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Chlodowsky variant of q-Bernstein-Schurer-Stancu operators

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Abstract

It was Chlodowsky who considered non-trivial Bernstein operators, which help to approximate bounded continuous functions on the unbounded domain. In this paper, we introduce the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. By obtaining the first few moments of these operators, we prove Korovkin-type approximation theorems in different function spaces. Furthermore, we compute the error of the approximation by using the modulus of continuity and Lipschitz-type functionals. Then we obtain the degree of the approximation in terms of the modulus of continuity of the derivative of the function. Finally, we study the generalization of the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators and investigate their approximations.

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1 Introduction

The classical Bernstein-Chlodowsky operators were defined by Chlodowsky [1] as

$$C_n(f;x) = \sum_{r=0}^{n} f\left(\frac{r}{n}b_n\right)\binom{n}{r}\left(\frac{x}{b_n}\right)^r\left(1-\frac{x}{b_n}\right)^{n-r},$$

where the function f is defined on $[0,\infty)$ and (b_n) is a positive increasing sequence with $b_n \to \infty$ and $\frac{b_n}{n} \to 0$ as $n \to \infty$.

In 1968, Stancu [2] constructed and studied the Bernstein-Stancu operators, which were defined as

$$P_m^{\alpha,\beta}(f;x) = \sum_{k=0}^{m} f\left(\frac{k+\alpha}{m+\beta}\right)\binom{m}{k}x^k(1-x)^{m-k},$$

where $\alpha,\beta \in \mathbb{R}$ such that $0 \le \alpha \le \beta$.

In 1987, Lupaş [3] introduced the q-based Bernstein operators and obtained the Korovkin-type approximation theorem. In 1996, other q-based Bernstein operators were defined by Phillips [4, 5]. During the last decade q-based operators have become an active research area (see [6–9]).

In 2011, the q-Bernstein-Schurer operators were defined by Muraru [10], for fixed $p \in \mathbb{N}_0$ and for all $x \in [0,1]$, by

$$B_n^p(f;q;x) = \sum_{k=0}^{n+p} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n+p \\ k \end{bmatrix} x^k \prod_{s=0}^{n+p-k-1} (1-q^s x), \tag{1.1}$$

where q is a positive real number and the function f is evaluated at the q-integers $\frac{[k]}{[n]}$. Recall that the q-integer of $k \in \mathbb{R}$ is [11]

$$[k] = \begin{cases} (1-q^k)/(1-q), & q \neq 1, \\ k, & q = 1, \end{cases}$$

the q-factorial is defined by

$$[k]! = \begin{cases} [k][k-1]\cdots[1], & k = 1,2,3,\ldots, \\ 1, & k = 0 \end{cases}$$

and q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[n-k]![k]!}$$

for $n \geq 0$, $k \geq 0$. Note that the case $q \to 1^-$ in (1.1), the operators defined by (1.1) were reduced to the operators considered by Schurer [12].

Some properties of the q-Bernstein-Schurer operators were investigated in [13]. We should notice that the case $p = 0$ reduces to the q-Bernstein operators.

It should be noted that complex approximation properties of some Schurer-type operators were investigated in [14, 15] and [16].

Recently, Barbosu investigated Schurer-Stancu operators $S_{m,p}^{\alpha,\beta} : C[0,1+p] \to C[0,1]$ which were defined by [17] (see also [18])

$$S_{m,p}^{\alpha,\beta}(f;x) = \sum_{k=0}^{m+p} f\left(\frac{k+\alpha}{m+\beta}\right) \binom{m+p}{k} x^k (1-x)^{m+p-k}, \tag{1.2}$$

where α and β are non-negative numbers which satisfy $0 \leq \alpha \leq \beta$ and also p is a non-negative integer.

In 2008, Karslı and Gupta [19] defined the q-analogue of Chlodowsky operators by

$$C_n(f;q;x) = \sum_{k=0}^{n} f\left(\frac{[k]}{[n]} b_n\right) \begin{bmatrix} n \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-k-1} \left(1 - q^s \frac{x}{b_n}\right), \quad 0 \leq x \leq b_n,$$

where (b_n) has the same property of Bernstein-Chlodowsky operators.

Lately, the q-analogues of Bernstein-Schurer-Stancu operators were introduced by Agrawal *et al.* as [20]

$$S_{n,p}^{\alpha,\beta}(f;q;x) = \sum_{k=0}^{n+p} f\left(\frac{[k]+\alpha}{[n]+\beta}\right) \begin{bmatrix} n+p \\ k \end{bmatrix} x^k \prod_{s=0}^{n+p-k-1} (1-q^s x),$$

where α, β and p are non-negative integers such that $0 \leq \alpha \leq \beta$. For the first few moments, we have the following lemma.

