4.2.12) as

$$\begin{aligned}
\varepsilon_h^1 &= S\varepsilon_h^1 + (Sv - v) \text{ on } D'_{1h}, \\
\varepsilon_h^1 &= S\varepsilon_h^1 \text{ on } D'_{2h}, \\
\varepsilon_h^1 &= 0 \text{ on } \gamma'_h, \qquad (4.2.14)
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon_h^2 &= S\varepsilon_h^2 \text{ on } D'_{1h}, \\
\varepsilon_h^2 &= S\varepsilon_h^2 + (Sv - v) \text{ on } D'_{2h}, \\
\varepsilon_h^2 &= 0 \text{ on } \gamma'_h. \qquad (4.2.15)
\end{aligned}$$

In order to obtain an estimation for $Sv - v$ on D'_{1h}, we use Taylor's formula. On the basis of Lemma 4.2.1, we have

$$|Sv - v| \le c_3 M_4 h^4 \text{ on } D'_{1h}, \qquad (4.2.16)$$

where

$$M_q = \sup_{(x,y)\in D'} \left\{ \left| \frac{\partial^q u(x,y)}{\partial x^p \partial y^{q-p}} \right|, p = 0, 1, ..., q \right\}.$$

Since at least two values of ε_h^1 in $S\varepsilon_h^1$ are lying on the boundary γ'_h, on which $\varepsilon_h^1 = 0$, from (4.2.14), (4.2.16) and the maximum principle (see [21]), we obtain

$$\max_{\overline{D'_h}} \left|\varepsilon_h^1\right| \leq \frac{2}{3} \max_{\overline{D'_h}} \left|\varepsilon_h^1\right| + c_3 M_4 h^4.$$

Hence

$$\max_{\overline{D'_h}} \left|\varepsilon_h^1\right| \leq c_4 h^4, \tag{4.2.17}$$

where $c_4 = 3c_3M_4$.

Next, we consider the estimation of ε_h^2. Let $D'_{2h,k}$ be the set of nodes whose distance from the point $P \in D'_{2h}$ to γ' is $\frac{\sqrt{3}}{2}kh$, $2 \leq k \leq a^*$, where $a^* = \left[\frac{d_t}{(\sqrt{3}h/2)}\right]$, $[c]$ denotes the integer part of c, and d_t is the minimum of the half-lengths of the sides of the parallelogram. Furthermore, $D'_{2h,1} \equiv D'_{1h}$ and $D'_{2h,0} \equiv \gamma'_h$. Since the vertices with $\alpha_j = \frac{1}{3}$ of the parallelogram D' are never used as a node of the hexagonal grid for the estimation of $|Sv - v|$ on $D'_{2h,k}$, $2 \leq k \leq a^*$, we use the inequalities

$$\max_{p+q=6} \left|\frac{\partial^6 v(x,y)}{\partial x^p \partial y^q}\right| \leq c_0 \rho^{\lambda-2} \text{ on } \overline{D'} \backslash \gamma'_m,$$

for the sixth order derivatives, where ρ is the distance from $(x,y) \in D'$ to γ'_m. Hence, we obtain

$$|Sv - v| \leq c_5 h^6/(kh)^{2-\lambda} \text{ on } D'_{2h,k},\ 2 \leq k \leq a^*. \tag{4.2.18}$$

Consider a majorant function of the form

$$Y_k = \begin{cases} 3m \text{ if } P \in D'_{2h,m},\ 0 \le m \le k, \\ 3k \text{ if } P \in D'_{2h,m},\ m > k. \end{cases} \tag{4.2.19}$$

Hence Y_k is a solution of the finite difference problem

$$\begin{aligned} Y_k &= SY_k + \mu_k \text{ on } D'_{2h,k}, \\ Y_k &= SY_k \text{ on } D'_h \setminus D'_{2h,k}, \\ Y_k &= 0 \text{ on } \gamma'_h. \end{aligned} \tag{4.2.20}$$

where $1 \le \mu_k \le 3$, $1 \le k \le a^*$.

We represent the solution of system (4.2.15) as the sum of the solution of the following subsystems:

$$\begin{aligned} \varepsilon^2_{h,k} &= S\varepsilon^2_{h,k} + \mu'_k \text{ on } D'_{2h,k}, \\ \varepsilon^2_{h,k} &= S\varepsilon^2_{h,k} \text{ on } D'_h \setminus D'_{2h,k}, \\ \varepsilon^2_{h,k} &= 0 \text{ on } \gamma'_h, \end{aligned} \tag{4.2.21}$$

where $1 \le k \le a^*$, $\mu'_k = 0$ when $k = 1$ and $\left|\mu'_k\right| \le c_6 \frac{h^{4+\lambda}}{k^{2-\lambda}}$ when $k = 2, 3, ..., a^*$.

By (4.2.20), (4.2.21) and Lemma 4.2.2, follows that

$$\left|\varepsilon^2_{h,k}\right| \le c_6 \frac{h^{4+\lambda}}{k^{2-\lambda}} Y_k. \tag{4.2.22}$$

Hence, by taking (4.2.21) and (4.2.22) into consideration, we have

$$\begin{aligned} \max_{D'_h} |\varepsilon_h^2| &\le \sum_{k=1}^{a^*} \varepsilon_{h,k}^2 \le \sum_{k=1}^{a^*} c_6 \frac{h^{4+\lambda}}{k^{2-\lambda}} Y_k \\ &\le 3c_6 h^{4+\lambda} \sum_{k=1}^{a^*} \frac{1}{k^{1-\lambda}} \le c_7 h^4. \end{aligned} \quad (4.2.23)$$

On the basis of (4.2.13), (4.2.17), and (4.2.23), we have estimation (4.2.10). ■

4.3 Error estimation of the Block-Grid equations on $\overline{D}$

Let

$$\varepsilon_h = u_h - u, \quad (4.3.1)$$

where u_h is the solution of the system (4.1.5)-(4.1.8), and u is the trace of the solution of problem (4.1.1), (4.1.2) on $\overline{D}_*^{h,n}$. It is easy to show that (4.3.1) satisfies the system of equations

$$\begin{aligned} \varepsilon_h &= S\varepsilon_h + r_h^1 \text{ on } D'_{lh}, \\ \varepsilon_h &= 0 \text{ on } \eta_{l1} \cap \gamma_m, \\ \varepsilon_h(r_j, \theta_j) &= \beta_j \sum_{k=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^k) \varepsilon_h(r_{j2}, \theta_j^k) + r_{jh}^2 \text{ on } t_j^h, \\ \varepsilon_h &= S^4(\varepsilon_h, 0) + r_h^3 \text{ on } \omega^{h,n}, \end{aligned} \quad (4.3.2)$$

where $1 \leq m \leq N$, $1 \leq l \leq M$, $j \in E$, and

$$r_h^1 = Su - u \text{ on } \cup_{l=1}^{M} D'_{lh}, \tag{4.3.3}$$

$$r_{jh}^2 = \beta_j \sum_{k=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^k) \left[u_h(r_{j2}, \theta_j^k) - Q_j(r_{j2}, \theta_j^k) \right] - (u(r_j, \theta_j) - Q_j(r_j, \theta_j)) \text{ on } \cup_{j \in E} t_j^h, \tag{4.3.4}$$

$$r_h^3 = S^4(u, \varphi) - u \text{ on } \omega^{h,n}. \tag{4.3.5}$$

Lemma 4.3.1 *Let the boundary functions φ_j, $j = 1,2,3,4$, in problem (4.1.1), (4.1.2) satisfy conditions (4.1.3), (4.1.4). Then*

$$\max_{\omega^{h,n}} \left| r_h^3 \right| \leq c_5 h^4, \tag{4.3.6}$$

where $\varphi = \cup_{j=1}^{N} \varphi_j$.

