Equistatistical Convergence

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ABSTRACT

In this thesis, we focus on different types of equistatistical convergences. We define some new type of convergences such as lacunary equistatistical convergence, λ-equistatistical convergence, $\mathcal{A}$-equistatistical convergence, $\mathcal{B}$-equistatistical convergence and $\alpha\beta$-equistatistical convergence. We also study properties of these new types of convergences. We construct examples for each case, to show that equistatistical convergence lies between point wise and uniform convergences. Moreover, we prove Korovkin type approximation theorems via lacunary equistatistical convergence, λ-equistatistical convergence, $\mathcal{A}$-equistatistical convergence, $\mathcal{B}$-equistatistical convergence and $\alpha\beta$-equistatistical convergence. In the last chapter we introduce $\alpha\beta$- statistical convergence of order γ and we prove Korovkin type approximation theorems in the sense of $\alpha\beta$- statistical convergence.

Keywords: Statistical convergence, lacunary statistical convergence, $\mathcal{A}$-statistical convergence, λ-statistical convergence, equistatistical convergence, Korovkin type approximation theorem.

ÖZ

Bu tezde esas olarak eşistatistiksel yakınsaklık kavramı ele alınacaktır. Eşistatistiksel yakınsaklık noktasal istatistiksel yakınsaklık ile düzgün istatistiksel yakınsaklık arasında yer alan bir yakınsama çeşididir. Bu doktora tezindeki esas amaç lacunary eşistatistiksel, A-eşistatistiksel, λ-eşistatistiksel, $\mathcal{B}$-eşistatistiksel ve $\alpha\beta$-eşistatistiksel yakınsaklık kavramlarını vermek ve herbiri için Korovkin Tipli Teoremler ispat etmektir. Bunun yanında bu yakınsama türlerinin daha anlaşılır olması için belli başlı özellikleride incelenecektir. Bu yakınsama türleri için elde edilecek Korovkin Tipli Teoremlerin Mevcut Korovkin Tipli Teoremlerle ilişkileri de verilecektir.

Anahtar Kelimeler: İstatistiksel yakınsaklık, lacunary istatistiksel yakinsaklik, A-istatistiksel yakınsaklık, λ-istatistiksel yakınsaklık, eşistatistiksel yakınsaklık, Korovkin Tipli Teorem.

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Chapter 1

INTRODUCTION

Statistical convergence (or $s-$convergence) is an extension of convergence which is based on a set function δ, called density. Recently the concept of s-convergence, which was defined in 1951 by Steinhaus (see [43]) at a conference in Poland and developed in the same year by Fast in [23], has gained popularity amongst researchers and used in different fields of mathematics (see [12], [13], [15], [19], [20], [21], [24] and [26]). After, $s-$convergence some other methods are introduced by different researchers. Fridy and Orhan [27] introduced and discussed the concept of lacunary statistical convergence (or $\theta-$convergence) using an arbitrary lacunary sequence θ. Freedman andSember [24] used a non-negative regular matrix (NNRM) instead of Cesáro matrix and extended $s-$convergence to A-statistical convergence (or $A-$convergence). Mursaleen [36] initiated the concept of $\lambda-$statistical convergence (or $\lambda-$convergence). For each suggested method it is shown that they are nontrivial extensions of ordinary convergence. Moreover, for each case, implication relations are also studied. For example, Fridy obtained sufficient conditions for implications between $s-$convergence and ordinary convergence. Similarly Fridy and Orhan introduced $\theta-$convergence and carry out differences and implication conditions between other type of conditions. Mursaleen did the same for $\lambda-$convergence as well. He proved that under some condition $s-$convergence implies $\lambda-$convergence. Later Çolak [12] brought another dimension to this theory by introducing the concept of $s-$convergence of order α (see [9]).

Gadjiev and Orhan [28] joined this theory with Korovkin Type Approximation Theory by proving a Korovkin type approximation thereom (KTAT) in the sense of $s-$convergence. After they prove a (KTAT) via $s-$convergence many researchers extend this idea to $A-$convergence, $\theta-$convergence and $\lambda-$convergence for different spaces (see [15],[16],[18],[20],[21], [22], [30] and [31]).

In 2007 Balcerzak, Dems and Komisarski [7] initiated and investigated a new type of convergence called equistatistical convergence. The most important feature of equistatistical convergence is that it lies between pointwise and uniform $s-$convergences. They also constructed examples to show that equistatistical convergence is different from both pointwise and uniform $s-$convergences. This was a new expansion for researchers and in [30] Karakuş, Demirci and Duman initiated KTAT by introducing KTAT via equistatistical convergence. This encourage us to study equistatistical convergence for different type of convergences such as $A-$convergence, $\theta-$convergence, $\lambda-$convergence and $\mathcal{B}-$convergence. This thesis covers our studies about equistatistical convergence and related KTAT. Additionally, by using a different point of view we introduced $\alpha\beta-$statistical and $\alpha\beta-$equistatistical convergences of order γ. We complete our thesis by KTAT's via $\alpha\beta-$statistical and $\alpha\beta-$equistatistical convergences. It should be mentioned that $\alpha\beta-$statistical and $\alpha\beta-$equistatistical convergences of order γ which is a non-trivial extension of $\alpha\beta-$statistical and $\alpha\beta-$equistatistical convergence is also considered in this thesis.

In Chapter 1, after a general introduction we give the brief description of the whole work.

Chapter 2, contains preliminary and auxilary results which will be needed in the rest of the thesis. The idea here is to give definition and other informations which will help readers to follow the rest of the thesis. At the begining of this chapter we explain briefly the main properties of regular matrices then we continue with the density function. Later we give definition and some important properties of the concept of $s-$convergence and relations between ordinary convergence. We also give definitions and properties of $\theta-$convergence and its relation between $s-$convergence. Finally, we discuss $\lambda-$convergence and its relations between $s-$convergence. At the end of the chapter give required informations about $A-$convergence, equistatistical convergence and (KTAT).

Our contribution starts from Chapter 3. In this Chapter, we introduce lacunary equistatistical convergence. By the definitions one can see that lacunary equistatistical convergence lies between lacunary uniform convergence and lacunary pointwise convergence. To point out the difference we construct examples and show that in general the converse implication does not hold. Moreover we discuss the conditions under which lacunary equistatistical convergence and equistatistical convergence implies each other. At the end of chapter we prove a (KTAT) for lacunary equistatistical convergence.

In Chapter 4, we introduce $\lambda-$equistatistical convergence and investigate the conditions under which equistatistical convergence and $\lambda-$equistatistical convergence implies each other. Also we show that $\lambda-$equistatistical convergence lies between pointwise and uniform convergences in the same sense. Moreover, we prove (KTAT) in the sense of $\lambda-$equistatistical convergence.

Chapter 5, is about A-equistatistical convergence. In this Chapter we introduce A-equistatistical convergence and (KTAT) for A-equistatistical convergence. We also discuss implications conditions between A-equistatistical convergence with equistatistical convergence, lacunary equistatistical convergence and $\lambda-$equistatistical convergence.

In Chapter 6, we introduce $\mathcal{B}$-equistatistical convergence by using a sequence of infinite matrices $\mathcal{B}_j$. We prove that $\mathcal{B}$-equistatistical convergence lies between pointwise and uniform convergences in the same sense. Chapter 6 is completed by a KTAT for $\mathcal{B}$-equistatistical convergence.

Chapter 7 is devoted to $\alpha\beta-$statistical and $\alpha\beta-$equistatistical convergences of order γ which were introduced in [5]. It is shown that for special cases of $\alpha(n)$, $\beta(n)$ and γ, $\alpha\beta-$equistatistical convergence of order γ ($\alpha\beta-$statistical convergence of order γ) reduces to equistatistical convergence ($s-$convergence of order γ), $\theta-$equistatistical convergence ($\theta-$convergence) and $\lambda-$equistatistical convergence ($\lambda-$statistical convergence, $\lambda-$statistical convergence of order γ). Finally we prove KTAT for both $\alpha\beta-$equistatistical convergence of order γ and $\alpha\beta-$statistical convergence of order γ.

Chapter 2

NOTATION AND BACKGROUND MATERIAL

The present chapter is devoted to fundamental notions and background materials about the theory of infinite matrices, density functions, statistical type convergences and KTAT. The idea here is to explain some notions that will help readers to follow the rest of the thesis.

2.1 Infinite Matrix and Matrix Transformation

Let $D \subset \mathbb{N}$. Then the characteristic function of D is represented by χ_D and defined by

$$\chi_D(k) := \begin{cases} 1; & k \in D \\ 0; & k \notin D \end{cases}.$$

Example 2.1.1 *If E and F denotes the odd and even natural numbers respectively then clearly we have*

$$\chi_E := (1, 0, 1, 0, 1, ...) \textit{ and } \chi_F := (0, 1, 0, ...).$$

An infinite matrix, $D := (d_{nk})$ is a matrix which has infinitely many rows and coloums. In the case of infinite matrices addition and scalar multiplications are defined componentwise, analogously to the case of sequences. More precisely if

$D = (d_{nk})$ and $E = (e_{nk})$ are two infinite matrices then

$$D + E := (d_{nk} + e_{nk})$$

$$\lambda D := (\lambda d_{nk}) .$$

Definition 2.1.2 *An infinite matrix* $E := (e_{nk})$, *with* $e_{nk} \geq 0$, *for all* $n, k \geq 0$ *is called a non-negative infinite matrix.*

Definition 2.1.3 *(See [10]) Let* $E = (e_{nk})$ *be an infinite matrix and* $x = (x_n)$ *be a sequence then the* E*-transform of* $x := (x_k)$ *is denoted by* $Ex := ((Ex)_n)$ *and defined as*

$$(Ex)_n = \sum_{k=1}^{\infty} e_{nk} x_{k,}$$

if it converges for each n.

Example 2.1.4 *Let* $x = (1, 1, 1, ...)$, $y = (0, 1, 0, 1, ...)$ *and* $z = (1, 0, 1, 0, ...)$. *Also let*

$$A = (a_{nk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$B = (b_{nk}) = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then simple calculations show that

$$(Ax) = x,$$

and

$$(Ay) = 0, \ (Bz) = \frac{1}{2}.$$

Definition 2.1.5 *If* $\lim_{n\to\infty}(Ax)_n = L$ *then* x *is said to be* $A-$*summable to* L*.*

Example 2.1.6 *Let* x *and* y *be the same as in Example 2.1.4. Also let*

$$A = (a_{nk}) = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then

$$(Ax) = (\frac{2}{3})$$

and

$$(Ay) = \left(0, \frac{2}{3}, 0, \frac{2}{3}. \cdots\right).$$

Hence x is $A-$summable to $\frac{2}{3}$ but y is not $A-$summable.

The above examples show that, for an infinite matrix it is possible to keep a sequence fix to transfer a divergent sequence to a convergent sequence.

Let A be an infinite matrix and let x be a sequence with limit L. In the present part we will try to answer the following questions. Under which conditions Ax is convergent and secondly under which conditions the limit of Ax is again L. The answer of the first question is known as the theorem of Kojima and Schur. The answer of the second theorem is known as the Silverman-Toeplitz conditions. Details about these theorems is given below.

Definition 2.1.7 *(see [10]) An infinite matrix A is said to be conservative if Ax is convergent for each convergent sequence x.*

Now we have the following well-known theorem which is given by Kojima-Schur which gives necessary and sufficient conditions for a matrix to be conservative.

Theorem 2.1.8 *(Kojima-Schur)(see [10])$A = (a_{nk})$ is conservative $\Leftrightarrow$*

(i) $\sup_n \sum_{k=1}^{\infty} |a_{nk}| \leq M < \infty$, *for some* $M > 0$,

(ii) $\lim_n a_{nk} = \delta_k$ *for all* k,

(iii) $\lim_n \sum_{k=1}^{\infty} a_{nk} = \delta$.

Example 2.1.9 *The following infinite matrices are conservative,*

$$D = (d_{nk}) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ 1-\frac{1}{n} & \frac{1}{n} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$E = (e_{nk}) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It is easly seen that conservative matrices may or may not preserve the limit of a convergent sequence.

Definition 2.1.10 *(see [10])An infinite matrix* A *is called regular if* $x_n \to L$

implies $(Ax)_n \to L$.

Theorem 2.1.11 *(Silverman-Toeplitz Conditions) (see [10], [44])* $A = (a_{nk})$ *is regular* $\Leftrightarrow$

(i) $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$,

(ii) *For all* k, $\lim_n a_{nk} = 0$,

(iii) $\lim_n \sum_{k=1}^{\infty} a_{nk} = 1$.

Example 2.1.12 *([10])The infinite matrix* $C_1 = (c_{nk})$ *where*

$$c_{nk} = \begin{cases} \frac{1}{n}, & \text{if} \quad 1 \le k \le n, \\ 0, & \text{otherwise} \end{cases}$$

is a NNRM which is known as Cesàro matrix of order one.

Example 2.1.13 *Identity matrix* I, *which has infinite number of rows and coloums is also a NNRM.*

Remark 2.1.14 *The matrix* E *in Example 2.1.9 is also a NNRM but the matrix* D, *in the same example is not a regular matrix.*

2.2 Densities

The objective of the present section is to introduce definition and basic properties of density functions. Basicly a density is a function from a subset of $\wp(\mathbb{N})$ to $[0,1]$.

Definition 2.2.1 *Let* $C,\ D \subseteq \mathbb{N}$, *the symmetric difference of this two sets is denoted by* $C \vartriangle D$ *and defined as*

$$C \vartriangle D = (C \backslash D) \cup (D \backslash C).$$

If the symmetric difference of two sets C and D is finite then we say that C and D has "$\sim$" relation, in other words

$$C \sim D \Longleftrightarrow C \vartriangle D \text{ is finite.}$$

Definition 2.2.2 *(See [24]) The lower asymptotic density (may be called just a density) is a function, defined for all sets of natural numbers taking values in* $[0,1]$ *and denoted by* δ *if it satisfies the following four axioms:*

$(d.1)$ $\ \ if\ \ F \sim G\ \ then\ \delta(F) = \delta(G);$

$(d.2)$ $\ \ if\ \ F \cap G = \emptyset,\ then\ \delta(F) + \delta(G) \leq \delta(F \cup G);$

$(d.3)$ $\ \ \forall\ F, G;\ \delta(F) + \delta(G) \leq 1 + \delta(F \cap G);$

$(d.4)$ $\ \ \delta(\mathbb{N}) = 1.$

Definition 2.2.3 *(See [24]) For a density* δ *we define* $\overline{\delta}$, *the upper density associated with* δ, *by*

$$\overline{\delta}(F) = 1 - \delta(\mathbb{N} \backslash F)$$

for any $F \subseteq \mathbb{N}$.

Definition 2.2.4 *(See [24]) We say that the set $C \subseteq \mathbb{N}$ has the natural density with respect to δ, if*

$$\delta(C) = \overline{\delta}\,(C)\,.$$

Definition 2.2.5 *(See [24]) The term "asymptotic density" (or natural density) is generally used for the function*

$$d(A) = \liminf_{n \to \infty} \frac{|A(n)|}{n}. \tag{2.2.1}$$

Here by $|A(n)|$,we mean the number of elements in $A \cap \{1, 2, ...n\}$. The function d satisfies the conditions, $(d1 - d4)$ therefore it is a density.

