Irreducible Representations of Some Finite Groups

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ABSTRACT

In this thesis we compute the irreducible representations and the characters of some certain finite groups.

We first provide the necessary overview on linear algebra, group theory and the representations theory.

Then, we compute the irreducible representations of finite cyclic groups, smaller symmetric groups and the direct products of the two groups.

Finally, we give a general method to compute the irreducible representations of S_n by using Young diagrams and provide the Frobenius formula to obtain the characters for these irreducible representations.

Keywords: Representation, Character, Cyclic groups, Symmetric groups,
Young diagram, Frobenius formula.

ÖZ

Bu tezde sonlu grupların indirgenemez reprezantasyonları ve karakterleri hesaplanmıştır.

İlk olarak Cebir, grup teorisi ve reprezantasyon teorileri hakkında ön bilgi verilmiştir. Daha sonra devirli ve simetrik grupların ve bunların direkt çarpımlarının indirgenemez reprezantasyonları hesaplanmıştır.

Son olarak simetrik grup S_n için genelleştirilmiş indirgenemez reprezantasyon metodu Young şeması kullanılarak verilmiş ve Frobenius formülüyle bu reprezantasyonların karakterleri bulunmuştur.

Anahtar Kelimeler: Reprezantasyon, Karakter, Devirli grup, Simetrik grup,
Young şeması, Frobenius formülü

To my Family

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Chapter 1

INTRODUCTION

Representation theory is a topic of pure mathematıcs that studies abstract algebraic structure. Instead of working on complex algebraic operations, this theory gives a chance to derive all of the operations from matrix addition and matrix multiplication. Since it reduces problems of abstract algebra to problem of linear algebra, it is regarded as an extremely useful branch of pure mathematics.

This thesis consists of 5 chapters. First chapter of the thesis is this introduction. The second chapter includes some background information on linear algebra and groups. In our thesis we mostly use the finite groups while working on representations. In third chapter we give the overview information on representation theory.

Linear representation of G in V is a homomorphism φ from G into $GL(V)$ where G is a multiplicative group with identity element 1 and $GL(V)$ is the group of ismorphisms of V onto itself, where V is a vector space over the field of complex numbers, $\mathbb{C}$.

The character of the representation φ is equal to the trace of the image matrix, where $\chi_{\varphi}(s) = Tr(\varphi_s)$ for all $s \in G$.

There are lots of important theorems and corollaries about representations. One of the most important corollaries is about deciding whether a given representation is irreducible or not.

We mostly compute the irreducible representations of the cyclic and symmetric groups in our thesis. The corollaries pointed out in chapter 3 are going to be used on examples in chapter 4.

In chapter 4 we give examples about the finite groups. One family of these finite groups are the symmetric groups, S_3 and S_4. From the chapter 3 we know that number of irreducible representations of symmetric groups is equal to the number of conjugacy classes and the first 3 irreducible representation are the trivial , alternating and the standard representations. Other examples in chapter 4 are the representations of cyclic groups C_2, C_4 and the generalized form C_n.

We also explained homomorphism in chapter 2 as background information and there are examples in chapter 4 that we use homomorphism rules to find the representations of the direct products of the finite groups.

In the last chapter, chapter 5, we give the general formula to find all the irreducible representations of the symmetric group S_n. In this chapter we explain what a partition is and how the Young diagrams can be formed. Young diagrams play important role while finding the irreducible representations of S_n. By using these diagrams we can find the group algebra and by using the group algebra we can reach the irreducible representations.

The brief definitions on group algebra and calculations by using Young diagrams are given in this chapter. After finding irreducible representations, we start to work on the character values of the elements of S_n, and we use the Frobenius formula below to calculate the character values,

$$\chi_\lambda(C_i) = \left[\Delta(x)\prod_j P_j(x)^{i_j}\right]_{l_1, l_2, \ldots, l_k}$$

where λ is the partition and all of the other notations are explained in chapter 5. With respect to these character values we can observe which irreducible representation this partition belongs.

Chapter 2

OVERVIEW

Part 1: Linear Algebra

2.1 Matrix Representation of a Linear Transformation

Definition 2.1.1 Let V be a vector space and $\beta = \{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\}$ be an ordered basis for V. The coordinate vector of $\underline{x} \in V$ associated with β is illustrated by the column vector $[\underline{x}]_\beta = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$, where $d_1, \dots, d_n$ are unique scalars such that $\underline{x} = d_1\underline{x}_1 + d_2\underline{x}_2 + \cdots + d_n\underline{x}_n$.

Suppose that V and W are two finite dimensional vector spaces , and let γ and β be the corresponding ordered bases such that $\gamma = \{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\}$ and $\beta = \{\underline{y}_1, \underline{y}_2, \dots, \underline{y}_m\}$ Let T be a linear mapping of V and W such that ; $T: V \to W$. Then , since T is linear there exist unique scalars d_{ij} , such that

$$T(\underline{x}_j) = \sum_{i=1}^{m} d_{ij}\underline{y}_i$$

where , for $1 \le j \le n$, $\underline{x}_j$belong to the ordered basis of V, and similarly for $1 \le i \le m$, $\underline{y}_i$ belong to the ordered basis of W.

The matrix representation of this linear transformation, T, corresponding to ordered bases β and γ is an $m \times n$ matrix and can be denoted by $[T]_\beta^\gamma$. If $V = W$ (that means if their dimensions and bases are same), then the matrix representation of T from V to W and from W to V are equal , so $[T]_\beta^\gamma = [T]_\gamma^\beta$. Hence if two bases are equal to each other, $[T]_\beta^\gamma$ can be written as $[T]_\beta$ since $[T]_\beta^\gamma = [T]_\beta^\beta = [T]_\beta$.

Theorem 2.1.2 [7] Suppose that T and U are two linear mappings from V to W, where V and W are finite dimensional vector spaces. Let β and γ be the corresponding ordered bases such that $\beta = \{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\}$ and $\gamma = \{\underline{y}_1, \underline{y}_2, \dots, \underline{y}_m\}$ Then,

$$(1)\ [T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$$

$$(2)\ c[T]_\beta^\gamma = \mathrm{c}[T]_\beta^\gamma$$

Proof: (1) Let that $\beta = \{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\}$ and $\gamma = \{\underline{y}_1, \underline{y}_2, \dots, \underline{y}_m\}$.

$$T(\underline{x}_j) = \sum_{i=1}^{m} a_{ij}\underline{y}_i \ , \ U(\underline{x}_j) = \sum_{i=1}^{m} b_{ij}\underline{y}_i \ \text{ and } \ (T+U)(\underline{x}_j) = \sum_{i=1}^{m}(a_{ij}+b_{ij})\underline{y}_i$$

$\Rightarrow ([T+U]_\beta^\gamma)_{ij} = a_{ij} + b_{ij} = ([T]_\beta^\gamma)_{ij} + ([U]_\beta^\gamma)_{ij}$ ■

Definition 2.1.3 (Inner product) Let $u = \langle \underline{u}_1, \underline{u}_2, \dots, \underline{u}_n \rangle$ and $v = \langle \underline{v}_1, \underline{v}_2, \dots, \underline{v}_n \rangle$ be 2 vectors and their inner product defined as $\langle u.v \rangle = \underline{u}_1\underline{v}_1 + \underline{u}_2\underline{v}_2 + \cdots + \underline{u}_n\underline{v}_n$

Remark 2.1.4 Two vectors are orthogonal if their inner product gives 0.

Eg: $u = \langle 1,1,1 \rangle$ and $= \langle 1,3,-4 \rangle$, so $\langle u.v \rangle = 1.1 + 1.3 + 1.-4 = 0$. That means u and v are orthogonal.

Definition 2.1.5 (Tensor product) Let $U = \begin{pmatrix} u_{1,1} & \cdots & u_{1,n} \\ \vdots & \ddots & \vdots \\ u_{n,1} & \cdots & u_{n,n} \end{pmatrix}$ and

$V = \begin{pmatrix} v_{1,1} & \cdots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{n,1} & \cdots & v_{n,n} \end{pmatrix}$ then the tensor product

$$U \otimes V = \begin{pmatrix} u_{1,1} \begin{pmatrix} v_{1,1} & \cdots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{n,1} & \cdots & v_{n,n} \end{pmatrix} & \cdots & u_{1,n} \begin{pmatrix} v_{1,1} & \cdots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{n,1} & \cdots & v_{n,n} \end{pmatrix} \\ \vdots & \ddots & \vdots \\ u_{n,1} \begin{pmatrix} v_{1,1} & \cdots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{n,1} & \cdots & v_{n,n} \end{pmatrix} & \cdots & u_{n,n} \begin{pmatrix} v_{1,1} & \cdots & v_{1,n} \\ \vdots & \ddots & \vdots \\ v_{n,1} & \cdots & v_{n,n} \end{pmatrix} \end{pmatrix}$$

2.2 Matrix Representation for Composition of Linear Transformations.

Theorem 2.2.1 [7] Suppose that $T: V \to W$ and $U: W \to Z$ are linear and let $\alpha = \{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\}$, $\beta = \{\underline{y}_1, \underline{y}_2, \dots, \underline{y}_m\}$ and $\gamma = \{\underline{z}_1, \underline{z}_2, \dots, \underline{z}_p\}$ be the corresponding ordered bases for V, W and Z respectively. Then UT is also linear and $[UT]_\alpha^\gamma = [U]_\beta^\gamma \, [T]_\alpha^\beta$.

Proof: Let $\alpha = \{\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n\}$ $\beta = \{\underline{y}_1, \underline{y}_2, \dots, \underline{y}_m\}$ and $\gamma = \{\underline{z}_1, \underline{z}_2, \dots, \underline{z}_p\}$.

$$(UT)(\underline{x}_j) = U\left(T(\underline{x}_j)\right) = U(\sum_{i=1}^{m} a_{ij}\, \underline{y}_i) = \sum_{i=1}^{m} a_{ij} U(\underline{y}_i) = \sum_{i=1}^{m} a_{ij} (\sum_{l=1}^{p} b_{li}\, \underline{z}_l) =$$

$$\sum_{l} \sum_{i} b_{li} a_{ij}\, \underline{z}_l = \sum_{l} (ba)_{lj}\, \underline{z}_l \quad ([UT]_\alpha^\beta)_{lj} = (ba)_{lj} = ([U]_\beta^\gamma)_{li} . ([T]_\alpha^\beta)_{ij}$$

■

Theorem 2.2.2 [7] Let β and γ be ordered bases for finite dimensional vector space V and W respectively , and let T be a linear map from V to W such that $T: V \to W$.Then , for all vector ,x, of V : $[T(\underline{x})]_\gamma = [T]_\beta^\gamma\,[\underline{x}]_\beta$

Part 2: Group Theory

2.3 Groups

Definition 2.3.1 A group is a non-empty set of elements with binary operation $*$. We usually denote a group G by $(G,*)$.There are 4 axioms for a group that need to be satisfied:

(1) **<u>Closure:</u>** A binary operation combines any two elements in the group G to collect the third one, and this third element collected is the element of the group G as well. Mathematically for all $g_1, g_2 \in G$, $g_1 * g_2 \in G$.

(2) **<u>Associativity:</u>** For a group $(G,*)$, binary operation $*$ is associative. That means, $(a * b) * c = a * (b * c)$ for all $a, b, c \in G$.

(3) **<u>Identity element</u>:** There exists an **<u>identity</u>** element e in G, such that $g * e = e * g = g$ for all $g \in G$.

(4) **<u>Inverse element:</u>** For each $g \in G$, we can find an **<u>inverse</u>** element $g^{-1} \in G$ such that, $g * g^{-1} = e = g^{-1} * g$, where e is the identity element of G.

Proposition 2.3.2 The identity element in axiom 3 is unique.

Proof: Assume that there are two identity elements a and b. Here, $ab = a$ (Since b is the identity element) and $ab = b$ (Since a is the identity element). Therefore $a = b$

■

Proposition 2.3.3 The inverse element in axiom 4 is unique.

Proof: Let a and b be two inverses of the elemet , where $a, b, c \in G$. Then ;

$$a = (a * e) = a * (c * b) = (a * c) * b = e * b = b$$

Since b is inverse of c   by the axiom (2)   since a is inverse of c

$\therefore \ a = b$ ■

2.3.4 Groups Examples :

1. $(\mathbb{Z}, +)$; The set of integers is a group under addition.
2. $(\mathbb{R}^*, \times)$; The set $\mathbb{R}^*$, of non-zero real numbers is a group under multiplication.
3. $(\mathbb{Z}, \times)$ The set of integers is not a group under multiplication because of the failure of axiom (4). For example, the inverse of $2 \in \mathbb{Z}$, under multiplication is $\frac{1}{2}$, which is not an integer number.
4. **The General Linear Groups:**

   The general linear group $GL_n = \{A \in M_n | \det(A) \neq 0\}$ is the group of $n \times n$ invertible matrices with operation of multiplication. Remember that a matrix A is invertible if and only if $\det A \neq 0$. So this group can also be identified with matrices A such that det $A \neq 0$.

