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Research Article

Quantitative Global Estimates for Generalized Double Szasz-Mirakjan Operators

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We introduce the generalized double Szász-Mirakjan operators in this paper. We obtain several quantitative estimates for these operators. These estimates help us to determine some function classes $\mathcal{S}$ (including some Lipschitz-type spaces) which provide uniform convergence on the whole domain $[0,\infty)\times[0,\infty)$.

1. Introduction

The well-known Szász-Mirakjan operators are defined on the space $\mathscr{A}_1$ as follows:

$$S_n(f;x)=e^{-nx}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!}, \tag{1}$$

where $\mathscr{A}_1$ is the set of all real functions on $[0,\infty)$ such that the right-hand side in (1) make sense for all $n>0$ and $x\in[0,\infty)$. By modifying the Szász-Mirakjan operators as

$$D_n(f;x)=e^{-nu_n(x)}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\frac{(nu_n(x))^k}{k!}, \tag{2}$$

where $\{u_n(x)\}$ is a sequence of real-valued, continuous functions defined on $[0,\infty)$ with $0\le u_n(x)<\infty$, it has been shown in [1] that if one let

$$u_n^*(x):=\frac{-1+\sqrt{4n^2x^2+1}}{2n},\quad n\in\mathbb{N}, \tag{3}$$

then the operators defined by

$$D_n^*(f;x):=S_n(f;u_n^*(x)) \tag{4}$$

preserve the test function $e_2(x)=x^2$ and provide a better error estimation than the operators $S_n(f;x)$ for all $f\in C_B([0,\infty))$ and for each $x\in[0,\infty)$. Note that $C_B([0,\infty))$ denotes the space of all bounded and continuous functions on $[0,\infty)$. On the other hand, by letting

$$v_n(x):=x-\frac{1}{2n};\quad n\in\mathbb{N}, \tag{5}$$

it has been shown in [2] that the operators defined by

$$V_n^*(f;x):=S_n(f;v_n(x)) \tag{6}$$

do not preserve the test functions $e_1(x)=x$ and $e_2(x)=x^2$ but provide the best error estimation among all the Szász-Mirakjan operators for all $f\in C_B([0,\infty))$ and for each $x\in[1/2,\infty)$. For the other linear positive operator families which preserve $e_2(x)=x^2$, we refer [3–9]. On the other hand, in [10, 11] the authors considered some operators preserving $e_1(x)=x$.

Favard was the first to introduce the double Szász-Mirakjan operators [12]:

$$S_n(f;x,y)=e^{-n(x+y)}\sum_{k,l=0}^{\infty}f\left(\frac{k}{n},\frac{l}{n}\right)\frac{(nx)^k}{k!}\frac{(ny)^l}{l!},\quad f\in\mathscr{A}_2, \tag{7}$$

where $\mathscr{A}_2$ is the set of all real functions on $[0,\infty)\times[0,\infty)$ such that the right-hand side in (7) has a meaning for all $n>0$ and $x,y\in[0,\infty)$. Recently, Dirik and Demirci have introduced and investigated different variants of the general double Szász-Mirakjan operators:

$$D_n(f;x,y) : S_n(f;u_n(x),v_n(y)) = e^{-n(u_n(x)+v_n(y))} \times \sum_{k,l=0}^{\infty} f\left(\frac{k}{n},\frac{l}{n}\right) \times \frac{(nu_n(x))^k}{k!}\frac{(nv_n(y))^l}{l!}, \quad f \in \mathscr{A}_2. \tag{8}$$

In [13], they considered the case of operators

$$u_n^{(1)}(x) := \frac{-1+\sqrt{4n^2x^2+1}}{2n}, \quad v_n^{(1)}(y) := \frac{-1+\sqrt{4n^2y^2+1}}{2n}, \quad n \in \mathbb{N}, \tag{9}$$

which preserve the test function $e_{2,0}(x,y)+e_{0,2}(x,y) := x^2+y^2$ and provide a better error estimation than the operators $S_n(f;x,y)$ for all $f \in C_B([0,\infty)\times[0,\infty))$ and for each $x, y \in [0,\infty)$. On the other hand, in [14], they considered the case

$$u_n^{(2)}(x,\alpha) := \frac{-(n\alpha+1)+\sqrt{4n^2(x^2+\alpha x)+(n\alpha+1)^2}}{2n}, \quad v_n^{(2)}(y,\beta) := \frac{-(n\beta+1)+\sqrt{4n^2(y^2+\beta y)+(n\beta+1)^2}}{2n}, \quad n \in \mathbb{N},\ \alpha,\beta \in \mathbb{R}. \tag{10}$$