The definition given in 2.2.1 also be given as

$$d(A) = \liminf_{n \to \infty} (C_1.\chi_A)_n.$$

Now the following question arises : Since Cesàro matrix is a NNRM is it possible to extend this idea to any NNRM. Answer is positive. For instance see the following definition which is given by Fredman and Sember (see [24]).

Definition 2.2.6 *(see [24]) Let M be a NNRM and $A \subseteq \mathbb{N}$. Then δ_M defined by*

$$\delta_M(A) = \liminf_{n \to \infty} (M.\chi_A)_n$$

Satisfies conditions, therefore δ_M is a density. Moreover

$$\overline{\delta}_M(A) = \limsup_{n\to\infty}(M.\chi_A)_n.$$

Definition 2.2.7 *Let $K \subset \mathbb{N}$ be an arbitrary subset of the natural numbers then the natural density of K is defined by*

$$\delta(K) = \lim_{n\to\infty} \frac{|K(n) := \{k \leq n : k \in K\}|}{n}$$

Example 2.2.8 *For the set $K := \{ak + b : k \in \mathbb{N}\}$ we have $\delta(K) = \frac{1}{a}$.*

Example 2.2.9 *Finite sets and natural numbers have density zero and one respectively.*

Example 2.2.10 *The set $K := \{k = m^2 : k \in \mathbb{N}\}$ has density zero. In fact, since $|K(n)| \leq \sqrt{n}$ we conclude that*

$$\lim_n \frac{\sqrt{n}}{n} = 0.$$

2.3 Statistical Convergence

This sections aims to give the definition of $s-$convergence and some properties that will be needed in the sequel. We are also aimed to give some examples to illusrate differences between ordinary convergence and $s-$convergence. Moreover we give definition of $s-$convergence of order α (see [9] and [12]).

Definition 2.3.1 *(see [23] and [43])* $x = (x_k)$ *is said to be statistically convergent to* L *if* $\forall \epsilon > 0,\ K_n(\epsilon) = \{k \leq n : |x_k - L| \geq \epsilon\}$ *has natural density zero i.e.* $\lim_{n\to\infty} \frac{|K_n(\epsilon)|}{n} = 0$. *Throughout this thesis we denote* $s-$*convergence of* x *to* L *by* $x_k \to L$ *(stat).*

Remark 2.3.2 *For "ordinary convergence", for all* $\varepsilon > 0$, $K_n(\epsilon)$ *is finite therefore*

$$x_k \to L \Rightarrow x_k \to L \ (stat).$$

The following example shows that the converse implication is not true in general.

Example 2.3.3 *Consider the sequence*

$$x_k := \begin{cases} 1; & if \quad k = m^2, \\ 0; & if \quad k \neq m^2. \end{cases}$$

Since $\{k^2 : k \in \mathbb{N}\}$ *has density zero we have* $x_k \to 0$ *(stat), but clearly* x *is not convergent in the ordinary sense.*

Another important difference between statistical and ordinary convergence is the following. Ordinary convergence implies boundedness, but we may have unbounded and statistical convergent sequences.

Example 2.3.4 *Consider the sequence*

$$x_k := \begin{cases} \sqrt{k}; & \text{if } k = m^2, \\ 0; & \text{if } k \neq m^2. \end{cases}$$

For the given sequence we have $x_k \to 0$ (stat), but x is not bounded.

Example 2.3.5 *The sequence*

$$x_k = (0, 1, 0, 1, \cdots)$$

is not statistically convergent.

Remark 2.3.6 *It is easily seen that when x_k is statistically convergent to L then it may have infinitely many terms at the outside of each ε-neigbourhood of L, but the density of its indices must be zero.*

We know that the ordinary convergence implies $s-$convergence. For the inverse implication Fridy proved the following theorem.

Theorem 2.3.7 *(see [26]) If $x_k \to L$ (stat) and $\triangle x_k = o(\frac{1}{k})$ then $x_k \to L$ where $\triangle x_k = x_k - x_{k+1}$.*

The following definition was firstly given by Gadjiev and Orhan for statistical convergence (see [28]). Later it was extended to $A-$statistical convergence by

Duman, Khan and Orhan (see [15]). Recently, $s-$convergence of order α was discussed by Çolak, in the following way (see also Bhunia, Das and Pal [9]).

Definition 2.3.8 *(see [9] and [12]) For $x = x_k$ and $0 < \alpha \leq 1$ then x_k is called statistically convergent to L of order α, if $\forall \varepsilon > 0$*

$$\lim_{n \to \infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{n^\alpha} = 0.$$

It is proved that for $0 < \alpha \leq \beta \leq 1$ $s-$convergence of order α implies s-convergence of order β. Moreover the inclusion is strict for any α, β with $0 < \alpha < \beta \leq 1$. For instance see the following example.

Example 2.3.9 *(see [9]) Given $0 < \alpha < \beta \leq 1$, we can pick $k \in \mathbb{N}$ such that $\alpha < \frac{1}{k} < \beta$. Define the sequence x by*

$$x_k = \begin{cases} 1, & k = j^n, \\ & \qquad j \in \mathbb{N}. \\ 0, & k \neq j^n, \end{cases}$$

Then x is statistically convergent to 0 of order β but not statistically convergent of order α.

2.4 Lacunary Statistical Convergence

An increasing sequence $\theta := \{k_r\} \subset \mathbb{N}$ with

$$k_0 = 0,$$

$$h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty \text{ (see [25])}.$$

is called lacunary sequence. In the rest of the thesis by I_r we mean the intervals $(k_{r-1}, k_r]$ determined by θ and by q_r we will represent the ratio $\frac{k_r}{k_{r-1}}$.

Example 2.4.1 *$\theta := \{k_r\} = 2^r - 1$, $\theta := \{k_r\} = r! - 1$ and $\theta := \{k_r\} = r^2$ are lacunary sequences.*

Fridy and Orhan [27] introduced the concept of $\theta-$convergence by using an arbitrary lacunary sequence. They also gave the conditions that lacunary statistical and $s-$convergence implies each other. The inclusion properties between lacunary sequences for $\theta-$convergence discussed by Jinlu (see [29]). For the following section our aim is to give a short summary of $\theta-$convergence.

Definition 2.4.2 *(see [27]) x is called $\theta-$convergent to L if $\forall\ \epsilon > 0$,*

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0.$$

$\theta-$convergence of x to L will be denoted by $x_k \to L$ $(\theta-stat)$. In other words, if we denote the characteristic function of the set $K(\epsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$

by the function $\chi_{K(\epsilon)}$ *and* A_θ *by*

$$A_\theta = a_{rk} = \begin{cases} \frac{1}{h_r}; & k \in I_r \\ 0; & k \notin I_r \end{cases}$$

we have $\frac{1}{h_r}\sum_{k\in I_r}\chi_{K(\epsilon)}(k) = \frac{1}{h_r}\left|\{k \in I_r : |x_k - L| \geq \epsilon\}\right|$, *that is x is said to be lacunary statistical convergent to L if and only if* $\lim_r (A_\theta \chi_{K(\epsilon)})_r = 0$.

Example 2.4.3 *Let*

$$x_k = \begin{cases} 1; & k = 2^r - 1 \\ 0; & k \neq 2^r - 1 \end{cases}$$

and let $\theta := \{k_r\} = 2^r - 1$. *Since* $|\{k \in I_r : |x_k| \geq \varepsilon\}| \leq 1$ *for each r we can show that*

$$\lim_r \frac{1}{h_r} \left|\{k \in I_r : |x_k| \geq \varepsilon\}\right| = 0.$$

Hence x is $\theta-$*convergent to* 0 *but non-convergent in the usual sense.*

In [27], Fridy and Orhan introduced the conditions, under which $s-$convergence and $\theta-$convergence implies each other. They proved the following theorem.

Theorem 2.4.4 *(see [27]) Let* θ *be a lacunary sequence; then* $x_k \to L \quad (stat)$ *and* $x_k \to L \quad (\theta-stat)$ *implies each other* $\Longleftrightarrow$

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty.$$

2.5 λ-Statistical Convergence

Mursaleen [36], investigated the concept of λ-statistically convergence (or $\lambda-$convergence) for sequences of numbers. He proved that, under some conditions $s-$convergence implies $\lambda-$convergence. Conditions for inverse implication are obtained by Aktuğlu, Gezer and Özarslan in [3]. We shall discuss details in Chapter 4, but here we will give a brief outline of $\lambda-$convergence.

Let $\lambda = (\lambda_r)$ be a sequence of non-decreasing and positive numbers such that

$$\lim_{r\to\infty} \lambda_r = \infty,$$

$$\lambda_{r+1} \leq \lambda_r + 1,$$

$$\lambda_1 = 1.$$

and M_r be the closed interval $[r - \lambda_r + 1, r]$. The set of all sequences satisfying above conditions will be represented by ω.

Example 2.5.1 *Sequences $\lambda_r = r$ and $\lambda_r = |[\sqrt{r}]|$ are elements of ω.*

Definition 2.5.2 *(see [36]) x is said to be $\lambda-$convergent to L and denoted by $x_k \to L \quad (\lambda-stat)$ if* $\forall \epsilon > 0$,

$$\lim_{r} \frac{1}{\lambda_r} \left|\{k \in M_r : |x_k - L| \geq \epsilon\}\right| = 0.$$

Example 2.5.3 *Let* $\lambda_r = |[\sqrt{r}]|$. *Then* x_k *is* $\lambda-$*convergent to* 1 *where*

$$x_k = \begin{cases} 0; & k = m^3 \\ 1; & k \neq m^3 \end{cases}.$$

Indeed for every $\varepsilon > 0$, *the set* $\{k \in M_r : |x_k - 1| \geq \epsilon\}$ *has cardinality at most one. Hence*

$$\lim_r \frac{1}{\lambda_r} |\{k \in M_r : |x_k - 1| \geq \epsilon\}| = 0.$$

Remark 2.5.4 *Taking* $\lambda_r = r$, $\lambda-$*convergence reduces to* $s-$*convergence.*

Remark 2.5.5 *Let* $\lambda = (\lambda_r) \in \omega$. *Then we can define a NNRM in the following way*

$$A_\lambda = a_{rk} := \begin{cases} \frac{1}{\lambda_r}, & \text{if } k \in M_r \\ 0, & \text{if } k \notin M_r \end{cases}.$$

Theorem 2.5.6 *(see [36])* $x_k \to L$ *(stat) implies* $x_k \to L$ $(\lambda-$*stat*$)$ $\Leftrightarrow$

$$\liminf_{n\to\infty} \frac{\lambda_r}{r} > 0.$$

Example 2.5.7 *Let* $\lambda_r = |[\sqrt{r}]|$ *then* $\liminf_{r\to\infty} \frac{\lambda_r}{r} = 0$ *and consider the subsequence* $r(j) = j^4$. *Then* $\frac{\lambda_{r(j)}}{r(j)} < \frac{1}{j}$. *Define the sequence* x_i *by*

$$x_i = \begin{cases} 1 & \text{if } i \in I_{n(j)},\ j = 1, 2, ... \\ 0 & \text{otherwise.} \end{cases}$$

Then x is not $\lambda-$statistical convergent.

Later Çolak and Bektaş (see [13]) extended the idea of $\lambda-$convergence to $\lambda-$convergence of order α in the following way.

Definition 2.5.8 *(see [13]) x_k is said to be $\lambda-$convergence to a complex number L of order α for $0 < \alpha \leq 1$, if $\forall \varepsilon > 0$*

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in M_r : |x_k - L| \geq \epsilon\}\right|.$$

They proved that for $0 < \alpha \leq \beta \leq 1$, $\lambda-$convergence of order α implies $\lambda-$convergence of order β. They also proved that for $0 < \alpha < \beta \leq 1$ the inclusion is strict.

2.6 $\mathcal{A}$-Statistical Convergence

Fredman and Sember (see [24]) extended the idea of $s-$convergence to A-convergence using an arbitrary NNRM A instead of C_1. We start this section by defining A-density of a subset K of $\mathbb{N}$ where A is a NNRM. Parallel to the other sections, A-convergence will not be given with details we just give definition and some important properties.

Definition 2.6.1 *(See [24]) Given $K \subset \mathbb{N}$ and a NNRM, A then the $A-$density of K is given by*

$$\delta_A(K) = \lim_n \sum_{k\in K} a_{nk} = \lim_n (A\chi_K(k))_n \tag{2.6.1}$$

provided that the limit exists.

Definition 2.6.2 *(see [24]) $x = (x_k)$ is called A-convergent to L if $\forall\epsilon > 0$, $K(\epsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has A-density zero.*

Example 2.6.3 *Given $x_k = (0, 1, 0, 1, \cdots)$ and*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

then x is not s-convergent but it is A-convergent to zero. Indeed if $\varepsilon > 1$ then the set $\{k : |x_k - 0| \geq \varepsilon\}$ is empty, so we claim that it has A-density zero. If $0 < \varepsilon \leq 1$ then $K = \{k : |x_k - 0| \geq \varepsilon\} = \{2, 4, 6, \cdots\}$. So $\chi_K = (0, 1, 0, 1, \cdots)$ and $A\chi_K = (0, 0, 0, \cdots)$. Hence $\delta_A \{K\} = 0$.

Remark 2.6.4 *Finite sets have $A-$density zero for any NNRM A.Thus every convergent sequence is A-convergent.*

2.7 Equistatistical Convergence

Balcerzak, Dems and Komisarski [7] initiated the concept of equistatistical convergence which lies between statistical uniform and statistical pointwise convergence. After Balcerzak, Dems and Komisarski, the concept of equistatistical converge, gained popularity among researchers (see [1], [2], [3], [30] and [31]). Now we will give related definitions and examples about equistatistical convergence.

Definition 2.7.1 *(see [7]) Let $K \subset \mathbb{N}$ be any subset of natural numbers then*

$$d_j(K) = \frac{|K \cap \{1, 2, ..., j\}|}{j}$$

is called the jth partial density of K.

The definitions of statistical uniform and statistical pointwise convergence has been initiated by Duman and Orhan in more general case (see [17]).

Definition 2.7.2 *(see [2] and [7]) (f_n) is called $s-$pointwise convergent to the function f on X and denoted by*

$$f_n \to f \quad \textit{(stat)}$$

if $\forall \epsilon > 0$, and each $x \in X$

$$\lim_{r \to \infty} \frac{|\{k \leq r : |f_k(x) - f(x)| \geq \epsilon\}|}{r} = 0.$$

Definition 2.7.3 *(see [2] and [7]) (f_n) is called s-uniform convergent to f on X and denoted by*

$$f_n \rightrightarrows f \quad \textit{(stat)}$$

if $\forall \epsilon > 0$,

$$\lim_{r\to\infty} \frac{\left|\left\{k \leq r : \|f_k(x) - f(x)\|_{C(X)} \geq \epsilon\right\}\right|}{r} = 0.$$

Definition 2.7.4 *(see [7]) (f_n) is called equistatistically convergent to f on X and denoted by $f_n \twoheadrightarrow f \quad (stat)$ if $\forall \epsilon > 0$, the sequence of real valued functions*

$$p_{r,\epsilon}(x) := \frac{1}{r} \left|\{k \leq r : |f_k(x) - f(x)| \geq \epsilon\}\right|$$

converges uniformly to the zero function on X i.e.

$$\lim_r \|p_{r,\epsilon}(.)\|_{C(X)} = 0.$$

Theorem 2.7.5 *(see [7]) It is obvious that*

$$f_n \twoheadrightarrow f \quad (stat) \text{ implies } f_n \to f \quad (stat),$$

$$f_n \rightrightarrows f \quad (stat) \text{ implies } f_n \twoheadrightarrow f \quad (stat).$$

Example 2.7.6 *(see [7]) Define $f, f_n : [0,1] \to \mathbb{R}$ in the following way,*

$$f \equiv 0,$$

$$f_n(x) = \chi_{\{\frac{1}{n}\}}.$$

Then

$$f_n \twoheadrightarrow f \ (stat)$$

but

$$f_n \rightrightarrows f \ \text{(stat)}$$

does not hold.