Proof. The function $S^4(u, \varphi)$ is defined as equation (3.14) in [31]. Keeping in mind the gluing depth $\varkappa_0$ for the positioning of the points in $\omega^{h,n}$, conditions (4.1.3), (4.1.4) and estimation (4.64) in [34], estimation (4.3.6) follows. ■

Lemma 4.3.2 *There exists a natural number n_0 such that for all $n \geq \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right] + 1\right\}$, $\varkappa > 0$ being a fixed number,*

$$\max_{j \in E} \left| r_{jh}^2 \right| \leq c_6 h^4.$$

Proof. The proof follows by analogy to the proof of Lemma 6.2 in [20]. ■

Theorem 4.3.3 *Assume that conditions (4.1.3), (4.1.4) hold. Then there exists a nat-*

ural number n_0 such that for all $n \geq \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right] + 1\right\}$, $\varkappa > 0$ being a fixed number,

$$\max_{\overline{D}_*^{h,n}} |u_h - u| \leq c_8 h^4. \tag{4.3.7}$$

Proof. Consider an arbitrary parallelogram D'_{l^*} and let $t^h_{l^*j} = \overline{D}'_{l^*} \cap t^h_j$. Assume that $t^h_{l^*j} \neq \emptyset$, z_h is the solution of system (4.3.2), and r^1_h, r^2_{jh}, r^3_h are defined the same as (4.3.3)-(4.3.5) on D'_{l^*}, but are zero on $\overline{D}_*^{h,n} \backslash D'_{l^*}$. Hence,

$$V = \max_{\overline{D}_*^{h,n}} |z_h| = \max_{\overline{D}'_{l^*}} |z_h|. \tag{4.3.8}$$

We represent the function z_h as

$$z_h = \sum_{q=1}^{4} z_h^q, \tag{4.3.9}$$

where

$$\begin{aligned} z_h^2 &= S z_h^2 + r_h^1 \text{ on } D'_{l^*}, \\ z_h^2 &= 0 \text{ on } \eta^h_{l^*1} \cap \gamma_m, \\ z_h^2 &= 0 \text{ on } t^h_{l^*j}, \\ z_h^2 &= 0 \text{ on } \omega^{h,n}, \end{aligned} \tag{4.3.10}$$

$$
\begin{aligned}
z_h^3 &= Sz_h^3 \text{ on } D'_{l^*}, \\
z_h^3 &= 0 \text{ on } \eta_{l^*1}^h \cap \gamma_m, \\
z_h^3 &= r_{jh}^2 \text{ on } t_{l^*j}^h, \\
z_h^3 &= 0 \text{ on } \omega^{h,n}, \qquad (4.3.11)
\end{aligned}
$$

$$
\begin{aligned}
z_h^4 &= Sz_h^4 \text{ on } D'_{l^*}, \\
z_h^4 &= 0 \text{ on } \eta_{l^*1}^h \cap \gamma_m, \\
z_h^4 &= 0 \text{ on } t_{l^*j}^h, \\
z_h^4 &= r_h^3 \text{ on } \omega^{h,n}, \qquad (4.3.12)
\end{aligned}
$$

and

$$
z_h^q = 0, q = 2,3,4, \text{ on } \overline{D}_*^{h,n} \backslash D'_{l^*}. \qquad (4.3.13)
$$

Hence by (4.3.9)-(4.3.13), z_h^1 satisfies the system of equations

$$
\begin{aligned}
z_h^1 &= Sz_h^1 \text{ on } D'_l, \\
z_h^1 &= 0 \text{ on } \eta_{l1}^h \cap \gamma_m, \\
z_h^1 &= \beta_j \sum_{k=1}^{n(j)} R_j(r_j, \theta_j, \theta_j^k) \sum_{q=1}^{4} z_h^q(r_{j2}, \theta_j^k) \text{ on } t_{lj}^h, \\
z_h^1 &= S^4 \left(\sum_{q=1}^{4} z_h^q \right) \text{ on } \omega^{h,n}, \qquad (4.3.14)
\end{aligned}
$$

where $1 \le m \le N$, $1 \le l \le M$, $j \in E$, and the functions z_h^q, $q = 2,3,4$, are assumed to be known.

As the solution of system (4.3.10), z_h^2, is the error function of the finite difference solution with step size $h_{l^*} \le h$, of system (4.2.3), (4.2.4), by (4.3.13), the maximum principle and Lemma 4.2.3, we have

$$V_2 = \max_{\overline{D}_*^{h,n}} \left| z_h^2 \right| \le c_9 h^4. \tag{4.3.15}$$

Also, for the solutions of systems (4.3.11) and (4.3.12), as the operator S has coefficients which are nonnegative and their sum do not exceed one, by the maximum principle, (4.3.13), Lemma 4.3.1 and Lemma 4.3.2, we obtain the inequalities

$$V_3 = \max_{\overline{D}_*^{h,n}} \left| z_h^3 \right| \le c_{10} h^4, \tag{4.3.16}$$

$$V_4 = \max_{\overline{D}_*^{h,n}} \left| z_h^4 \right| \le c_{11} h^4. \tag{4.3.17}$$

Now we consider the solution of v_h^1. Taking into consideration (2.1.25), (4.3.14), the maximum principle, and the gluing condition of D'_l, $l = 1,2,...,M$, T_j^2, $j \in E$, for all $n \ge \max\left\{n_0, \left[\ln^{1+\varkappa} h^{-1}\right] + 1\right\}$, $\varkappa > 0$ being a fixed number, we have the inequality

$$V_1 = \max_{\overline{D}_*^{h,n}} \left| z_h^1 \right| \le \lambda^* V + \sum_{q=2}^{4} \max_{\overline{D}_*^{h,n}} \left| z_h^q \right|, \tag{4.3.18}$$

where $0 < \lambda^* < 1$. By (4.3.8), (4.3.9), (4.3.15), (4.3.16), (4.3.17) and (4.3.18), we have

$$V = \max_{\overline{D}_*^{h,n}} |z_h| \le c_{12} h^4.$$

Hence (4.3.7) follows. ■

Remark 4.3.4 *Let u_h be the solution of the system of equations (4.1.5)-(4.1.8) and let an approximate solution of problem (4.1.1),(4.1.2) be found on blocks $\overline{T}_j^3$, $j \in E$, by (2.1.26). Then Theorem 2.4.2 holds and the proof of the Theorem follows by analogy to the proof of Theorem 2 in[15], by taking estimation* (4.3.7) *into account.*

4.4 Schwarz's alternating method for the solution of block-grid equations in D

For the approximation of the solution of problem (4.1.1), (4.1.2), we first of all consider the solution in $\overline{D}_*^{h,n}$. Thus, we need to apply Schwarz's alternating procedure for the numerical solution of the system of equations (4.1.5)-(4.1.8). The procedure follows by analogy to the method described in Section 2.5, with the following system under consideration:

$$\begin{aligned} u_h^{(m)}(r_j,\theta_j) &= Q_j(r_j,\theta_j)+ \\ &\quad +\beta_j \sum_{k=1}^{n(j)} R_j(r_j,\theta_j,\theta_j^k)\left[S^4(u_h^{(m-1)}(r_{j2},\theta_j^k),\varphi)-Q_j(r_{j2},\theta_j^k)\right] \\ &\quad \text{on } t_j^h, \end{aligned} \tag{4.4.1}$$

$$u_h^{(m)} = S^4 u_h^{(m-1)} \text{ on } \omega^{h,n} \tag{4.4.2}$$

$$u_h^{(m)} = S u_h^{(m)} \text{ on } D'_{l,h}, \tag{4.4.3}$$

$$u_h^{(m)} = \varphi \text{ on } \eta_{l1}, \tag{4.4.4}$$

where $1 \le l \le M$, $j \in E$, $m = 1,2,\ldots$.