Note that for this case, the operators $D_n(f;x,y)$ do not preserve any test function (i.e., $e_{0,0}(x,y) = 1$, $e_{1,0}(x,y) = x$, $e_{0,1}(x,y) = y$, and $e_{2,0}(x,y)+e_{0,2}(x,y) = x^2+y^2$) but provide a better error estimation than the operators $S_n(f;x,y)$ for all $f \in C_B([0,\infty)\times[0,\infty))$ and $x, y \in [0,1]$.

Finally, we should note that, following the similar arguments as used in [2], the best error estimation among all the general double Szász-Mirakjan operators can be obtained from the case:

$$u_n^{(3)}(x) := x - \frac{1}{2n}, \quad v_n^{(3)}(y) := y - \frac{1}{2n}, \quad n \in \mathbb{N}, \tag{11}$$

for all $f \in C_B([0,\infty)\times[0,\infty))$ and $x, y \in [1/2,\infty)$.

For the operators $D_n(f;x,y)$ the following Lemma is straightforward.

Lemma 1. *Let* $\mathbf{x} = (x,y)$, $\mathbf{t} = (t,s)$, $e_{i,j}(\mathbf{x}) = x^i y^j$, $i, j = 0,1,2$, *and* $\psi_{\mathbf{x}}^2(\mathbf{t}) = \|\mathbf{t}-\mathbf{x}\|^2$. *Then, for each* $x, y \geq 0$ *and* $n > 1$, *one has*

(a) $D_n(e_{0,0};x,y) = 1$,

(b) $D_n(e_{1,0};x,y) = u_n(x), D_n(e_{0,1};x,y) = v_n(y)$,

(c) $D_n(e_{2,0}+e_{0,2};x,y) = u_n^2(x)+v_n^2(y)+((u_n(x)+v_n(y))/n)$,

(d) $D_n(\psi_{\mathbf{x}}^2(\mathbf{t});x,y) = (u_n(x)-x)^2 + (v_n(y)-y)^2 + ((u_n(x)+v_n(y))/n)$.

2. Global Results

In this section we first introduce the following Lipschitz-type space:

$$\mathrm{Lip}_M^*(\alpha) := \left\{ f \in C([0,\infty)\times[0,\infty)) : \left|f(\mathbf{t})-f(\mathbf{x})\right| \leq M\frac{\|\mathbf{t}-\mathbf{x}\|^\alpha}{(\|\mathbf{t}\|+x+y)^{\alpha/2}}; \ t,s;x,y \in (0,\infty) \right\}, \tag{12}$$

where $\mathbf{t} = (t,s)$, $\mathbf{x} = (x,y)$, M is any positive constant, and $0 < \alpha \leq 1$.

We should note that this space is the bivariate extension of Lipschitz-type space considered earlier by Szasz [15]. For the space $\mathrm{Lip}_M^*(\alpha)$ with $0 < \alpha \leq 1$, we have the following approximation result.

Theorem 2. *For any* $f \in Lip_M^*(\alpha)$, $\alpha \in (0,1]$ *and for each* $x, y \in (0,\infty)$, $n \in \mathbb{N}$, *one has*

$$\left|D_n(f;x,y)-f(x,y)\right| \leq \frac{M}{(x+y)^{\alpha/2}}\left[(u_n(x)-x)^2+(v_n(y)-y)^2 + \frac{u_n(x)+v_n(y)}{n}\right]^{\alpha/2}. \tag{13}$$

Proof. Take $\alpha = 1$. Then, for $f \in \mathrm{Lip}_M^*(1)$ and for each $x, y \in (0,\infty)$, we get

$$\begin{aligned}\left|D_n(f;x,y)-f(x,y)\right| &\leq D_n\left(\left|f(t,s)-f(x,y)\right|;x,y\right)\\ &\leq MD_n\left(\frac{\|\mathbf{t}-\mathbf{x}\|}{(\|\mathbf{t}\|+x+y)^{1/2}};x,y\right)\\ &\leq \frac{M}{(x+y)^{1/2}}D_n\left(\|\mathbf{t}-\mathbf{x}\|;x,y\right).\end{aligned} \tag{14}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\left|D_n(f;x,y)-f(x,y)\right| \\
&\quad\leq \frac{M}{(x+y)^{1/2}}\sqrt{D_n\left(\psi_{\mathbf{x}}^2(\mathbf{t});x,y\right)} \\
&\quad= \frac{M}{(x+y)^{1/2}} \\
&\qquad\times\sqrt{\left(u_n(x)-x\right)^2+\left(v_n(y)-y\right)^2+\frac{u_n(x)+v_n(y)}{n}}.
\end{aligned}\tag{15}$$