5. **Special Linear Groups:**

   The special linear group $SL_n = \{A \in GL_n \mid \det A = 1\}$ is the group of invertible $m \times m$ (square) matrices with determinant 1. Again this set forms a group under multiplication.

6. **Modulo Group $\mathbb{Z}_n$:**

   The modulo operation is denoted as $r = m(mod\ n)$ where $m = np + r$**.** And let $n > 0$, the modulo set $\mathbb{Z}_n = \{0,1,2,\dots,n-1\}$**.**

7. Let $n \in \mathbb{N},\ n > 1$, $U(n) = \{x \in \{1,2,\dots,n-1\} |\ hcf(x,n) = 1\}$ is a group under multiplication modulo n. For example, if $n = 12$, $U(12) = \{1,5,7,11\}$

Definition 2.3.5 A group G is Abelian if the operation $*$ is commutative, such that $ab = ba$ for all $a, b \in G$.

Example 2.3.6 The set of non-zero rational numbers is an Abelian group under multiplication whereas GL_n given above is not Abelian.

Remark 2.3.7 Note that we can use the following simplified notations for the given statements.

i. "G is a group"$= \langle G,*\rangle$

ii. $a * b = ab$,

iii. order of $G = |G|$

2.4 SUBGROUPS

Definition 2.4.1 (Order of a Group) the number of elements of a group (finite or infinite) is called its *order*. We will use $|G|$ or sometimes $n(G)$ to denote the order of G.

Example 2.4.2 Order of $\mathbb{Z}$ (under addition) is infinite, whereas $U(10) = \{1,3,7,9\}$ has order 4.

Definition 2.4.3 The order of an element g in G is the smalles positive integer n such that $g^n = e$. If there is no n values for g that means g has infinite order.

Example2.4.4 Consider $U(15) = \{1,2,4,7,8,11,13,14\}$ under multiplication modulo 15. The order of the element 7 is 4 since $7^4 = 1(mod\ 15)$ where 1 is an identity element of multiplication.

Definition 2.4.5 If a subset H of a group G is itself a group under the operation defined on G, then H is said to be a *subgroup* of G $(H \leq G)$. If we write $H < G$, then it means that H is a proper subgroup of G.

Theorem 4.2.6 (The Subgroup Criterion) [8] Let G be a group and H is a non-empty subset of G. H is a subgroup of G if and only if the following axioms hold:

(1) For all a and $b \in H$, $ab \in H$.

(2) For all $a \in H$, $a^{-1} \in H$.

Example 2.4.7

1. $SL_n(n, \mathbb{R})$ is a subgroup of GL_n . The subset SL_ncontains the identity matrix so it is nonempty. Since A and B $\in SL_n$ that means det$A$ = det$B$ = 1, then det(AB) =det$A$det$B$=1, which means $AB \in SL_n$. And det $A^{-1} = {}^{1}\!/_{detA} = 1$ that measn $A^{-1} \in SL_n$. So 2 axioms of subgroup criterion was satisfied.

2. $\mathbb{Q}$ is subgroup of $\mathbb{R}$.

Theorem 2.4.8 [4] Let H be a non-empty subset of G. Then, H is a subgroup of G if and only if for all a and b in H, ab^{-1} is also in H.

Proof : Since H is a subgroup, one way of the theorem is clear. For the reverse direction, since H is non-empty, let some $a \in H$. Then by the statement of the theorem, aa^{-1}= e is in H. Now by using e and a, $ea^{-1} = a^{-1}$ is also in H. Finally given a and b in H, since a^{-1} and b^{-1} are in H, $a(b^{-1})^{-1} = ab$ is also in H. ■

Theorem 2.4.9 (Finite Subgroup Test) [7]: If H is a non-empty finite subset of G and H is closed under the operation of G, then H is a subgroup of G.

Example 2.4.10 The subset $\{1, -1, i, -i\}$ is a group under complex multiplication.

2.5 Symmetric Groups$(S_n,*)$.

Definition 2.5.1 The set of all 1-1 functions from the set $\{1,2,\dots,n\}$ onto itself is called the symmetric group of degree n and its denoted by S_n. Since the elements are functions, operation is the composition of functions. Note that S_n has $n!$ elements.

Remark 2.5.2 The elements of S_n can be represented by using the matrices. For example if

$\rho \in S_n$, then ρ can be represented as $\rho = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \rho(1) & \rho(2) & \rho(3) & \dots & \rho(n) \end{pmatrix}$. The identity element of this group S_n is $I = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$. Finally the inverse of the element ρ given above is $\rho^{-1} = \begin{pmatrix} \rho(1) & \rho(2) & \rho(3) & \dots & \rho(n) \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$. It is clear that $S_n$has a group structure as compositions of functions is associative.

Remark 2.5.3 It is sometimes easier to represent elements of S_n in cycle form rather than using matrix notation.

For example, if $\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 4 & 6 & 5 \end{pmatrix}$, in cycle notation , we write this as

$$\rho = (123)(4)(56) = (123)(56)$$

Theorem 2.5.5 [8] Let ρ and σ be any two cycles under S_n. If these two cycles are disjoint (have no entries in common), then $\rho\sigma = \sigma\rho$.

Theorem 2.5.3 [8] The order of a permutation given in disjoint cycle form is the least common multiple of the lengths of its disjoint cycles.

Theorem 2.5.4 [8] Every permutation in S_n is a product of 2-cycles.

Proof: $\{x_1, x_2, \dots, x_n\}\{y_1, y_2, \dots, y_m\}\{z_1, z_2, \dots, z_p\} =$

$(x_1 x_2)(x_2 x_3) \dots (x_{n-1} x_n)(y_1 y_2)(y_2 y_3) \dots (y_{m-1} y_m)(z_1 z_2)(z_2 z_3) \dots (z_{p-1} z_p)$.

Definition 2.5.5 A cycle of length 2 is called transposition.

Definition 2.5.6 (Odd and Even permutation) Let $n \geq 2$, for σ in S_n, if we can write σ as a product of even (respectively odd) number of transpositions, then σ is an even (respectively odd) permutation.

Remark 2.5.6 For $n \geq 2$ and for σ in $S_{n,}$ σ is either an even or odd permutation and is only one or other.

Definition 2.5.7 (Alternating Group A_n) The group of even permutations of n symbols is called alternating group of degree n.

Definition 2.5.8 (Conjugate) Let G be a group and a, b and $c \in G$. We say that a and b are conjugate in G if b can be expressed as cac^{-1}.

Definition 2.5.9 (Conjugacy Classes) Conjugacy classes are represented by the cycle types. If two permutation σ and τ are conjugate each other if and only if they belong to same conjugacy classes.

Example 2.5.10 $\sigma = (123)$, then the conjugacy classes are

$\{(1)(2)(3)\}, \{(12)(13)(23)\}$ and $\{(132)(123)\}$.

2.6 Cyclic Groups

Definition 2.6.1 Let G be a group. Then we say that G is a *cyclic* group if there exists a generator b such that the powers of the generator gives all the elements in the group. We denote this group as $G = \langle b \rangle$. Order of a cyclic group G, is the order of one of its generators. Note that the inverse of a generator is also a generator.

Example 2.6.2 $\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$ is a cyclic group and is denoted by $\langle \mathbb{Z}, + \rangle$.

Theorem 2.6.3 [8] Every subgroup of a cyclic group is again a cyclic group.

Theorem 2.6.4 (Fundamental theorem of Cyclic groups) [8] Let G be a cyclic group ($G = \langle b \rangle$) and let the order of G be n. Then the order of any subgroup of G is a divisor of n. The group $\langle b \rangle$ has exactly one subgroup of order k, if k is the positive divisor of n. This subgroup is namely $\langle a^{n/k} \rangle$

Example 2.6.5 If $|G|$ (where $G = \langle a \rangle$) is 20, the, we can list all the subgroups of G as, $\langle a \rangle, \langle a^2 \rangle, \langle a^4 \rangle, \langle a^5 \rangle, \langle a^{10} \rangle, \langle a^{20} \rangle$. So subgroups of $\mathbb{Z}_{20}$ are

$\langle 20 \rangle = \langle 0 \rangle$, $\langle 10 \rangle = \langle 0,10 \rangle$, $\langle 5 \rangle = \langle 0,5,10,15 \rangle$, $\langle 4 \rangle = \langle 1,4,8,12,16 \rangle$, $\langle 2 \rangle = \langle 0,2,4,6,8,10,12,14,16,18 \rangle$, $\langle 1 \rangle$.

Before finishing cyclic groups, there are some corollaries that we are going to point out.

Corollary 2.6.6 Let G be a group and let a be an element of order n in G. If $a^k = e$, then $n|k$.

Corollary 2.6.7 (Generators of Cyclic Groups) Let $G = \langle a \rangle$ be a cyclic group of order n, then $G = \langle a^k \rangle$ if and only if $gcd(n, k) = 1$.

Corollary 2.6.8 (Generator of $\mathbb{Z}_n$) An integer $k \in \mathbb{Z}_n$ if and only if $\gcd(n, k) = 1$.

2.7 Cosets and Normal Groups:

Definition 2.7.1 (Coset) Let G be a group and H be a subset of G. For $g \in G$, the set $gH = \{gh : h \in H\}$ is named as the left coset of H in G. Similarly the set $Hg = \{hg : h \in H\}$ is called the right coset of H in G.

Example 2.7.2 Let $G = \mathbb{Z}_{12}$ and $H = \langle 3 \rangle = \{0,3,6,9\}$. The left cosets of H in G are

$0 + \langle 3 \rangle$, $1 + \langle 3 \rangle$, $2 + \langle 3 \rangle$. Therefore, it has 3 distinct left cosets.

Remark 2.7.3 Note that the set $\mathbb{Z}_n$ forms a group under addition mod n, whereas $U(n) = \mathbb{Z}_n{}^*$ forms a group under multiplication.

Example 2.7.4 Find the coset of $H = \{1,7\}$ in

$G = U(26) = \{1,3,5,7,9,11,15,17,19,21,23,25\}$.

The left cosets of H in G are ; $H = \{1,7\}$, $3H = \{3,21\}$, $5H = \{5,9\}$, $7H = \{7,23\}$, $9H = \{9,11\}$, $11H = \{11,25\}$, $15H = \{1,15\}$, $17H = \{17,15\}$, $19H = \{3,19\}$, $21H = \{17,21\}$, $23H = \{5,23\}$, $25H = \{19,25\}$.

Theorem 2.7.5 (Properties of Cosets) [4] Let H be a subgroup of G, and let a and b be elements in G. Then,

(1) $a \in aH$

(2) $aH = H$ if and only if $a \in H$.

(3) $aH = bH$ or $aH \cap bH = \emptyset$

(4) $aH = bH$ if and only if $a^{-1}b \in H$

(5) $|aH| = |bH|$

(6) $aH = Ha$ iff $H = aHa^{-1}$

(7) $aH \leq G$ iff $a \in H$

Definition 2.7.6 The number of distinct left cosets of H in G is called the *index* of H in G.

Definition 2.7.7 (Normal Subgroups) Let H be a subgroup of G. If the right and left cosets are the same, that is to say if $gH = Hg$ for all $g \in G$, then H is said to be a normal subgroup of G. We denote this by $H \lhd G$. ($ghg^{-1} \in H$ for all $h \in H$ and $g \in G$)

Example 2.7.8

1. For any group ,idendity element is a normal subgroup of G.

2. $SL_n \lhd GL_n$. Let the matrix $A \in GL_n$ and matrix $B \in SL_n$, then

   $\det(ABA^{-1}) = \det(A)\det(B)\det(A^{-1}) = \det(A) . 1. \det(A^{-1}) =$
   $\det(A)\det(A^{-1}) = 1$.Therefore $ABA^{-1} \in SL_n$.

3. $D_4 \lhd SL_n$. Let $K \in D_4$ and $L \in SL_n$.

Theorem 2.7.9 (Lagrange Theorem)[4] Let G be a finite group and H be a subgroup of G. Then, the order of H divides the order of G. Thus, we have $|G| = |G: H||H|$, where, $|G: H|$ denotes the index of H in G.

Corollary 2.7.10 For a finite group G, the order of the element divides the order of the group.