Example 2.7.7 *(see [7])Let* $f_n(x) = x^n,\ x \in [0,1]$, *then* $f_n \to f \quad \text{(stat)}$ *but* $f_n \twoheadrightarrow f \quad \text{(stat)}$ *does not hold.*

2.8 Korovkin's Theorem

Approximation theory has important applications for different areas of Functional Analaysis, approximation of polinomials, numerical solutions for differential and integral equations. KTAT is a base for approximation theory (see [6], [11] and [34]). Gadjiev and Orhan [28] obtained a KTAT for $s-$convergence of positive linear operators which is defined on a function space of closed, bounded and continuous intervals of real numbers. Also Duman, Khan and Orhan investigate the KTAT in $A-$statistical sense (see [15]). Moreover Duman and Orhan investigate the KTAT in statistical and $A-$statistical sense for different spaces (see [15], [16] and [18]).

Definition 2.8.1 *A mapping* $L : X \to Y$ *is called a linear operator if*

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$$

for $\forall f, g \in X$ *and* $\forall \alpha,\ \beta \in \mathbb{R}$ *where* X *and* Y *are linear spaces of functions. If*

$Lf \geq f$ when $f \geq 0$ then L is said to be positive operator.

Proposition 2.8.2 *Let $L : X \to Y$ be a positive and linear operator, then*

1. *If $f, g \in X$ with $f \leq g$ then $Lf \leq Lg$ which means that L is monotonic.*

2. *For every $f \in X$ we have $|Lf| \leq L|f|$.*

Theorem 2.8.3 *(Bohman-Korovkin Theorem) (See [34]) Let $L_r : C[a,b] \to C[a,b]$ be a sequence of positive linear operator. If the sequence of operators L_r satisfy*

$$\lim_r \|L_r(1;x) - 1\|_{C[a,b]} = 0$$

$$\lim_r \|L_r(t;x) - x\|_{C[a,b]} = 0$$

$$\lim_r \left\|L_r(t^2;x) - x^2\right\|_{C[a,b]} = 0$$

for $\forall f \in C[a,b]$, we have

$$\lim_r \|L_r(f;x) - f(x)\|_{C[a,b]} = 0$$

Example 2.8.4 *(see [35]) The Bernstein polinomials is an example for the linear operators for $I = [0,1]$, which is defined by*

$$B_r(f,x) := \sum_{k=0}^{r} f(\frac{k}{r}) \binom{r}{k} x^k (1-x)^{r-k}, \ f \in C[0.1].$$

Direct calculations show that $B_n(1;x) = 1$, $B_n(t;x) = x$ and $B_n(t^2;x) = x^2 + \frac{x-x^2}{n}$. In other words Bernstein polinomials satisfies the conditions of KTAT.

The following KTAT is given by Gadjiev and Orhan.

Theorem 2.8.5 *(see [28]) Let* $L_r : C_M[a,b] \to B[a,b]$ *be a sequence of positive linear operator. If the sequence of operators satisfy*

$$st - \lim_r \|L_r(1;x) - 1\|_B = 0$$

$$st - \lim_r \|L_r(t;x) - x\|_B = 0$$

$$st - \lim_r \left\|L_r(t^2;x) - x^2\right\|_B = 0$$

then for any function f *in* $C_M[a,b]$,

$$st - \lim_r \|L_r(f;x) - f(x)\|_B = 0.$$

Later the following KTAT in A-statistical sense is given by Duman, Khan and Orhan.

Theorem 2.8.6 *(see [15]) Let* $\{L_r\}$ *be a sequence of positive linear operators from* $C[a,b]$ *into* $C[a,b]$, *then the following statements are equivalent*

$$(i)\ st_A - \lim_r \|L_r(f;x) - f(x)\|_{C[a,b]} = 0,\ \forall\ f \in C[a,b],$$

$$(ii)\ st_A - \lim_r \|L_r(f_i;x) - f_i(x)\|_{C[a,b]} = 0\ \ \text{for } f_i(x) = x^i,\ i = 0,1,2.$$

Recently, Balcerzak, Dems and Komisarski introduce the concept of equistatistical convergence and after them Karakuş, Demirci and Duman (see [30]) proved the following KTAT in equistatistical sense.

Theorem 2.8.7 *(see [30])Let* $\{L_r\}$ *be a sequence of linear positive operators from* $C(X)$ *to* $C(X)$ *where* X *is a compact subset of the real numbers. Then* $\forall\ f \in C(X)$,

$$L_r(f;x) \twoheadrightarrow f \ \textit{(stat) on } X$$

$\Longleftrightarrow$

$$L_r(e_i) \twoheadrightarrow e_i \ \textit{(stat) on } X$$

where $e_i(x) = x^i,\ i = 0, 1, 2.$

Chapter 3

LACUNARY EQUISTATISTICAL CONVERGENCE

3.1 Lacunary Equistatistical Convergence

Fridy and Orhan introduced the concept of $\theta-$convergence [27] by using an arbitrary lacunary sequence. They also showed that under some conditions, $\theta-$convergence implies $s-$convergence. Moreover they give necessary and sufficient conditions so that $\theta-$convergence and $s-$convergence are equivalent to each other.

In this chapter we mainly focus on $\theta-$convergence and the concept of lacunary equistatistical convergence which lies between pointwise and uniform $\theta-$convergence (see [2]). We also construct examples of function sequences to point out that in general the converse implications does not hold. We started to this chapter with the following definitions.

Definition 3.1.1 *(see [2]) Let θ be a lacunary sequence. (f_r) is said to be lacunary statistical pointwise convergent to f on X if $\forall\epsilon > 0$, and each $x \in X$,*

$$\lim_{r\to\infty} \frac{|\{m \in I_r : |f_m(x) - f(x)| \geq \epsilon\}|}{h_r} = 0.$$

Lacunary statistical pointwise convergence of f_r to f is denoted by

$$f_r \to f \quad (\theta - stat)$$

Definition 3.1.2 *(see [2]) Let θ be a lacunary sequence. (f_r) is said to be lacunary statistical uniform convergent to f on X if $\forall\ \epsilon > 0$*

$$\lim_{r\to\infty} \frac{\left|\left\{m \in I_r : \|f_m - f\|_{C(X)} \geq \epsilon\right\}\right|}{h_r} = 0.$$

Lacunary statistical uniform convergence of f_r to f is denoted by

$$f_r \rightrightarrows f \quad (\theta - stat).$$

Definition 3.1.3 *(see [2]) Let θ be a lacunary sequence. $(f_r)_{r\in\mathbb{N}}$ is said to be lacunary equistatistical convergent to f on X if $\forall\ \epsilon > 0$, the sequence of real valued functions $(s_{r,\epsilon})_{r\in\mathbb{N}}$, defined by*

$$s_{r,\epsilon}(x) = \frac{1}{h_r} \left|\{m \in I_r : |f_m(x) - f(x)| \geq \epsilon\}\right|$$

uniformly converges to zero function on X, that is

$$\lim_{r\to\infty} \|s_{r,\epsilon}(.)\|_{C(X)} = 0$$

Lacunary equistatistical convergence of f_r to f is denoted by

$$f_r \twoheadrightarrow f \quad (\theta - stat).$$

After these definitions we have following lemma which can be proved easly.

Lemma 3.1.4 *(see [2]) For a lacunary sequence θ we have*

$$f_r \twoheadrightarrow f \ (\theta - stat) \Longrightarrow f_r \rightarrow f \ (\theta - stat),$$

$$f_r \rightrightarrows f \ (\theta - stat) \Longrightarrow f_r \twoheadrightarrow f \ (\theta - stat).$$

In the previous lemma, it clear that lacunary equistatistical convergence lies between pointwise and uniform $\theta-$convergence. But one can ask the following question. "Does the converse implications hold?" Example 3.1.5 and Example 3.1.6 show that in general the inverse implications are not true. Firstly we will show that there exists a function sequence (f_r) such that it is lacunary equistatistical convergent but not uniformly lacunary statistical convergent.

Example 3.1.5 *(see [2]) Let θ be a lacunary sequence. Consider the sequence of continuous functions*

$$f_n = \begin{cases} -\left[2n(n+1)\right]^2 \left(x - \frac{1}{n}\right)\left(x - \frac{1}{n+1}\right) & \text{if } x \in \left(\frac{1}{n+1}, \frac{1}{n}\right] \\ 0 & \text{otherwise} \end{cases}$$

then $\forall \varepsilon > 0$,

$$\begin{aligned} s_{r,\varepsilon}(x) &= \frac{1}{h_r} \left|\{m \in I_r : |f_m(x) \geq \varepsilon|\}\right| \\ &\leq \frac{1}{h_r}. \end{aligned}$$

Hence $\frac{1}{h_r} \to 0$ *as* r *approaches to* ∞ *uniformly in* x *that is* $f_r \twoheadrightarrow 0$ *(*$\theta-$*stat). But* $f_r \rightrightarrows 0$ *(*$\theta-$*stat) does not hold since* $\sup_{x\in[0,1]} |f_r(x)| = 1$, *for all* $r \in \mathbb{N}$.

Secondly we introduce a function sequence such that f_r is lacunary statistical pointwise convergent but not lacunary equistatistical convergence.

Example 3.1.6 *(see [2]) Let the sequence of functions and the lacunary sequence be as in the following*

$$f_r : [0,1] \to \mathbb{R},\ \ f_r(x) = x^r,$$

$$\theta = \left\{2^k\right\}\ \ k > 1,\ \ for\ k = 1.$$

Clearly f_r *is pointwise convergent to the function*

$$f(x) = \begin{cases} 0; & 0 \leq x < 1 \\ 1; & \quad x = 1 \end{cases}$$

in the ordinary sense, then obviously $f_r \to f \quad (\theta-stat)$*. To see that* $f_r \twoheadrightarrow f \quad (\theta$ *stat) does not hold choose* $\varepsilon = \frac{1}{2}$*. Then* $\forall K \in \mathbb{N}$, $\exists n > K$ *such that* $m \in [2^{n-1}, 2^n)$ *and* $x \in \left(\sqrt[2^n]{\frac{1}{2}}, 1\right)$,

$$|f_m(x)| = |x^m| \geq \left|\left(\sqrt[2^n]{\frac{1}{2}}\right)^m\right| \geq \left|\left(\sqrt[2^n]{\frac{1}{2}}\right)^{2^n}\right| = \frac{1}{2}.$$

Now the following question arises "under which conditions does lacunary equistatistical convergence and equistatistical convergence implies each other?" The following two lemmas give the answer to this qustion.

Lemma 3.1.7 *(see [2]) Let θ be a lacunary sequence then equistatistical convergence implies lacunary equistatistical convergence if and only if*

$$\liminf_r q_r > 1.$$

Proof. Assume $f_n \twoheadrightarrow f$ (stat) on X and $\liminf_r q_r > 1$. Then $\exists \alpha > 0$ such that $1 + \alpha \leq q_r$, for large r and

$$\frac{1}{k_r} \geq \frac{\alpha}{(\alpha+1)h_r}.$$

$\forall\, \varepsilon > 0$, we have

$$\begin{aligned} p_{k_r,\varepsilon}(x) &= \frac{1}{k_r} \left|\{m \leq k_r : |f_m(x) - f(x)| \geq \varepsilon\}\right| \\ &\geq \frac{1}{k_r} \left|\{m \in I_r : |f_m(x) - f(x)| \geq \varepsilon\}\right| \\ &\geq \frac{\alpha}{(\alpha+1)h_r} \left|\{m \in I_r : |f_m(x) - f(x)| \geq \varepsilon\}\right| \\ &\geq \frac{\alpha}{(\alpha+1)h_r} s_{r,\varepsilon}(x) \end{aligned}$$

uniformly in x. This proves the sufficiency. For the converse, consider the lacunary sequence and the subsequence

$$\theta = \{k_r\} = \left\{r^2\right\},$$

$$r(j) = j^j \ (j > 2).$$

Then $\liminf_r q_r = 1$, proceeding as in the proof of Lemma 2 of [27] (or as in [25]; p. 510) and take

$$f_i(x) = \begin{cases} 1, & \text{if } i \in I_{r(j)} \text{ for some } j = 3, 4, ... \\ 0, & \text{otherwise} \end{cases}, \ x \in X$$

then we have

$$f_n \twoheadrightarrow 0 \quad (\text{stat}).$$

But since

$$\frac{1}{h_r} \left|\{n \in I_r : |f_n| \geq \varepsilon\}\right| = \begin{cases} 1, & \text{if } r = j^j \text{ for some } j = 3, 4, ... \\ 0, & \text{otherwise} \end{cases}$$

$f_r \twoheadrightarrow f \quad (\theta-\text{stat})$ does not hold. ■

Secondly we consider the following lemma which gives conditions, under which lacunary equistatistical convergence implies equistatistical convergence.

Lemma 3.1.8 *(see [2]) Let θ be a lacunary sequence. Lacunary equistatistical convergence implies equistatistical convergence if and only if*

$$\limsup_r q_r < \infty.$$

Proof. Assume $\limsup_r q_r < \infty$ and $f_r \twoheadrightarrow f \quad (\theta-\text{stat})$, then $\exists\ M > 0$ such that

$q_r < M$, $\forall r$. Given $\varepsilon > 0$, by the assumtion we have

$$\lim_{r\to\infty} \|s_{r,\varepsilon}(.)\| = 0.$$

That is $\exists$ an integer $r_0 > 0$ such that

$$s_{r,\varepsilon}(x) < \varepsilon,$$

for all $r > r_0$, uniformly in x. Let n be an arbitrary positive integer, then $\exists r > 0$ such that $n \in I_r$. We can write that

$$\begin{aligned} p_{n,\varepsilon}(x) &= \frac{1}{n} \left|\{m \le n : |f_m(x) - f(x)| \ge \varepsilon\}\right| \\ &\le \frac{1}{k_{r-1}} \left|\{m \le k_r : |f_m(x) - f(x)| \ge \varepsilon\}\right| \\ &= \frac{1}{k_{r-1}} \left\{ \sum_{i=1}^{r_0} h_i s_{i,\varepsilon}(x) + \sum_{i=r_0+1}^{r} h_i s_{i,\varepsilon}(x) \right\}. \end{aligned}$$

Since $s_{i,\varepsilon}(x) \le 1$, we conclude that

$$p_{n,\varepsilon}(x) \le \frac{T_{r_0}}{k_{r-1}} + \frac{1}{k_{r-1}} \sum_{i=r_0+1}^{r} h_i s_{i,\varepsilon}(x),$$

where $T := Max\{h_1, h_2, ..., h_{r_0}\}$. Hence

$$\begin{aligned} p_{n,\varepsilon}(x) &\le \frac{Tr_0}{k_{r-1}} + \frac{\varepsilon(k_r - k_{r_0})}{k_{r-1}} \\ &\le \frac{Tr_0}{k_{r-1}} + \varepsilon M, \end{aligned}$$

which proves the sufficiency. For the converse, consider the lacunary sequence

$$\theta = \{k_r\} = \left\{r^{r+1}\right\}.$$

Then

$$\lim_r q_r = \infty$$

and

$$\lim_r \frac{h_r}{k_{r-1}} = \infty.$$

Define function sequence in the following way:

$$f_i(x) = \begin{cases} 1, & \text{if} \quad k_{r-1} < i \leq 2k_{r-1}, \text{ for some } r = 1, 2, ... \\ 0, & \text{otherwise,} \end{cases} \qquad x \in X.$$

Then

$$s_{r,\varepsilon}(x) = \frac{1}{h_r} \left|\{m \in I_r : |f_m| \geq \varepsilon\}\right| \leq \frac{1}{h_r} k_{r-1}.$$

Hence we have $f_r \twoheadrightarrow f \quad (\theta-\text{stat})$. But $f_n \twoheadrightarrow f \quad (\text{stat})$ does not hold since

$$\lim_r \frac{1}{r} \left|\{m \leq r : |f_m(x) - f(x)| \geq \varepsilon\}\right|$$

does not exists. ■

As a consequence of Lemma 3.1.7 and Lemma 3.1.8 we can state the following theorem.