Lemma 1.1 [20, p.7755] *For the operator* $S_{n,p}^{\alpha,\beta}(f;q;x)$, *we have the following moments*:

(i) $S_{n,p}^{\alpha,\beta}(1;q;x) = 1$,

(ii) $S_{n,p}^{\alpha,\beta}(t;q;x) = \dfrac{[n+p]x+\alpha}{[n]+\beta}$,

(iii) $S_{n,p}^{\alpha,\beta}(t^2;q;x) = \dfrac{1}{([n]+\beta)^2}\big([n+p]^2x^2 + [n+p]x(1-x) + 2\alpha[n+p]x + \alpha^2\big)$.

The organization of the paper as follows.

In Section 2, we introduce the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators and investigate the moments of the operator. In Section 3, we study several Korovkin-type theorems in different function spaces. In Section 4, we obtain the order of convergence of the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators by means of Lipschitz class functions and the first modulus of continuity. In addition, we calculate the degree of convergence of the approximation process in terms of the first modulus of continuity of the derivative of the function. In Section 5, we study the generalization of the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators and investigate their approximations.

2 Construction of the operators

We construct the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators as

$$C_{n,p}^{(\alpha,\beta)}(f;q;x) := \sum_{k=0}^{n+p} f\left(\frac{[k]+\alpha}{[n]+\beta}b_n\right)\begin{bmatrix} n+p \\ k \end{bmatrix}\left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right), \tag{2.1}$$

where $n \in \mathbb{N}$ and $p \in \mathbb{N}_0$ and $\alpha, \beta \in \mathbb{R}$ with $0 \leq \alpha \leq \beta$, $0 \leq x \leq b_n$, $0 < q < 1$. Clearly, $C_{n,p}^{(\alpha,\beta)}$ is a linear and positive operator. Note that the cases $q \to 1$ and $p = 0$ in (2.1) reduce to the Stancu-Chlodowsky polynomials [21].

Firstly, we obtain the following lemma, which will be used throughout the paper.

Lemma 2.1 *Let* $C_{n,p}^{(\alpha,\beta)}(f;q;x)$ *be given in* (2.1). *Then the first few moments of the operators are*

(i) $C_{n,p}^{(\alpha,\beta)}(1;q;x) = 1$,

(ii) $C_{n,p}^{(\alpha,\beta)}(t;q;x) = \dfrac{[n+p]x+\alpha b_n}{[n]+\beta}$,

(iii) $C_{n,p}^{(\alpha,\beta)}(t^2;q;x) = \dfrac{1}{([n]+\beta)^2}\big\{[n+p-1][n+p]qx^2 + (2\alpha+1)[n+p]b_nx + \alpha^2b_n^2\big\}$,

(iv) $C_{n,p}^{(\alpha,\beta)}((t-x);q;x) = \left(\dfrac{[n+p]}{[n]+\beta} - 1\right)x + \dfrac{\alpha b_n}{[n]+\beta}$,

(v)
$$C_{n,p}^{(\alpha,\beta)}\big((t-x)^2;q;x\big) = \left(\frac{[n+p-1][n+p]q}{([n]+\beta)^2} - 2\frac{[n+p]}{[n]+\beta} + 1\right)x^2 + \left(\frac{(2\alpha+1)[n+p]}{([n]+\beta)^2} - \frac{2\alpha}{[n]+\beta}\right)b_nx + \frac{\alpha^2b_n^2}{([n]+\beta)^2}.$$

Proof Using Lemma 1.1 and considering the following facts:

$$C_{n,p}^{\alpha,\beta}(1;q;x) = S_{n,p}^{\alpha,\beta}\left(1;q;\frac{x}{b_n}\right),$$
$$C_{n,p}^{\alpha,\beta}(t;q;x) = b_n S_{n,p}^{\alpha,\beta}\left(t;q;\frac{x}{b_n}\right),$$
$$C_{n,p}^{\alpha,\beta}\left(t^2;q;x\right) = b_n^2 S_{n,p}^{\alpha,\beta}\left(t^2;q;\frac{x}{b_n}\right),$$

we get the assertions (i), (ii), and (iii).

Using the linearity of the operators, we have

$$\begin{aligned} C_{n,p}^{(\alpha,\beta)}\left((t-x);q;x\right) &= C_{n,p}^{(\alpha,\beta)}(t;q;x) - xC_{n,p}^{(\alpha,\beta)}(1;q;x) \\ &= \left(\frac{[n+p]}{[n]+\beta} - 1\right)x + \frac{\alpha b_n}{[n]+\beta}. \end{aligned}$$

So, we get (iv).