2.8 HOMOMORPHISM

Definition 2.8.1 Homomorphism is a mapping $f: G \to H$ between two groups G and H such that

$$f(ab) = f(a)f(b) \text{ for all } a, b \in G$$

So we can say that there are two operations (for two different groups) in a homomorphism and these operations may be different from one another.

Definition 2.8.2 (Kernel) If there is a homomorphism from G to H $[f: G \to H]$, Kernel of f, K_f, is the set defined as $K_f = \{x \in G |\, f(x) = e_H\}$. The Kernel of f is denoted by $Ker(f)$.

Examples 2.8.3

1. Let $f: \mathbb{R} \to GL(2; \mathbb{R})$ be defined by $(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$. The map f is a homomorphism as $f(x+y) = \begin{pmatrix} 1 & 0 \\ x+y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = f(x)f(y)$.

   Note that Ker $f = \{0\}$

2. Let A and $B \in GL(n, \mathbb{R})$ and let $\varphi: GL(n, \mathbb{R}) \to \mathbb{R}^*$ is defined by $\varphi(A) = \det(A)$. Then φ is homomorphism;

   $$\varphi(AB) = \det(AB) = \det A \det B = \varphi(A)\varphi(B). \text{ Here } Ker(\varphi) = SL(n, \mathbb{R})$$

3. Let G be the group of positive real numbers, and let H be the group of all real numbers. Define $h: \mathbb{R}^+ \to \mathbb{R}$ by $h(x) = log_{10}x$. Then φ is a homomorphism since $h(xy) = log_{10}(xy) = log_{10}(x) + log_{10}(y) = h(x)h(y)$. Note again that $Ker(h) = \{1\}$.

Theorem 2.8.4 [7] Let θ be a homomorphism from G to H $(\theta: G \rightarrow H)$ and let $g \in G$.

1. θ carries the identity of G to the identity of H.
2. $\theta(g^n) = (\theta(g))^n$ for all $n \in \mathbb{Z}$
3. If $|g|$ is finite , then $|\theta(g)|$ divides $|g|$.
4. $Ker\theta$ is a subgroup of G.
5. $\theta(a) = \theta(b) \Leftrightarrow a\,Ker\theta = b\,Ker\theta$.
6. If $\theta(g) = g'$, then $\theta^{-1}(g') = \{x \in G \mid \theta(x) = g'\} = gKer\theta$.

Proofs

1. Let e_G be the identity element of G and e_H be the identity element of H.

   $\theta(e_G) = \theta(e_G e_G) = \theta(e_G)\theta(e_G)$, get $\theta(e_G)^{-1}$ of both sides

   $\theta(e_G)^{-1}.\theta(e_G) = \theta(e_G)\theta(e_G).\theta(e_G)^{-1}$, $e_H = \theta(e_G)$

2. We can prove it by induction; n=0 : $\theta(e) = e$. n=1 : $\theta(g) = \theta(g)$.

   Assume its true for all n and prove it for n+1:

   $\theta(g^{n+1}) = \theta(g^n g) = \theta(g^n)\theta(g) = \theta(g)^n\theta(g) = \theta(g)^{n+1}$.

   Since n is negative :That means –n is positive .

   $\theta(g^{-n}g^n) = \theta(g^{-n})\theta(g^n)$. $e_H = \theta(g)^{-n}\theta(g^n)$ so that $\theta(g)^n$= $\theta(g^n)$

3. If $|g|$ is finite, $g^n = e$, $\theta(g^n) = (\theta(g))^n = e_H$, so $|\theta(g)|$ divides $n = |g|$.

4. $e \in Ker\theta$ by property 1, so $Ker\theta$ is non-empty. If $g \in Ker\theta$, then $\theta(g) = e_H$. $\theta(g^{-1}) = \theta(g)^{-1} = e^{-1} = e,$ so $g^{-1} \in Ker\theta.$ If $a, b \in Ker\theta,$ $\theta(ab) = \theta(a)\theta(b) = e_H$ so $ab \in Ker\theta.$

5. ($\Longrightarrow$) If $\theta(a) = \theta(b)$. We show that $a^{-1}b \in Ker\theta.$

   $\theta(a^{-1}b) = \theta(a^{-1})\theta(b) = \theta(a)^{-1}\theta(b) = e_H$ $(since\ \theta(a) = \theta(b))$ so $a^{-1}b \in Ker\theta.$

   ($\Longleftarrow$) If $a^{-1}b \in Ker\theta.$ Show that $\theta(a) = \theta(b).\theta(a^{-1}b) = \theta(a^{-1})\theta(b) = \theta(a)^{-1}\theta(b) = e_H \quad \Longrightarrow \theta(a) = \theta(b).$

6. Let $x \in Ker\theta.$ Then, $\theta(gx) = \theta(g)\theta(x) = \theta(g)e_H = \theta(g) = g' \Longrightarrow gx \in \theta^{-1}(g')$. If $\theta(x) = g'$, then we show $x \in gKer\ \theta$. By using property 5 we can say that; $\theta(x) = \theta(g) \Longrightarrow xKer\theta = gKer\theta$

Theorem 2.8.5 [7] Let θ be a homomorphism from a group G to a group H $(\theta: G \to H)$, and let $\bar{G}$ be a subgroup of G such that;

1. $\theta(\bar{G}) = \{\theta(\bar{g}) \mid \bar{g} \in \bar{G}\}$ is a subgroup of H.
2. If $\bar{G}$ is cyclic , then $\theta(\bar{G})$ is cyclic.
3. If $\bar{G}$ is Abelian, $\theta(\bar{G})$ is Abelian.
4. If $\bar{G}$ is normal in G, then , $\theta(\bar{G})$ is normal in , $\theta(G)$.
5. If $|Ker\theta| = n$, then , θ is an $n - 1$ mapping from G onto $\theta(G)$.
6. If $|\bar{G}| = n$, then $|\theta(\bar{G})|$ divides n.
7. If K is a subgroup of H, then $\theta^{-1}(K)$ is a subgroup of G.

8. If K is a normal subgroup of H, then $\theta^{-1}(K)$ is a normal subgroup of G.

9. If θ is onto and $Ker\theta = \{e\}$, then θ is an isomorphism from G to H.

Corollary 2.8.6 Let θ be a group homomorphism from G to H. Then $Ker\theta$ is a normal subgroup of G.

Exercise 2.8.7 Determine all homomorphisms from $\mathbb{Z}_{18}$ to $\mathbb{Z}_{45}$.

Such a homomorphism is completely determined by the image of 1. As if 1 goes a, x goes ax.

By property (3) of theorem 1, $|a|$ divides 18, but by Lagrange`s theorem $|a|$ also divides 45.

So $|a| = 1,3,9$

*If $|a| = 1$ $\quad a = 0$.

*If $|a| = 3$, a $= 15$ and 30

*If $|a|$ =9 , a = 5,10,20,25,35 and 40

Corollary 2.8.8 A homomorphism θ of G onto H is an isomorphism if and only if it is onto and Ker $\theta = \{e\}$.

Chapter 3

Representation Theory

3.1 Basic Definitions of Representation Theory

Definition 3.1.1 Let $GL(V)$ be the group of ismorphisms of V onto itself, where V is a vector space over the field of complex numbers, $\mathbb{C}$.

By the definition, each element of $GL(V)$ is linear and has a linear inverse. Now let $T \in GL(V)$, if V has a finite basis $\{e_i\}$ of n elements, then each linear map T is defined by a square matrix a_{ij} of order n.

$$T\left(\underline{e}_j\right) = \sum_i a_{ij}\underline{e}_i$$

where the coefficients are complex numbers.

We know that T is an isomorphism, therefore we can say that A is invertible which implies that $T \neq 0$. Since det(T) =det(A), the set $GL(V)$ can be identified with the group of invertible square matrices of order n.

Definition 3.1.2 Let G be a finite group. Linear representation of G in V is a homomorphism φ from G into $GL(V)$. G is a group with identity element 1, which means that the operation of homomorphism is multiplication.

If $\varphi: G \to GL(V)$ is a homomorphism then , $\varphi(ab) = \varphi(a).\varphi(b)$, where $a, b \in G$ and $\varphi(a), \varphi(b) \in GL(V)$.

Note that V is said to be the representation space or sometimes just the representation of the group G.

Definition 3.1.3 Let φ and φ' be the two representations of the same group G in vector spaces V and V'. We can say that these 2 representations are similar (isomorphic) if we can find a linear isomorphism $\delta: V \to V'$ satisfying

$$\delta \circ \varphi(s) = \varphi'(s) \circ \delta \text{ , for all } s \in G.$$

Basic example 3.1.4:

The homomorphism $\varphi: G \to \mathbb{C}^*$ is a representation of the group G with degree 1. The matrix here is 1x1. Here $\mathbb{C}^*$ illustrates the multiplicative group of nonzero complex numbers.

Let $g \in G$ and $z = \varphi(g) \in \mathbb{C}^*$. G has finite order. Let us say $|G| = n$. By Lagrange's Theorem we know that $|g| \mid n$. Since the order of an element of $\mathbb{C}^*$ divides the order of an element of G, $|z|$ is finite. Hence the values of z are the roots of unity.

Definition 3.1.5 (The character of a Representation) Let $\varphi: G \to GL(V)$ be the linear representation of a finite group G in the vector space V. The character of the representation φ is equal to the trace of the image matrix. So,

$$\chi_\varphi: G \to \mathbb{C} \text{ , where } \chi_\varphi(s) = Tr(\varphi_s) \text{ for all } s \in G.$$

Proposition 3.1.6 Let the character of the representation φ of degree m be χ. Then the following properties hold;

1. $\chi(1) = m$,
2. $\chi(g^{-1}) = \overline{\chi(g)}$ for $g \in G$,
3. $\chi(hgh^{-1}) = \chi(g)$ for all $g, h \in G$.

Definition 3.1.7 (Invariant Subspace) Suppose that there is a linear map and let W be a subspace of V. We say that W is an *invariant* supsbace of V relative to φ if for every element $w \in W$, $\varphi(w) \in W$.

Definition 3.1.8 (Subrepresentation)[Ref:Graduate Text in Mathematics 42] Let $\varphi: G \to GL(V)$ be a linear representation and let W be a vector subspace of V. Suppose that W is invariant under the action of G, or in other words, suppose that $w \in W$ implies $\varphi_s(w) \in W$ for all $s \in G$. The restriction $\varphi^{\boldsymbol{W}}$ of $\varphi_{\boldsymbol{s}}$ to W is then an isomorphism of W onto itself. Thus, $\varphi^W : G \to GL(W)$ is a linear representation of G in W; and W is said to be a *subrepresentation* of G.

Definition 3.1.9 (Irreducibility and Indecomposable) Let $\rho: G \to GL(V)$ be a non-zero linear representation of G.It is said to be *irreducible* when it contains no proper invariant subspaces.

Representation is *indecomposable* if we cannot write it as a direct sum of any two nonzero subrepresentations.

Definition 3.1.10 (G-invariant inner product) Let G be a finite froup and V be a representation of G. For any $g \in G$ and $\underline{v}_1, \underline{v}_2 \in V$, G-invariant inner product can be defined as $\langle \underline{v}_1, \underline{v}_2 \rangle = \langle g\underline{v}_1, g\underline{v}_2 \rangle$

Definition 3.1.11 We can say that representation V of finite group G is the direct sum of W and $W^\perp$ where W and $W^\perp$ are both sub representations of V where $W \cap W^\perp = \emptyset$.

Definition 3.1.12 (Hermitian inner product)

$\langle \underline{v}_1, \underline{v}_2 \rangle = \frac{1}{|G|} \sum_g \langle g\underline{v}_1, g\underline{v}_2 \rangle$ where $\underline{v}_1, \underline{v}_2 \in V$.

Definition 3.1.13 (The space of class function) Let V be a representation of G and $h: G \to \mathbb{C}$ be a class function on G. Then for any 2 class functions h and h', $\langle h, h' \rangle = \frac{1}{|G|} \sum_g h(g)\overline{h'(g)}$ where $g \in G$.

3.2 Types of Representations [6]

All groups we consider in this section will be finite and all vector spaces will be finite dimensional over $\mathbb{C}$.

Definition 3.2.1 (Trivial representation): Let the trivial action of G be defined by $g\underline{x} = \underline{x}$ for every $x \in \mathbb{C}$ and for every $g \in G$. Since $\mathbb{C}$ is equipped with the trivial action of G we say that it is the trivial representation of the finite group G. We can point out that the every finite group has the trivial representation and since this is 1-dimensional, then $\mathbb{C}$ has no proper nontrivial subspaces; in other words it is irreducible.