Theorem 3.1.9 *(see [2]) For any lacunary sequence θ, $f_r \twoheadrightarrow f \quad (\theta{-}stat)$ and $f_n \twoheadrightarrow f \quad (stat)$ implies each other $\Leftrightarrow$*

$$1 < \liminf_r \leq \limsup_r q_r < \infty.$$

3.2 Korovkin Type Theorem for Lacunary Equistatistical Convergence

In this section we prove a KTAT via lacunary equistatistical convergence.

Theorem 3.2.1 *(see [2]) Let $X \subset \mathbb{R}$ be compact subset, and let $\{L_r\}$ be a sequence of linear positive operators acting from $C(X)$ into $C(X)$. Also let θ be a lacunary sequence. If*

$$L_r(e_i, x) \twoheadrightarrow e_i(x) \ (\theta - stat) \ \text{on } X \ \text{where } e_i(x) = x^i, \ i = 0, 1, 2,$$

then $\forall f \in C(X)$ we have

$$L_r(f, x) \twoheadrightarrow f \ (\theta - stat).$$

Proof. Let f be a continuous function on X and let x be a fix point of X. For every $\varepsilon > 0$ $\exists \delta$ such that $|f(y) - f(x)| < \varepsilon$, $\forall y \in X$ with $|y - x| < \delta$. Now define

$K_\delta = \{y \in \mathbb{R} : |y - x| < \delta\}$ and let $X_\delta = X \cap K_\delta$. Then we have,

$$|f(y) - f(x)| \leq |f(y) - f(x)|\, \chi_{X_\delta}(y) + |f(y) - f(x)|\, \chi_{X \setminus X_\delta}(y) \leq \varepsilon + 2M\chi_{X\setminus X_\delta}(y),$$

where $M = \|f\|_{C(X)}$. After some easy calculations we have

$$\chi_{X\setminus X_\delta}(y) \leq \frac{1}{\delta^2}(y - x)^2.$$

Now for all $y \in X$ we may write that

$$|f(y) - f(x)| \leq \varepsilon + \frac{2M}{\delta^2}(y - x)^2.$$

Since $\{L_r\}$ is linear and positive, we have

$$\begin{aligned}
|L_r(f, x) - f(x)| &\leq L_r(|f(y) - f(x)e_0|\,; x) + |f(x)|\, |L_r(f_0; x) - e_0(x)| \\
&\leq \varepsilon L_r(e_0; x) + \frac{2M}{\delta^2}\left\{L_r\left((y-x)^2\,; x\right)\right\} + M\,|L_r(e_0; x) - e_0(x)| \\
&\leq \varepsilon + \left(\varepsilon + M + \frac{2M\,\|e_2\,(x)\|_{C(X)}}{\delta^2}\right)|L_r(e_0; x) - e_0(x)| \\
&+ \frac{4M\,\|e_1\,(x)\|_{C(X)}}{\delta^2}\,\{|L_r(e_1; x) - e_1(x)|\} \\
&+ \frac{2M}{\delta^2}\,|L_r(e_2; x) - e_2(x)| \\
&\leq \varepsilon + B\sum_{i=0}^{2}|L_r(e_i; x) - e_i(x)|\,, \qquad (3.2.1)
\end{aligned}$$

where $B = \varepsilon + M + \frac{4M}{\delta^2}\left(\|e_2(x)\|_{C(X)} + \|e_1\,(x)\|_{C(X)} + 1\right)$.

Now given $s > 0$, choose $0 < \varepsilon < s$ and define the following sets:

$$D_s(x) = \{m \in \mathbb{N} : |L_m(f;x) - f(x)| \geq s\}$$
$$D_s^i(x) = \left\{m \in \mathbb{N} : |L_m(e_i;x) - e_i(x)| \geq \frac{s-\varepsilon}{3B}\right\}, \ i = 0, 1, 2.$$

Using (3.2.1) we have

$$D_s(x) \subset \bigcup_{i=0}^{2} D_s^i(x). \tag{3.2.2}$$

Now define the following real valued functions

$$s_{r,s}(x) = \frac{1}{h_r} \left|\{m \in I_r : |L_m(f;x) - f(x)| \geq s\}\right|$$

and

$$s_{r,s}^i(x) = \frac{1}{h_r} \left|\left\{m \in I_r : |L_m(e_i;x) - e_i(x)| \geq \frac{s-\varepsilon}{3B}\right\}\right|, \ i = 0, 1, 2.$$

Since the operators are monotonic, together with (3.2.2), we have

$$s_{r,s}(x) \leq \sum_{i=0}^{2} s_{r,s}^i(x), \text{ for all } x \in X.$$

Hence we get

$$\|s_{r,s}(.)\|_{C(X)} \leq \sum_{i=0}^{2} \left\|s_{r,s}^i(.)\right\|_{C(X)}. \tag{3.2.3}$$

Taking limit in (3.2.3) as $r \to \infty$ and combining the hypothesis of the theorem we get

$$\lim_{r} \|s_{r,s}(.)\|_{C(X)} = 0,$$

which complites the proof. $\blacksquare$

Chapter 4

λ-EQUISTATISTICAL CONVERGENCE

4.1 λ-Equistatistical Convergence

Mursaleen [36] initiated the concept of $\lambda-$convergence for sequences. He also proved that under some conditions statistical convergence implies $\lambda-$statistical convergence. In this chapter firstly, conditions under which λ-statistical convergence implies statistical convergence are given. Secondly we introduce pointwise and uniform convergences in $\lambda-$statistical sense (see [3]). Also we introduce the concept of λ-equistatistical convergence and showed that it lies between pointwise and uniform convergences in the same sense. Moreover we constracted some examples to support the idea that in general λ-equistatistical convergence is different from $\lambda-$pointwise and $\lambda-$uniform convergences.

Example 4.1.1 *(see [3])* $\lambda_r = \ln(re)$ *is in* ω. *Indeed* $\lambda_1 = 1$, $\lambda_r > 0$, $\lambda_r \to \infty$ *as* $r \to \infty$ *and*

$$\lambda_{r+1} = \ln((r+1)e) = \ln(r+1) + 1 \leq \ln(re) + 1 = \lambda_r + 1.$$

we conclude that $\lambda_r = \ln(re) \in \omega$.

Lemma 4.1.2 *Given* $\lambda = (\lambda_r) \in \omega$ *then for each* r, $\lambda_r \leq r$.

Proof. We will use the matematical induction. For $r = 1$, it is obvious that

$$\lambda_1 \leq 1.$$

Let the inequality be true for $r = k$, that is

$$\lambda_k \leq k. \tag{4.1.1}$$

Then we need to show that

$$\lambda_{k+1} \leq k + 1.$$

Since $\lambda \in \omega$ we have

$$\lambda_{k+1} \leq \lambda_k + 1. \tag{4.1.2}$$

Combining (4.1.1) with (4.1.2) we can write that

$$\lambda_{k+1} \leq \lambda_k + 1 \leq k + 1.$$

Hence we get the result. ■

Definition 4.1.3 *(see [3]) Given* $\lambda = (\lambda_r) \in \omega$, $S \subset \mathbb{N}$ *and* $k \in \mathbb{N}$. *The ratio*

$$d_r^{\lambda}(S) = \frac{|M_r \cap S|}{\lambda_r}$$

is called the $r-th$ *partial,* $\lambda-$*density of* S *where* $M_r = [r - \lambda_r + 1, r]$.

Some simple properties of d_r^λ are given in the following lemma.

Lemma 4.1.4 *(see [3]) For each $\lambda = (\lambda_r) \in \omega$ and $r \in \mathbb{N}$, d_r^λ satisfies,*

i) $d_r^\lambda(\emptyset) = 0$.

ii) $d_r^\lambda(\mathbb{N}) = 1$.

iii) $A,\ B \subset \mathbb{N},\ A \cap B = \emptyset \Rightarrow d_r^\lambda(A \cup B) = d_r^\lambda(A) + d_r^\lambda(B)$.

In the following lemma we give the conditions under which $\lambda-$convergence implies $s-$convergence.

Lemma 4.1.5 *(see [3]) Assume that $\lambda = (\lambda_r) \in \omega$ and that the sequence $(r-\lambda_r)$ is bounded, then $\lambda-$convergence implies $s-$convergence.*

Proof. By Lemma (4.1.2) we have

$$\frac{1}{r} \leq \frac{1}{\lambda_r} \text{ for each } r.$$

Given $\varepsilon > 0$, there exists $K > 0$, such that, $r - \lambda_r \leq K$, for all r. Therefore

$$\begin{aligned}
\frac{1}{r}\left|\{k \leq r : |x_k - L|\} \geq \varepsilon\right| &\leq \frac{1}{\lambda_r}\left|\{k \leq r : |x_k - L|\} \geq \varepsilon\right| \\
&\leq \frac{1}{\lambda_r}\left|\{k \in M_r : |x_k - L|\} \geq \varepsilon\right| \\
&+ \frac{1}{\lambda_r}\left|\{k \leq r - \lambda_r + 1 : |x_k - L|\} \geq \varepsilon\right| \\
&\leq \frac{1}{\lambda_r}\left|\{k \in M_r : |x_k - L|\} \geq \varepsilon\right| + \frac{K+1}{\lambda_r}
\end{aligned}$$

for each r. Taking limit as $r \to \infty$, completes the proof. ■

Example 4.1.6 *(see [3]) Consider the sequence*

$$\lambda_r = \begin{cases} 1, & r = 1 \\ r - \frac{1}{2}, & r > 1, \end{cases}$$

then $\lambda-$convergence implies $s-$convergence.

Theorem 4.1.7 *(see [3]) Let $\lambda = (\lambda_r) \in \omega$ with*

$$\lim_{r\to\infty} \inf \frac{\lambda_r}{r} > 0 \text{ and } (r - \lambda_r) \text{ is bounded}$$

then $s-$convergence and $\lambda-$convergence implies each other.

Proof. Combining Lemma 4.1.5 with Theorem 2.5.6 completes the proof. ∎

Let $\lambda = (\lambda_r) \in \omega$ and let (f_r) be a sequence of real valued functions then we can give the following definitions;

Definition 4.1.8 *(see [3]) (f_r) is called $\lambda-$pointwise convergent to f on X and denoted by*

$$f_r \to f \quad (\lambda - stat)$$

if $\forall \varepsilon > 0$, and for each $x \in X$

$$\lim_{r\to\infty} d_r^{\lambda} \left(\{m : |f_m(x) - f(x)| \geq \varepsilon\}\right) = 0.$$

Definition 4.1.9 *(see [3])* (f_r) *is called* $\lambda-$*uniform convergent to* f *on* X *and denoted by*

$$f_r \rightrightarrows f \quad (\lambda - stat)$$

if $\forall \varepsilon > 0$,

$$\lim_{r\to\infty} d_r^{\lambda}\left(\left\{m : \|f_m(x) - f(x)\|_{C(X)} \geq \varepsilon\right\}\right) = 0.$$

Definition 4.1.10 *(see [3])* (f_r) *is called* λ*-equistatistically convergent to* f *on* X *and denoted by*

$$f_r \twoheadrightarrow f \quad (\lambda - stat)$$

if $\forall \varepsilon > 0$,

$$u_{r,\varepsilon}(x) = d_r^{\lambda}\left(\{m : |f_m(x) - f(x)| \geq \varepsilon\}\right)$$

converges uniformly to 0 *on* X, *i.e.*

$$\lim_{r\to\infty} \|u_{r,\varepsilon}(.)\|_{C(X)} = 0.$$

Remark 4.1.11 *In the case* $\lambda_r = r$, $\lambda-$*pointwise,* $\lambda-$*uniform and* $\lambda-$*equistatistical convergence reduce to* $s-$*pointwise,* $s-$*uniform and equistatistical convergence, respectively.*

As a consequence of above definitions we can state the following lemma which proves that, $\lambda-$equistatistical convergence lies between $\lambda-$pointwise and $\lambda-$uniform convergences.

Lemma 4.1.12 *(see [3]) Let $\lambda = (\lambda_r) \in \omega$. Then we have*

$$f_r \twoheadrightarrow f \ (\lambda - stat) \Longrightarrow f_r \to f \ (\lambda - stat),$$

$$f_r \rightrightarrows f \ (\lambda - stat) \Longrightarrow f_r \twoheadrightarrow f \ (\lambda - stat).$$

In general the inverse implications does not hold. The following two examples show that $\exists$ (f_r) such that, f_r is $\lambda-$equistatistical convergence but not $\lambda-$statistically uniform convergent.

Example 4.1.13 *(see [3]) Let (λ_r) be as in the Example 4.1.1 . Define $f(x) = 0$, $x \in [0,1]$ and $f_r : [0,1] \to \mathbb{R}$, $f_r = \chi\left(\frac{2}{2r-\lambda_r+1}\right)$. Let $\varepsilon > 0$, and $x \in [0,1]$. Then we have*

$$\begin{aligned} u_{r,\varepsilon}(x) &= d_r^{\lambda}\left(\{m : |f_m(x) - f(x)| \geq \varepsilon\}\right) \\ &= \frac{1}{\lambda_r}\left|\{m \in M_r : |f_m(x) - f(x)| \geq \varepsilon\}\right| \\ &\leq \frac{1}{\lambda_r} \leq \varepsilon, \end{aligned}$$

which means that $f_r \twoheadrightarrow f$ $(\lambda-stat)$ but since $\sup_{x\in[0,1]} |f_r(x) - f(x)| = 1$, $f_r \rightrightarrows f$ $(\lambda-stat)$ does not hold.

Example 4.1.14 *(see [3]) Let* $\lambda = (\lambda_r) \in \omega$ *and* $f_r(x)$ *be as in the Example 3.1.5. Then for every* $\varepsilon > 0$

$$\begin{aligned} u_{r,\varepsilon}(x) &:= d_r^{\lambda}\left(\{m : |f_m(x) - f(x)| \geq \varepsilon\}\right) \\ &\leq \frac{1}{\lambda_r} \left|\{m \in M_r : |f_m(x)| \geq \varepsilon\}\right| \\ &\leq \frac{1}{\lambda_r} \to 0 \text{ as } r \to \infty \end{aligned}$$

uniformly in x. *This implies that* $f_r \twoheadrightarrow 0$ *($\lambda-$stat). But* $f_r \rightrightarrows f$ *($\lambda-$stat) does not hold since*

$$\sup_{x \in [0,1]} |f_r(x)| = 1, \text{ for all } r.$$

Theorem 4.1.15 *(see [3]) Consider the real valued functions* f_r, f *on* X *and let* x_0 *be a fixed point in* X. *If* $f_r \twoheadrightarrow f$ *($\lambda-$stat) and* f_r *is continuous at* x_0, *then* f *is continuous at* x_0.