Similar computations give

$$C_{n,p}^{(\alpha,\beta)}\left((t-x)^2;q;x\right) = C_{n,p}^{(\alpha,\beta)}\left(t^2;q;x\right) - 2xC_{n,p}^{(\alpha,\beta)}(t;q;x) + x^2 C_{n,p}^{(\alpha,\beta)}(1;q;x).$$

Then we have

$$\begin{aligned} C_{n,p}^{(\alpha,\beta)}\left((t-x)^2;q;x\right) &= \left(\frac{[n+p-1][n+p]}{([n]+\beta)^2}q - 2\frac{[n+p]}{[n]+\beta} + 1\right)x^2 \\ &\quad + \left(\frac{(2\alpha+1)[n+p]}{([n]+\beta)^2} - 2\frac{\alpha}{[n]+\beta}\right)b_n x + \frac{\alpha^2 b_n^2}{([n]+\beta)^2}. \end{aligned}$$

This proves (v). □

Lemma 2.2 *For each fixed* $q \in (0,1)$, *we have*

$$\frac{[n+p-1][n+p]}{([n]+\beta)^2}q - 2\frac{[n+p]}{[n]+\beta} + 1 \le \frac{(q^n[p]-\beta)^2}{([n]+\beta)^2}.$$

Proof Since $[n+p-1][n+p]q \le [n+p]^2$ we get

$$\frac{[n+p-1][n+p]}{([n]+\beta)^2}q - 2\frac{[n+p]}{[n]+\beta} + 1 \le \left(\frac{[n+p]}{[n]+\beta} - 1\right)^2.$$

If we calculate the right-hand side of the above inequality, we get the following:

$$\begin{aligned} &\frac{[n+p-1][n+p]}{([n]+\beta)^2}q - 2\frac{[n+p]}{[n]+\beta} + 1 \\ &\quad \le \frac{1}{([n]+\beta)^2}\left\{\frac{(1-q^{n+p})^2}{(1-q)^2} - 2\frac{(1-q^{n+p})(1-q^n)}{(1-q)^2} - 2\beta\frac{(1-q^{n+p})}{1-q} + \frac{(1-q^n)^2}{(1-q)^2}\right. \\ &\qquad \left. + 2\beta\frac{(1-q^n)}{1-q} + \beta^2\right\} \end{aligned}$$

$$= \frac{1}{([n]+\beta)^2}\left\{\frac{q^{2n}(q^{2p}-2q^p+1)}{(1-q)^2}+2\beta\frac{q^n(q^p-1)}{1-q}+\beta^2\right\}$$
$$= \frac{1}{([n]+\beta)^2}\left\{\frac{q^{2n}(1-q^p)^2}{(1-q)^2}-2\beta\frac{q^n(1-q^p)}{1-q}+\beta^2\right\}$$
$$= \frac{1}{([n]+\beta)^2}\left\{q^{2n}[p]^2-2\beta q^n[p]+\beta^2\right\}=\frac{(q^n[p]-\beta)^2}{([n]+\beta)^2}. \qquad \square$$

Remark 2.1 As a consequence of Lemma 2.1 and Lemma 2.2, we have

$$C_{n,p}^{(\alpha,\beta)}\big((t-x)^2;q;x\big)\le\frac{(q^n[p]-\beta)^2}{([n]+\beta)^2}x^2+\left(\frac{(2\alpha+1)[n+p]}{([n]+\beta)^2}\right)b_nx+\frac{\alpha^2b_n^2}{([n]+\beta)^2}. \tag{2.2}$$

Lemma 2.3 *For the second central moment we have the following estimate*:

$$\sup_{0\le x\le b_n}C_{n,p}^{(\alpha,\beta)}\big((t-x)^2;q;x\big)$$
$$\le\frac{(q^n[p]-\beta)^2}{([n]+\beta)^2}b_n^2+\left(\frac{(2\alpha+1)[n+p]}{([n]+\beta)^2}\right)b_n^2+\frac{\alpha^2b_n^2}{([n]+\beta)^2}.$$

Proof Taking supremum over $[0,b_n]$ in (2.2) we get the result. $\square$

3 Korovkin-type approximation theorem

In this section, we prove Korovkin-type approximation theorem for the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. Denote by C_ρ the space of all continuous functions f, satisfying the condition

$$|f(x)|\le M_f\rho(x),\quad -\infty<x<\infty.$$

Obviously, C_ρ is a linear normed space with the norm

$$\|f\|_\rho=\sup_{-\infty<x<\infty}\frac{|f(x)|}{\rho(x)}.$$

In studying the weighted approximation, the following theorem is crucial.

Theorem 3.1 (See [22]) *There exists a sequence of positive linear operators T_n, acting from C_ρ to C_ρ, satisfying the conditions*

$$\lim_{n\to\infty}\big\|T_n(1;x)-1\big\|_\rho=0, \tag{3.1}$$
$$\lim_{n\to\infty}\big\|T_n(\phi;x)-\phi\big\|_\rho=0, \tag{3.2}$$
$$\lim_{n\to\infty}\big\|T_n\big(\phi^2;x\big)-\phi^2\big\|_\rho=0, \tag{3.3}$$

where $\phi(x)$ is continuous and increasing function on $(-\infty,\infty)$ such that $\lim_{x\to\pm\infty}\phi(x)=\pm\infty$ *and $\rho(x)=1+\phi^2$ and there exists a function $f^*\in C_\rho$ for which $\overline{\lim}_{n\to\infty}\|T_nf^*-f^*\|_\rho>0$.*

The following theorem has been given in [22] and will be used in the investigation of the approximation properties of $C_{n,p}^{(\alpha,\beta)}(f;q;x)$ in weighted spaces.