Definition 3.2.2 (Permutation representation of): Let V be a vector space generated by the basis $\{\underline{e}_y \,|y \in Y\}$ where Y is a finite G-set and let

$$g\left(a_1\underline{e}_{y_1} + a_2\underline{e}_{y_2} + \cdots + a_n\underline{e}_{y_n}\right) = a_1\underline{e}_{gy_1} + a_2\underline{e}_{gy_2} + \cdots + a_n\underline{e}_{gy_n}$$

be the action of G on V. We name V as the *permutation representation* of G.

It's easily noticed that each member of G only changes the place of a basis element which is added to another and because of the commutative property of addition that doesn't change the sum. Therefore the subspace spanned by the basis of the vector space V is invariant under the action of G. As a result , every permutation representation has nontrivial subrepresentation and is therefore reducible.

- If we replace the G-set Y by G itself, we say that V is the *regular representation* of G.

Definition 3.2.3 (Permutation representation of S_n): Let $\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n\}$ be the standard basis for $\mathbb{C}^n$ for any n, and let

$$\sigma\left(a_1\underline{e}_1 + a_2\underline{e}_2 + \cdots + a_n\underline{e}_n\right) = a_1\underline{e}_{\sigma(1)} + a_2\underline{e}_{\sigma(2)} + \cdots + a_n\underline{e}_{\sigma(n)}$$

be the action of S_n on $\mathbb{C}^n$. This action is called the *permutation representation* of S_n. We already explained why the subspace is invariant under the action of G by our earlier definition of permutation representation on G. Because of the same reason we note that 1-dimensional subspace of $\mathbb{C}$ spanned by $\underline{e}_1 + \underline{e}_2 + \cdots + \underline{e}_n$ is invariant under the action of S_n. Therefore, $\langle \underline{e}_1 + \underline{e}_2 + \cdots + \underline{e}_n \rangle$ is a subrepresentation of $\mathbb{C}^n$.

At that point, we can say that its orthogonal complement $V = \{\langle x_1 + x_2 + \cdots + x_n\rangle | x_1 + x_2 + \cdots + x_n = 0\}$ is a subrepresentation as well since this orthogonal complement is also invariant. V is called the *standard representation* of S_n.

Remark 3.2.4 $\mathbb{C}^n = V \oplus \langle e_1 + e_2 + \cdots + e_n\rangle$, where $\langle e_1 + e_2 + \ldots + e_n\rangle$ under the permutation action of S_n is isomorphic to $\mathbb{C}$ under the trivial action of S_n, so it is the trivial representation, let say U. By using the corollary of character we can say that;

$$\chi_{\mathbb{C}^n} = \chi_{V\oplus\langle e_1+e_2+\cdots+e_n\rangle} \to \chi_{\mathbb{C}^n} = \chi_{\langle e_1+e_2+\cdots+e_n\rangle} + \chi_V$$

Definition 3.2.5 (Alternating representation): Let σ be a permutation in S_n and for every , we define the alternating representation by $\rho(\sigma) = sign(\sigma)I$. We can illustrate this representation by the action

$$\sigma.w = \begin{Bmatrix} \mathrm{w} & \text{if } \sigma \text{ is an even permutation} \\ -\mathrm{w} & \text{if } \sigma \text{ is an odd permutation} \end{Bmatrix}$$

For $n \geq 2$, any S_n has the alternating representation and since this representation is 1-dimensional it is irreducible.

Theorem 3.2.6 (Fixed Point Formula) Let V be the representation of a finite group G as described in definition 3.1.2 and X be a finite G-set. Then the number of left fixed elements (by the action of g) in X is $\chi_V(g)$ for every $g \in G$.

3.3 Important Theorems and Corollaries about Representation Theory [1]

Theorem 3.3.1 The number of conjugacy classes of G is same as the number of irreducible representations of G (up to isomorphism) .

Theorem 3.3.2 G is Abelian if and only if all irreducible representations have degree 1.

Theorem 3.3.3 The sum of the squares of the dimensions of distinct irreducible representations is same as the order of the given group G.

$$\sum_{i=1}^{k} (\dim(W_i))^2 = |G|$$

Example 3.3.4 (The cyclic group C_2) Since $C_2$is Abelian, from theorem 3.2.2 , all irreducible representations have degree 1. Sum of the squares of dimensions of distinct irreducible representations is 2, so C_2 has two irreducible representations of degree 1.

Let r be the generator of C_2 and $\rho: C_2 \to \mathbb{C}$. Then $\rho(r) = w$. Since $r^2 = 1$, $\rho(r^2) = w^2 = 1$. Hence, w is 1 or -1.

Lemma 3.3.5 Let W be a subrepresentation of V where V is a representation of a finite group G. The orthogonal complement $W^{\perp}$ of W, is also a subrepresentation of V under the G-invariant inner product.

Proof. For a fixed x in $W^{\perp}$, $W^{\perp} = \{x \in V \mid \langle x, w \rangle = 0 \,, \forall w \in W\}$ and because we have G $-$invariant inner product, $\langle x, w \rangle = \langle gx, w \rangle = 0$ for any $g \in G$ and $w \in W$. Therefore $gx \in W^{\perp}$ and this implies that $W^{\perp}$ is also invariant under the action of G.

Definition 3.3.6 let ρ_1 and ρ_2 be linear representations of G_1 and G_2 respectively then the tensor prduct of these 2 representation is defined by $\rho_1 \otimes \rho_2(s,t) = \rho_1(s) \otimes \rho_2(t)$.

Theorem 3.3.7

(a) Let τ_1 and τ_2 be irreducible representations of G_1 and G_2 respectively. Then $\tau_1 \otimes \tau_2$ be an irreducible representation of $G_1 \times G_2$.

(b) Each irreducible representation of $G_1 \times G_2$ is isomorphic to a representation $\rho_1 \otimes \rho_2$, where ρ_i is an irreducible representation of G_i.

Proposition 3.3.8 $\chi_{V \oplus W} = \chi_V + \chi_W$ and $\chi_{V \otimes W} = \chi_V \cdot \chi_W$ where V and W are the representations of group G.

Theorem 3.3.9 For any given representation V of G, we can break it up to subrepresentations and we can write V as $V = W_1 \oplus W_2 \oplus \ldots \ldots \oplus W_k$ where W_i doesn't break up into smaller pieces, which means W_i is irreducible for all i.

Theorem 3.3.10 Let G be a group. The set of character functions of the irreducible representations is orthonormal with respect to inner product.

Let χ and χ' be character sets for W and W' respectively , where W and W' are irreducible representations. Then , $\langle \chi, \chi' \rangle = \frac{1}{|G|} \sum \chi(g)\, \chi'(g) = 1$, where $g \in G$.

If χ is isomorphic to χ' their inner product will be equal to 1. If they are not isomorphic, their inner product will be equal to 0.

Corollary 3.3.11 (a) Let G be a group and V be a representation of G. Then V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$. $\langle \chi_V, \chi_W \rangle = \sum_{g \in G} \chi_V(g) \chi_W(g)$

(b) Let V be a representation of G with character χ_V, and let $\chi_{W_1}, \chi_{W_2}, \dots, \chi_{W_k}$ be irreducible characters of $W_1, W_2, \dots, W_k$ respectively where $W_1, W_2, \dots, W_k$ are irreducible representations for V.

- We can write χ_V as $\chi_V = c_1 \chi_{W_1} + c_2 \chi_{W_2} + \dots + c_k \chi_{W_k}$, where $c_1, c_2, \dots, c_k \in \mathbb{N}$. Hence, $\chi_V = \sum c_i \chi_{W_i}$, where $c_i = \langle \chi_V, \chi_{W_i} \rangle$
- $\langle \chi_V, \chi_V \rangle = \sum c_i{}^2$.
- $\sum_{i=1}^{k} \dim(W_i).\chi_{W_i}(g) = 0$ if $g \in G$ is not the identity.

Corollary 3.3.12 Number of irreducible representations of a finite group G is equal to the number of cycle types.

Chapter 4

EXAMPLES

Example 4.1 (The cyclic group C_4)

C_4 is an Abelian group, and we know that for all Abelian groups, all irreducible representations have degree 1. C_4 has 4 irreducible representations since the order is 4. (Since the sum of the squares of the dimensions of irreducible representations is equal to the order of the group : $1^2 + 1^2 + 1^2 + 1^2 = 4$).

Let the generator of C_4 be r, then r^4 need to be equal to 1. Then $\rho(r) = w$ which implies that $\rho(r^4) = w^4 = 1$. Hence, $w = e^{2.\pi.i.k/n}$ where $k = 0,1,2,3$. So w (irreducible representations) is $1, -1, \mathrm{i}$ or $-\mathrm{i}$.

Remark 4.2 (Representation of C_n) In general cyclic (Abelian) groups has n irreducible representations which are given by $w = e^{2.\pi.i.k/n}$, where $k = 0,1, \dots, n-1$.

Example 4.3 : Representation of S_3

From earlier theory the number of irreducible representations of a finite group G is equal to the number of conjugacy classes in G. Since there are 3 conjugacy classes there exists 3 irreducible representations. One of this is trivial representation U. It is 1×1 idendity matrix where the character value is equal to 1. Second irreducible representation is the alternate representation, U'. By the homomorphism $\varphi(\sigma) = sign(\sigma)$.

Then, character of (1) is 1,character of odd permutations [(12),(23),(13)] is -1 and the character of even permutations [(123),(132)] is 1. Irreducibility of U' can also be proved by

$$\langle \chi_{U'}, \chi_{U'} \rangle = \frac{1}{6} \sum 1.1 + -1.-1 + -1.-1 + -1.-1 + 1.1 + 1.1 = \frac{1}{6}(6) = 1$$

Standard representation is the remaining irreducible representation. By remark 3.2.4 $\mathbb{C}^3 = V \oplus \langle e_1 + e_2 + e_3 \rangle$. Note that $\langle e_1 + e_2 + e_3 \rangle$ under the permutation action of $\boldsymbol{S_3}$ is isomorphic to $\mathbb{C}$ under the trivial action . We know that $\chi_U + \chi_V = \chi_{\mathbb{C}^3}$. Since the character of $\mathbb{C}^3$ with respect to (1) , (12) and (123) are 3 , 1 and 0 [from the fixed point theorem], then the corresponding characters of V (standard representation) are 2 , 0 and -1.

$$\langle \chi_V, \chi_V \rangle = \frac{1}{6} \sum 2.2 + 0.0 + 0.0 + 0.0 + -1.-1 + -1.-1 = \frac{1}{6}(6) = 1$$

Then V is also irreducible . Permutation representation is not irreducible representation of $\boldsymbol{S_3}$.

Example 4.4 : Representation of D_3(application of S_3)

D_3 is the group of symmetries of an equilateral triangle.

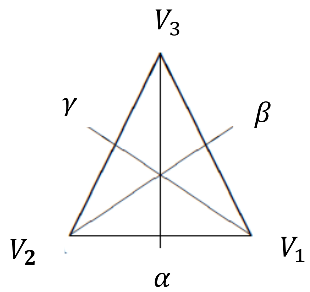

Figure 1: Sketch of Triangle with Reflection Lines

Let α, β and γ be the line of reflections (at the end of each reflection there is 1 fixed point) and r_1 and r_2 be the rotation of 120^o and 240^o (not any fix point) respectively and e be the identity.

Let $\mathrm{K}_o, \mathrm{K}_1$ and K_2 be the representations of vertices , edges and faces respectively.

(1) $\mathbf{K_o}$ (vertices)

$\chi_{K_o}(e) = 3$ $\qquad$ $\chi_{K_o}(\beta) = 1$ $\qquad$ $\chi_{K_o}(r_1) = 0$

$\chi_{K_o}(\alpha) = 1$ $\qquad$ $\chi_{K_o}(\gamma) = 1$ $\qquad$ $\chi_{K_o}(r_2) = 0$

Since $\langle \chi_{K_o}, \chi_{K_o} \rangle = \frac{1}{|G|}(3.3 + 1.1 + 1.1 + 1.1 + 0.0 + 0.0) = \frac{1}{6}(12) = 2$. K_o is the direct sum of 2 distinct irreducible representations. To decide which two representations, we need to take the product of χ_{K_o} with respect to each of the irreducible representations of S_3.