Proof. Let $\varepsilon > 0$ be given, $\forall x \in X$ define

$$D_{\varepsilon}(x) = \{m \in \mathbb{N} : |f_m(x) - f(x)| \geq \varepsilon\}.$$

Since $f_r \twoheadrightarrow f$ ($\lambda-$stat), $\exists k \in \mathbb{N}$, with

$$d_k^{\lambda}(D_{\frac{\varepsilon}{3}}(x)) = \frac{1}{\lambda_k} \left|\left\{m \in M_k : |f_m(x) - f(x)| \geq \frac{\varepsilon}{3}\right\}\right| < \frac{1}{2};\ \forall x \in X.$$

Now define

$$R_{\varepsilon}(x) = \{m : |f_m(x) - f(x)| < \varepsilon\}, \ x \in X.$$

As a consequence of Lemma 4.1.4 *(ii)* and *(iii)*,

$$d_k^{\lambda}(R_{\varepsilon}(x)) > \frac{1}{2}.$$

Since f_i is continuous at x_0 $\exists \delta > 0$ with

$$|f_i(x) - f_i(x_0)| < \frac{\varepsilon}{3}, \forall i \in M_k \text{ and } x \in B(x_0, \delta).$$

Fix $x \in B(x_0, \delta)$. Combining $d_k^{\lambda}(R_{\frac{\varepsilon}{3}}(x)) > \frac{1}{2}$ and $d_k^{\lambda}(R_{\frac{\varepsilon}{3}}(x_0)) > \frac{1}{2}$ with Lemma 4.1.4, $R_{\frac{\varepsilon}{3}}(x) \cap R_{\frac{\varepsilon}{3}}(x_0) \neq \emptyset$.

Now let an arbitrary element p in $R_{\frac{\varepsilon}{3}}(x) \cap R_{\frac{\varepsilon}{3}}(x_0)$, then we have

$$|f(x) - f(x_0)| \leq \varepsilon,$$

which completes the proof. ■

Example 4.1.16 *(see [3]) Let* (λ_r) *be as in the Example 4.1.1 and* $f_r(x) = x^r$, $x \in [0,1]$, $r \in \mathbb{N}$. *Then obviously* f_r *is* $\lambda-$*statistical pointwise convergent to the* *function*

$$f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1, \end{cases}$$

Since f is not continuous at 1, $f_r \twoheadrightarrow f \quad (\lambda-stat)$ *does not hold from the previous theorem.*

Implication relationships between λ equi-statistical and equistatistical convergence will be given in next theorem and remark.

Theorem 4.1.17 *(see [3])* $f_r \twoheadrightarrow f \quad (stat)$ *implies* $f_r \twoheadrightarrow f \quad (\lambda-stat) \Longleftrightarrow$

$$\liminf_{r\to\infty} \frac{\lambda_r}{r} > 0. \tag{4.1.3}$$

Proof. Given $\varepsilon > 0$. Since $M_r \subset [1, r]$, we can write that

$$\{m \le r : |f_m(x) - f(x)| \ge \varepsilon\} \supset \{m \in M_r : |f_m(x) - f(x)| \ge \varepsilon\}.$$

Using the above inclusion we get

$$\begin{aligned} \frac{1}{r}\left|\{m \le r : |f_m(x) - f(x)| \ge \varepsilon\}\right| &\ge \frac{1}{r}\left|\{m \in M_r : |f_m(x) - f(x)| \ge \varepsilon\}\right| \\ &\ge \frac{\lambda_r}{r}\frac{1}{\lambda_r}\left|\{m \in M_r : |f_m(x) - f(x)| \ge \varepsilon\}\right| \\ &\ge \frac{\lambda_r}{r} d_r^{\lambda}\left(\{m : |f_m(x) - f(x)| \ge \varepsilon\}\right). \end{aligned}$$

Considering 4.1.3 and taking the limit when n tends to infinity the implication follows.

To show the converse, assume that $\liminf_{r\to\infty} \frac{\lambda_r}{r} = 0$. Then $\exists$a subsequence $n(r)$ with

$$\frac{\lambda_{n(r)}}{n(r)} < \frac{1}{r}.$$

Using the choosen subsequence we can define the following function sequence

$$f_i(x) := \begin{cases} 1, & \text{if } i \in M_{n(r)}, \text{ for some } r = 1, 2, 3, ... \\ 0, & \text{otherwise} \end{cases} \quad x \in X$$

then we have $f_r \twoheadrightarrow f \ \ (\text{stat})$. But since

$$\frac{1}{\lambda_k} \left|\{m \in M_k : |f_m(x)| \geq \varepsilon\}\right| = \begin{cases} 1, & \text{if } k \in M_{n(r)}, \text{ for some } r \\ 0, & \text{otherwise,} \end{cases}$$

$f_r \twoheadrightarrow f \ \ (\lambda\text{−stat})$ does not hold. ■

Remark 4.1.18 *(see [3]) If* $(r - \lambda_r)$ *is bounded then*

$$f_r \twoheadrightarrow f \ (\lambda - stat) \Rightarrow f_r \twoheadrightarrow f \ (stat).$$

4.2 Korovkin Type Theorem for λ-Equistatistical Convergence

Now our aim is to prove a KTAT for λ-equistatistical convergence.

Theorem 4.2.1 *(see [3]) Let $X \subset R$ be compact and $C(X)$ be the space of all continuous real valued functions from X to X. Also let $\lambda \in \omega$. Suppose that $\{L_r\}$ is a sequence of positive linear operators defined on $C(X)$. If*

$$L_r(e_i; x) \twoheadrightarrow e_i(x) \ (\lambda\text{-}stat), \ \ i = 0, 1, 2$$

where $e_i(x) = x^i$. Then $\forall\ f \in C(X)$,

$$L_r(f; x) \twoheadrightarrow f \ \ (\lambda - stat).$$

Proof. Let f be a continuous function on X and let $x \in X$ be fixed, $\forall \varepsilon > 0$ $\exists\ \delta > 0$ such that $|f(y) - f(x)| < \varepsilon$, $\forall y \in X$ with $|y - x| < \delta$. Now define $K_\delta = \{y \in \mathbb{R} : |y - x| < \delta\}$ and let $X_\delta = X \cap K_\delta$. Then,

$$|f(y) - f(x)| \leq |f(y) - f(x)|\, \chi_{X_\delta}(y) + |f(y) - f(x)|\, \chi_{X \setminus X_\delta}(y) \leq \varepsilon + 2M \chi_{X \setminus X_\delta}(y),$$

where $M = \|f\|_{C(X)}$. After some easy calculations we have

$$\chi_{X \setminus X_\delta}(y) \leq \frac{1}{\delta^2}(y - x)^2.$$

Now for all $y \in X$ we may write that

$$|f(y) - f(x)| \leq \varepsilon + \frac{2M}{\delta^2}(y - x)^2.$$

Since $\{L_r\}$ is linear and positive, we have

$$\begin{aligned} |L_r(f,x) - f(x)| &\leq L_r(|f(y) - f(x)e_0|\,; x) + |f(x)|\,|L_r(f_0;x) - e_0(x)| \\ &\leq \varepsilon + B\sum_{i=0}^{2} |L_r(e_i;x) - e_i(x)|\,, \end{aligned} \tag{4.2.1}$$

where $B = \varepsilon + M + \frac{4M}{\delta^2}\left(\|e_2\| + \|e_1(x)\| + 1\right).$

$\forall s > 0$, choose $0 < \varepsilon < s$ and define

$$D_s(x) = \{m \in \mathbb{N} : |L_m(f;x) - f(x)| \geq s\}$$

$$D_s^i(x) = \left\{m \in \mathbb{N} : |L_m(e_i;x) - e_i(x)| \geq \frac{s-\varepsilon}{3B}\right\},\ \ i = 0, 1, 2.$$

Using (4.2.1) we can have

$$D_s(x) \subset \bigcup_{i=0}^{2} D_s^i(x). \tag{4.2.2}$$

Now define the following real valued functions

$$u_{r,s}(x) = \frac{1}{\lambda_r}\left|\{m \in M_r : |L_m(f;x) - f(x)| \geq s\}\right|$$

and

$$u_{r,s}^i(x) = \frac{1}{\lambda_r}\left|\left\{m \in M_r : |L_m(e_i;x) - e_i(x)| \geq \frac{s-\varepsilon}{3B}\right\}\right|,\ \ i = 0, 1, 2.$$

Since the operators are monotonic, together with (4.2.2) we have

$$u_{r,s}(x) \leq \sum_{i=0}^{2} u_{r,s}^{i}(x), \ \forall x \in X.$$

Hence we get

$$\|u_{r,s}(.)\|_{C(X)} \leq \sum_{i=0}^{2} \left\|u_{r,s}^{i}(.)\right\|_{C(X)}. \tag{4.2.3}$$

Taking limit in 4.2.3 as $r \to \infty$ and combining the hypothesis of the theorem we get

$$\lim_{r} \|u_{r,s}(.)\|_{C(X)} = 0.$$

■

Chapter 5

A-EQUISTATISTICAL CONVERGENCE

5.1 $\mathcal{A}$-Equistatistical Convergence

In this chapter our aim is to extend the idea of equistatistical convergence to A-equistatistical convergence by using an arbitrary NNRM A (see [1]). We will also discuss the relations between A-statistical pointwise, A-statistical uniform and A-equistatistical convergence.

Definition 5.1.1 *(see [1]) Let $K \subset \mathbb{N}$ and A be a NNRM, then*

$$\delta_A^m(K) = \sum_{k=1}^{\infty} a_{mk} \chi_K(k)$$

is called the mth partial A-density of K. When m tends to infinity and the limit exists this definition coincises with the Definition 2.6.1.

Definition 5.1.2 *(see [1]) Let $A = (a_{mk})$ be a NNRM. Then (f_n) is called A-statistically pointwise convergent to f on X and denoted by*

$$f_n \to f \quad (A - stat)$$

if $\forall\varepsilon > 0$ *and* $\forall x \in X$,

$$\delta_A\left(\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon\}\right) = 0.$$

Definition 5.1.3 *(see [1]) Let* $A = (a_{mk})$ *be a NNRM.* (f_n) *is called A-statistically uniform convergent to* f *on* X *and denoted by*

$$f_n \rightrightarrows f \quad (A - stat)$$

if $\forall\ \varepsilon > 0$,

$$\delta_A\left(\left\{n \in \mathbb{N} : \|f_n(x) - f(x)\|_{C(X)} \geq \varepsilon\right\}\right) = 0.$$

Definition 5.1.4 *(see [1]) Let* $A = (a_{mk})$ *be a NNRM. Then* (f_n) *is called A-equistatistically convergent to* f *on* X *if* $\forall\varepsilon > 0$,

$$h_{m,\varepsilon}(x) = \delta_A^m(\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon\}), \quad x \in X$$

converges uniformly to the function zero on X, *i.e,*

$$\lim_{m\to\infty} \|h_{m,\varepsilon}(.)\|_{C(X)} = 0.$$

The A-equistatistical convergence of f_n *to* f *will be denoted by* $f_n \twoheadrightarrow f \quad (A{-}stat)$.

Remark 5.1.5 *Taking* $A = A_\lambda$ *as in Remark 2.5.5,* $A-$*equistatistical convergence includes* $\lambda-$*equistatistical convergence . Also if we take* $A = A_\theta$ *which is de-*

fined in Definition 2.4.2 lacunary equistatistical convergence is obtained as a special case of $A-$equistatistical convergence. Moreover taking $A = C_1$, $A-$equistatistical convergence reduces to equistatistical convergence.

Lemma 5.1.6 *(see [1])Let $X \subset \mathbb{R}$ and $f_n,\ f : X \to \mathbb{R}$, for all $n \in \mathbb{N}$, then we have*

$$i)\ f_n \twoheadrightarrow f\ (A-stat) \Rightarrow f_n \to f\ (A-stat)$$

$$ii)\ f_n \rightrightarrows f\quad (A-stat) \Rightarrow f_n \twoheadrightarrow f\quad (A-stat).$$

In general, reverse implications are not true. For instance see examples below.

Example 5.1.7 *(see [1]) Let $A = (a_{mk})$ be the NNRM with the following conditions;*

$$a_{mk} \leq b_m, \quad k = 1, 2, ...\ and\ \lim_{m\to\infty} b_m = 0.$$

Also let (f_n) be as in Example 3.1.5 .Then we have $f_n \twoheadrightarrow 0\ \ (A-stat)$ but $f_n \rightrightarrows 0$ $(A-stat)$ fails to hold. To see that $f_n \rightrightarrows 0\ \ (A-stat)$ does not hold choose $\varepsilon = 1$. Then we have

$$\|f_n\|_{C[0,1]} = \sup_{x\in[0,1]} |f_n(x)| = 1,\ \forall n.$$

Hence

$$\delta_A \left\{ n \in \mathbb{N} : \|f_n\|_{C[0,1]} \geq 1 \right\} = \delta_A \left\{ \mathbb{N} \right\} = 1 \neq 0.$$

Now we need to verify that $f_n \twoheadrightarrow 0 \quad (A-stat)$*.* $\forall \varepsilon > 0$ *and* $\forall\ x \in [0,1]$ *it is easly seen that*

$$|\{n \in \mathbb{N} : |f_n(x)| \geq \varepsilon\}| \leq 1.$$

Thus for every $\varepsilon > 0$ *and* $x \in [0,1]$

$$h_{m,\varepsilon}(x) = \delta_A^m(\{n \in \mathbb{N} : |f_n(x)| \geq \varepsilon\}) \leq b_m.$$

Hence

$$\lim_{m\to\infty} \|h_{m,\varepsilon}(x)\| \leq \lim_{m\to\infty} b_m = 0.$$

Example 5.1.8 *(see [1]) Let* $A = (a_{nk})$ *be the NNRM*

$$A = \begin{cases} \frac{1}{2n} & n \leq k \leq 3n-1 \\ 0 & \mathit{otherwise.} \end{cases}$$

Also let $f_n : [0,1] \to \mathbb{R}$*, defined by*

$$f_n(x) = \chi_{\{\frac{1}{2n}\}}.$$

Then for each $\varepsilon > 0$ and for every $x \in [0,1]$,

$$h_{m,\varepsilon}(x) = \delta_A^m(\{n \in \mathbb{N} : |f_n(x)| \geq \varepsilon\}) \leq \frac{1}{2m}.$$

Thus $f_n \twoheadrightarrow 0$ $(A-stat)$. But it is obvious that $f_n \rightrightarrows 0$ $(A-stat)$ does not hold.