Theorem 3.2 (See [22]) *The conditions* (3.1), (3.2), (3.3) *imply* $\lim_{n\to\infty}\|T_nf-f\|_\rho=0$ *for any function* f *belonging to the subset* C_ρ^0 *of* C_ρ *for which*

$$\lim_{|x|\to\infty}\frac{f(x)}{\rho(x)}$$

exists finitely.

Particularly, choosing $\rho(x)=1+x^2$ and applying Theorem 3.2 to the operators

$$T_n^{\alpha,\beta}(f;q;x)=\begin{cases}C_{n,p}^{(\alpha,\beta)}(f;q;x) & \text{if } 0\le x\le b_n,\\ f(x) & \text{if } x\notin[0,b_n],\end{cases}$$

we can state the following theorem.

Theorem 3.3 *For all* $f\in C_{1+x^2}^0$ *we have*

$$\lim_{n\to\infty}\sup_{0\le x\le b_n}\frac{|T_n^{\alpha,\beta}(f;q_n;x)-f(x)|}{1+x^2}=0$$

provided that $q:=(q_n)$ *with* $0<q_n<1$, $\lim_{n\to\infty}q_n=1$ *and* $\lim_{n\to\infty}\frac{b_n}{[n]}=0$ *as* $n\to\infty$.

Proof In the proof we directly use Theorem 3.2. Obviously, by Lemma 2.1(i), (ii), and (iii) we get the following inequalities, respectively:

$$\begin{aligned}\sup_{0\le x\le b_n}\frac{|T_n^{\alpha,\beta}(t;q_n;x)-x|}{1+x^2}&\le\sup_{0\le x\le b_n}\frac{|(\frac{[n+p]}{[n]+\beta}-1)|x+\frac{\alpha b_n}{[n]+\beta}}{(1+x^2)}\\&\le\left|\frac{[n+p]}{[n]+\beta}-1\right|+\frac{\alpha b_n}{[n]+\beta}\to0\end{aligned}$$

and

$$\begin{aligned}&\sup_{0\le x\le b_n}\frac{|T_n^{\alpha,\beta}(t^2;q_n;x)-x^2|}{1+x^2}\\&\quad\le\sup_{0\le x\le b_n}\frac{|[n+p-1][n+p]q_n-([n]+\beta)^2|x^2+(2\alpha+1)[n+p]b_nx+\alpha^2b_n^2}{([n]+\beta)^2(1+x^2)}\\&\quad\le\frac{|[n+p-1][n+p]q_n-([n]+\beta)^2|+(2\alpha+1)[n+p]b_n+\alpha^2b_n^2}{([n]+\beta)^2}\to0\end{aligned}$$

is satisfied since $\lim_{n\to\infty}q_n=1$ and $\frac{b_n}{[n]}\to0$ as $n\to\infty$. □

Lemma 3.4 *Let* A *be a positive real number independent of* n *and* f *be a continuous function which vanishes on* $[A,\infty]$. *Assume that* $q:=(q_n)$ *with* $0<q_n<1$, $\lim_{n\to\infty}q_n^n=K<\infty$ *and* $\lim_{n\to\infty}\frac{b_n^2}{[n]}=0$. *Then we have*

$$\lim_{n\to\infty}\sup_{0\le x\le b_n}\left|C_{n,p}^{(\alpha,\beta)}(f;q_n;x)-f(x)\right|=0.$$

Proof From the hypothesis on f, one can write $|f(x)| \leq M$ $(M > 0)$. For arbitrary small $\varepsilon > 0$, we have

$$\left|f\left(\frac{[k]+\alpha}{[n]+\beta}b_n\right) - f(x)\right| < \varepsilon + \frac{2M}{\delta^2}\left(\frac{[k]+\alpha}{[n]+\beta}b_n - x\right)^2,$$

where $x \in [0, b_n]$ and $\delta = \delta(\varepsilon)$ are independent of n. With the help of the following equality:

$$\sum_{k=0}^{n+p}\left(\frac{[k]+\alpha}{[n]+\beta}b_n - x\right)^2 \begin{bmatrix} n+p \\ k \end{bmatrix}\left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n+p-k-1}\left(1 - q^s\frac{x}{b_n}\right) = C_{n,p}^{(\alpha,\beta)}\big((t-x)^2; q_n; x\big),$$

we get by Theorem 3.3 and Remark 2.1

$$\begin{aligned}
&\sup_{0\leq x\leq b_n}\left|C_{n,p}^{(\alpha,\beta)}(f; q_n; x) - f(x)\right| \\
&\quad\leq \varepsilon + \frac{2M}{\delta^2}\left[\frac{(q_n^n[p]-\beta)^2}{([n]+\beta)^2}b_n^2 + \left(\frac{(2\alpha+1)[n+p]}{([n]+\beta)^2}\right)b_n^2 + \frac{\alpha^2 b_n^2}{([n]+\beta)^2}\right].
\end{aligned}$$