1- Trivial : $\langle \chi_{K_o}, \chi_U \rangle = \frac{1}{|G|}(3.1 + 1.1 + 1.1 + 1.1 + 0.1 + 0.1) = \frac{1}{6}(6) = 1$

2- Alternating : $\langle \chi_{K_o}, \chi_{U'} \rangle = \frac{1}{|G|}(3.1 + 1.-1 + 1.-1 + 1.-1 + 0.1 + 0.1) = 0$

3- Standard : $\langle \chi_{K_o}, \chi_V \rangle = \frac{1}{|G|}(3.2 + 1.0 + 1.0 + 1.0 + 0.-1 + 0.-1) = \frac{1}{6}(6) = 1$

As a result $K_o = U \oplus V$

(2) K_1 (edges)

$\chi_{K_1}(e) = 3$	$\chi_{K_1}(\beta) =$ -1	$\chi_{K_1}(r_1) = 0$
$\chi_{K_1}(\alpha) =$ -1	$\chi_{K_1}(\gamma) =$ -1	$\chi_{K_1}(r_2) = 0$

Since $\langle \chi_{K_1}, \chi_{K_1} \rangle = \frac{1}{|G|}(3.3 + -1.-1 + -1.-1 + -1.-1 + 0.0 + 0.0) = 2$, again there exists 2 distinct irreducible representations. To decide which two, we need to take the product of χ_{K_1} with respect to each of the irreducible representations of S_3.

1- Trivial: $\langle \chi_{K_1}, \chi_U \rangle = \frac{1}{|G|}(3.1 + -1.1 + -1.1 + -1.1 + 0.1 + 0.1) = \frac{1}{6}(0) = 0$

2- Alternating: $\langle \chi_{K_1}, \chi_{U'} \rangle = \frac{1}{|G|}(3.1 + -1.-1 + -1.-1 + -1.-1 + 0.1 + 0.1) = 1$

3- Standard: $\langle \chi_{K_1}, \chi_V \rangle = \frac{1}{|G|}(3.2 + -1.0 + -1.0 + -1.0 + 0.-1 + 0.-1) = 1$

Hence, $K_1 = U' \oplus V$

(3) K_2 (faces)

$\chi_{K_2}(e) = 1$	$\chi_{K_2}(\beta) =$ -1	$\chi_{K_2}(r_1) = 1$
$\chi_{K_2}(\alpha) =$ -1	$\chi_{K_2}(\gamma) =$ -1	$\chi_{K_2}(r_2) = 1$

Since $\langle \chi_{K_2}, \chi_{K_2} \rangle = \frac{1}{|G|}(1.1 + -1.-1 + -1.-1 + -1.-1 + 1.1 + 1.1) = \frac{1}{6}(6) = 1$, K_2 is a direct sum of only 1 irreducible representation.

1- Trivial : $\langle \chi_{K_1}, \chi_U \rangle = \frac{1}{|G|}(1.1 + -1.1 + -1.1 + -1.1 + 1.1 + 1.1) = \frac{1}{6}(0) = 0$

2- Alternating: $\langle \chi_{K_1}, \chi_{U'} \rangle = \frac{1}{|G|}(1.1 + -1.-1 + -1.-1 + -1.-1 + 1.1 + 1.1) = 1$

3- Standard : $\langle \chi_{K_1}, \chi_{V} \rangle = \frac{1}{|G|}(1.2 + -1.0 + -1.0 + -1.0 + 1.-1 + 1.-1) = 0$

So, $K_2 = U'$

Example 4.5 Representation of $C_2 \times C_2$:

We know that $C_2 \times C_2$ is Abelian so all the irreducible representations have degree 1 this implies that there exists 4 irreducible representations. We know that C_2 has 2 irreducible representations which are 1 and -1.

Let I, τ_1, τ_2 and τ_3 be 4 irreducible representations of $C_2 \times C_2$ where the elements of the latter are $(0,0), (1,0), (0,1)$ and $(1,1)$.

By using the rules of homomorphism, we can find the representations, as follows;

$(\mathbf{1})$ I: $(0,0) \rightarrow 1$

$(1,0) \rightarrow 1$ (Trivial representation)

$(0,1) \rightarrow 1$

$(1,1) \rightarrow 1$

$(\mathbf{2})$ τ_1 : $(0,0) \rightarrow 1$

$(1,0) \rightarrow 1$

$(0,1) \rightarrow -1$

$(1,1) \rightarrow -1$

$(\mathbf{3})$ $\tau_2 : (0,0) \rightarrow 1$

$(1,0) \rightarrow -1$

$(0,1) \rightarrow -1$

$(1,1) \rightarrow 1$

$(\mathbf{4})$ $\tau_3 : (0,0) \rightarrow 1$

$(1,0) \rightarrow -1$

$(0,1) \rightarrow 1$

$(1,1) \rightarrow -1$

Example 4.6 : Representation of $C_2 \times C_2$: (By using Tensor Product)

We know that $(\rho_1 \otimes \rho_2)(s,t) = \rho_1(s).\rho_2(t)$**.** Let ρ_1 and ρ_2 be the irreducible representations of C_2 and ρ_3 and ρ_4 be the irreducible representations of C_2 again , where

$C_2 \times C_2$

$\rho_1 : 0 \rightarrow 1$ $\quad\quad$ $\rho_3 : 0 \rightarrow 1$

$1 \rightarrow 1$ $\quad\quad$ $1 \rightarrow 1$

$\rho_2 : 0 \rightarrow 1$ $\quad\quad$ $\rho_4 : 0 \rightarrow 1$

$1 \rightarrow -1$ $\quad\quad$ $1 \rightarrow -1$

The elements of $C_2 \times C_2$ are $(0,0), (1,0), (0,1)$ and $(1,1)$ and Representation are,

(1) Trivial, *I*

$(\rho_1 \otimes \rho_3)(0,0) = \rho_1(0).\rho_3(0) = 1.1 = 1$

$(\rho_1 \otimes \rho_3)(0,1) = \rho_1(0).\rho_3(1) = 1.1 = 1$

$(\rho_1 \otimes \rho_3)(1,0) = \rho_1(1).\rho_3(0) = 1.1 = 1$

$(\rho_1 \otimes \rho_3)(1,1) = \rho_1(1).\rho_3(1) = 1.1 = 1$

(2) $\boldsymbol{\tau_1}$

$(\rho_2 \otimes \rho_3)(0,0) = \rho_2(0).\rho_3(0) = 1.1 = 1$

$(\rho_2 \otimes \rho_3)(0,1) = \rho_2(0).\rho_3(1) = 1.1 = 1$

$(\rho_2 \otimes \rho_3)(1,0) = \rho_2(1).\rho_3(0) = -1.1 = -1$

$(\rho_2 \otimes \rho_3)(1,1) = \rho_2(1).\rho_3(1) = -1.1 = -1$

(3) $\boldsymbol{\tau_2}$

$(\rho_2 \otimes \rho_4)(0,0) = \rho_2(0).\rho_4(0) = 1.1 = 1$

$(\rho_2 \otimes \rho_4)(0,1) = \rho_2(0).\rho_4(1) = 1.-1 = -1$

$(\rho_2 \otimes \rho_4)(1,0) = \rho_2(1).\rho_4(0) = -1.1 = -1$

$(\rho_2 \otimes \rho_4)(1,1) = \rho_2(1).\rho_4(1) = -1.-1 = 1$

(4) $\boldsymbol{\tau_3}$

$(\rho_1 \otimes \rho_4)(0,0) = \rho_1(0).\rho_4(0) = 1.1 = 1$

$(\rho_1 \otimes \rho_4)(0,1) = \rho_1(0).\rho_4(1) = 1.-1 = -1$

$(\rho_1 \otimes \rho_4)(1,0) = \rho_1(1).\rho_4(0) = 1.1 = 1$

$(\rho_1 \otimes \rho_4)(1,1) = \rho_1(1).\rho_4(1) = 1.-1 = -1$

Example 4.6 : $C_2 \times C_2$ acting on an octahedron as rotations through π about the three axes through opposite vertices.

The group $C_2 \times C_2$ has four elements: e, r_1, r_2, r_3, where r_1 is the axis joining the vertices 5 and 6, r_2 is joining the vertices 3 and 4, and r_3 is joining the vertices 1 and 2.

K$_o$: (vertices)

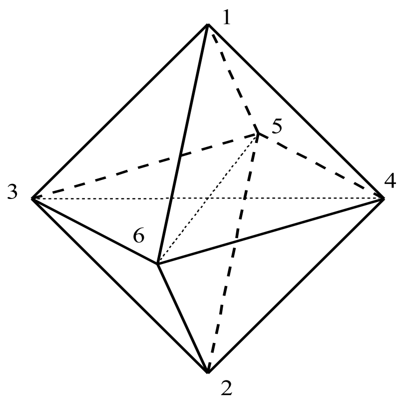

Figure 2: Octahedron

$\chi_{K_o}(I) = 6$

$\chi_{K_o}(r_1) = 2$

$\chi_{K_o}(r_2) = 2$

$\chi_{K_o}(r_3) = 2$

$\langle \chi_{K_o}, \chi_{K_o} \rangle = \frac{1}{|G|}(6.6 + 2.2 + 2.2 + 2.2) = \frac{1}{4}(48) = 12.$

To find which representations, we need to take the product of χ_{K_o} with respect to each of the irreducible representations of $C_2 \times C_2$.

1- Trivial : $\langle \chi_{K_o}, \chi_I \rangle = \frac{1}{|G|}(6.1 + 2.1 + 2.1 + 2.1) = \frac{1}{4}(12) = 3.$

2- τ_1 : $\langle \chi_{K_o}, \chi_{\tau_1} \rangle = \frac{1}{|G|}(6.1 + 2.1 + 2.-1 + 2.-1) = \frac{1}{4}(4) = 1.$

3- τ_2 : $\langle \chi_{K_o}, \chi_{\tau_2} \rangle = \frac{1}{|G|}(6.1 + 2.-1 + 2.-1 + 2.1) = \frac{1}{4}(4) = 1.$

4- τ_3 : $\langle \chi_{K_o}, \chi_{\tau_3} \rangle = \frac{1}{|G|}(6.1 + 2.-1 + 2.1 + 2.-1) = \frac{1}{4}(4) = 1.$

Since $3^2 + 1^2 + 1^2 + 1^2 = 12$, $K_o = 3I \oplus \tau_1 \oplus \tau_2 \oplus \tau_3$.

$\mathrm{K_1}$ (edges)

$\chi_{K_1}(I) = 12$

$\chi_{K_1}(r_1) = 0$

$\chi_{K_1}(r_2) = 0$

$\chi_{K_1}(r_3) = 0$

$\langle \chi_{K_1}, \chi_{K_1} \rangle = \frac{1}{|G|}(12.12 + 0.0 + 0.0 + 0.0) = \frac{1}{4}(144) = 36.$

To find which representations, we need to take the product of χ_{K_1} with respect to each of the irreducible representations of $C_2 \times C_2$.

1- Trivial : $\langle \chi_{K_1}, \chi_I \rangle = \frac{1}{|G|}(12.1 + 0.1 + 0.1 + 0.1) = \frac{1}{4}(12) = 3.$

2- τ_1 : $\langle \chi_{K_1}, \chi_{\tau_1} \rangle = \frac{1}{|G|}(12.1 + 0.1 + 0.-1 + 0.-1) = \frac{1}{4}(12) = 3.$

3- τ_2 : $\langle \chi_{K_1}, \chi_{\tau_2} \rangle = \frac{1}{|G|}(6.1 + 0.-1 + 0.-1 + 0.1) = \frac{1}{4}(12) = 3.$

4- τ_3 : $\langle \chi_{K_1}, \chi_{\tau_3} \rangle = \frac{1}{|G|}(12.1 + 0.-1 + 0.1 + 0.-1) = \frac{1}{4}(4) = 3.$

$(\langle \chi_{K_1}, \chi_I \rangle = 3^2 + 3^2 + 3^2 + 3^2 = 36)$

$\mathrm{K}_1 = 3I \oplus 3\tau_1 \oplus 3\tau_2 \oplus 3\tau_3$.

$\mathrm{K_2}$ (faces):

$\chi_{K_2}(I) = 8$

$\chi_{K_2}(r_1) = 0$

$\chi_{K_2}(r_2) = 0$

$\chi_{K_2}(r_3) = 0$

$\langle \chi_{K_1}, \chi_{K_1} \rangle = \frac{1}{|G|}(8.8 + 0.0 + 0.0 + 0.0) = \frac{1}{4}(64) = 16.$

Similarly, we need to take the product of χ_{K_2} with respect to each of the irreducible representations of $C_2 \times C_2$.