Remark 4.4.1 *Theorem 2.5.1 remains valid and is proved by analogy to Theorem 3 in [15] for system (4.4.1)-(4.4.4).*

Chapter 5

NUMERICAL EXPERIMENTS

5.1 Examples solved in a hexagonal grid for the Dirichlet problem

5.1.1 Examples on a rectangular domain

Consider the rectangular domain

$$\Pi = \left\{ (x,y) \in D : 0 < x < 1, 0 < y < \frac{\sqrt{3}}{2} \right\},$$

with the boundary γ. The hexagonal grid (2.2.5), denoted Π^h, is assigned to Π, and γ^h denotes the set of nodes on the boundary γ.

Example 5.1.1 *We consider the problem*

$$\begin{aligned} \Delta u &= 0 \textit{ on } \Pi, \\ u &= v(x,y) \textit{ on } \gamma, \end{aligned}$$

where

$$v(x,y) = e^y \sin x \qquad (5.1.1)$$

is the exact solution of the problem in the rectangular domain $\overline{\Pi}$.

This example is solved using Incomplete LU-Decomposition Method (see [29], Chapter 5), and all the calculations are carried out in double precision. As a convergence

h	$\|\varepsilon_h\|_{\overline{\Pi}^h}$	$R^m_{\overline{\Pi}^h}$
2^{-3}	1.15727×10^{-7}	
2^{-4}	7.33698×10^{-9}	15.7731
2^{-5}	4.58658×10^{-10}	15.9966
2^{-6}	2.89896×10^{-11}	15.4765
2^{-7}	2.02482×10^{-12}	14.3171

Table 5.1. Approximations in a rectangle with smooth exact solution

test, we request the maximum residual error to be 10^{-12} and $v_h^{(0)} = 0$ is used as the initial value.

Table 5.1 gives the values obtained in the maximum norm of the difference between the exact and approximate solutions, for the values of $h = 2^{-k}, k = 3,4,5,6,7$, i.e., $\|\varepsilon_h\|_{\overline{\Pi}^h} = \max_{\overline{\Pi}^h} |v - v_h|$. The order of convergence $R^m_{\overline{\Pi}^h} = \frac{\|v - v_{2^{-m}}\|_{\overline{\Pi}^h}}{\left\|v - v_{2^{-(m+1)}}\right\|_{\overline{\Pi}^h}}$ has also been included, where $O(h^4)$ order of accuracy corresponds to 2^4 of the value $R^m_{\overline{\Pi}^h}$.

Example 5.1.2 *We consider the same problem as in Example 5.1.1 with the exact solution*

$$v(x,y) = \frac{1}{2}\ln(x^2 + y^2)\,\mathrm{Re}\,z^7 - \tan^{-1}(\frac{y}{x})\,\mathrm{Im}\,z^7. \tag{5.1.2}$$

which is less smooth than (5.1.1). The results obtained are consistent with the theoretical results and are summarized in Table 5.2.

h	$\|\varepsilon_h\|_{\Pi^h}$	$R^m_{\Pi^h}$
2^{-3}	1.9285677×10^{-4}	
2^{-4}	1.1998304×10^{-5}	16.0737
2^{-5}	7.4809403×10^{-7}	16.0385
2^{-6}	$4.67808169 \times 10^{-8}$	15.9915
2^{-7}	2.922653×10^{-9}	16.0063

Table 5.2. Approximations in a rectangle with less smooth exact solution

S^4u	1.56912199976621
Exact	1.56912199014188
$\|\varepsilon_h(P_1)\|$	$9.624329x10^{-9}$

Table 5.3. Results for approximation of inner points with the matching operator

5.1.2 The matching operator

Examples of the matching operator have also been considered in the domain Π. The coordinate $P_1(0.55, 0.4387)$ is chosen, where $P_1 \in \Pi_0$, and

$$u(x,y) = e^x \cos y, \tag{5.1.3}$$

is assumed to be the exact solution. The result in Table 5.3 is obtained using $h = 2^{-4}$ and demonstrates the high accuracy of the above constructed matching operator.

The second coordinate considered demonstrates the accuracy of the approximate solution at near-boundary points. The point chosen is $P_2(0.195938, 0.02)$, where $P_2 \in \Pi_{01}$ and equation (2.3.23) is used for approximation. Again, the harmonic function (5.1.3) is used as the exact solution. As a third example, a point near one of the corners of the domain, $P_3\,(0.005, 0.005)$ has been considered, where the nodes of evaluation emerge

h	$\lvert\varepsilon_h(P_2)\rvert$	$\lvert\varepsilon_h(P_3)\rvert$
2^{-4}	$1.716286x10^{-8}$	$3.85255412x10^{-8}$
2^{-5}	$5.385032x10^{-10}$	$4.3619541x10^{-9}$
2^{-6}	$2.2436186x10^{-10}$	$3.41679468x10^{-10}$
2^{-7}	$2.4942270x10^{-11}$	$2.8927971x10^{-12}$

Table 5.4. Results for approximation of near boundary points with the matching operator

through both adjacent sides of the corner. The function

$$u(x,y) = e^y \cos x$$

is used as the exact solution. The results obtained are summarized in Table 5.4.

5.1.3 Solutions in an L-Shaped domain

An example is solved in an L-Shaped domain with an angle singularity at the origin, where $\alpha_1\pi = 3\pi/2$. The domain is defined by

$$\Omega = \left\{ (x,y) : -1 \leq x \leq 1, -\frac{\sqrt{3}}{2} \leq y \leq \frac{\sqrt{3}}{2} \right\} \backslash \Omega_1,.$$

where $\Omega_1 = \left\{(x,y) : 0 \leq x \leq 1, -\frac{\sqrt{3}}{2} \leq y \leq 0\right\}$ and is covered by four overlapping rectangles and a sector. The singular part is defined to be the region

$$\Omega^S = \left\{ (x,y) : -\frac{1}{2} \leq x \leq \frac{1}{2}, -\frac{\sqrt{3}}{4} \leq y \leq \frac{\sqrt{3}}{4} \right\} \backslash \Omega_1^S$$

where $\Omega_1^S = \left\{(x,y) : 0 \leq x \leq \frac{1}{2}, -\frac{\sqrt{3}}{4} \leq y \leq 0\right\}$, and the nonsingular part is $\Omega^{NS} = \Omega/\Omega^S$. The system of Block-Grid equations in the "nonsingular" part of the domain is solved by Schwarz's alternating method. The solution at the quadrature nodes lying on the curvilinear part of the boundary of sector $\overline{T}_1^2$, whose radius is taken as 0.75, and the overlapping boundaries of the rectangles are renewed after each Schwarz's iteration. The solution at the nodes on the circular arc, the inner boundaries of the overlapping rectangles and the nodes in the set $\cup_{k=1}^{4}\eta_{k0}^{*h}$ are renewed using the matching operator constructed above. Since the boundary functions are harmonic polynomials on the sides γ_1 and $\gamma_0 \equiv \gamma_6$, the approximation of the solution at the points whose neighbouring nodes emerge through these sides are approximated using the function $u - Q_1$. Finally, the solution on the singular part is approximated using the integral representation.

The problem considered is

$$\begin{aligned} \Delta u &= 0 \text{ on } \Omega, \\ u &= v(x,y) \text{ on } \gamma, \end{aligned}$$

where

$$v(x,y) = \theta + r^{2/3}\sin\left(\frac{2}{3}\theta\right) + \operatorname{Re} z^5 + \operatorname{Im} z^5,$$

is the exact solution. Accordingly, the function $Q_1(r_1,\theta_1)$ used in the integral representation is constructed as

$$Q_1(r_1,\theta_1) = \theta_1 + r_1^5(\cos(5\theta_1) + \sin(5\theta_1)).$$

The results in Table 5.5 and 5.6 show the solution for different pairs of (h,N), where N is the number of quadrature nodes and h is the mesh size of the hexagonal grid.