Secondly let $0<\alpha<1$. Then, for $f\in \mathrm{Lip}_M^*(\alpha)$ and for each $x,y\in(0,\infty)$, we have

$$\begin{aligned}
\left|D_n(f;x,y)-f(x,y)\right| &\leq D_n\left(\left|f(t,s)-f(x,y)\right|;x,y\right) \\
&\leq MD_n\left(\frac{\|\mathbf{t}-\mathbf{x}\|^\alpha}{(\|\mathbf{t}\|+x+y)^{\alpha/2}};x,y\right) \\
&\leq \frac{M}{(x+y)^{\alpha/2}}D_n\left(\|\mathbf{t}-\mathbf{x}\|^\alpha;x,y\right).
\end{aligned}\tag{16}$$

Applying the Hölder inequality with $p=2/\alpha$ and $q=2/(2-\alpha)$, we have, for any $f\in\mathrm{Lip}_M^*(\alpha)$,

$$\begin{aligned}
\left|D_n(f;x,y)-f(x,y)\right| &\leq \frac{M}{(x+y)^{\alpha/2}}\left[D_n\left(\psi_{\mathbf{x}}^2(\mathbf{t});x,y\right)\right]^{\alpha/2} \\
&= \frac{M}{(x+y)^{\alpha/2}} \\
&\quad\times\left[\left(u_n(x)-x\right)^2+\left(v_n(y)-y\right)^2\right. \\
&\qquad\left.+\frac{u_n(x)+v_n(y)}{n}\right]^{\alpha/2},
\end{aligned}\tag{17}$$

Hence, the result. □

The following lemma will be used in the rest of the paper.

Lemma 3. *One has, for each $x,y>0$,*

$$\begin{aligned}
&D_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right) \\
&\quad\leq \frac{1}{\sqrt{x}}\sqrt{\left(u_n(x)-x\right)^2+\frac{u_n(x)}{n}} \\
&\qquad+\frac{1}{\sqrt{y}}\sqrt{\left(v_n(y)-y\right)^2+\frac{v_n(y)}{n}}.
\end{aligned}\tag{18}$$

Proof. Using the fact that $\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$ $(a,b\geq 0)$, we get

$$\begin{aligned}
&D_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right) \\
&\quad= e^{-n(u_n(x)+v_n(y))}\sum_{k,l=0}^{\infty}\sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2} \\
&\qquad\times\frac{(nu_n(x))^k}{k!}\frac{(nv_n(y))^l}{l!} \\
&\quad\leq e^{-nu_n(x)}\sum_{k=0}^{\infty}\left|\sqrt{\frac{k}{n}}-\sqrt{x}\right|\frac{(nu_n(x))^k}{k!} \\
&\qquad+e^{-nv_n(y)}\sum_{l=0}^{\infty}\left|\sqrt{\frac{l}{n}}-\sqrt{y}\right|\frac{(nv_n(y))^l}{l!} \\
&\quad= e^{-nu_n(x)}\sum_{k=0}^{\infty}\frac{|k/n-x|}{\sqrt{k/n}+\sqrt{x}}\frac{(nu_n(x))^k}{k!} \\
&\qquad+e^{-nv_n(y)}\sum_{l=0}^{\infty}\frac{|l/n-y|}{\sqrt{l/n}+\sqrt{y}}\frac{(nv_n(y))^l}{l!} \\
&\quad\leq \frac{e^{-nu_n(x)}}{\sqrt{x}}\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|\frac{(nu_n(x))^k}{k!} \\
&\qquad+\frac{e^{-nv_n(y)}}{\sqrt{y}}\sum_{l=0}^{\infty}\left|\frac{l}{n}-y\right|\frac{(nv_n(y))^l}{l!}.
\end{aligned}\tag{19}$$