1- Trivial : $\langle \chi_{K_2}, \chi_I \rangle = \frac{1}{|G|}(8.1 + 0.1 + 0.1 + 0.1) = \frac{1}{4}(8) = 2.$

2- τ_1 : $\langle \chi_{K_2}, \chi_{\tau_1} \rangle = \frac{1}{|G|}(8.1 + 0.1 + 0.-1 + 0.-1) = \frac{1}{4}(8) = 2.$

3- τ_2 : $\langle \chi_{K_2}, \chi_{\tau_2} \rangle = \frac{1}{|G|}(8.1 + 0.-1 + 0.-1 + 0.1) = \frac{1}{4}(8) = 2.$

4- τ_3 : $\langle \chi_{K_2}, \chi_{\tau_3} \rangle = \frac{1}{|G|}(8.1 + 0.-1 + 0.1 + 0.-1) = \frac{1}{4}(8) = 2.$

$(\langle \chi_{K_2}, \chi_I \rangle = 2^2 + 2^2 + 2^2 + 2^2 = 16)$,

$\mathrm{K}_1 = 2I \oplus 2\tau_1 \oplus 2\tau_2 \oplus 2\tau_3$.

Example 4.7: Representation of $C_2 \times C_2 \times C_2$: (By using Tensor Product)

We know that $\boldsymbol{C_2} \times \boldsymbol{C_2} \times \boldsymbol{C_2}$ is Abelian, so by theory all the irreducible representations have degree 1. Let I_i and ρ_i be the corresponding irreducible representations of each $\boldsymbol{C_i}$.

$\boldsymbol{C_2} \times \boldsymbol{C_2} \times \boldsymbol{C_2}$

$\boldsymbol{I_1}$: $0 \to 1, 1 \to 1$ | $\boldsymbol{I_2}$: $0 \to 1, 1 \to 1$ | $\boldsymbol{I_3}$: $0 \to 1, 1 \to 1$

$\boldsymbol{\rho}_1$: $0 \to 1, 1 \to -1$ | $\boldsymbol{\rho}_2$: $0 \to 1, 1 \to -1$ | $\boldsymbol{\rho}_3$: $0 \to 1, 1 \to -1$

Since this group is Abelian, there exists 8 irreducible representations for this group, let σ_i for $1 \leq i \leq 8$ denotes these representations. There are 8 elements in the group which are $(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1)$ and $(1,1,1)$. Below, we list the character values for these representations;

(1) Trivial ($\boldsymbol{\sigma_1}$)

$(I_1 \otimes I_2 \otimes I_3)(0,0,0) = I_1(0) \otimes I_2(0) \otimes I_3(0) = 1.1.1 = 1$

$(I_1 \otimes I_2 \otimes I_3)(1,0,0) = I_1(1) \otimes I_2(0) \otimes I_3(0) = 1.1.1 = 1$

$(I_1 \otimes I_2 \otimes I_3)(0,1,0) = I_1(0) \otimes I_2(1) \otimes I_3(0) = 1.1.1 = 1$

$(I_1 \otimes I_2 \otimes I_3)(0,0,1) = I_1(0) \otimes I_2(0) \otimes I_3(1) = 1.1.1 = 1$

$(I_1 \otimes I_2 \otimes I_3)(1,1,0) = I_1(1) \otimes I_2(1) \otimes I_3(0) = 1.1.1 = 1$

$(I_1 \otimes I_2 \otimes I_3)(1,0,1) = I_1(1) \otimes I_2(0) \otimes I_3(1) = 1.1.1 = 1$

$(I_1 \otimes I_2 \otimes I_3)(0,1,1) = I_1(0) \otimes I_2(1) \otimes I_3(1) = 1.1.1 = 1$

$(I_1 \otimes I_2 \otimes I_3)(1,1,1) = I_1(1) \otimes I_2(1) \otimes I_3(1) = 1.1.1 = 1$

$(\mathbf{2})\ \boldsymbol{\sigma_2}$:

$(\rho_1 \otimes I_2 \otimes I_3)(0,0,0) = \rho_1(0) \otimes I_2(0) \otimes I_3(0) = 1.1.1 = 1$

$(\rho_{1_1} \otimes I_2 \otimes I_3)(1,0,0) = \rho_1(1) \otimes I_2(0) \otimes I_3(0) = -1.1.1 = -1$

$(\rho_1 \otimes I_2 \otimes I_3)(0,1,0) = \rho_1(0) \otimes I_2(1) \otimes I_3(0) = 1.1.1 = 1$

$(\rho_1 \otimes I_2 \otimes I_3)(0,0,1) = \rho_1(0) \otimes I_2(0) \otimes I_3(1) = 1.1.1 = 1$

$(\rho_1 \otimes I_2 \otimes I_3)(1,1,0) = \rho_1(1) \otimes I_2(1) \otimes I_3(0) = -1.1.1 = -1$

$(\rho_1 \otimes I_2 \otimes I_3)(1,0,1) = \rho_1(1) \otimes I_2(0) \otimes I_3(1) = -1.1.1 = -1$

$(\rho_1 \otimes I_2 \otimes I_3)(0,1,1) = \rho_1(0) \otimes I_2(1) \otimes I_3(1) = 1.1.1 = 1$

$(\rho_1 \otimes I_2 \otimes I_3)(1,1,1) = \rho_1(1) \otimes I_2(1) \otimes I_3(1) = -1.1.1 = -1$

(3) $\boldsymbol{\sigma_3}$:

$(I_1 \otimes \rho_2 \otimes I_3)(0,0,0) = I_1(0) \otimes \rho_2(0) \otimes I_3(0) = 1.1.1 = 1$

$(I_1 \otimes \rho_2 \otimes I_3)(1,0,0) = I_1(1) \otimes \rho_2(0) \otimes I_3(0) = 1.1.1 = 1$

$(I_1 \otimes \rho_2 \otimes I_3)(0,1,0) = I_1(0) \otimes \rho_2(1) \otimes I_3(0) = 1.-1.1 = -1$

$(I_1 \otimes \rho_2 \otimes I_3)(0,0,1) = I_1(0) \otimes \rho_2(0) \otimes I_3(1) = 1.1.1 = 1$

$(I_1 \otimes \rho_2 \otimes I_3)(1,1,0) = I_1(1) \otimes \rho_2(1) \otimes I_3(0) = 1.-1.1 = -1$

$(I_1 \otimes \rho_2 \otimes I_3)(1,0,1) = I_1(1) \otimes \rho_2(0) \otimes I_3(1) = 1.1.1 = 1$

$(I_1 \otimes \rho_2 \otimes I_3)(0,1,1) = I_1(0) \otimes \rho_2(1) \otimes I_3(1) = 1.-1.1 = -1$

$(I_1 \otimes \rho_2 \otimes I_3)(1,1,1) = I_1(1) \otimes \rho_2(1) \otimes I_3(1) = 1.-1.1 = -1$

(4) $\boldsymbol{\sigma_4}$:

$(I_1 \otimes I_2 \otimes \rho_3)(0,0,0) = I_1(0) \otimes I_2(0) \otimes \rho_3(0) = 1.1.1 = 1$

$(I_1 \otimes I_2 \otimes \rho_3)(1,0,0) = I_1(1) \otimes I_2(0) \otimes \rho_3(0) = 1.1.1 = 1$

$(I_1 \otimes I_2 \otimes \rho_3)(0,1,0) = I_1(0) \otimes I_2(1) \otimes \rho_3(0) = 1.1.1 = 1$

$(I_1 \otimes I_2 \otimes \rho_3)(0,0,1) = I_1(0) \otimes I_2(0) \otimes \rho_3(1) = 1.1.-1 = -1$

$(I_1 \otimes I_2 \otimes \rho_3)(1,1,0) = I_1(1) \otimes I_2(1) \otimes \rho_3(0) = 1.1.1 = 1$

$(I_1 \otimes I_2 \otimes \rho_3)(1,0,1) = I_1(1) \otimes I_2(0) \otimes \rho_3(1) = 1.1.-1 = -1$

$(I_1 \otimes I_2 \otimes \rho_3)(0,1,1) = I_1(0) \otimes I_2(1) \otimes \rho_3(1) = 1.1.-1 = -1$

$(I_1 \otimes I_2 \otimes \rho_3)(1,1,1) = I_1(1) \otimes I_2(1) \otimes \rho_3(1) = 1.1.-1 = -1$

(5) $\boldsymbol{\sigma_5}$:

$(\rho_1 \otimes \rho_2 \otimes I_3)(0,0,0) = \rho_1(0) \otimes \rho_2(0) \otimes I_3(0) = 1.1.1 = 1$

$(\rho_1 \otimes \rho_2 \otimes I_3)(1,0,0) = \rho_1(1) \otimes \rho_2(0) \otimes I_3(0) = -1.1.1 = -1$

$(\rho_1 \otimes \rho_2 \otimes I_3)(0,1,0) = \rho_1(0) \otimes \rho_2(1) \otimes I_3(0) = 1.-1.1 = -1$

$(\rho_1 \otimes \rho_2 \otimes I_3)(0,0,1) = \rho_1(0) \otimes \rho_2(0) \otimes I_3(1) = 1.1.1 = 1$

$(\rho_1 \otimes \rho_2 \otimes I_3)(1,1,0) = \rho_1(1) \otimes \rho_2(1) \otimes I_3(0) = -1.-1.1 = 1$

$(\rho_1 \otimes \rho_2 \otimes I_3)(1,0,1) = \rho_1(1) \otimes \rho_2(0) \otimes I_3(1) = -1.1.1 = -1$

$(\rho_1 \otimes \rho_2 \otimes I_3)(0,1,1) = \rho_1(0) \otimes \rho_2(1) \otimes I_3(1) = 1.-1.1 = -1$

$(\rho_1 \otimes \rho_2 \otimes I_3)(1,1,1) = \rho_1(1) \otimes \rho_2(1) \otimes I_3(1) = -1.-1.1 = 1$

(6) $\boldsymbol{\sigma_6}$:

$(\rho_1 \otimes I_2 \otimes \rho_3)(0,0,0) = \rho_1(0) \otimes I_2(0) \otimes \rho_3(0) = 1.1.1 = 1$

$(\rho_{1_1} \otimes I_2 \otimes \rho_3)(1,0,0) = \rho_1(1) \otimes I_2(0) \otimes \rho_3(0) = -1.1.1 = -1$

$(\rho_1 \otimes I_2 \otimes \rho_3)(0,1,0) = \rho_1(0) \otimes I_2(1) \otimes \rho_3(0) = 1.1.1 = 1$

$(\rho_1 \otimes I_2 \otimes \rho_3)(0,0,1) = \rho_1(0) \otimes I_2(0) \otimes \rho_3(1) = 1.1.-1 = -1$

$(\rho_1 \otimes I_2 \otimes \rho_3)(1,1,0) = \rho_1(1) \otimes I_2(1) \otimes \rho_3(0) = -1.1.1 = -1$

$(\rho_1 \otimes I_2 \otimes \rho_3)(1,0,1) = \rho_1(1) \otimes I_2(0) \otimes \rho_3(1) = -1.1.-1 = 1$

$(\rho_1 \otimes I_2 \otimes \rho_3)(0,1,1) = \rho_1(0) \otimes I_2(1) \otimes \rho_3(1) = 1.1.-1 = -1$

$(\rho_1 \otimes I_2 \otimes \rho_3)(1,1,1) = \rho_1(1) \otimes I_2(1) \otimes \rho_3(1) = -1.1.-1 = 1$

(7) $\boldsymbol{\sigma_7}$:

$(I_1 \otimes \rho_2 \otimes \rho_3)(0,0,0) = I_1(0) \otimes \rho_2(0) \otimes \rho_3(0) = 1.1.1 = 1$

$(I_1 \otimes \rho_2 \otimes \rho_3)(1,0,0) = I_1(1) \otimes \rho_2(0) \otimes \rho_3(0) = 1.1.1 = 1$

$(I_1 \otimes \rho_2 \otimes \rho_3)(0,1,0) = I_1(0) \otimes \rho_2(1) \otimes \rho_3(0) = 1.-1.1 = -1$

$(I_1 \otimes \rho_2 \otimes \rho_3)(0,0,1) = I_1(0) \otimes \rho_2(0) \otimes \rho_3(1) = 1.1.-1 = -1$

$(I_1 \otimes \rho_2 \otimes \rho_3)(1,1,0) = I_1(1) \otimes \rho_2(1) \otimes \rho_3(0) = 1.-1.1 = -1$

$(I_1 \otimes \rho_2 \otimes \rho_3)(1,0,1) = I_1(1) \otimes \rho_2(0) \otimes \rho_3(1) = 1.1.-1 = -1$

$(I_1 \otimes \rho_2 \otimes \rho_3)(0,1,1) = I_1(0) \otimes \rho_2(1) \otimes \rho_3(1) = 1.-1.-1 = 1$