Example 5.1.9 *(see [1]) Consider C_1 and $f_n : [0,1] \to \mathbb{R}$, where $f_n(x) = x^n$. Taking $\varepsilon = \frac{1}{4}$, then $\forall n \in \mathbb{N}$, $\exists m \geq n$ such that for any $x \in \left(\sqrt[m]{\frac{1}{4}}, 1\right)$,*

$$\{1, 2, ..., m\} \subset \left\{n \in \mathbb{N} : |f_n(x)| \geq \frac{1}{4}\right\}$$

it follows that

$$1 = \delta_{C_1}^m(\{1, 2, ..., m\}) \leq \delta_{C_1}^m\left(\left\{n \in \mathbb{N} : |f_n(x)| \geq \frac{1}{4}\right\}\right)$$

and hence f_n is not equistatistically convergent to the zero function.

5.2 Korovkin Type Theorem for $\mathcal{A}$-Equistatistical Convergence

KTAT is proved for A-equistatistical convergence in the following theorem.

Theorem 5.2.1 *(see [1])Let $X \subset R$, be compact, and let $\{L_n\}$ be a sequence of linear positive operators from $C(X)$ into $C(X)$. If*

$$L_r(e_i; x) \twoheadrightarrow e_i(x) \ (A - stat) \text{ on } X \text{ where } e_i(x) = x^i, \ i = 0, 1, 2,$$

then for all $f \in C(X)$

$$L_r(f;x) \twoheadrightarrow f \ (A - stat) \ on \ X.$$

Proof. Let $f \in C(X)$ and $x \in X$ be fixed, $\forall \varepsilon > 0$, $\exists \delta > 0$ with $|f(y) - f(x)| < \varepsilon$, $\forall y \in X$ satisfying $|y - x| < \delta$ since f is continuous at x. For $X_\delta = [x - \delta, x + \delta] \cap X$ we can write that

$$|f(y) - f(x)| \leq \varepsilon + 2M \frac{(y-x)^2}{\delta^2}.$$

$\forall y \in Y$, where $M := \|f\|_{C(X)}$. By the positivity of L_r,

$$\begin{aligned} |L_r(f,x) - f(x)| &\leq L_r(|f(y) - f(x)e_0|\,; x) + |f(x)|\,|L_r(f_0;x) - e_0(x)| \\ &\leq \varepsilon + B \sum_{i=0}^{2} |L_r(e_i;x) - e_i(x)|\,, \end{aligned} \tag{5.2.1}$$

where $B = \varepsilon + M + \frac{4M}{\delta^2}\left(\|e_2\| + \|e_1(x)\| + 1\right)$.

$\forall\ s > 0$, take $\varepsilon > 0$ with $\varepsilon < s$ and define

$$\begin{aligned} \Phi_s(x) &:= \{m \in \mathbb{N} : |L_m(f;x) - f(x)| \geq s\} \\ \Phi_s^i(x) &:= \left\{ m \in \mathbb{N} : |L_m(e_i;x) - e_i(x)| \geq \frac{s - \varepsilon}{3B} \right\} \quad (i = 0, 1, 2) \end{aligned}$$

Using (5.2.1) we have

$$\Phi_s(x) \subset \bigcup_{i=0}^{2} \Phi_s^i(x). \tag{5.2.2}$$

Also define the following real valued functions:

$$h_{r,s}(x) = \delta_A^r \left(\{m \in \mathbb{N} : |L_m(f,x) - f(x)| \geq s\}\right)$$

and

$$h_{r,s}^i(x) = \delta_A^r \left(\left\{m \in \mathbb{N} : |L_m(e_i,x) - e_i(x)| \geq \frac{s-\varepsilon}{3B}\right\}\right)$$

$i = 0, 1, 2$. Then by the monotonicity and (5.2.2) we have

$$h_{r,s}(x) \leq \sum_{i=0}^{2} h_{r,s}^i(x), \forall x \in X.$$

and

$$\|h_{r,s}(.)\|_{C(X)} \leq \sum_{i=0}^{2} \left\|h_{r,s}^i(.)\right\|_{C(X)}. \tag{5.2.3}$$

Taking limit in (5.2.3) and using the hypothesis of the theorem we conclude that

$$\lim_r \|h_{r,s}(.)\|_{C(X)} = 0$$

whence the result. ■

Remark 5.2.2 *If we take $A = C_1$ in the previous Therorem then we reduce to the result of Karakus, Demirci and Duman (see [30]) which is given in Theorem 2.8.7. Also if we take $A = A_\theta$ then it reduces to the result of Aktuğlu and Gezer (see [2]) which is given in Theorem 3.2.1. Moreover letting $A = A_\lambda$ then we obtain* *the results of Özarslan, Aktuğlu and Gezer (see [3]) which is given in Theorem 4.2.1.*

Chapter 6

$\mathcal{B}$-EQUISTATISTICAL CONVERGENCE

6.1 $\mathcal{B}$-Equistatistical Convergence

Up to here, we discuss some type of convergences and among them $A-$convergence has an important role because it is the most general method and includes all other methods. In fact all other methods considered in this thesis can be obtained from $A-$convergence for different choice of A. In this view of point, it seems $A-$convergence is large enough and can not be extended. But using [33] and [37], it is shown that by using sequences of NNRM we can take one more step to extend this type of convergences. By using this idea $A-$convergence is extended to $\mathcal{B}-$ statistical convergence (or $\mathcal{B}-$convergence) by Mursaleen and Edely in [37]. Let $\mathcal{B} = (B_j)$ be a sequence of infinite matrices $B_j = (b_{mj}(j))$. A bounded sequence x is said to be $\mathcal{B}$-summable to L if

$$\lim_{m\to\infty}(B_j x)_m = \lim_{m\to\infty}\sum_s b_{ms}(j)x_s = L, \text{ uniformly in } j.$$

The method $\mathcal{B}$ is called regular method (RM) if it preserves the limit of each convergent sequence. Necessary and sufficient conditions for regular methods is given as in the following theorem.

Theorem 6.1.1 *(see [42] and [8]) The method $\mathcal{B} = (B_j)$ is regular* $\Leftrightarrow$

i) $\sup_{m,j} \sum_s |b_{ms}(j)| < \infty$

ii) $\lim_m \sum_s b_{ms}(j) = 1,$ *uniformly in* $j,$

iii) $\lim_m b_{ms}(j) = 0,\ \forall s \geq 1,$ *uniformly in* $j.$

If $b_{ms}(j) \geq 0, \forall m,\ s$ and j then the method $\mathcal{B}$ is called non-negative (NN).

A subset $S = \{s_1 \leq s_2 \leq \cdots\} \subset \mathbb{N}$, is said to have $\mathcal{B}$-density L if

$$\delta_{\mathcal{B}}(S) = \lim_m \sum_{s \in S} b_{ms}(j) = L,\ \text{uniformly in } j.$$

Definition 6.1.2 *Let $\mathcal{B}$ be a NN and RM, then x is called $\mathcal{B}$-statistical convergent to L if* $\forall \varepsilon > 0,$

$$\delta_{\mathcal{B}}(\{s : |x_s - L| \geq \varepsilon\}) = 0.$$

Obviously, $s-$convergence, $A-$convergence and $\mathcal{B}-$convergence are all different from each other (see for example [1] and [14]). But taking $B_j = A$ for each j where A is a NNRM then $\mathcal{B}-$convergence reduces to $A-$convergence. Similarly taking $B_j = C_1$ for each j, then $\mathcal{B}$-convergence coincides with $s-$convergence.

Recently, the concept of $\mathcal{B}$-convergence and their applications are studied in (see [14] and [37]).

Let $X \subset \mathbb{R}, \quad f,\ f_n : X \to \mathbb{R}$ and $\mathcal{B} = (B_j)$ be NN and RM then;

Definition 6.1.3 (f_n) *is said to be $\mathcal{B}$-statistically pointwise convergent to f on*

X and denoted by $f_n \to f$ ($\mathcal{B}-$stat) if $\forall\varepsilon > 0$ *and* $\forall x \in X$,

$$\delta_{\mathcal{B}}(\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

Definition 6.1.4 *(f_n) is said to be $\mathcal{B}$-statistically uniform convergent to f on X and denoted by $f_n \rightrightarrows f$ ($\mathcal{B}-$stat) if* $\forall\varepsilon > 0$,

$$\delta_{\mathcal{B}}(\left\{n \in \mathbb{N} : \|f_n(x) - f(x)\|_{C(X)} \geq \varepsilon\right\}) = 0.$$

Definition 6.1.5 *(f_n) is said to be $\mathcal{B}$-equistatistically convergent to f on X and denoted by $f_n \twoheadrightarrow f$ ($\mathcal{B}-$stat) if $\forall\varepsilon > 0$, the sequence of real valued functions $\left(\Psi^j_{m,\varepsilon}\right)_{m\in\mathbb{N}}$ where*

$$\Psi^j_{m,\varepsilon}(x) = \delta^m_{B_j}\left(\{n \in \mathbb{N} : |f_n(x) - f(x)| \geq \varepsilon\}\right),\ x \in X$$

uniformly converges to the zero function on X for each j, i.e.

$$\lim_{m\to\infty} \left\|\Psi^j_{m,\varepsilon}(.)\right\|_{C(X)} = 0 \text{ for all } j.$$

Remark 6.1.6 *Note that taking $\mathcal{B}_j = A$ for each j, where A is a NNRM, then the above definitions reduce to $A-$statistical pointwise, $A-$statistical uniform and A-equistatistical convergence (see [1]) respectively. If we take $\mathcal{B}_j = C_1$ then the above definitions reduce to statistical pointwise, statistical uniform and equ-istatistical convergence (see [7]) respectively. Also if we take $\mathcal{B}_j = A_\theta$ for each j, then the above definitions reduce to lacunary statistical pointwise, lacunary statistical uni-*

form and lacunary equ-statistical convergence (see [2]) respectively. Moreover if we take $\mathcal{B}_j = A_\lambda$ *for each* j*, then the above definitions reduce to* $\lambda-$*statistical pointwise,* $\lambda-$*statistical uniform and* $\lambda-$*equistatistical convergence (see [3]) respectively.*

Lemma 6.1.7 *i)* $f_n \rightrightarrows f$ *(*$\mathcal{B}-stat$*)* $\Rightarrow f_n \twoheadrightarrow f$ *(*$\mathcal{B}-stat$*),*

ii) $f_n \twoheadrightarrow f$ *(*$\mathcal{B}-stat$*)* $\Rightarrow f_n \to f$ *(*$\mathcal{B}-stat$*).*

The following examples shows that, in general the inverse implications does not hold, for instance see the following examples.

Example 6.1.8 *Consider the NN and RM such that*

$$\mathcal{B} = (B_j) = b_{ms}(j) = \begin{cases} \frac{1}{jm}, & mj \leq s < 2mj \\ 0, & otherwise \end{cases}$$

and let $f_n : [0,1] \to \mathbb{R}$*, defined as*

$$f_n(x) = \begin{cases} -4^{n+1}(x - \frac{1}{2^n})(x - \frac{1}{2^{n-1}}), & if\ x \in \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \\ 0, & otherwise. \end{cases}$$

Then $f_n \twoheadrightarrow 0$ *(*$\mathcal{B}-stat$*) but* $f_n \rightrightarrows 0$ *(*$\mathcal{B}-stat$*) does not hold. In fact,* $\forall\ \varepsilon > 0$ *and* $\forall x \in [0,1]$*, the set* $|\{n \in \mathbb{N} : \ |f_n(x)| \geq \varepsilon\}| \leq 1$*. Thus* $\forall\ \varepsilon > 0$ *and* $x \in [0,1]$

$$\Psi^j_{m,\varepsilon}(x) = \delta^m_{B_j}\left(\{n \in \mathbb{N} : |f_n(x)| \geq \varepsilon\}\right) \leq \frac{1}{mj}, \forall\ j.$$

When m tends to infinity on both sides we conclude that

$$\lim_{m\to\infty} \left\| \Psi^{j}_{m,\varepsilon}(x) \right\|_{C[0,1]} = 0, \ \forall j.$$

Hence $f_n \twoheadrightarrow 0$ ($\mathcal{B}-$stat). But $\|f_n\|_{C[0,1]} = \sup_{x\in[0,1]} |f_n(x)| = 1 \ \forall n,$ and choose $\varepsilon = 1,$

$$\delta_{\mathcal{B}} \left\{ n \in \mathbb{N} : \|f_n\|_{C[0,1]} \geq 1 \right\} = \delta_{\mathcal{B}} \left\{ \mathbb{N} \right\} = 1.$$

Therefore $f_n \rightrightarrows 0$ ($\mathcal{B}-$stat) does not hold.

Example 6.1.9 *Consider $f_n : [0,1] \to \mathbb{R}, \ \forall n,$ defined by*

$$f_n(x) = x^n$$

and the function by

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}.$$

Let $\mathcal{B} = (B_j)$ where

$$b_{ms}(j) = \begin{cases} \frac{1}{m}, & 1 \leq s \leq m \\ 0, & \text{otherwise} \end{cases} \quad j = 1, 2, \cdots .$$

Then $f_n \to f$ ($\mathcal{B}-$stat). But $f_n \twoheadrightarrow f$ ($\mathcal{B}-$stat) does not hold. To see that take

$\varepsilon = \frac{1}{2}$, then $\forall n \in \mathbb{N}$, $\exists m \geq n$ such that $x \in \left(\sqrt[m]{\frac{1}{2}}, 1 \right)$, implies that

$$\{1, 2, ..., m\} \subset \left\{ n \in \mathbb{N} : |f_n(x)| \geq \frac{1}{2} \right\}$$

which gives that for each j

$$1 = \delta^m_{B_j} \left(\{1, 2, ..., m\} \right) \leq \delta^m_{B_j} \left(\left\{ n \in \mathbb{N} : |f_n(x)| \geq \frac{1}{2} \right\} \right).$$

This proves that $f_n \twoheadrightarrow f$ ($\mathcal{B}-$stat) does not hold.

6.2 Korovkin Type Theorem for $\mathcal{B}$-Equistatistical Convergence

Dirik and Demirci (see [14]) introduce the concept of KTAT in the sense of $\mathcal{B}-$convergence. They also show that KTAT given in $\mathcal{B}-$statistical sense and statistical sense are different from each other. Our aim is to give KTAT in the sense of $\mathcal{B}-$equistatistical convergence.