Finally, applying the Cauchy-Schwarz inequality, we write

$$\begin{aligned}
&D_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right) \\
&\quad\leq \frac{1}{\sqrt{x}}\sqrt{e^{-nu_n(x)}\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^2\frac{(nu_n(x))^k}{k!}} \\
&\qquad+\frac{1}{\sqrt{y}}\sqrt{e^{-nv_n(y)}\sum_{l=0}^{\infty}\left(\frac{l}{n}-y\right)^2\frac{(nv_n(y))^l}{l!}} \\
&\quad= \frac{1}{\sqrt{x}}\sqrt{\left(u_n(x)-x\right)^2+\frac{u_n(x)}{n}} \\
&\qquad+\frac{1}{\sqrt{y}}\sqrt{\left(v_n(y)-y\right)^2+\frac{v_n(y)}{n}}.
\end{aligned}\tag{20}$$

Using Lemma 1, we get the result. □

Recall that, for all $f \in C_B([0,\infty)\times[0,\infty))$, the modulus of f denoted by $\omega(f;\delta)$ is defined as

$$\omega(f;\delta) := \sup\Big\{ |f(t,s)-f(x,y)| : \sqrt{(t-x)^2+(s-y)^2} < \delta, \quad (t,s),(x,y) \in [0,\infty)\times[0,\infty) \Big\}. \tag{21}$$

Theorem 4. *Let $f^*(x,y) = f(x^2,y^2)$. Then one has, for each $x,y>0$,*

$$|D_n(f;x,y) - f(x,y)| \le 2\omega(f^*;\delta_n(x,y)), \tag{22}$$

where

$$\delta_n(x,y) := \frac{1}{\sqrt{x}}\sqrt{(u_n(x)-x)^2 + \frac{u_n(x)}{n}} + \frac{1}{\sqrt{y}}\sqrt{(v_n(y)-y)^2+\frac{v_n(y)}{n}}. \tag{23}$$

Proof. We directly have

$$\begin{aligned}
&|D_n(f;x,y)-f(x,y)| \\
&\le D_n(|f(t,s)-f(x,y)|;x,y) \\
&= D_n(|f^*(\sqrt{t},\sqrt{s}) - f^*(\sqrt{x},\sqrt{y})|;x,y) \\
&\le D_n\left(\omega\left(f^*;\sqrt{(\sqrt{t}-\sqrt{x})^2+(\sqrt{s}-\sqrt{y})^2}\right);x,y\right) \\
&= e^{-n(u_n(x)+v_n(y))} \\
&\quad\times \sum_{k,l=0}^{\infty} \omega\left(f^*;\sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2};x,y\right) \\
&\quad\times \frac{(nu_n(x))^k}{k!}\frac{(nv_n(y))^l}{l!}.
\end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned}
&|D_n(f;x,y)-f(x,y)| \\
&= e^{-n(u_n(x)+v_n(y))}\sum_{k,l=0}^{\infty}\frac{(nu_n(x))^k}{k!}\frac{(nv_n(y))^l}{l!} \\
&\quad\times\omega\left(f^*;\frac{\sqrt{(\sqrt{k/n}-\sqrt{x})^2+(\sqrt{l/n}-\sqrt{y})^2}}{D_n\left(\sqrt{(\sqrt{t}-\sqrt{x})^2+(\sqrt{s}-\sqrt{y})^2};x,y\right)}\right. \\
&\quad\times D_n\Big(\big((\sqrt{t}-\sqrt{x})^2 \\
&\qquad\left.+(\sqrt{s}-\sqrt{y})^2\big)^{1/2};x,y\Big);x,y\right).
\end{aligned} \tag{25}$$

Because of the fact that

$$\omega(f;\lambda\delta) \le (1+\lambda)\,\omega(f;\delta), \tag{26}$$

we have

$$\begin{aligned}
&|D_n(f;x,y)-f(x,y)| \\
&\le \omega\left(f^*;D_n\left(\sqrt{(\sqrt{t}-\sqrt{x})^2+(\sqrt{s}-\sqrt{y})^2};x,y\right)\right) \\
&\quad\times e^{-n(u_n(x)+v_n(y))} \\
&\quad\times\sum_{k,l=0}^{\infty}\left[1+\frac{\sqrt{(\sqrt{k/n}-\sqrt{x})^2+(\sqrt{l/n}-\sqrt{y})^2}}{D_n\left(\sqrt{(\sqrt{t}-\sqrt{x})^2+(\sqrt{s}-\sqrt{y})^2};x,y\right)}\right] \\
&\quad\times\frac{(nu_n(x))^k}{k!}\frac{(nv_n(y))^l}{l!},
\end{aligned} \tag{27}$$

and hence

$$\begin{aligned}
&|D_n(f;x,y)-f(x,y)| \\
&\quad\le 2\omega\left(f^*;D_n\left(\sqrt{(\sqrt{t}-\sqrt{x})^2+(\sqrt{s}-\sqrt{y})^2};x,y\right)\right).
\end{aligned} \tag{28}$$