$(I_1 \otimes \rho_2 \otimes \rho_3)(1,1,1) = I_1(1) \otimes \rho_2(1) \otimes \rho_3(1) = 1.-1.-1 = 1$

(8) $\boldsymbol{\sigma_8}$:

$(\rho_1 \otimes \rho_2 \otimes \rho_3)(0,0,0) = \rho_1(0) \otimes \rho_2(0) \otimes \rho_3(0) = 1.1.1 = 1$

$\rho_1 \otimes \rho_2 \otimes \rho_3)(1,0,0) = \rho_1(1) \otimes \rho_2(0) \otimes \rho_3(0) = -1.1.1 = -1$

$(\rho_1 \otimes \rho_2 \otimes \rho_3)(0,1,0) = \rho_1(0) \otimes \rho_2(1) \otimes \rho_3(0) = 1.-1.1 = -1$

$(\rho_1 \otimes \rho_2 \otimes \rho_3)(0,0,1) = \rho_1(0) \otimes \rho_2(0) \otimes \rho_3(1) = 1.1.-1 = -1$

$(\rho_1 \otimes \rho_2 \otimes \rho_3)(1,1,0) = \rho_1(1) \otimes \rho_2(1) \otimes \rho_3(0) = -1.-1.1 = 1$

$(\rho_1 \otimes \rho_2 \otimes \rho_3)(1,0,1) = \rho_1(1) \otimes \rho_2(0) \otimes \rho_3(1) = -1.1.-1 = 1$

$(\rho_1 \otimes \rho_2 \otimes \rho_3)(0,1,1) = \rho_1(0) \otimes \rho_2(1) \otimes \rho_3(1) = 1.-1.-1 = 1$

$(\rho_1 \otimes \rho_2 \otimes \rho_3)(1,1,1) = \rho_1(1) \otimes \rho_2(1) \otimes \rho_3(1) = -1.-1.-1 = -1$

Example 4.7 : $C_2 \times C_2 \times C_2$ acting on an octahedron as reflections in the 3 coordinate planes.

The group $C_2 \times C_2 \times C_2$ has eight elements: $I,\ \alpha, \beta, \gamma, \alpha\beta, \alpha\gamma, \beta\gamma$ and $\alpha\beta\gamma$ where α is the reflection in the xy plane β in the yz plane and γ in the xz plane.

By the fixed point formula

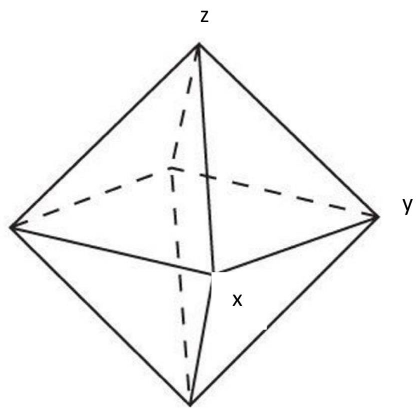

Figure 3: Octahedron

K_o : (vertices)

$\chi_{K_o}(I) = 6$

$\chi_{K_o}(\alpha) = 4$

$\chi_{K_o}(\beta) = 4$

$\chi_{K_o}(\gamma) = 4$

$\chi_{K_o}(\alpha\beta) = 2$

$\chi_{K_o}(\alpha\gamma) = 2$

$\chi_{K_o}(\beta\gamma) = 2$

$\chi_{K_o}(\alpha\beta\gamma) = 0$

$\langle \chi_{K_o}, \chi_{K_o} \rangle = \frac{1}{|G|}(6.6 + 4.4 + 4.4 + 4.4 + 2.2 + 2.2 + 2.2) = \frac{1}{8}(96) = 12 \neq 1,$

Showing that this representation is not irreducible. To find which representations it reduces to, we need to take the inner product of χ_{K_o} with respect to each of the irreducible representations of $C_2 \times C_2 \times C_2$ computed earlier.

While doing these calculations note that α, β and γ correspond to the elements $(1,0,0), (0,1,0)$ and $(0,0,1)$ of $C_2 \times C_2 \times C_2$ respectively.

1- Trivial σ_1: $\langle \chi_{K_o}, \chi_{\sigma_1} \rangle = \frac{1}{|G|}(6.1 + 4.1 + 4.1 + 4.1 + 2.1 + 2.1 + 2.1 + 0.1) = \frac{1}{8}(24) = 3.$

2- σ_2: $\langle \chi_{K_o}, \chi_{\sigma_2} \rangle = \frac{1}{|G|}(6.1 + 4.-1 + 4.1 + 4.1 + 2.-1 + 2.-1 + 2.1 + 0) = 1.$

3- σ_3: $\langle \chi_{K_o}, \chi_{\sigma_3} \rangle = \frac{1}{|G|}(6.1 + 4.1 + 4.-1 + 4.1 + 2.-1 + 2.1 + 2.-1 + 0) = 1.$

4- σ_4: $\langle \chi_{K_o}, \chi_{\sigma_4} \rangle = \frac{1}{|G|}(6.1 + 4.1 + 4.1 + 4.-1 + 2.1 + 2.-1 + 2.-1 + 0) = 1.$

5- σ_5: $\langle \chi_{K_o}, \chi_{\sigma_5} \rangle = \frac{1}{|G|}(6.1 + 4.-1 + 4.-1 + 4.1 + 2.1 + 2.-1 + 2.-1 + 0) = 0.$

6- σ_6: $\langle \chi_{K_o}, \chi_{\sigma_6} \rangle = \frac{1}{|G|}(6.1 + 4.-1 + 4.1 + 4.-1 + 2.-1 + 2.1 + 2.-1 + 0) = 0.$

7- σ_7: $\langle \chi_{K_o}, \chi_{\sigma_7} \rangle = \frac{1}{|G|}(6.1 + 4.1 + 4.-1 + 4.-1 + 2.-1 + 2.-1 + 2.1 + 0) = 0.$

8- σ_8: $\langle \chi_{K_o}, \chi_{\sigma_8} \rangle = \frac{1}{|G|}(6.1 + 4.-1 + 4.-1 + 4.-1 + 2.1 + 2.1 + 2.1 + 0) = 0.$

Since $3^2 + 1^2 + 1^2 + 1^2 = 12$, then $K_o = 3\,\sigma_1 \oplus \sigma_2 \oplus \sigma_3 \oplus \sigma_4$

K_1 : (edges)

$\chi_{K_1}(I) = 12$

$\chi_{K_1}(\alpha) = 4$

$\chi_{K_1}(\beta) = 4$

$\chi_{K_1}(\gamma) = 4$

$\chi_{K_1}(\alpha\beta) = 0$

$\chi_{K_1}(\alpha\gamma) = 0$

$\chi_{K_1}(\beta\gamma) = 0$

$\chi_{K_1}(\alpha\beta\gamma) = 0$

$\langle \chi_{K_1}, \chi_{K_1} \rangle = \frac{1}{|G|}(6.6 + 4.4 + 4.4 + 4.4 + 2.2 + 2.2 + 2.2) = \frac{1}{8}(192) = 24 \neq 1,$

Again showing that this representation is irreducible. To find the irreducibles it decomposes to, we need to take the inner product of χ_{K_1} with respect to each of the irreducible representations of $C_2 \times C_2 \times C_2$.

1.Trivial σ_1: $\langle \chi_{K_1}, \chi_{\sigma_1} \rangle = \frac{1}{|G|}(12.1 + 4.1 + 4.1 + 4.1 + 0.1 + 0.1 + 0.1 + 0.1) = \frac{1}{8}(24) =$ 3.

2. σ_2 : $\langle \chi_{K_1}, \chi_{\sigma_2} \rangle = \frac{1}{|G|}(12.1 + 4.-1 + 4.1 + 4.1 + 0.-1 + 0.-1 + 0.1 + 0.-1) = 2.$

3- σ_3 : $\langle \chi_{K_1}, \chi_{\sigma_3} \rangle = \frac{1}{|G|}(12.1 + 4.1 + 4.-1 + 4.1 + 0.-1 + 0.1 + 0.-1 + 0.-1) = 2.$

4- σ_4 : $\langle \chi_{K_1}, \chi_{\sigma_4} \rangle = \frac{1}{|G|}(12.1 + 4.1 + 4.1 + 4.-1 + 0.1 + 0.-1 + 0.-1 + 0.-1) = 2.$

5- σ_5: $\langle \chi_{K_1}, \chi_{\sigma_5} \rangle = \frac{1}{|G|}(12.1 + 4.-1 + 4.-1 + 4.1 + 0.1 + 0.-1 + 0.-1 + 0.1) = 1.$

6- σ_6: $\langle \chi_{K_1}, \chi_{\sigma_6} \rangle = \frac{1}{|G|}(12.1 + 4.-1 + 4.1 + 4.-1 + 0.-1 + 0.1 + 0.-1 + 0.1) = 1.$

7- σ_7: $\langle \chi_{K_1}, \chi_{\sigma_7} \rangle = \frac{1}{|G|}(12.1 + 4.1 + 4.-1 + 4.-1 + 0.-1 + 0.-1 + 0.1 + 0.1) = 1.$

8- σ_8: $\langle \chi_{K_1}, \chi_{\sigma_8} \rangle = \frac{1}{|G|}(12.1 + 4.-1 + 4.-1 + 4.-1 + 0.1 + 0.1 + 0.1 + 0.-1) = 0.$

Since $3^2 + 2^2 + 2^2 + 2^2 + 1^2 + 1^2 + 1^2 = 24$, then

$\mathrm{K}_1 = 3\,\sigma_1 \oplus 2\,\sigma_2 \oplus 2\,\sigma_3 \oplus 2\,\sigma_4 \oplus \sigma_5 \oplus \sigma_6 \oplus \sigma_7$

Finally for K_2 : (Faces)

$\chi_{K_2}(I) = 8$

$\chi_{K_2}(\alpha) = 0$

$\chi_{K_2}(\beta) = 0$

$\chi_{K_2}(\gamma) = 0$

$\chi_{K_2}(\alpha\beta) = 0$

$\chi_{K_2}(\alpha\gamma) = 0$

$\chi_{K_2}(\beta\gamma) = 0$

$\chi_{K_2}(\alpha\beta\gamma) = 0$

Similar calculations show that

$K_2 = \sigma_1 \oplus \sigma_2 \oplus \sigma_3 \oplus \sigma_4 \oplus \sigma_5 \oplus \sigma_6 \oplus \sigma_7 \oplus \sigma_8.$

Example 4.10: Representation of $\boldsymbol{C_2 \times C_4}$: (By using Tensor Product)

This group is again Abelian, so the procedure follows exactly same as other Abelian groups. The representations of $\boldsymbol{C_2}$ with elements 0 and 1, are given by I_1 and ϑ_1 and the representations of $\boldsymbol{C_4}$ with elements 0,1,2 and 3 are given by $I_2, \vartheta_2, \vartheta_3$ and ϑ_4 where ;

$I_1: 0 \to 1,\ \ 1 \to 1$

$\vartheta_1: 0 \to 1,\ \ 1 \to -1$

$I_2: 0 \to 1,\ \ 1 \to 1,\ \ 2 \to 1,\ \ 3 \to 1$

$\vartheta_2: 0 \to 1,\ \ 1 \to i,\ \ 2 \to -1,\ \ 3 \to -i$

$\vartheta_3: 0 \to 1,\ \ 1 \to -i,\ \ 2 \to -1,\ \ 3 \to i$

$\vartheta_4: 0 \to 1,\ \ 1 \to -1,\ \ 2 \to 1,\ \ 3 \to -1$

There exist 8 irreducible representations, which are $I_1 \otimes I_2$, $I_1 \otimes \vartheta_2$, $I_1 \otimes \vartheta_3$, $I_1 \otimes \vartheta_4$, $\vartheta_1 \otimes I_2$, $\vartheta_1 \otimes \vartheta_2$, $\vartheta_1 \otimes \vartheta_3$ and $\vartheta_1 \otimes \vartheta_4$.