Since $\frac{b_n^2}{[n]} \to 0$ as $n \to \infty$, we have the desired result. □

Theorem 3.5 *Let f be a continuous function on the semi-axis $[0,\infty)$ and*

$$\lim_{x\to\infty} f(x) = k_f < \infty.$$

Assume that $q := (q_n)$ with $0 < q_n < 1$, $\lim_{n\to\infty} q_n = 1$, $\lim_{n\to\infty} q_n^n = K < \infty$, and $\lim_{n\to\infty}\frac{b_n^2}{[n]} = 0$. Then

$$\lim_{x\to\infty}\sup_{0\leq x\leq b_n}\left|C_{n,p}^{(\alpha,\beta)}(f; q_n; x) - f(x)\right| = 0.$$

Proof The proof will be given along the lines of the proof of Theorem 2.5 in [23]. Clearly, it is sufficient to prove the theorem for the case $k_f = 0$. Since $\lim_{x\to\infty} f(x) = 0$, given any $\varepsilon > 0$ we can find a point x_0 such that

$$\left|f(x)\right| \leq \varepsilon, \quad x \geq x_0. \tag{3.4}$$

For any fixed $c > 0$, define an auxiliary function as follows:

$$g(x) = \begin{cases} f(x), & 0 \leq x \leq x_0, \\ f(x_0) - \frac{f(x_0)}{c}(x - x_0), & x_0 \leq x \leq x_0 + c, \\ 0, & x \geq x_0 + c. \end{cases}$$

Then for sufficiently large n in such a way that $b_n \geq x_0 + c$ and in view of $\sup_{x_0\leq x\leq x_0+c}|g(x)| = |f(x_0)|$, we have

$$\begin{aligned}
\sup_{0\leq x\leq b_n}\left|f(x) - g(x)\right| &\leq \sup_{x_0\leq x\leq x_0+c}\left|f(x) - g(x)\right| + \sup_{b_n\geq x\geq x_0+c}\left|f(x)\right| \\
&\leq 2\sup_{x_0\leq x\leq x_0+c}\left|f(x)\right| + \sup_{b_n\geq x\geq x_0+c}\left|f(x)\right|.
\end{aligned}$$

We have from (3.4)

$$\sup_{0\leq x\leq b_n}\left|f(x)-g(x)\right|\leq 3\varepsilon.$$

Now, we can write

$$\begin{aligned}
&\sup_{0\leq x\leq b_n}\left|C_{n,p}^{(\alpha,\beta)}(f;q_n;x)-f(x)\right| \\
&\quad\leq \sup_{0\leq x\leq b_n}\left|C_{n,p}^{(\alpha,\beta)}\left(|f-g|;q_n;x\right)\right| + \sup_{0\leq x\leq b_n}\left|C_{n,p}^{(\alpha,\beta)}(g;q_n;x)-g(x)\right| \\
&\qquad + \sup_{0\leq x\leq b_n}\left|f(x)-g(x)\right| \\
&\quad\leq 6\varepsilon + \sup_{0\leq x\leq b_n}\left|C_{n,p}^{(\alpha,\beta)}(g;q_n;x)-g(x)\right|,
\end{aligned}$$

where $g(x)=0$ for $x_0+c\leq x\leq b_n$. By Lemma 3.4, we obtain the result. □

4 Order of convergence

In this section, we compute the rate of convergence of the operators in terms of the elements of Lipschitz classes and the modulus of continuity of the function. Additionally, we calculate the order of convergence in terms of the first modulus of continuity of the derivative of the function.

Now, we give the rate of convergence of the operators $C_{n,p}^{(\alpha,\beta)}$ in terms of the Lipschitz class $\mathrm{Lip}_M(\gamma)$, for $0<\gamma\leq 1$. Let $C_B[0,\infty)$ denote the space of bounded continuous functions on $[0,\infty)$. A function $f\in C_B[0,\infty)$ belongs to $\mathrm{Lip}_M(\gamma)$ if

$$\left|f(t)-f(x)\right|\leq M|t-x|^{\gamma}\quad\left(t,x\in[0,\infty)\right)$$

is satisfied.

Theorem 4.1 *Let* $f\in\mathrm{Lip}_M(\gamma)$

$$\left|C_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right|\leq M\left(\lambda_n(x)\right)^{\gamma/2},$$

where $\lambda_{n,q}(x)=(\frac{(q_n^n[p]-\beta)^2}{([n]+\beta)^2})x^2+(\frac{(2\alpha+1)[n+p]}{([n]+\beta)^2}-\frac{2\alpha}{[n]+\beta})b_nx+\frac{\alpha^2b_n^2}{([n]+\beta)^2}$.