Finally, using Lemma 3, the proof is completed. □

Theorem 5. *Let $f^*(x,y) = f(x^2,y^2)$. Let*

$$\begin{aligned}
f^* \in Lip_M(\alpha) := \{ & f^* \in C_B([0,\infty)\times[0,\infty)) : \\
& |f^*(\mathbf{t}) - f^*(\mathbf{x})| \le M\|\mathbf{t}-\mathbf{x}\|^{\alpha}; \\
& t,s;x,y \in (0,\infty)\},
\end{aligned} \tag{29}$$

where $\mathbf{t} = (t,s)$, $\mathbf{x} = (x,y)$, M is any positive constant, and $0<\alpha\le 1$. Then

$$|D_n(f;x,y)-f(x,y)| \le M\delta_n^{\alpha}(x,y), \tag{30}$$

where $\delta_n(x,y)$ is the same as in Theorem 4.

Proof. We directly have

$$\begin{aligned}
&\left|D_n(f;x,y)-f(x,y)\right| \\
&\quad\le D_n\left(\left|f(t,s)-f(x,y)\right|;x,y\right) \\
&\quad= D_n\left(\left|f^*\left(\sqrt{t},\sqrt{s}\right)-f^*\left(\sqrt{x},\sqrt{y}\right)\right|;x,y\right) \\
&\quad\le MD_n\left(\left(\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2\right)^{\alpha/2};x,y\right) \\
&\quad= Me^{-n(u_n(x)+v_n(y))} \\
&\qquad\times\sum_{k,l=0}^{\infty}\left(\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2\right)^{\alpha/2} \\
&\qquad\times\frac{(nu_n(x))^k}{k!}\frac{(nv_n(y))^l}{l!}.
\end{aligned} \tag{31}$$

Applying the Hölder inequality with $p=1/\alpha$ and $q=1/(1-\alpha)$, we have

$$\begin{aligned}
&\left|D_n(f;x,y)-f(x,y)\right| \\
&\quad\le M\left[D_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)\right]^{\alpha}.
\end{aligned} \tag{32}$$

Using Lemma 3, we get the result. □

3. Concluding Remarks

In this section we show that taking $u_n(x)=x$ and $v_n(y)=y$ or $u_n(x)=u_n^{(i)}(x)$ and $v_n(y)=v_n^{(i)}(y)$, $i=1,3$, in Theorems 2, 4, and 5 gives global results. Also we present the results obtained by Theorems 2, 4, and 5 for $u_n(x)=u_n^{(2)}(x)$ and $v_n(y)=v_n^{(2)}(y)$.

Corollary 6. *For any $f\in Lip_M^*(\alpha)$, $\alpha\in(0,1]$ and for all $x,y\in(0,\infty)$, $n\in\mathbb{N}$, one has*

$$\left|D_n(f;x,y)-f(x,y)\right|\le\frac{M}{n^{\alpha/2}}, \tag{33}$$

uniformly as $n\to\infty$, for the following pairs of $u_n(x)$ and $v_n(x)$:

(i) *$u_n(x)=x$ and $v_n(y)=y$,*

(ii) *$u_n(x)=(-1+\sqrt{4n^2x^2+1})/2n$ and $v_n(y)=(-1+\sqrt{4n^2y^2+1})/2n$,*

(iii) *$u_n(x)=x-1/2n$ and $v_n(y)=y-1/2n$.*

Proof. (i) Taking $u_n(x)=x$ and $v_n(y)=y$ in (13), we directly have

$$\left|D_n(f;x,y)-f(x,y)\right|\le\frac{M}{n^{\alpha/2}}. \tag{34}$$

(ii) Taking $u_n(x)=(-1+\sqrt{4n^2x^2+1})/2n$ and $v_n(y)=(-1+\sqrt{4n^2y^2+1})/2n$ in (13) gives