(h,N)	$\|\varepsilon_h\|_{\Omega^{NS}}$	$R^m_{\Omega^{NS}}$
$(2^{-4},40)$	5.2742×10^{-4}	14.7656
$(2^{-5},60)$	3.57195×10^{-5}	
$(2^{-5},100)$	8.2649×10^{-7}	15.3796
$(2^{-6},100)$	5.373923×10^{-8}	
$(2^{-6},100)$	5.373923×10^{-8}	15.7215
$(2^{-7},125)$	3.418192×10^{-9}	

Table 5.5. Results obtained in "nonsingular" part of the L-shaped domain for the Dirichlet problem

5.2 Examples solved in a hexagonal grid for Laplace's equation with mixed boundary conditions

5.2.1 Examples on a rectangular domain

To demonstrate the accuracy of the approximate solution obtained by the system of equations (3.1.6)-(3.1.9), three examples have been solved in the domain

$$\Pi=\left\{(x,y):0<x<1,0<y<\frac{\sqrt{3}}{2}\right\}, \tag{5.2.1}$$

where γ_j, $j=1,2,3,4$, denotes the boundary of Π, numbered in the positive direction starting from left, and $\gamma=\cup_{j=1}^{4}\gamma_j$. These examples are solved using block Gauss-Seidel method, where each block is solved by Gaussian elimination, and all the calculations are carried out in double precision. As a stopping criteria, it is required that the successive error is more than $\varepsilon=10^{-15}$, and zero is taken as the initial approximation.

(h,N)	$\|\varepsilon_h\|_{\Omega^S}$	$R^m_{\Omega^S}$
$(2^{-4},40)$	9.9742×10^{-4}	11.7569
$(2^{-5},60)$	3.57195×10^{-5}	
$(2^{-5},100)$	8.2649×10^{-7}	14.9731
$(2^{-6},100)$	5.373923×10^{-8}	
$(2^{-6},100)$	5.373923×10^{-8}	16.5665
$(2^{-7},125)$	3.418192×10^{-9}	

Table 5.6. Results obtained in "singular" part of the L-shaped domain for the Dirichlet problem

First, we consider the boundary value problem

$$\Delta u_1 = 0 \text{ on } \Pi, \quad (5.2.2)$$

$$u_1 = v_n^{(1)} \text{ on } \gamma_2, \quad (5.2.3)$$

$$u_1 = v(x,y) \text{ on } \gamma\setminus\gamma_2, \quad (5.2.4)$$

where we have Neumann boundary conditions on the side $y=0$. The second example considers the numerical solution of the problem

$$\Delta u_2 = 0 \text{ on } \Pi, \quad (5.2.5)$$

$$u_2 = v_n^{(1)} \text{ on } \gamma_1 \text{ and } \gamma_3, \quad (5.2.6)$$

$$u_2 = v(x,y) \text{ on } \gamma_2 \text{ and } \gamma_4, \quad (5.2.7)$$

which has Neumann boundary conditions on the parallel sides, $x=0$ and $x=1$, and

h	$\left\|\varepsilon_h^1\right\|_{\overline{\Pi}^h}$	$R_{\overline{\Pi}^h}^{1,m}$	$\left\|\varepsilon_h^2\right\|_{\overline{\Pi}^h}$	$R_{\overline{\Pi}^h}^{2,m}$
2^{-3}	1.7922×10^{-7}		1.8699×10^{-7}	
2^{-4}	1.1238×10^{-8}	15.948	1.1721×10^{-8}	15.953
2^{-5}	7.0395×10^{-10}	15.964	7.3452×10^{-10}	15.957
2^{-6}	4.4033×10^{-11}	15.987	4.5937×10^{-11}	15.989
2^{-7}	2.7868×10^{-12}	15.801	2.9047×10^{-12}	15.814

Table 5.7. Solutions on a rectangular domain with mixed boundary conditions

the final problem is

$$\Delta u_3 = 0 \text{ on } \Pi, \quad (5.2.8)$$

$$u_3 = v_n^{(1)} \text{ on } \gamma_1 \text{ and } \gamma_2, \quad (5.2.9)$$

$$u_3 = v(x,y) \text{ on } \gamma_3 \text{ and } \gamma_4, \quad (5.2.10)$$

which has Neumann boundary conditions on the adjacent sides, $x = 0$ and $y = 0$. For the solution of problems (5.2.2)-(5.2.10), the function

$$v(x,y) = e^x \cos y$$

is taken as the exact solution. Numerical results are obtained by solving the system of equations (3.1.6)-(3.1.9) in the hexagonal grid (2.2.5), on the domain (5.2.1). Results are given in Table 5.7 and Table 5.8, and are presented with the notation $\left\|\varepsilon_h^i\right\|_{\overline{\Pi}^h} = \max_{\overline{\Pi}^h} \left|u_i - u_h^i\right|$ and $R_{\overline{\Pi}^h}^{i,m} = \frac{\left\|u_i - u_{2^{-m}}^i\right\|_{\overline{\Pi}^h}}{\left\|u_i - u_{2^{-(m+1)}}^i\right\|_{\overline{\Pi}^h}}$, $i = 1,2,3$.

h	$\|\varepsilon_h^3\|_{\overline{\Pi}^h}$	$R_{\overline{\Pi}^h}^{3,m}$
2^{-3}	4.2635×10^{-7}	
2^{-4}	2.6688×10^{-8}	15.975
2^{-5}	1.6687×10^{-9}	15.993
2^{-6}	1.0434×10^{-10}	15.991
2^{-7}	6.5528×10^{-12}	15.923

Table 5.8. Solutions on a rectangular domain with Neumann boundary conditions on adjacent sides

5.2.2 Solutions in an L-Shaped domain

To demonstrate the accuracy of the block-grid method for the solution of a problem with mixed boundary conditions, an example has been solved in an L-shaped domain, with a corner singularity at the origin, where the interior angle is $\frac{3\pi}{2}$. Four overlapping rectangles Π_k, $k=1,2,3,4$, covered the "nonsingular" part of the domain. As a stopping criteria for the Schwarz's iterations, it is requested that the successive error on the sides of the overlapping rectangles is 10^{-15}. The system of finite-difference equations in the rectangles are solved by using block Gauss-Seidel method, and the blocks are solved by Gaussian elimination. All the calculations are carried out in double precision and $u_h^{(0)}=0$ is taken as the initial value. Finally the harmonic function (2.1.26) is applied for the approximation of the solution in the "singular" part of the domain.

Let

$$G=\left\{(x,y):-1<x<1,\ -\frac{\sqrt{3}}{2}<y<\frac{\sqrt{3}}{2}\right\}\setminus G_1,$$

where $G_1=\left\{(x,y):0\le x\le 1,\ -\frac{\sqrt{3}}{2}\le y\le 0\right\}$, and γ_i, $i=1,2,...,6$ be the sides of

(h,N)	$\|\varepsilon_h\|_{G_{NS}}$	$R^m_{G_{NS}}$
$(2^{-4},60)$	$9.9569476946\times10^{-6}$	17.3095
$(2^{-5},75)$	$5.7523082852\times10^{-7}$	
$(2^{-5},125)$	$4.04386071850\times10^{-9}$	16.2625
$(2^{-6},135)$	$2.4866231296272\times10^{-10}$	
$(2^{-6},95)$	$1.485055436535\times10^{-8}$	16.3341
$(2^{-7},115)$	$9.091693509155\times10^{-10}$	

Table 5.9. Results obtained in the "nonsingular" part of the L-shaped domain with mixed boundary conditions

G, enumerated counterclockwise, starting from left. We consider the problem

$$\begin{aligned} \Delta u &= 0 \text{ on } G, \\ u &= v_n^{(1)} \text{ on } \gamma_4, \\ u &= v(r,\theta) \text{ on } \gamma\setminus\gamma_4, \end{aligned}$$

where

$$v(r,\theta)=\frac{1}{4}r^{1/3}\cos\left(\frac{\theta}{3}\right),$$

is the exact solution. Let $\Pi^*=\overline{G}\setminus(\cup_{k=1}^4\overline{\Pi}_k)$, and $G_{NS}=\overline{G}\setminus\Pi^*$, $G_S=\overline{G}\cap\Pi^*$ denote the "nonsingular" and "singular" parts of G, respectively. We use the notation $\|\varepsilon_h\|_{G_{NS}}=\max_{G_{NS}}|u-u_h|$ and $R^m_{G_{NS}}=\frac{\|u-u_{2^{-m}}\|_{G_{NS}}}{\|u-u_{2^{-(m+1)}}\|_{G_{NS}}}$ to denote the error approximation and order of convergence in the "nonsingular" part of G, and $\|\varepsilon_h\|_{G_S}$, $R^m_{G_S}$ denote the error approximation and the order of convergence in the "singular" part of G. The results are presented in Table 5.9 and Table 5.10.