Proof Considering the monotonicity and the linearity of the operators, and taking into account that $f\in\mathrm{Lip}_M(\gamma)$,

$$\begin{aligned}
&\left|C_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right| \\
&\quad=\left|\sum_{k=0}^{n+p}\left(f\left(\frac{[k]+\alpha}{[n]+\beta}b_n\right)-f(x)\right)\begin{bmatrix}n+p\\k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right)\right| \\
&\quad\leq\sum_{k=0}^{n+p}\left|f\left(\frac{[k]+\alpha}{[n]+\beta}b_n\right)-f(x)\right|\begin{bmatrix}n+p\\k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\quad\leq M\sum_{k=0}^{n+p}\left|\frac{[k]+\alpha}{[n]+\beta}b_n-x\right|^{\gamma}\begin{bmatrix}n+p\\k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right).
\end{aligned}$$

Using Hölder's inequality with $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we get by the statement (2.2)

$$\begin{aligned}
&\left|C_{n,p}^{(\alpha,\beta)}(f;q;x) - f(x)\right| \\
&\quad\le M\sum_{k=0}^{n+p}\left\{\left[\left(\frac{[k]+\alpha}{[n]+\beta}b_n - x\right)^2\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right)\right]^{\frac{\gamma}{2}}\right. \\
&\qquad\left.\times\left[\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right)\right]^{\frac{2-\gamma}{2}}\right\} \\
&\quad\le M\left[\left\{\sum_{k=0}^{n+p}\left[\left(\frac{[k]+\alpha}{[n]+\beta}b_n - x\right)^2\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right)\right]\right\}^{\frac{\gamma}{2}}\right. \\
&\qquad\left.\times\left\{\sum_{k=0}^{n+p}\left[\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right)\right]\right\}^{\frac{2-\gamma}{\gamma}}\right] \\
&\quad= M\left[C_{n,p}^{(\alpha,\beta)}\left((t-x)^2;q;x\right)\right]^{\frac{\gamma}{2}} \\
&\quad\le M\left(\lambda_{n,q}(x)\right)^{\frac{\gamma}{2}}. \qquad\square
\end{aligned}$$

Now we give the rate of convergence of the operators by means of the modulus of continuity which is denoted by $\omega(f;\delta)$. Let $f \in C_B[0,\infty)$ and $x \ge 0$. Then the definition of the modulus of continuity of f is given by

$$\omega(f;\delta) = \max_{\substack{|t-x|\le\delta \\ t,x\in[0,\infty)}}\left|f(t) - f(x)\right|. \tag{4.1}$$

It follows that for any $\delta > 0$ the inequality

$$\left|f(x) - f(y)\right| \le \omega(f;\delta)\left(\frac{|x-y|}{\delta} + 1\right) \tag{4.2}$$

is satisfied [24].

Theorem 4.2 *If $f \in C_B[0,\infty)$, we have*

$$\left|C_{n,p}^{(\alpha,\beta)}(f;q;x) - f(x)\right| \le 2\omega\left(f;\sqrt{\lambda_{n,q}(x)}\right),$$

where $\omega(f;\cdot)$ is modulus of continuity of f which is defined in (4.1) *and $\lambda_{n,q}(x)$ be the same as in Theorem* 4.1.

Proof By the triangular inequality, we get

$$\begin{aligned}
&\left|C_{n,p}^{(\alpha,\beta)}(f;q;x) - f(x)\right| \\
&\quad= \left|\sum_{k=0}^{n+p} f\left(\frac{[k]+\alpha}{[n]+\beta}b_n\right)\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right) - f(x)\right| \\
&\quad\le \sum_{k=0}^{n+p}\left|f\left(\frac{[k]+\alpha}{[n]+\beta}b_n\right) - f(x)\right|\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right).
\end{aligned}$$

Now using (4.2) and the Hölder inequality, we can write

$$\begin{aligned}
&\left|C_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right| \\
&\quad\le \sum_{k=0}^{n+p}\left(\frac{\left|\frac{[k]+\alpha}{[n]+\beta}b_n-x\right|}{\lambda}+1\right)\omega(f;\lambda)\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\quad= \omega(f;\lambda)\sum_{k=0}^{n+p}\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\qquad+\frac{\omega(f;\lambda)}{\lambda}\sum_{k=0}^{n+p}\left|\frac{[k]+\alpha}{[n]+\beta}b_n-x\right|\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\quad= \omega(f;\lambda)+\frac{\omega(f;\lambda)}{\lambda}\left\{\sum_{k=0}^{n+p}\left(\frac{[k]+\alpha}{[n]+\beta}b_n-x\right)^2\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right)\right\}^{1/2} \\
&\quad= \omega(f;\lambda)+\frac{\omega(f;\lambda)}{\lambda}\left\{C_{n,p}^{(\alpha,\beta)}\left((t-x)^2\right);q;x\right\}^{1/2}.
\end{aligned}$$

Now choosing $\lambda_{n,q}(x)$ the same as in Theorem 4.1, we have

$$\left|C_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right|\le 2\omega\left(f;\sqrt{\lambda_{n,q}(x)}\right). \qquad \square$$

Now, we compute the rate of convergence of the operators $C_{n,p}^{(\alpha,\beta)}$ in terms of the modulus of continuity of the derivative of the function.