$$\begin{aligned}
&\left|D_n(f;x,y)-f(x,y)\right| \\
&\quad=\frac{M}{(x+y)^{\alpha/2}}\left[\frac{1}{n}\left(x+y-x\sqrt{4n^2x^2+1}\right.\right. \\
&\qquad\left.\left.-y\sqrt{4n^2y^2+1}+2nx^2+2ny^2\right)\right]^{\alpha/2} \\
&\quad\le\frac{M}{(x+y)^{\alpha/2}}\left[\frac{1}{n}\left(x+y-x\sqrt{4n^2x^2}\right.\right. \\
&\qquad\left.\left.-y\sqrt{4n^2y^2}+2nx^2+2ny^2\right)\right]^{\alpha/2} \\
&\quad=\frac{M}{n^{\alpha/2}}.
\end{aligned} \tag{35}$$

(iii) Taking $u_n(x)=x-1/2n$ and $v_n(y)=y-1/2n$ in (13), we have

$$\begin{aligned}
&\left|D_n(f;x,y)-f(x,y)\right| \\
&\quad=\frac{M}{(x+y)^{\alpha/2}}\left[\left(-\frac{1}{2n}\right)^2+\left(-\frac{1}{2n}\right)^2+\frac{x+y-1/n}{n}\right]^{\alpha/2} \\
&\quad\le\frac{M}{(x+y)^{\alpha/2}}\left[\frac{1}{2n^2}(2nx+2ny-1)\right]^{\alpha/2} \\
&\quad=\frac{M}{(x+y)^{\alpha/2}}\left[\frac{1}{n}(x+y)\right]^{\alpha/2} \\
&\quad=\frac{M}{n^{\alpha/2}}.
\end{aligned} \tag{36}$$

□

Corollary 7. *Let $f^*(x,y)=f(x^2,y^2)$. Then one has*

$$\left|D_n(f;x,y)-f(x,y)\right|\le2\omega\left(f^*;\frac{2}{\sqrt{n}}\right), \tag{37}$$

uniformly as $n\to\infty$, for the following pairs of $u_n(x)$ and $v_n(x)$:

(i) *$u_n(x)=x$ and $v_n(y)=y$,*

(ii) *$u_n(x)=(-1+\sqrt{4n^2x^2+1})/2n$ and $v_n(y)=(-1+\sqrt{4n^2y^2+1})/2n$,*

(iii) *$u_n(x)=x-1/2n$ and $v_n(y)=y-1/2n$.*

Proof. (i) Taking $u_n(x)=x$ and $v_n(y)=y$ in (23), we directly have

$$\left|D_n(f;x,y)-f(x,y)\right|\le2\omega\left(f^*;\frac{2}{\sqrt{n}}\right). \tag{38}$$

(ii) Taking $u_n(x) = (-1 + \sqrt{4n^2x^2+1})/2n$ and $v_n(y) = (-1 + \sqrt{4n^2y^2+1})/2n$ in (23) gives

$$\left|D_n(f;x,y) - f(x,y)\right| \le 2\omega\left(f^*;\delta_n(x,y)\right), \tag{39}$$

where

$$\begin{aligned}\delta_n(x,y) &:= \frac{1}{\sqrt{x}\sqrt{y}}\left(\sqrt{y}\sqrt{\frac{1}{n}\left(x - x\sqrt{4n^2x^2+1} + 2nx^2\right)}\right.\\ &\quad\left. + \sqrt{x}\sqrt{\frac{1}{n}\left(y - y\sqrt{4n^2y^2+1} + 2ny^2\right)}\right)\\ &\le \frac{1}{\sqrt{x}\sqrt{y}}\left(\sqrt{y}\sqrt{\frac{1}{n}x} + \sqrt{x}\sqrt{\frac{1}{n}y}\right)\\ &= \frac{2}{\sqrt{n}}.\end{aligned} \tag{40}$$