Theorem 6.2.1 *Let $\mathcal{B} = (B_j)$ be a NN and RM, and let X be a compact subset of $\mathbb{R}$. Suppose that $\{L_r\}$ is a sequence of positive linear operators define on $C(X)$. If*

$$L_r(e_i; x) \twoheadrightarrow e_i(x) \ (\mathcal{B} - stat) \text{ on } X \text{ where } e_i(x) = x^i, \ i = 0, 1, 2,$$

then $\forall f \in C(X)$

$$L_r(f; x) \twoheadrightarrow f \ (\mathcal{B} - stat) \text{ on } X.$$

Proof. Let $f \in C(X)$ and $x \in X$ be fixed, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(y) - f(x)| < \varepsilon$, $\forall y \in X$ satisfying $|y - x| < \delta$. For $X_\delta = [x - \delta, x + \delta] \cap X$ we can write that

$$|f(y) - f(x)| \leq \varepsilon + 2M\frac{(y-x)^2}{\delta^2}.$$

$\forall y \in Y$, where $M := \|f\|_{C(X)}$. Since L_r is positive and linear

$$\begin{aligned} |L_r(f,x) - f(x)| &\leq L_r(|f(y) - f(x)e_0|\,; x) + |f(x)|\,|L_r(f_0; x) - e_0(x)| \\ &\leq \varepsilon + B\sum_{i=0}^{2} |L_r(e_i; x) - e_i(x)| \end{aligned} \tag{6.2.1}$$

where $B = \varepsilon + M + \frac{4M}{\delta^2}\left(\|x^2\| + \|x\| + 1\right).$

On the other hand, $\forall s > 0$, take $\varepsilon > 0$ with $\varepsilon < s$ and define,

$$\begin{aligned} \Phi_s(x) &:= \{m \in \mathbb{N} : |L_m(f; x) - f(x)| \geq s\} \\ \Phi_s^i(x) &:= \left\{m \in \mathbb{N} : |L_m(e_i; x) - e_i(x)| \geq \frac{s-\varepsilon}{3B}\right\} \quad (i = 0, 1, 2) \end{aligned}$$

Using (6.2.1) we have

$$\Phi_s(x) \subset \bigcup_{i=0}^{2} \Phi_s^i(x). \tag{6.2.2}$$

Also for each j, define the following real valued functions

$$\Psi_{r,s}^j(x) = \delta_{B_j}^r\left(\{m \in \mathbb{N} : |L_m(f,x) - f(x)| \geq s\}\right)$$

and

$$\Psi_{r,s,i}^{j}(x) = \delta_{B_j}^{r}\left(\left\{m \in \mathbb{N} : |L_m(e_i, x) - e_i(x)| \geq \frac{s-\varepsilon}{3B}\right\}\right)$$

$i = 0, 1, 2$. Then by the monotonicity of the operatos $\delta_{B_j}^{m}$ and (6.2.2) we have

$$\Psi_{r,s}^{j}(x) \leq \sum_{i=0}^{2} \Psi_{r,s,i}^{j}(x),\ j = 1, 2, \cdots,\ \forall x \in X,$$

which implies the inequality

$$\left\|\Psi_{r,s}^{j}(.)\right\|_{C(X)} \leq \sum_{i=0}^{2} \left\|\Psi_{r,s,i}^{j}(.)\right\|_{C(X)},\ j = 1, 2, \cdots\ . \tag{6.2.3}$$

Taking limit in (6.2.3) as $r \to \infty$ and using the hypothesis of the theorem we conclude that

$$\lim_{r} \left\|\Psi_{r,s}^{j}(.)\right\|_{C(X)} = 0,\ j = 1, 2, \cdots\ .$$

whence the result. ■

Remark 6.2.2 *If $\mathcal{B} = (B_j) = A$ for each j then we reduce to the Theorem 5.2.1. If $\mathcal{B} = (B_j) = C_1$ for each j then we set Theorem 2.8.7. Also if $\mathcal{B} = (B_j) = A_\theta$ for each j then we obtain Theorem 3.2.1. Finally if $\mathcal{B} = (B_j) = A_\lambda$ for each j then we get Theorem 4.2.1.*

Chapter 7

$\alpha\beta$-EQUISTATISTICAL CONVERGENCE

In this Chapter we consider $\alpha\beta$-statistical and $\alpha\beta$-equistatistical convergences which were initiated by Aktuğlu in [5]. A careful observation shows that most of the convergence methods considered so far have some common points. First of all each methods is based on a set function called density. Secondly, for each method there is a sequence of intervals that are effecting convergence of the sequences, for example $s-$convergence uses intervals $[1, n]$. Moreover, end points of this intervals can be considered as a sequence of positive integers such as $\alpha(n)$ and $\beta(n)$ with $\beta(n) - \alpha(n) \to \infty$ as $n \to \infty$. An other observation was that, each density function has deneminator which is the lenght of the interval used for this method. By considering these common points we decided to define a convergence method which depends on intervals $[\alpha(n), \beta(n)]$. In this chapter, by combining all these observations and we introduce a new type convergence which is called $\alpha\beta$-statistical convergence (or $\alpha\beta$-convergence). Using parallel idea for the existing methods also we introduce $\alpha\beta$-equistatistical convergence for sequence of functions. We show that $\alpha\beta$-convergence is a non-trivial extension of ordinary and statistical convergences.

We also show that, for special choices of $\alpha(n)$ and $\beta(n)$, $\alpha\beta$-convergence reduces to some well known methods such as $s-$convergence etc. Moreover we prove two different KTAT's via $\alpha\beta$-convergence and $\alpha\beta$-equistatistical convergence. Finally, we compare our results with other KTAT which are already given by different authors.

Let $\alpha(n)$ and $\beta(n)$ be two sequences of positive numbers satisfying following conditions;

$$P_1 : \alpha \text{ and } \beta \text{ are both non-decreasing.}$$

$$P_2 : \beta(n) \geq \alpha(n).$$

$$P_3 : \beta(n) - \alpha(n) \to \infty \text{ as } n \to \infty$$

and let Λ denotes the set of pairs (α, β) satisfying P_1, P_2 and P_3.

7.1 $\alpha\beta$-Statistical Convergence

Definition 7.1.1 *(see [5])For $K \subset \mathbb{N}$, $0 < \gamma \leq 1$ and each pair $(\alpha, \beta) \in \Lambda$, we define $\delta^{\alpha,\beta}(K, \gamma)$ in the following way,*

$$\delta^{\alpha,\beta}(K, \gamma) = \lim_{n\to\infty} \frac{\left|K \cap P_n^{\alpha,\beta}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}}$$

where $|S|$ represents the cardinality of S and $P_n^{\alpha,\beta}$ is the closed interval $[\alpha(n), \beta(n)]$.

After the above definition we can state the following lemma.

Lemma 7.1.2 *(see [5]) Let K and M be two subsets of $\mathbb{N}$ and $0 < \gamma \leq \delta \leq 1$*

then for all (α, β) *we have*

i) $\delta^{\alpha,\beta}(\emptyset, \gamma) = 0.$

ii) $\delta^{\alpha,\beta}(\mathbb{N}, 1) = 1.$

iii) If K *is finite then* $\delta^{\alpha,\beta}(K, \gamma) = 0.$

iv) If $K \subset M \Rightarrow \delta^{\alpha,\beta}(K, \gamma) \leq \delta^{\alpha,\beta}(M, \gamma).$

v) $\delta^{\alpha,\beta}(K, \delta) \leq \delta^{\alpha,\beta}(K, \gamma).$

Now we are ready to give the following definition.

Definition 7.1.3 *(see [5])* x *is said to be* $\alpha\beta-$*statistically convergent of order* γ *to* L *and denoted by* $x_n \to L$ $(\alpha\beta^{\gamma}-stat)$, *if* $\forall \varepsilon > 0$,

$$\delta^{\alpha,\beta}(\{k : |x_k - L| \geq \varepsilon\}, \gamma) = \lim_{n\to\infty} \frac{\left|\left\{k \in P_n^{\alpha,\beta} : |x_k - L| \geq \varepsilon\right\}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} = 0.$$

When $\gamma = 1$, we say that x is $\alpha\beta-$statistical convergent to L and denoted by $x_n \to L$ $(\alpha\beta-\text{stat})$.

Remark 7.1.4 *(see [5]) If* $0 < \gamma \leq \delta \leq 1$ *and*

$$x_n \to L \ (\alpha\beta^{\gamma} - stat)$$

then

$$x_n \to L \ (\alpha\beta^{\alpha} - stat).$$

As a direct consequence of Lemma 7.1.2 (iii) we have the following lemma.

Lemma 7.1.5 *(see [5])Assume that* $x \to L$ *(ordinary sense) and* $(\alpha, \beta) \in \Lambda$ *then* $x_n \to L$ *(*$\alpha\beta-$*stat).*

The following example shows that Definition 7.1.3 is a non-trivial generalisation of both ordinary and $s-$convergence.

Example 7.1.6 *(see [5])Let* $0 < \gamma < 1$ *be fixed. Taking* $\alpha(n) = 1$ *and* $\beta(n) = n^{\frac{1}{\gamma}}$*, then*

$$\delta^{\alpha,\beta}(\{k : |x_k - L| \geq \varepsilon\}, \gamma) = \lim_{n\to\infty} \frac{\left|\left\{k \in \left[1, n^{\frac{1}{\gamma}}\right] : |x_k - L| \geq \varepsilon\right\}\right|}{n}.$$

Especially for $\gamma = \frac{1}{2}$*, we have*

$$\delta^{\alpha,\beta}(\{k : |x_k - L| \geq \varepsilon\}, \frac{1}{2}) = \lim_{n\to\infty} \frac{|\{k \in [1, n^2] : |x_k - L| \geq \varepsilon\}|}{n}.$$

Now consider the sequence

$$x_n := \begin{cases} 1 : & n = k^2 \text{ for some } k \\ 0 : & \text{otherwise.} \end{cases}$$

It is obvious that $x_n \to 0$ *(stat) but since*

$$\delta^{\alpha,\beta}(\{k : |x_k| \geq \varepsilon\}, \frac{1}{2}) = \lim_{n\to\infty} \frac{|\{k \in [1, n^2] : |x_k| \geq \varepsilon\}|}{n} = 1,$$

for all $\varepsilon > 0$*,* $x_n \to 0$ *(*$\alpha\beta^{\frac{1}{2}}-$*stat) does not hold.*

7.2 $\alpha\beta$-Equistatistical Convergence

The main aim of this section is to introduce $\alpha\beta-$equistatistical convergence of order γ which lies between $\alpha\beta-$pointwise convergence of order γ and $\alpha\beta-$statistical uniform convergence of order γ.

Definition 7.2.1 *(see [5]) A function sequence f_r is said to be $\alpha\beta-$statistically pointwise convergent to f on X of order γ and denoted by $f_k \to f$ $(\alpha\beta^{\gamma}-stat)$ if for every $\varepsilon > 0,$ and for each $x \in X$*

$$\begin{aligned}
&\delta^{\alpha,\beta}(\{k : |f_k(x) - f(x)| \geq \varepsilon\}, \gamma) \\
&= \lim_{n\to\infty} \frac{\left|\left\{k \in P_n^{\alpha,\beta} : |f_k(x) - f(x)| \geq \varepsilon\right\}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} = 0.
\end{aligned}$$

For $\gamma = 1,$ f_r is said to be $\alpha\beta-$statistically pointwise convergent to f on X and denoted by $f_k \to f$ $(\alpha\beta-\text{stat})$.

Definition 7.2.2 *(see [5]) A function sequence f_r is said to be $\alpha\beta-$uniform convergent to f on X of order γ and denoted by $f_k \rightrightarrows f$ $(\alpha\beta^{\gamma}-stat)$ if for every $\varepsilon > 0,$*

$$\begin{aligned}
&\delta^{\alpha,\beta}(\left\{k : \|f_k(x) - f(x)\|_{C(X)} \geq \varepsilon\right\}, \gamma) \\
&= \lim_{n\to\infty} \frac{\left|\left\{k \in P_n^{\alpha,\beta} : \|f_k(x) - f(x)\|_{C(X)} \geq \varepsilon\right\}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} = 0.
\end{aligned}$$

For $\gamma = 1$, f_r is said to be $\alpha\beta-$statistically uniform convergent to f on X and denoted by $f_k \rightrightarrows f$ ($\alpha\beta-$stat).

Definition 7.2.3 *(see [5]) A function sequence f_r is said to be $\alpha\beta-$equistatistically convergent to f on X of order γ and denoted by $f_k \twoheadrightarrow f$ ($\alpha\beta^{\gamma}-$stat) if for every $\varepsilon > 0$, the sequence of real valued functions $(g^{\gamma}_{r,\,\varepsilon})$, defined by*

$$g^{\gamma}_{r,\,\varepsilon}(x) = \frac{\left|\left\{m \in P_r^{\alpha,\beta} : |f_m(x) - f(x)| \geq \varepsilon\right\}\right|}{(\beta(r) - \alpha(r) + 1)^{\gamma}}$$

uniformly converges to the zero function on X, i.e.

$$\lim_{r\to\infty} \left\| g^{\gamma}_{r,\,\varepsilon}(.) \right\|_{C(X)} = 0.$$

For $\gamma = 1$, f_r is said to be $\alpha\beta-$equistatistically convergent to f on X and denoted by $f_k \twoheadrightarrow f$ ($\alpha\beta-$stat).

As a direct consequence of the definitions we have the following lemma.

Lemma 7.2.4 *(see [5]) For $0 < \gamma \leq 1$ and each pair $(\alpha, \beta) \in \Lambda$ we have*

$$f_k \rightrightarrows f(\alpha\beta^{\gamma} - stat) \Rightarrow f_k \twoheadrightarrow f(\alpha\beta^{\gamma} - stat) \Rightarrow f_k \to f(\alpha\beta^{\gamma} - stat).$$

The following examples shows that the converse implications does not hold in general.

Example 7.2.5 *(see [5]) Let* $(\alpha, \beta) \in \Lambda$ *and* $0 < \gamma \leq 1$ *and consider the sequence of continuous functions which is defined in Example 3.1.5. Then for every* $\varepsilon > 0$, $0 < \gamma \leq 1$,

$$\begin{aligned} g_{r,\,\varepsilon}^{\gamma}(x) &= \frac{\left|\left\{m \in P_r^{\alpha,\beta} : |f_m(x) - f(x)| \geq \varepsilon\right\}\right|}{(\beta(r) - \alpha(r) + 1)^{\gamma}} \\ &\leq \frac{1}{(\beta(r) - \alpha(r) + 1)^{\gamma}} \to 0 \text{ as } r \to \infty, \end{aligned}$$

uniformly in x *which gives that* $f_k \twoheadrightarrow 0$ $(\alpha\beta^{\gamma}-stat)$. *But* $f_k \rightrightarrows 0$ $(\alpha\beta^{\gamma}-stat)$ *does not hold since* $\sup_{x\in[0,1]} |f_r(x)| = 1$ *for all* r.

Example 7.2.6 *(see [5]) Consider the functions* $f_r : [0,1] \to \mathbb{R}$, $r \in \mathbb{R}$, *with* $f_r(x) = x^r$, $\alpha(r) = 2^{r-1} + 1$ *and* $\beta(r) = 2^r$. *Also let*

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases} .$$

Then it is obvious that $f_r \to f$ $(\alpha\beta^{\gamma}-stat)$ *for any* $0 < \gamma \leq 1$, *but* $f_k \twoheadrightarrow f$ $(\alpha\beta^{\gamma}-stat)$ *does not hold for any* $0 < \gamma \leq 1$. *Indeed take* $\varepsilon = 1$ *then for all* $N_0 \in \mathbb{N}$, *there exists* $r > N_0$ *such that for all* $m \in P_r^{\alpha,\beta} := [2^{r-1} + 1, 2^r]$ *and* $x \in \left(\sqrt[2^r]{\frac{1}{2}}, 1\right)$

$$|f_m(x)| = |x^m| \geq \left(\sqrt[2^r]{\frac{1}{2}}\right)^m \geq \left(\sqrt[2^r]{\frac{1}{2}}\right)^{2^r} = \frac{1}{2}$$

in other words for all $x\left(\sqrt[2^r]{\frac{1}{2}},1\right)$ *and* $0<\gamma\leq 1$,

$$\begin{aligned} g^{\gamma}_{r,\,\frac{1}{2}}(x) &= \frac{\left|\left\{m\in P_r^{\alpha,\beta} : |x^m|\geq \frac{1}{2}\right\}\right|}{(2^{r-1})^{\gamma}} \\ &= \frac{2^{r-1}-1}{(2^{r-1})^{\gamma}}. \end{aligned}$$

Hence

$$\lim_{r\to\infty}\left\|g^{\gamma}_{r,\,\varepsilon}()\right\|_{C(X)} \nrightarrow 0.$$

7.3 Korovkin Type Theorem for $\alpha\beta$-Equistatistical Convergence and $\alpha\beta$ - Statistical Convergence

In this section our aim is to give a KTAT theorem for $\alpha\beta-$convergence and $\alpha\beta-$equistatistical convergence. We will also explain for the different choises of $\alpha(n)$, $\beta(n)$ and γ, $\alpha\beta-$convergence and $\alpha\beta-$equistatistical convergence are non trivial extensions of $s-$convergence, $s-$convergence of order γ, $\lambda-$convergence, $\lambda-$convergence of order γ and $\theta-$convergence.