(h,N)	$\|\varepsilon_h\|_{G_S}$	$R^m_{G_S}$
$(2^{-4},60)$	$6.90196081185\times10^{-6}$	23.7601
$(2^{-5},75)$	$2.90485160286\times10^{-7}$	
$(2^{-5},125)$	$1.35236603096\times10^{-9}$	25.2062
$(2^{-6},135)$	$5.365211053209\times10^{-11}$	
$(2^{-6},95)$	$5.3680998049\times10^{-9}$	26.7517
$(2^{-7},115)$	$2.00663609129\times10^{-10}$	

Table 5.10. Results obtained in the "singular" part of the L-shaped domain with mixed boundary conditions

(h^{-1},N)	$(16,80)$	$(32,100)$	$(64,115)$	$(128,125)$
$\left\|\varepsilon^{(1)}_{h,x}\right\|_{G_S}$	1.32313×10^{-6}	1.80245×10^{-7}	7.43531×10^{-8}	1.20714×10^{-9}

Table 5.11. $\varepsilon^{(1)}_{h,x}=r^{2/3}\left(\frac{\partial U_h}{\partial x}-\frac{\partial u}{\partial x}\right)$ in the "singular" part of the L-shaped domain with mixed boundary conditions

The derivatives of the solution have also been approximated in the “singular” part of the domain. The errors $\varepsilon^{(1)}_{h,x}=r^{2/3}\left(\frac{\partial U_h}{\partial x}-\frac{\partial u}{\partial x}\right)$, $\varepsilon^{(2)}_{h,xx}=r^{5/3}\left(\frac{\partial^2 U_h}{\partial x^2}-\frac{\partial^2 u}{\partial x^2}\right)$ in the maximum norm, are presented in Tables 5.11, 5.12, respectively. Furthermore, Figures 5.1 and 5.2 are given in order to demostrate the exact and approximate solutions obtained for the derivatives.

(h^{-1},N)	$(16,100)$	$(32,125)$	$(64,150)$	$(128,170)$
$\left\|\varepsilon^{(2)}_{h,xx}\right\|_{G_S}$	5.18289×10^{-6}	5.33195×10^{-7}	9.2546×10^{-9}	3.07158×10^{-9}

Table 5.12. $\varepsilon^{(2)}_{h,xx}=r^{5/3}\left(\frac{\partial^2 U_h}{\partial x^2}-\frac{\partial^2 u}{\partial x^2}\right)$ in the "singular" part of the L-shaped domain with mixed boundary conditions

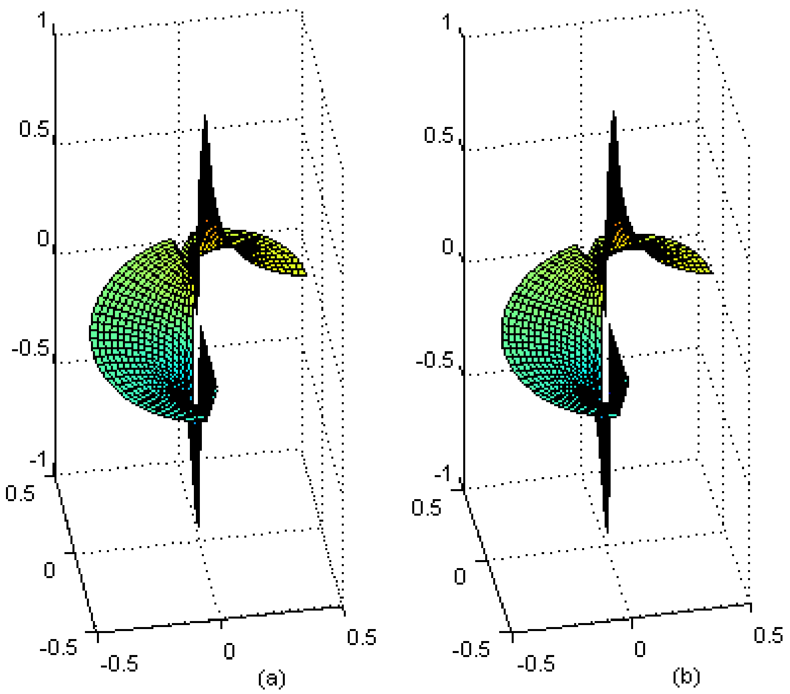

Figure 5.1. The approximate solution (a) and exact solution (b) of $\frac{\partial u}{\partial x}$, respectively, in the "singular" part of the L-shaped domain with mixed boundary conditions using polar coordinates

5.3 Examples solved on a special type of polygon

Two examples have been solved in the polygons defined in Chapter 4 in order to test the effectiveness of the proposed method. In Example 5.3.1, it is assumed that there is a slit in the domain D, thus causing a strong singularity at the origin. The vertex $\dot{\gamma}_1$ containing the singularity, has an interior angle of $\alpha_1\pi = 2\pi$. In Example 5.3.2, we consider a problem with two singularities. The vertices which contain the singularities have interior angles of $\alpha_j\pi = \frac{2}{3}\pi$, $j = 2,4$. In this example, the exact solution is not known.

After separating the "singular" part, in Example 5.3.1, the remaining part of the domain is covered by 5 overlapping parallelograms, whereas in Example 5.3.2, the "nonsin-

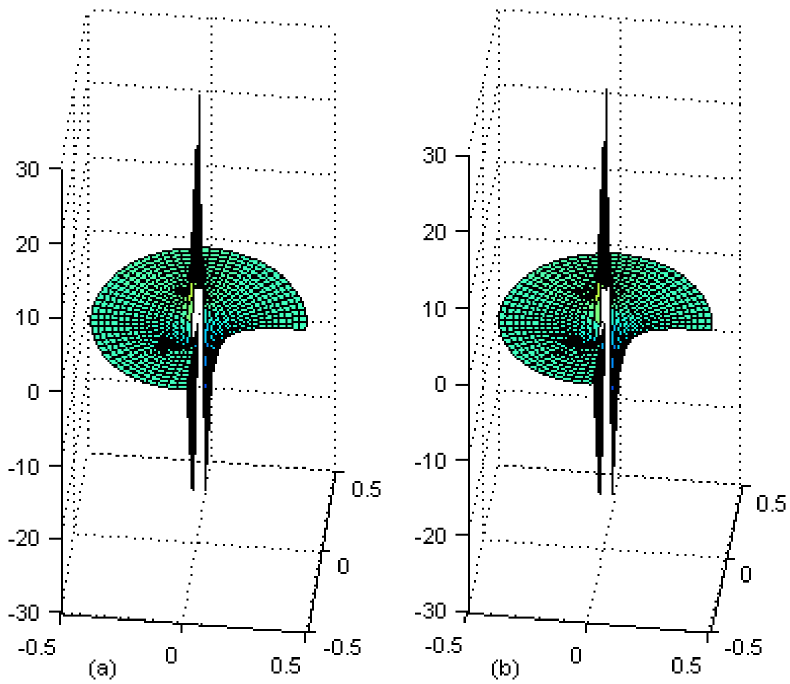

Figure 5.2. The approximate solution (a) and exact solution (b) of $\frac{\partial^2 u}{\partial x^2}$, respectively, in the "singular" part of the L-shaped domain with mixed boundary conditions using polar coordinates

gular" part of the domain is covered by only two parallelograms. For the solution of the block-grid equations, Schwarz's alternating procedure is used. In each Schwarz's iteration the system of equations on the parallelograms is solved by the block Gauss-Seidel method. The function $Q_j(r_j, \theta_j)$ is constructed for each example, taking into consideration the boundary conditions given on the adjacent sides of the vertices in the "singular" parts, and equation (2.1.6) introduced in Section 2.1. Furthermore, the derivatives are approximated in the "singular" parts for both of the examples.