(iii) Taking $u_n(x) = x - 1/2n$ and $v_n(y) = y - 1/2n$ in (23), we have

$$\left|D_n(f;x,y) - f(x,y)\right| \le 2\omega\left(f^*;\delta_n(x,y)\right), \tag{41}$$

where

$$\begin{aligned}\delta_n(x,y) &:= \frac{1}{\sqrt{x}}\sqrt{\left(\frac{1}{2n}\right)^2 + \frac{x}{n} - \frac{1}{2n^2}}\\ &\quad + \frac{1}{\sqrt{y}}\sqrt{\left(\frac{1}{2n}\right)^2 + \frac{y}{n} - \frac{1}{2n^2}}\\ &= \frac{1}{2\sqrt{x}\sqrt{y}}\left(\sqrt{x}\sqrt{\frac{1}{n^2}(4ny-1)}\right.\\ &\quad\left. + \sqrt{y}\sqrt{\frac{1}{n^2}(4nx-1)}\right)\\ &\le \frac{1}{\sqrt{x}\sqrt{y}}\left(\sqrt{x}\sqrt{\frac{1}{n}y} + \sqrt{y}\sqrt{\frac{1}{n}x}\right)\\ &= \frac{2}{\sqrt{n}}.\end{aligned} \tag{42}$$

□

Corollary 8. *Let $f^*(x,y) = f(x^2,y^2)$, and let*

$$\begin{aligned} f^* \in Lip_M(\alpha) := \{ & f^* \in C_B([0,\infty)\times[0,\infty)) :\\ & \left|f^*(\mathbf{t}) - f^*(\mathbf{x})\right| \le M\|\mathbf{t}-\mathbf{x}\|^{\alpha};\\ & t,s;x,y \in (0,\infty)\},\end{aligned} \tag{43}$$

where $\mathbf{t} = (t,s)$, $\mathbf{x} = (x,y)$, M is any positive constant, and $0 < \alpha \le 1$. Then

$$\left|D_n(f;x,y) - f(x,y)\right| \le M\left(\frac{4}{n}\right)^{\alpha/2}, \tag{44}$$

uniformly as $n \to \infty$, for the following pairs of $u_n(x)$ and $v_n(y)$:

(i) *$u_n(x) = x$ and $v_n(y) = y$,*

(ii) *$u_n(x) = (-1 + \sqrt{4n^2x^2+1})/2n$ and $v_n(y) = (-1 + \sqrt{4n^2y^2+1})/2n$,*

(iii) *$u_n(x) = x - 1/2n$ and $v_n(y) = y - 1/2n$.*

Proof. (i) Taking $u_n(x) = x$ and $v_n(y) = y$ in (23), we directly have

$$\left|D_n(f;x,y) - f(x,y)\right| \le M\left(\frac{4}{n}\right)^{\alpha/2}. \tag{45}$$

(ii) Taking $u_n(x) = (-1 + \sqrt{4n^2x^2+1})/2n$ and $v_n(y) = (-1 + \sqrt{4n^2y^2+1})/2n$ in (23) gives

$$\left|D_n(f;x,y) - f(x,y)\right| \le M\delta_n^{\alpha/2}(x,y), \tag{46}$$

where

$$\begin{aligned}\delta_n(x,y) &:= \delta_n(x,y)\\ &:= \frac{1}{\sqrt{x}\sqrt{y}}\\ &\quad\times\left(\sqrt{y}\sqrt{\frac{1}{n}\left(x - x\sqrt{4n^2x^2+1} + 2nx^2\right)}\right.\\ &\quad\left. + \sqrt{x}\sqrt{\frac{1}{n}\left(y - y\sqrt{4n^2y^2+1} + 2ny^2\right)}\right)\\ &\le \frac{1}{\sqrt{x}\sqrt{y}}\left(\sqrt{y}\sqrt{\frac{1}{n}x} + \sqrt{x}\sqrt{\frac{1}{n}y}\right)\\ &= \frac{2}{\sqrt{n}}.\end{aligned} \tag{47}$$

(iii) Taking $u_n(x) = x - 1/2n$ and $v_n(y) = y - 1/2n$ in (23), we have

$$\left|D_n(f;x,y) - f(x,y)\right| \le M\delta_n^{\alpha/2}(x,y), \tag{48}$$

where

$$
\begin{aligned}
\delta_n(x,y) &:= \frac{1}{\sqrt{x}}\sqrt{\left(\frac{1}{2n}\right)^2+\frac{x}{n}-\frac{1}{2n^2}} \\
&\quad+\frac{1}{\sqrt{y}}\sqrt{\left(\frac{1}{2n}\right)^2+\frac{y}{n}-\frac{1}{2n^2}} \\
&= \frac{1}{2\sqrt{x}\sqrt{y}}\left(\sqrt{x}\sqrt{\frac{1}{n^2}(4ny-1)}\right. \\
&\qquad\left.+\sqrt{y}\sqrt{\frac{1}{n^2}(4nx-1)}\right) \\
&\le \frac{1}{\sqrt{x}\sqrt{y}}\left(\sqrt{x}\sqrt{\frac{1}{n}y}+\sqrt{y}\sqrt{\frac{1}{n}x}\right) \\
&= \frac{2}{\sqrt{n}}.
\end{aligned}
\tag{49}
$$