Theorem 4.3 *If $f(x)$ has a continuous bounded derivative $f'(x)$ and $\omega(f';\delta)$ is the modulus of continuity of $f'(x)$ in $[0,A]$, then*

$$\begin{aligned}
&\left|f(x)-C_{n,p}^{(\alpha,\beta)}(f;q;x)\right| \\
&\quad\le M\left(\left|\frac{[n+p]}{[n]+\beta}-1\right|A+\frac{\alpha b_n}{[n]+\beta}\right)+2\left(B_{n,q}(\alpha,\beta)\right)^{1/2}\omega\left(f';\left(B_{n,q}(\alpha,\beta)\right)^{1/2}\right),
\end{aligned}$$

where M is a positive constant such that $|f'(x)|\le M$.

Proof Using the mean value theorem, we have

$$\begin{aligned}
f\left(\frac{[k]+\alpha}{[n]+\beta}b_n\right)-f(x) &= \left(\frac{[k]+\alpha}{[n]+\beta}b_n-x\right)f'(\xi) \\
&= \left(\frac{[k]+\alpha}{[n]+\beta}b_n-x\right)f'(x)+\left(\frac{[k]+\alpha}{[n]+\beta}b_n-x\right)\left(f'(\xi)-f'(x)\right),
\end{aligned}$$

where ξ is a point between x and $\frac{[k]+\alpha}{[n]+\beta}b_n$. By using the above identity, we get

$$\begin{aligned}
&C_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x) \\
&\quad= f'(x)\sum_{k=0}^{n+p}\left(\frac{[k]+\alpha}{[n]+\beta}b_n-x\right)\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\qquad+\sum_{k=0}^{n+p}\left(\frac{[k]+\alpha}{[n]+\beta}b_n-x\right)\left(f'(\xi)-f'(x)\right)\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|C_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right| \\
&\quad\leq \left|f'(x)\right|\left|C_{n,p}^{(\alpha,\beta)}\big((t-x);q;x\big)\right| \\
&\qquad+\sum_{k=0}^{n+p}\left|\frac{[k]+\alpha}{[n]+\beta}b_n-x\right|\left|f'(\xi)-f'(x)\right|\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\quad\leq M\left(\left|\frac{[n+p]}{[n]+\beta}-1\right|A+\frac{\alpha b_n}{[n]+\beta}\right) \\
&\qquad+\sum_{k=0}^{n+p}\left|\frac{[k]+\alpha}{[n]+\beta}-x\right|\left|f'(\xi)-f'(x)\right|\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\quad\leq M\left(\left|\frac{[n+p]}{[n]+\beta}-1\right|A+\frac{\alpha b_n}{[n]+\beta}\right) \\
&\qquad+\sum_{k=0}^{n+p}\omega\big(f';\delta\big)\left(\frac{\left|\frac{[k]+\alpha}{[n]+\beta}b_n-x\right|}{\delta}+1\right)\left|\frac{[k]+\alpha}{[n]+\beta}b_n-x\right|\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k \\
&\qquad\times\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right),
\end{aligned}$$

since

$$|\xi-x|\leq\left|\frac{[k]+\alpha}{[n]+\beta}b_n-x\right|.$$

Using the above inequality, we have

$$\begin{aligned}
&\left|C_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right| \\
&\quad\leq M\left(\left|\frac{[n+p]}{[n]+\beta}-1\right|A+\frac{\alpha b_n}{[n]+\beta}\right) \\
&\qquad+\omega\big(f';\delta\big)\sum_{k=0}^{n+p}\left|\frac{[k]+\alpha}{[n]+\beta}b_n-x\right|\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right) \\
&\qquad+\frac{\omega(f';\delta)}{\delta}\sum_{k=0}^{n+p}\left(\frac{[k]+\alpha}{[n]+\beta}b_n-x\right)^2\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right).
\end{aligned}$$

Therefore, applying the Cauchy-Schwarz inequality for the second term, we get

$$\begin{aligned}
&\left|C_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right| \\
&\quad\leq M\left(\left|\frac{[n+p]}{[n]+\beta}-1\right|A+\frac{\alpha b_n}{[n]+\beta}\right) \\
&\qquad+\omega\big(f';\delta\big)\left(\sum_{k=0}^{n+p}\left(\frac{[k]+\alpha}{[n]+\beta}b_n-x\right)^2\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right)\right)^{1/2} \\
&\qquad+\frac{\omega(f';\delta)}{\delta}\sum_{k=0}^{n+p}\left(\frac{[k]+\alpha}{[n]+\beta}b_n-x\right)^2\begin{bmatrix} n+p \\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right)
\end{aligned}$$

$$
\begin{aligned}
&= M\left(\left|\frac{[n+p]}{[n]+\beta}-1\right|A+\frac{\alpha b_n}{[n]+\beta}\right)\\
&\quad+\omega\big(f';\delta\big)\sqrt{C_{n,p}^{(\alpha,\beta)}\big((t-x)^2;q;x\big)}+\frac{\omega(f';\delta)}{\delta}C_{n,p}^{(\alpha,\beta)}\big((t-x)^2;q;x\big).
\end{aligned}
$$