The results are provided in Table 5.13-Table 5.17, and Figure 5.3-Figure 5.11.

Example 5.3.1 *Consider the open parallelogram*

$D=\left\{(x,y):-\frac{\sqrt{3}}{2}<y<\frac{\sqrt{3}}{2},-1-\frac{y}{\sqrt{3}}<x<1-\frac{y}{\sqrt{3}}\right\}$. *We assume that there is a slit along the straight line* $y=0,\ 0\leq x\leq 1$. *Let* $\gamma_j,\ j=1,2,...,7$, *be the sides of* D, *including the ends, enumerated counterclockwise starting from the upper side of the slit* $(\gamma_0\equiv\gamma_7)$, $\gamma=\cup_{j=1}^{7}\gamma_j$, *and* $\dot{\gamma}_j=\gamma_j\cap\gamma_{j-1}$ *be the vertices of* D.

The application of the method in the parallelogram is demonstrated in Figure 5.3

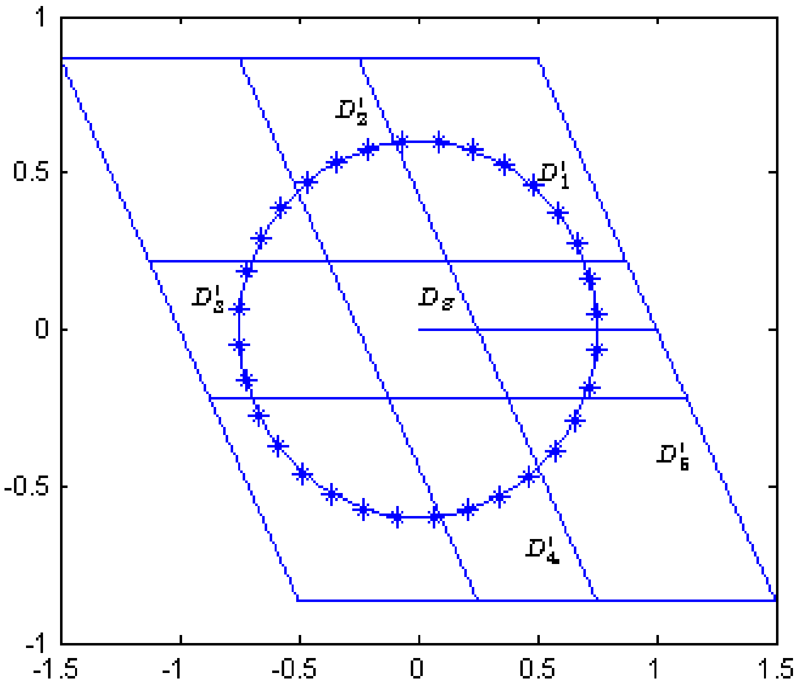

Figure 5.3. Domain of the slit problem with the applicaiton of BGM

Let $(\theta_1,r_1)\equiv(\theta,r)$ *be a polar system of coordinates with pole in* $\dot{\gamma}_1$, *where the angle* θ *is taken counterclockwise from the side* γ_1

We consider the boundary value problem

$$\begin{aligned}\Delta u &= 0 \text{ on } D,\\ u &= \varphi_j \text{ on } \gamma_j,\ j=1,2,...,7,\end{aligned} \tag{5.3.1}$$

where φ_j is the value of the function

$v(r,\theta) = 0.5r^{1/2}\sin\frac{\theta}{2} + 0.8r^{3/2}\sin\frac{3\theta}{2} + 2r^2\cos 2\theta + 2.5r^3\cos 3\theta + 2\theta$ *on* γ_j.

As $\varphi_0 = 2x^2 + 2.5x^3 + 4\pi$ and $\varphi_1 = 2x^2 + 2.5x^3$, we obtain the carrier function in the form

$$
\begin{aligned}
Q_1(r,\theta) &= 2\theta + 2\left(\xi_2(r,\theta) + \xi_2(r, 2\pi - \theta)\right) + \\
&\quad 2.5\left(\xi_3(r,\theta) + \xi_3(r, 2\pi - \theta)\right),
\end{aligned}
$$

where $\xi_2(r,\theta) = r^2\left((2\pi - \theta)\cos 2(2\pi - \theta) + \ln r \sin 2(2\pi - \theta)\right)/2\pi$ and $\xi_3(r,\theta) = r^3\left((2\pi - \theta)\cos 3(2\pi - \theta) + \ln r \sin 3(2\pi - \theta)\right)/2\pi$.

The following notation is used in the Table 5.13. Let $D_l^{\prime}, l = 1,2,...,5$, be the open overlapping parallelograms, $D_{NS} = \cup_{l=1}^{5}\overline{D}_l^{\prime}$ be the "nonsingular" part and $D_S = \overline{D} \backslash D_{NS}$ denote the "singular" part of D. In Table 5.13, the values are obtained in the maximum norm of the difference between the exact and the approximate solutions, for the values of $h = 2^{-k}$, $k = 4,5,6,7$, and n, which is the number of quadrature nodes on V_j. The order of convergence, $R_D^m = \frac{\left\| v - v_{2^{-m}} \right\|_D}{\left\| v - v_{2^{-(m+1)}} \right\|_D}$ have also been included. Figures 5.4, 5.5 illustrate the approximate solution u_h, and the exact solution u in the “singular” part of the domain, respectively. We also present the error obtained between the derivatives of the exact and the block-grid solutions $\varepsilon_h^{(1)} = r^{1/2}\left(\frac{\partial u}{\partial x} - \frac{\partial U_h}{\partial x}\right)$ and $\varepsilon_h^{(2)} = r^{3/2}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 U_h}{\partial x^2}\right)$, in the maximum norm, in Tables 5.14 and 5.15, respectively. Figures 5.6 and 5.7, 5.8 illustrate the shapes of the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ of the obtained approximate and exact solutions. These figures demonstrate also the highly accurate approximation of the derivatives.