Theorem 7.3.1 *(see [5]) Let* $(\alpha,\beta)\in\Lambda$, $0<\gamma\leq 1$ *and let* $L_r : C(X)\to C(X)$ *be a sequence of positive linear operators satisfying*

$$L_r(e_i,x)\twoheadrightarrow e_i(x)\ \ (\alpha\beta^{\gamma}-stat),\ i=0,1,2$$

where $e_i(x)=x^i$, *then for all* $f\in C(X)$,

$$L_r(f,x)\twoheadrightarrow f\ \ (\alpha\beta^{\gamma}-stat).$$

Proof. Let $f \in C(X)$ and $x \in X$ be fixed. We can write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ for all $y \in X$ satisfying $|y - x| < \delta$ since f is continuous at x. For $X_\delta = [x - \delta, x + \delta] \cap X$ we can write that

$$|f(y) - f(x)| = |f(y) - f(x)|\, \chi_{X_\delta(y)} + |f(y) - f(x)|\, \chi_{X \setminus X_\delta(y)}.$$

Then we have

$$|f(y) - f(x)| \leq \varepsilon + 2M \frac{(y - x)^2}{\delta^2}$$

For all $y \in Y$, where $M := \|f\|_{C(X)}$. Using the fact that L_r is positive and linear we have

$$\begin{aligned} |L_r(f, x) - f(x)| &\leq L_r(|f(y) - f(x)e_0|\,; x) + |f(x)|\, |L_r(f_0; x) - e_0(x)| \\ &\leq \varepsilon L_r(e_0; x) + \frac{2M}{\delta^2} \left\{ L_r\left((y - x)^2 ; x \right) \right\} + M\, |L_r(e_0; x) - e_0(x)| \\ &\leq \varepsilon + B \sum_{i=0}^{2} |L_r(e_i; x) - e_i(x)| \qquad (7.3.1) \end{aligned}$$

where $B = \varepsilon + M + \frac{4M}{\delta^2} \left(\|e_2(x)\|_{C(X)} + \|e_1(x)\|_{C(X)} + 1 \right)$.

For a given $s > 0$, choose $\varepsilon > 0$ such that $\varepsilon < s$ and define the following sets:

$$\begin{aligned} K_s(x) &:= \{ r \in \mathbb{N} : |L_r(f; x) - f(x)| \geq s \} \\ K_s^i(x) &:= \left\{ r \in \mathbb{N} : |L_r(e_i; x) - e_i(x)| \geq \frac{s - \varepsilon}{3B} \right\} \quad (i = 0, 1, 2) \end{aligned}$$

Using (7.3.1) we have

$$K_s(x) \subset \bigcup_{i=0}^{2} K_s^i(x). \tag{7.3.2}$$

Also define the following real valued functions

$$u_{m,s}^{\gamma}(x) := \frac{1}{(\beta(r) - \alpha(r) + 1)^{\gamma}} \left\{ k \in P_m^{\alpha,\beta} : |L_r(f,x) - f(x)| \geq s \right\}$$

and

$$g_{m,s}^{\gamma,i}(x) = \frac{1}{(\beta(r) - \alpha(r) + 1)^{\gamma}} \left\{ k \in P_m^{\alpha,\beta} : |L_r(e_i,x) - e_i(x)| \geq \frac{s-\varepsilon}{3B} \right\}$$

$i = 0, 1, 2$. Then by the monotonicity and (7.3.2) we have

$$g_{m,s}^{\gamma}(x) \leq \sum_{i=0}^{2} g_{m,s}^{\gamma,i}(x), \text{ for all } x \in X,$$

which implies the inequality

$$\left\| g_{m,s}^{\gamma}(.) \right\|_{C(X)} \leq \sum_{i=0}^{2} \left\| g_{m,s}^{\gamma,i}(.) \right\|_{C(X)}. \tag{7.3.3}$$

Taking limit in (7.3.3) as $m \to \infty$ and using the hpothesis of the theorem we conclude that

$$\lim_{m} \left\| g_{m,s}^{\gamma}(.) \right\|_{C(X)} = 0$$

which completes the proof. ■

Taking $\gamma = 1$ in the previous theorem, we have the following corollary.

Corollary 7.3.2 *(see [5])Let* $(\alpha, \beta) \in \Lambda$ *and let* $L_r : C(X) \to C(X)$ *be a sequence of positive linear operators satisfying*

$$L_r(e_i, x) \twoheadrightarrow e_i(x) \ \ (\alpha\beta - stat) \, , \, i = 0, 1, 2$$

where $e_i(x) = x^i$, *then for all* $f \in C(X)$,

$$L_r(f, x) \twoheadrightarrow f \quad (\alpha\beta - stat) \, .$$

Corollary 7.3.3 *(see [5]) Let* $(\alpha, \beta) \in \Lambda$ *and let* $L_r : C(X) \to C(X)$ *be a sequence of positive linear operators satisfying*

$$L_r(e_i, x) \twoheadrightarrow e_i(x) \ \ (\alpha\beta^{\gamma} - stat) \, , \, i = 0, 1, 2$$

where $e_i(x) = x^i$, *then for all* $f \in C(X)$,

$$L_r(f, x) \twoheadrightarrow f \quad (\alpha\beta - stat) \, .$$

Proof. Using Theorem 7.3.1 and the fact that $L_r(e_i, x) \twoheadrightarrow e_i(x)$ $(\alpha\beta^{\gamma}-\text{stat})$ implies $L_r(e_i, x) \twoheadrightarrow e_i(x)$ $(\alpha\beta-\text{stat})$, completes the proof. ■

Theorem 7.3.4 *(see [5]) Let* $(\alpha, \beta) \in \Lambda$, $0 < \gamma \leq 1$ *and let* $L_r : C(X) \to C(X)$

be a sequence of positive linear operators satisfying

$$L_r(e_i, x) \rightrightarrows e_i(x) \;\; (\alpha\beta^{\gamma} - stat) \;\; i = 0, 1, 2$$

where $e_i(x) = x^i$, *then for all* $f \in C(X)$,

$$L_r(f, x) \rightrightarrows f \quad (\alpha\beta^{\gamma} - stat) \, .$$

Proof. Let $f \in C(X)$ and $x \in X$ be fixed. We can write that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ for all $y \in X$ satisfying $|y - x| < \delta$ since f is continuous at x. For $X_\delta = [x - \delta, x + \delta] \cap X$ we can write that

$$|f(y) - f(x)| = |f(y) - f(x)| \, \chi_{X_\delta(y)} + |f(y) - f(x)| \, \chi_{X \setminus X_\delta(y)}.$$

Then we have

$$|f(y) - f(x)| \leq \varepsilon + 2M \frac{(y - x)^2}{\delta^2}$$

For all $y \in Y$, where $M := \|f\|_{C(X)}$. Using the fact that L_r is positive and linear we have

$$\begin{aligned} |L_r(f, x) - f(x)| &\leq L_r(|f(y) - f(x)e_0| \, ; x) + |f(x)| \, |L_r(f_0; x) - e_0(x)| \\ &\leq \varepsilon + B \sum_{i=0}^{2} |L_r(e_i; x) - e_i(x)| \end{aligned} \tag{7.3.4}$$

where $B = \varepsilon + M + \frac{4M}{\delta^2}\left(\|x^2\|_{C(X)} + \|x\|_{C(X)} + 1\right)$. By taking supremum over X we have

$$\|L_r(f;.) - f(.)\| \le \varepsilon + B\sum_{i=0}^{2} \|L_r(e_i;.) - e_i(.)\|.$$

$\forall s > 0$, choose $\varepsilon > 0$ such that $\varepsilon < s$ and define the following sets:

$$K_s(x) := \{r \in \mathbb{N} : \|L_r(f;x) - f(x)\| \ge s\}$$

$$K_s^i(x) := \left\{r \in \mathbb{N} : \|L_r(e_i;.) - e_i(.)\| \ge \frac{s-\varepsilon}{3B}\right\} \quad (i = 0, 1, 2) \tag{7.3.5}$$

Using (7.3.5) we have

$$K_s(x) \subset \bigcup_{i=0}^{2} K_s^i(x)$$

which completes the proof. ■

Taking $\gamma = 1$ in the previous theorem, we have the following corollary.

Corollary 7.3.5 *(see [5])Let $(\alpha, \beta) \in \Lambda$, $0 < \gamma \le 1$ and let $L_r : C(X) \to C(X)$ be a sequence of positive linear operators satisfying*

$$L_r(e_i, x) \rightrightarrows e_i(x) \ \ (\alpha\beta - stat) \ \ i = 0, 1, 2$$

where $e_i(x) = x^i$, *then for all* $f \in C(X)$,

$$L_r(f, x) \rightrightarrows f \quad (\alpha\beta - stat)\,.$$

In the following remarks we will explain that the results obtained in this chapter are new results. Also they are non-trivial extensions of results which are done by different authors in the past.

Remark 7.3.6 *(see [5]) Taking* $\alpha(n) = 1$ *and* $\beta(n) = n$, *and* $\gamma = 1$, *then* $P_n^{\alpha,\beta} = [1, n]$ *and*

$$\delta^{\alpha,\beta}\left(\{k : |x_k - L| \geq \varepsilon\}, 1\right) = \lim_{n\to\infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{n}$$

This shows that the case of $\alpha(n) = 1$, $\beta(n) = n$ *and* $\gamma = 1$, $\alpha\beta-$*convergence reduces to* $s-$*convergence. Therefore if we take* $\alpha(n) = 1$, $\beta(n) = n$ *and* $\gamma = 1$ *then Theorem 7.3.1 reduces to Theorem* 2.1 *of [30] and Theorem 7.3.4 reduces to Theorem* 1 *of [28].*

Remark 7.3.7 *(see [5]) For* $\alpha(n) = 1$ *and* $\beta(n) = n$, *and* $0 < \gamma < 1$, *then* $P_n^{\alpha,\beta} = [1, n]$,

$$\delta^{\alpha,\beta}\left(\{k : |x_k - L| \geq \varepsilon\}, \gamma\right) = \lim_{n\to\infty} \frac{|\{k \leq n : |x_k - L| \geq \varepsilon\}|}{n^\gamma},$$

which the definition coincides with the definition of $s-$*convergence of order* γ. *Therefore Theorem 7.3.1 reduces to a KTAT via equistatistical convergence of* *order γ and Theorem 7.3.4, reduces to a KTAT via statistical uniform convergence of order γ.*

Remark 7.3.8 *(see [5]) Let $\lambda_n \in \omega$ and choose $\alpha(n) = n - \lambda_n + 1$, $\beta(n) = n$ and $\gamma = 1$ then*

$$P_n^{\alpha,\beta} = [n - \lambda_n + 1, n] .$$

Moreover

$$\begin{aligned} &\delta^{\alpha,\beta}\left(\{k : |x_k - L| \geq \varepsilon\}, 1\right) \\ &= \lim_{n\to\infty} \frac{|\{k \in [n - \lambda_n + 1, n] : |x_k - L| \geq \varepsilon\}|}{\lambda_n} \end{aligned}$$

which shows that $\alpha\beta-$convergence of order γ reduces to $\lambda-$convergence for the case of $\alpha(n) = n - \lambda_n + 1$, $\beta(n) = n$ and $\gamma = 1$. Therefore if we take $\alpha(n) = n - \lambda_n + 1$, $\beta(n) = n$ and $\gamma = 1$ then Theorem 7.3.1, reduces to Theorem 1 of [41]. Similarly Theorem 7.3.4, will be a special case of Theorem 3.1 of [1].

Remark 7.3.9 *(see [5]) Let $\lambda_n \in \omega$ and choose $\alpha(n) = n - \lambda_n + 1$, $\beta(n) = n$ and $0 < \gamma < 1$ then we have*

$$\begin{aligned} &\delta^{\alpha,\beta}\left(\{k : |x_k - L| \geq \varepsilon\}, \gamma\right) \\ &= \lim_{n\to\infty} \frac{|\{k \in [n - \lambda_n + 1, n] : |x_k - L| \geq \varepsilon\}|}{\lambda_n^\gamma} \end{aligned}$$

which shows that $\alpha\beta-$convergence of order γ reduces to $\lambda-$convergence of order

γ. Therefore Theorem 7.3.1, reduces to a KTAT via $\lambda-$equistatistical convergence of order γ and Theorem 2, reduces to a KTAT via $\lambda-$convergence of order γ.

Remark 7.3.10 *(see [5]) Take $\alpha(r) = k_{r-1} + 1$, $\beta(r) = k_r$ and $\gamma = 1$ then $P_r^{\alpha,\beta} = [k_{r-1} + 1, k_r]$ where k_r is a lacunary sequence. Since*

$$(k_{r-1}, k_r] \cap \mathbb{N} = [k_{r-1} + 1, k_r] \cap \mathbb{N},$$

we have

$$\begin{aligned}
&\delta^{\alpha,\beta}\left(\{k : |x_k - L| \geq \varepsilon\}, \gamma\right) \\
&= \lim_{r\to\infty} \frac{|\{k \in [k_{r-1} + 1, k_r] : |x_k - L| \geq \varepsilon\}|}{h_r^{\gamma}} \\
&= \lim_{r\to\infty} \frac{|\{k \in (k_{r-1}, k_r] : |x_k - L| \geq \varepsilon\}|}{\lambda_r^{\gamma}}.
\end{aligned}$$

We conclude that in the case $\alpha(r) = k_{r-1}+1$, $\beta(r) = k_r$ and $\gamma = 1$, $\alpha\beta-$convergence of order γ reduces to $\theta-$convergence. Therefore if we take $\alpha(r) = k_{r-1} + 1$, $\beta(r) = k_r$ and $\gamma = 1$ then Theorem 7.3.1, reduces to Theorem 3.1 of [2] and Theorem 7.3.4, will be a special case of Theorem 3.1 of [1].

Remark 7.3.11 *(see [5]) For $\alpha(r) = k_{r-1} + 1$, $\beta(r) = k_r$, and $0 < \gamma < 1$ we* *have*

$$
\begin{aligned}
&\delta^{\alpha,\beta}\left(\left\{k : |x_k - L| \geq \varepsilon\right\}, \gamma\right) \\
&= \lim_{r\to\infty} \frac{|\{k \in [k_{r-1}+1, k_r] : |x_k - L| \geq \varepsilon\}|}{h_r^{\gamma}} \\
&= \lim_{r\to\infty} \frac{|\{k \in (k_{r-1}, k_r] : |x_k - L| \geq \varepsilon\}|}{\lambda_r^{\gamma}}.
\end{aligned}
$$

Therefore Theorem 7.3.1, reduces to a KTAT via $\theta-$*equistatistical convergence of order* γ *and Theorem 7.3.4, reduces to a KTAT via lacunary statistical uniform convergence of order* γ.

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