Therefore, using (2.2) we see that

$$
\begin{aligned}
&\sup_{0\le x\le A} C_{n,p}^{(\alpha,\beta)}\big((t-x)^2;q;x\big)\\
&\le \sup_{0\le x\le A}\frac{(q^n[p]-\beta)^2}{([n]+\beta)^2}x^2+\left(\frac{(2\alpha+1)[n+p]}{([n]+\beta)^2}-\frac{2\alpha}{[n]+\beta}\right)b_n x+\frac{\alpha^2 b_n^2}{([n]+\beta)^2}\\
&\le \frac{(q^n[p]-\beta)^2}{([n]+\beta)^2}A^2+\left|\frac{(2\alpha+1)[n+p]}{([n]+\beta)^2}-\frac{2\alpha}{[n]+\beta}\right|Ab_n+\frac{\alpha^2 b_n^2}{([n]+\beta)^2}\\
&:= B_{n,q}(\alpha,\beta).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left|C_{n,p}^{(\alpha,\beta)}(f;q;x)-f(x)\right| &\le M\left(\left|\frac{[n+p]}{[n]+\beta}-1\right|A+\frac{\alpha b_n}{[n]+\beta}\right)\\
&\quad+\omega\big(f';\delta\big)\left[\big(B_{n,q}(\alpha,\beta)\big)^{1/2}+\frac{1}{\delta}B_{n,q}(\alpha,\beta)\right].
\end{aligned}
$$

Choosing $\delta := \delta_{n,q}(p) = (B_{n,q}(\alpha,\beta))^{1/2}$, we obtain the desired result. □

5 Generalization of the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators

In this section, we introduce generalization of Chlodowsky variant of q-Bernstein-Schurer-Stancu operators. The generalized operators help us to approximate the continuous functions on more general weighted spaces. Note that this kind of generalization was considered earlier for the Bernstein-Chlodowsky polynomials [22] and q-Bernstein-Chlodowsky polynomials [25].

For $x \ge 0$, consider any continuous function $\omega(x) \ge 1$ and define

$$
G_f(t) = f(t)\frac{1+t^2}{\omega(t)}.
$$

Let us take into account the generalization of the $C_{n,p}^{(\alpha,\beta)}(f;q;x)$ as follows:

$$
L_{n,p}^{\alpha,\beta}(f;q;x) = \frac{\omega(x)}{1+x^2}\sum_{k=0}^{n+p} G_f\left(\frac{[k]+\alpha}{[n]+\beta}b_n\right)\begin{bmatrix} n+p\\ k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right),
$$

where $0 \le x \le b_n$ and (b_n) has the same properties as the Chlodowsky variant of the q-Bernstein-Schurer-Stancu operators.

Theorem 5.1 *For the continuous functions satisfying*

$$
\lim_{x\to\infty}\frac{f(x)}{\omega(x)} = K_f < \infty,
$$

we have

$$\lim_{n\to\infty}\sup_{0\le x\le b_n}\frac{|L_{n,p}^{\alpha,\beta}(f;q;x)-f(x)|}{\omega(x)}=0.$$

Proof Clearly,

$$\begin{aligned}&L_{n,p}^{\alpha,\beta}(f;q;x)-f(x)\\&\quad=\frac{\omega(x)}{1+x^2}\left(\sum_{k=0}^{n+p}G_f\left(\frac{[k]+\alpha}{[n]+\beta}b_n\right)\begin{bmatrix}n+p\\k\end{bmatrix}\left(\frac{x}{b_n}\right)^k\prod_{s=0}^{n+p-k-1}\left(1-q^s\frac{x}{b_n}\right)-G_f(x)\right),\end{aligned}$$

thus

$$\sup_{0\le x\le b_n}\frac{|L_{n,p}^{\alpha,\beta}(f;q;x)-f(x)|}{\omega(x)}=\sup_{0\le x\le b_n}\frac{|C_{n,p}^{(\alpha,\beta)}(G_f;q;x)-G_f(x)|}{1+x^2}.$$

By using $|f(x)|\le M_f\omega(x)$ and continuity of the function f, we get $|G_f(x)|\le M_f(1+x^2)$ for $x\ge 0$ and $G_f(x)$ is continuous function on $[0,\infty)$. Thus, from Theorem 3.2 we get the result. □

Finally note that, taking $\omega(x)=1+x^2$, the operators $L_{n,p}^{\alpha,\beta}(f;q;x)$ reduce to $C_{n,p}^{(\alpha,\beta)}(G_f;q;x)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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