□

Remark 9. Corollaries 7 and 8 conclude that f is a real continuous and bounded function on $[0,\infty)\times[0,\infty)$ and if $f^*(x,y)=f(x^2,y^2)$ is uniformly continuous on $[0,\infty)\times[0,\infty)$, then $D_n(f)$ converges uniformly to f as $n\to\infty$. Note that the one variable version of Corollary 7 was given in [16].

Corollary 10. *Take*

$$
\begin{aligned}
u_n(x) &= u_n^{(2)}(x,\gamma) \\
&= \frac{-(n\gamma+1)+\sqrt{4n^2(x^2+\gamma x)+(n\gamma+1)^2}}{2n}, \\
v_n(y) &= v_n^{(2)}(y,\beta) \\
&= \frac{-(n\beta+1)+\sqrt{4n^2(y^2+\beta y)+(n\beta+1)^2}}{2n},
\end{aligned}
\tag{50}
$$

where α, $\beta\in\mathbb{R}$. Then

(i) *for any $f\in Lip_M^*(\alpha)$, $\alpha\in(0,1]$ and for each $x,y\in(0,\infty)$, $n\in\mathbb{N}$, one has*

$$
\begin{aligned}
&\left|D_n(f;x,y)-f(x,y)\right| \\
&\quad\le \frac{M}{[2n(x+y)]^{\alpha/2}}[\delta(x,\gamma)+\delta(y,\beta)]^{\alpha/2},
\end{aligned}
\tag{51}
$$

where

$$
\begin{aligned}
\delta(x,\gamma) &= \Big[(2x+\gamma) \\
&\quad\times\left(n\gamma-\sqrt{n^2(2x+\gamma)^2+2n\gamma+1}+2nx+1\right)\Big],
\end{aligned}
\tag{52}
$$

(ii) *let $f^*(x,y)=f(x^2,y^2)$. Then one has for each $x,y>0$*

$$
\left|D_n(f;x,y)-f(x,y)\right|\le 2\omega\left(f^*;\delta(x,\gamma)+\delta(y,\beta)\right),
\tag{53}
$$

where

$$
\begin{aligned}
&\delta(x,\gamma) \\
&\quad=\frac{1}{\sqrt{2x}} \\
&\qquad\times\left(\frac{1}{n}(2x+\gamma)\left(n\gamma-\sqrt{n^2(2x+\gamma)^2+2n\gamma+1}\right.\right. \\
&\qquad\qquad\left.\left.+2nx+1\right)\right)^{1/2},
\end{aligned}
\tag{54}
$$

(iii) *let $f^*(x,y)=f(x^2,y^2)$, and let*

$$
\begin{aligned}
f^*\in \mathrm{Lip}_M(\alpha) := \{ & f^*\in C_B([0,\infty)\times[0,\infty)) : \\
&\left|f^*(\mathbf{t})-f^*(\mathbf{x})\right|\le M\|\mathbf{t}-\mathbf{x}\|^{\alpha}; \\
&t,s;x,y\in(0,\infty)\},
\end{aligned}
\tag{55}
$$

where $\mathbf{t}=(t,s)$, $\mathbf{x}=(x,y)$, M is any positive constant, and $0<\alpha\le1$. Then

$$
\left|D_n(f;x,y)-f(x,y)\right|\le M[\delta(x,\gamma)+\delta(y,\beta)]^{\alpha/2},
\tag{56}
$$

where $\delta(x,\gamma)$ is the same given in Corollary 10(ii).

It should be mentioned that, for $\alpha=0$ and $\beta=0$, $u_n^{(2)}(x,0)=(-1+\sqrt{4n^2x^2+1})/2n$ and $v_n^{(2)}(y,0)=(-1+\sqrt{4n^2y^2+1})/2n$. Therefore, Corollary 10(i), Corollary 10(ii), and Corollary 10(iii) reduce to Corollary 6(ii), Corollary 7(ii), and Corollary 8(ii), respectively.

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