(h^{-1},n)	$\|u-u_h\|_{\overline{D}_{NS}}$	$\|u-u_h\|_{\overline{D}_S}$	$R^m_{D_{NS}}$	$R^m_{D_S}$
$(16,70)$	5.924280×10^{-5}	5.191270×10^{-7}		
$(32,70)$	3.910378×10^{-6}	4.794595×10^{-8}	15.1501	10.8273
$(64,110)$	2.478126×10^{-7}	2.558563×10^{-9}	15.7796	18.7394
$(128,130)$	1.56560×10^{-8}	1.27915×10^{-10}	15.8286	20.0021

Table 5.13. Results obtained for the slit problem

(h^{-1},n)	$(16,70)$	$(32,70)$	$(64,110)$	$(128,130)$
$\left\|\varepsilon_h^{(1)}\right\|_{\overline{D}_S}$	7.89831×10^{-7}	9.78871×10^{-8}	4.29502×10^{-9}	2.94108×10^{-10}

Table 5.14. $\varepsilon_h^{(1)}=r^{1/2}\left(\frac{\partial u}{\partial x}-\frac{\partial U_h}{\partial x}\right)$ in the "singular" part of the parallelogram with a slit

(h^{-1},n)	$(16,70)$	$(32,70)$	$(64,110)$	$(128,130)$
$\left\|\varepsilon_h^{(2)}\right\|_{\overline{D}_S}$	3.7119×10^{-6}	9.736×10^{-7}	2.03211×10^{-8}	9.30597×10^{-10}

Table 5.15. $\varepsilon_h^{(2)}=r^{3/2}\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 U_h}{\partial x^2}\right)$ in the "singular" part of the parallelogram with a slit

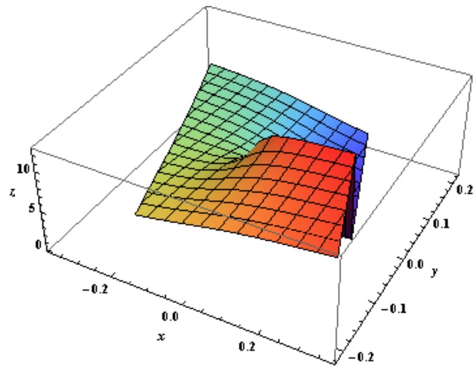

Figure 5.4. Approximate solution in the “singular” part of the slit problem

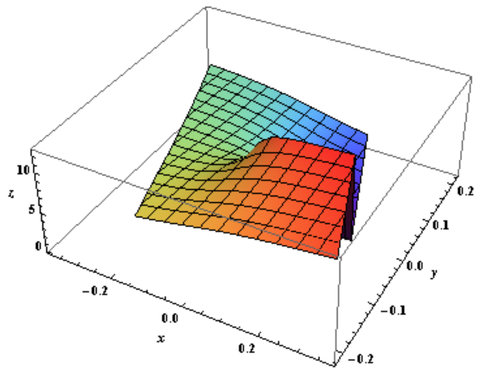

Figure 5.5. Exact solution in the “singular” part of the slit problem

Example 5.3.2 *Let P be the open parallelogram*
$P = \left\{(x,y) : 0 < y < \frac{\sqrt{3}}{2}, -\frac{y}{\sqrt{3}} < x < 1 - \frac{y}{\sqrt{3}}\right\}$, *let* γ_j, $j = 1,2,3,4$, *be the sides of P, including the ends, numbered in the positive direction, starting from the left-hand side* ($\gamma_0 \equiv \gamma_4, \gamma_1 \equiv \gamma_5$), $\gamma = \cup_{j=1}^{4}\gamma_j$, *and* $\dot{\gamma}_j = \gamma_j \cap \gamma_{j-1}$ *represents the jth vertex of P. We consider a problem with two corner singularities at the vertices* $\dot{\gamma}_2$ *and* $\dot{\gamma}_4$, *where* $\alpha_j\pi = \frac{2}{3}\pi$, $j = 2,4$. *The two “singular” corners of P are covered by sectors and these areas are denoted by* P_S^i, $i = 1,2$, *and two overlapping parallelograms cover the “nonsingular” part of the domain, denoted by* P_{NS}^i, $i = 1,2$. *Application of the method for this example is demonstrated in Figure 5.9.*

We consider the boundary value problem

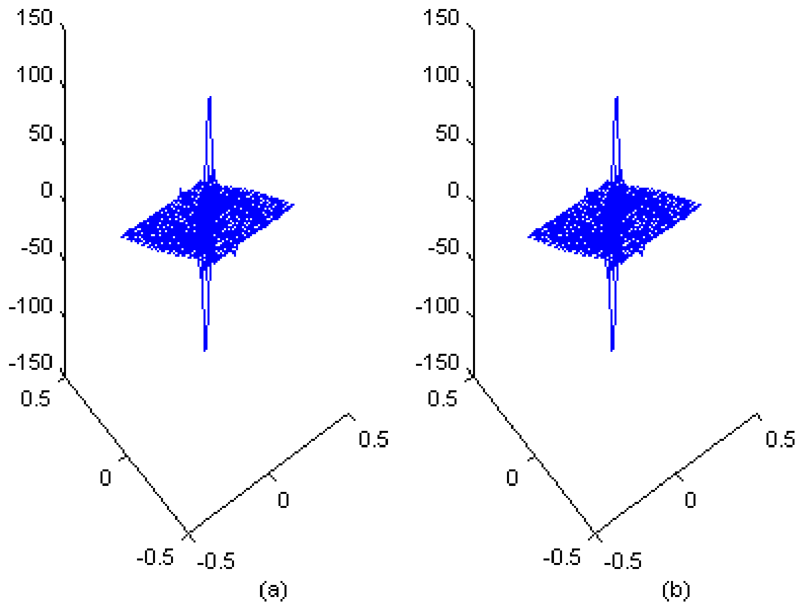

Figure 5.6. The approximate solution (a) and exact solution (b) of $\frac{\partial u}{\partial x}$ in the "singular" part, respectively, of the slit problem.

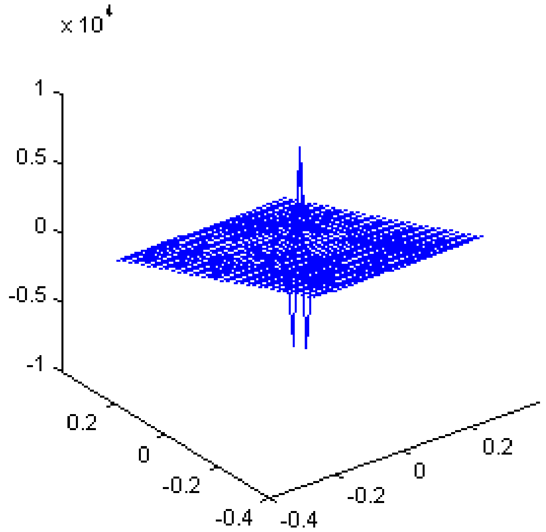

Figure 5.7. The approximate solution of $\frac{\partial^2 u}{\partial x^2}$ in the "singular" part of the slit problem.

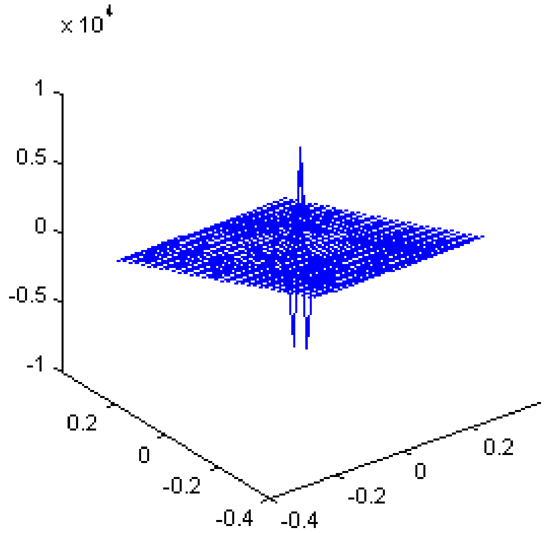

Figure 5.8. The exact solution of $\frac{\partial^2 u}{\partial x^2}$ in the "singular" part of the slit problem.

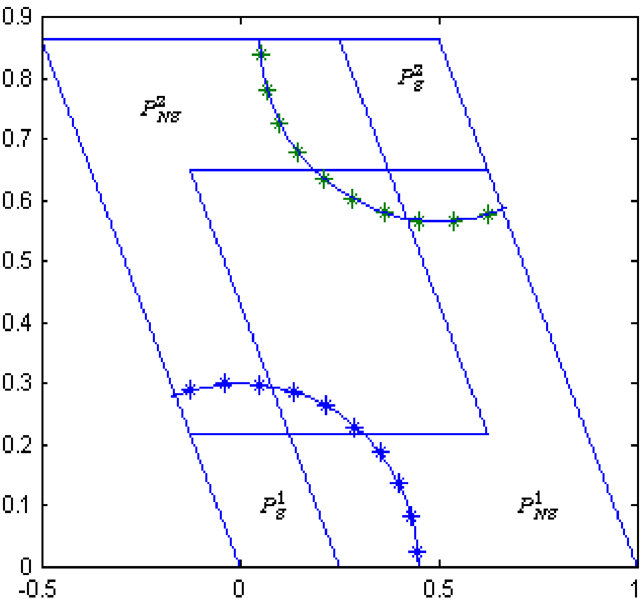

Figure 5.9. Domain of the problem with double singularities

$$\begin{aligned} \Delta u &= 0 \; on \; P, \\ u &= 0 \; on \; \gamma_j, \;\; j = 1,4, \\ u &= 1 \; on \; \gamma_j, \;\; j = 2,3. \end{aligned} \tag{5.3.2}$$

2^{-m}	2^{-5}	2^{-6}
$\widetilde{R}^m_{P^1_{NS}}$	16.257	15.9884
$\widetilde{R}^m_{P^2_{NS}}$	16.2387	16.0086
$\widetilde{R}^m_{P^1_S}$	19.3268	12.7771
$\widetilde{R}^m_{P^2_S}$	18.2604	14.0755

Table 5.16. Order of convergence for problem with double singularities

2^{-m}	2^{-5}	2^{-6}
$\widetilde{R}^m_{P^1_S}$	13.8404	19.6426
$\widetilde{R}^m_{P^2_S}$	13.7489	19.6505

Table 5.17. Order of convergence of derivatives in the "singular" parts of the parallelogram with double singularities

The functions $Q_j(r_j,\theta_j), j=2,4$, constructed for each singularity are $Q_2(r_2,\theta_2)=1-\frac{3\theta_2}{2\pi}$ and $Q_4(r_4,\theta_4)=\frac{3\theta_4}{2\pi}$.We have checked the accuracy of the obtained approximate results u_h by looking at the order of convergence using the formula $\widetilde{R}^m_P=\frac{\left\|u_{2^{-m}}-u_{2^{-m+1}}\right\|_P}{\left\|u_{2^{-m-1}}-u_{2^{-m}}\right\|_P}$, which corresponds to 2^4, for the pairs $(h,n)=(2^{-4},80)$, $(2^{-5},100)$, $(2^{-6},100)$, $(2^{-7},90)$. The results are presented in Table 5.16. Moreover, $\frac{\partial^2 u}{\partial x^2}$ has been approximated in the "singular" part, where u is the unknown exact solution of problem (5.3.2). The results are presented in Table 5.17 and illustrated further in Figures 5.10, 5.9.

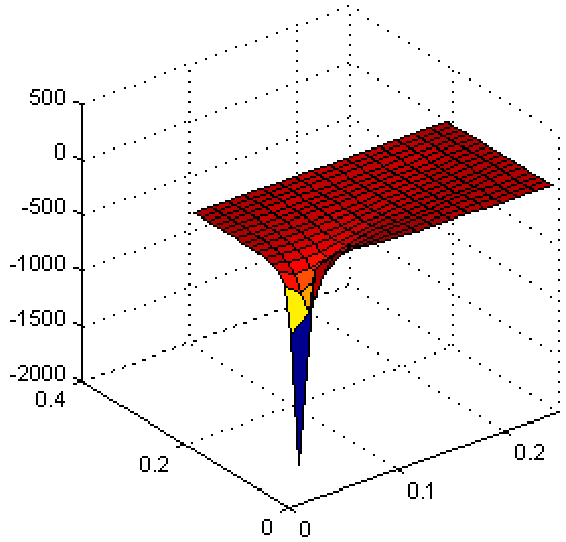

Figure 5.10. $\frac{\partial^2 U_h}{\partial x^2}$ in the “singular” part P_S^1 of the parallelogram with double singularities

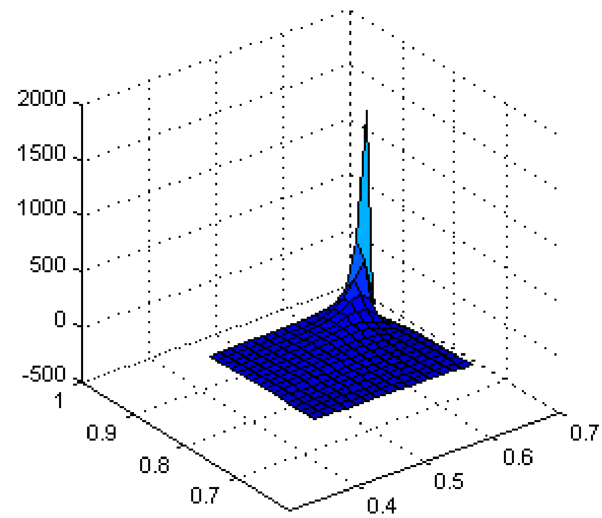

Figure 5.11. $\frac{\partial^2 U_h}{\partial x^2}$ in the “singular” part P_S^2 of the parallelogram with double singularities

Chapter 6

Conclusion

A matching operator of fourth-order accuracy is constructed on the closed rectangle, for the numerical solution of Laplace's equation in hexagonal grids. With the use of this matching operator the Block-Grid method (BGM) has been applied and analysed on staircase polygons, with a hexagonal grid, for the approximation of the Dirichlet and mixed boundary-value problem of Laplace's equation. The difficulties of having neighbouring nodes emerge through the side of the domain while approximating the solution on near-boundary nodes, and nodes lying on sides with Neumann boundary conditions, are overcome by the construction of fourth-order accurate finite-difference operators.

It is justified that an accuracy of $O(h^4)$ is obtained everywhere in the domain, where h is the step size, when the boundary functions away from the singular points are from the Hölder classes $C^{6,\lambda}$, $0 < \lambda < 1$.

The approximation of the Dirichlet problem of Laplace's equation has also been considered on special type of polygons with interior angles of $\alpha_j\pi$, $\alpha_j \in \left\{\frac{1}{3}, \frac{2}{3}, 1, 2\right\}$. It has been justified that in these polygons, with the application of BGM, the smoothness of the boundary functions away from the singular points can be lowered down to $C^{4,\lambda}$, $0 < \lambda < 1$, in order to obtain fourth-order accuracy everywhere in the domain.

In order to demonstrate the accuracy of these results, the L-shaped problem has been considered, where the first example solved had Dirichlet boundary conditions on all sides of the domain, and in the second example Neumann boundary conditions were assumed on one of the adjacent sides of the singular vertex.

For the realization of BGM on polygons with interior angles of $\alpha_j\pi$, $\alpha_j \in \left\{\frac{1}{3}, \frac{2}{3}, 1, 2\right\}$, first of all the slit problem has been considered. As a second example, the computation of BGM was carried out in a domain with double singularities. All the solutions obtained are consistent with the theoretical results.

As an extension of the results obtained in this thesis, it will be interesting to investigate the solution of the biharmonic equation. Eventhough harmonic functions satisfy the biharmonic equation, it does not always follow that biharmonic functions are harmonic. Hence BGM can not always be applied directly for the approximation of biharmonic problems, but can be used by reducing them to two problems for the Laplace and Poisson equations.

Furthemore, for the generalization of the results in this thesis, it will also be worthwhile to analyze BGM with nonanalytic boundary conditions, thus removing the restriction of the boundary functions on the adjacent sides of the singular points to be algebraic polynomials. Eventhough this restriction has been removed for the application of BGM on staircase polygons with square grids (see [23]), the extension to hexagonal grids has not been investigated.

Finally, considering BGM on three-dimansional domains will also be of interest. This application will require a new construction of the matching operator with the use of a different method than the one used in this thesis, and also a new definition of the integral representation of the solution will be needed.

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