As an example, we list below the character values for $\vartheta_1 \otimes \vartheta_2$;

$(\vartheta_1 \otimes \vartheta_2)(0,0) = \vartheta_1(0) \otimes \vartheta_2(0) = 1.1 = 1$

$(\vartheta_1 \otimes \vartheta_2)(0,1) = \vartheta_1(0) \otimes \vartheta_2(1) = 1.i = i$

$(\vartheta_1 \otimes \vartheta_2)(0,2) = \vartheta_1(0) \otimes \vartheta_2(2) = 1.-1 = -1$

$(\vartheta_1 \otimes \vartheta_2)(0,3) = \vartheta_1(0) \otimes \vartheta_2(3) = 1.-i = -i$

$(\vartheta_1 \otimes \vartheta_2)(1,0) = \vartheta_1(1) \otimes \vartheta_2(0) = -1.1 = -1$

$(\vartheta_1 \otimes \vartheta_2)(1,1) = \vartheta_1(1) \otimes \vartheta_2(1) = -1.i = i$

$(\vartheta_1 \otimes \vartheta_2)(1,2) = \vartheta_1(1) \otimes \vartheta_2(2) = -1.-1 = 1$

$(\vartheta_1 \otimes \vartheta_2)(1,3) = \vartheta_1(1) \otimes \vartheta_2(3) = -1.-i = i$

Example 4.11: Representation of $C_2 \times S_3$

We did both of the representations separately before, so we know that $\boldsymbol{C_2}$ has 2 representations, the trivial I, which takes both of the elements to 1, and ρ, taking the generator to -1. The group $\boldsymbol{S_3}$ has 3 representations, first one is the trivial , U, second one is the standard, V, and the last one is the alternating representation , U'. The figure below shows the character values of these representations.

$\boldsymbol{C_2 \times S_3}$

$I : 0 \to 1$, $1 \to 1$

$\rho : 0 \to 1$, $1 \to -1$

U : everything goes to 1

$V : (1) \to 2$, $(12) \to 0$, $(123) \to -1$

$U' : (1) \to 1$, $(12) \to -1$, $(123) \to 1$

Hence irreducible representations are given by $I\otimes U, I\otimes U^{'}$, $\rho\otimes U$ and $\rho\otimes U'$ of degree 1 and $I\otimes V$ and $\rho\otimes V$ of degree 2. Note that $4.1^2 + 2.2^2 = 12 = |C_2 \times S_3|$.

As an example we list below the character values of one of the degree 2`s.

$\chi_{(\rho\otimes V)}\big(0,(1)\big) = \chi_\rho(0).\chi_V(1) = 1.2 = 2$

$\chi_{(\rho\otimes V)}\big(0,(12)\big) = \chi_\rho(0).\chi_V(12) = 1.0 = 0$

$\chi_{(\rho\otimes V)}\big(0,(123)\big) = \chi_\rho(0).\chi_V(123) = 1.-1 = -1$

$\chi_{(\rho\otimes V)}\big(1,(1)\big) = \chi_\rho(1).\chi_V(1) = -1.2 = -2$

$\chi_{(\rho\otimes V)}\big(1,(12)\big) = \chi_\rho(1).\chi_V(12) = -1.0 = 0$

$\chi_{(\rho\otimes V)}(1,(123)) = \chi_\rho(1).\chi_V(123) = -1.-1 = 1$

Example 4.12: Representations of $\boldsymbol{S_3} \times \boldsymbol{C_n}$:

Irreducible representations of S_3 and C_n are given before. By using tensor products we can compute the irreducible representations of $S_3 \times C_n$.

Irreducible representations of S_3 are U , V and U', which are the trivial, standard and the alternating ones, and the irreducibles of C_n are μ_k with $\mu_k\,(1) = e^{2i\pi k/n}$ where $0 \leq k \leq n - 1$.

There are $3n$ many tensor product representations for $\boldsymbol{S_3} \times \boldsymbol{C_n}$, with $2n$ 1 dimensional and n 2 dimensional ones. Finally note that

$2n.1^2 + n.2^2 = 6n = |\boldsymbol{S_3} \times \boldsymbol{C_n}|$.

As an example we list below the character values for the representation $V \otimes \mu_1$.

$$\chi_{(V\otimes\mu_1)}\big((1),0\big) = \chi_V\big((1)\big).\chi_{\mu_1}(0) = 2.1 = 2$$

$$\chi_{(V\otimes\mu_1)}\big((1),1\big) = \chi_V\big((1)\big).\chi_{\mu_1}(1) = 2.\,e^{.2i\pi.1/n} = 2\,e^{.2i\pi/n}$$

$$\chi_{(V\otimes\mu_1)}\big((1),2\big) = \chi_V\big((1)\big).\chi_{\mu_1}(2) = 2.\,e^{2.2i\pi.1/n} = 2\,e^{4i\pi./n}$$

$\vdots$

$$\chi_{(V\otimes\mu_1)}\big((1),n\big) = \chi_V\big((1)\big).\chi_{\mu_1}(n) = 2.\,e^{n.2i\pi 1/n} = 2.\,e^{2ni\pi/n}$$

$$\chi_{(V\otimes\mu_1)}\big((12),0\big) = \chi_V\big((12)\big).\chi_{\mu_1}(0) = 0.1 = 0$$

$$\chi_{(V\otimes\mu_1)}\big((12),1\big) = \chi_V\big((12)\big).\chi_{\mu_1}(1) = 0.\,e^{.2i\pi.1/n} = 0$$

$\vdots$

$$\chi_{(V\otimes\mu_1)}\big((12),n\big) = \chi_V\big((12)\big).\chi_{\mu_1}(n) = 0.\,e^{n.2i\pi 1/n} = 0$$

$$\chi_{(V\otimes\mu_1)}\big((123),0\big) = \chi_V\big((123)\big).\chi_{\mu_1}(0) = -1.1 = -1$$

$$\chi_{(V\otimes\mu_1)}\big((123),1\big) = \chi_V\big((123)\big).\chi_{\mu_1}(1) = -1.\,e^{.2i\pi.1/n} = -1\,e^{.2i\pi/n}$$

$\vdots$

$$\chi_{(V\otimes\mu_1)}\big((123),n\big) = \chi_V\big((123)\big).\chi_{\mu_1}(n) = -1.\,e^{n.2i\pi 1/n} = -1\,e^{2ni\pi/n}$$

Example 4.13: Representation of S_4.

S_4 is a symmetric group on four elements and it has 5 cycle types which are (1),(12),(123) ,(1234) and(12)(34), where there exists 1,6,8,6 and 3 different cycles in the same cycle type respectively. We know that the number of cycle types gives the number of irreducible representations, so there are 5 irreducible representations for S_4.

First of the irreducible representations is the trivial one, U. The second one is the alternating representation, U'. We can easily show that they are irreducible by the corollary 3.3.11 since $< \chi_U, \chi_U > = 1$ and $< \chi_{U'}, \chi_{U'} >= 1$.

Now we will check the standard representation. We know by earlier remarks that $C_4 = V \oplus < e_1 + e_2 + e_3 + e_4 >$ under the action of S_4. By the fixed point formula, we can find the character of each element under standard representation of S_4.

Table 1: Character Values of each element under standard representation S_4

	(1)	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1

By the fixed point formula , at (1), there are 4 fixed elements , since the character of U is 1, character of V should be 3 to make sum equal to 4. Just like that for (12) there are 2 fixed elements, for (123) there are 1 and for (1234) and (12)(34) there are 0 fixed elements. So by considering characters of conjugacy classes at U, we will find the characters of them with respect to the standard representation.

Until now we find first 3 irreducible representations of S_4, that means we need 2 more. By the theorem 3.3.7, taking tensor product of 2 irreducible representations produces another irreducible representation, and we also know that $\chi_{\rho\otimes\tau} = \chi_\rho.\chi_\tau$. so let us say that the 4th irreducible representation will be W. Now we need to try which 2 will give an irreducible representation, since trivial representation doesn't change anything we will not consider this, so we need to consider either U' and V , U' and U' or V and V together. By trying these pairs we can easily find that the inner product of U' and V gives an irreducible representation since $<\chi_w, \chi_w >= \frac{1}{24}(3.3 + (-1)(-1) + (-1)(-1) + (-1)(-1) + (-1)(-1) + (-1)(-1) + (-1)(-1) + (-1)(-1) + 8.0.0 + (-1)(-1) + (-1)(-1) + (-1)(-1) + (-1)(-1) + (-1)(-1) + (-1)(-1) + (-1)(-1) + (-1)(-1) + (-1)(-1)) = \frac{1}{24}.24 = 1$

Table 2: Character Values of each element under standard representation S_4

	(1)	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1
W	3	-1	0	1	-1

At last step we need to find the last irreducible representation. For finding that one we need to use earlier corollaries. Theorem 3.3.3 says that $\sum(\dim(x_i))^2 = |G| = 24$ where

x_i are distinct irreducible representations, so $\sum(\dim(x_i))^2 = 1^2 + 1^2 + 3^2 + 3^2 + x^2 = 24$ which implies that $x = 2$. So that character of the idendity element with respect to this representation need to be eqaul to 2 since dimension gives the character of the idendity element. After finding character of the idendity with respect to the last irreducible representation, say W', its so easy to find the characters again by using earlier corollaries.

By corollary 3.3.11 $\sum \dim(x_i).\chi_{x_i}(g)$ is equal to zero. So from the table below we can compare the characters and the dimensions to find the missing enteries.

Table 3: Character Values of each element under standard representation S_4

	(1)	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1
W	3	-1	0	1	-1
W'	2	a	b	c	d

$$\sum \dim(x_i).\chi_{x_i}(12) = 1.1 + 1.-1 + 3.1 + 3.-1 + 2.a = 0 + 2a = 0 \quad \Rightarrow a = 0$$

$$\sum \dim(x_i).\chi_{x_i}(123) = 1.1 + 1.1 + 3.0 + 3.0 + 2.b = 2 + 2b = 0 \quad \Rightarrow b = -1$$

$$\sum \dim(x_i).\chi_{x_i}(1234) = 1.1 + 1.-1 + 3.-1 + 3.1 + 2.c = 0 + 2c = 0 \quad \Rightarrow c = 0$$

$\sum \dim(x_i).\chi_{x_i}(12)(34) = 1.1 + 1.1 + 3.-1 + 3.-1 + 2.a = -4 + 2d = 0 \Rightarrow$

$d = 2$

So the irreducible representations are U, U', V, W and W' where $< \chi_w, \chi_w >= 1$ and $< \chi_{W'}, \chi_{W'} >= 1$.

Table 4: Character Values of each element under standard representation S_4

	(1)	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1
W	3	-1	0	1	-1
W'	2	0	-1	0	2

Chapter 5

REPRESENTATION OF S_n

Remember that the number of conjugacy classes of the symmetric group is equal to the number of irreducible representations. In this chapter we are going to form a relationship between the set of cycle types and the ways to write n (number of elements of the symmetric group) as the sum of positive integers.

Definition 5.1 (Group Algebra) Let G be a group and F be any field. Then the vector space over F generated by the basis $\{e_x \mid x \in G\}$ is the group algebra of G over the field F with multiplication defined by

$$\left(\sum_{x\in G} a_x.x\right)\left(\sum_{y\in G} b_y.y\right) = \sum_{x,y\in G} a_x b_y xy$$

Where a_x`s and $b_y$`s are scalars in the field.

Remark 5.2 If $F = \mathbb{C}$, then $\mathbb{C}G$ becomes permutation representation under the group action , $g.e_x = e_{gx}$.

5.3 Representation algorithm for S_n.

Recall that S_n has as many irreducible representations as the number of distinct cycle types. For finding the irreducible representations, the algorithm works in the order of the following steps.

Definition 5.3.1 A partition for S_n is a set of integers $(\lambda_1, \lambda_2, \dots, \lambda_n)$ such that

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = n$

Step 1 : Write corresponding sums for each of the cycle types .

Eg: (123)(45)$\in S_6$, corresponding sum for this cycle type is 3+2+1 and the particular partition for this cycle type is $\lambda = (3,2,1)$.

Step 2 : Form corresponding Young diagram for each partition.

Definition 5.3.2 (Young diagram) Young diagram is a diagram of boxes arranged with respect to the partition. For each partition a different Young diagram is produced as a row in decreasing order.

Eg: Since partition is $\lambda = (3,2,1)$

Corresponding Young diagram is:

1	2	3
4	5	
6		

Step 3 With respect to the young diagram , for each diagram find the following sets

$$A_\lambda = \{\sigma \in S_n | \, preserves\ the\ set\ of\ numbers\ in\ each\ row\}$$

$$B_\lambda = \{\sigma \in S_n | \, preserves\ the\ set\ of\ numbers\ in\ each\ column\}$$

Step 4 Find following 3 elements of a group algebra

1. $x_\lambda = \sum_{\sigma \in A_\lambda} e_\sigma$
2. $y_\lambda = \sum_{\sigma \in B_\lambda} sign(\sigma) e_\sigma$
3. $z_\lambda = x_\lambda y_\lambda$

So irreducible representations of S_n are $\mathbb{C}S_n . z_\lambda$

Example 5.4 We can find all irreducible representations of S_4 by using the algorithm.

First of all we need to produce a young diagram for each partition of S_4.

$\lambda = (4)$ $\quad$ $\lambda = (3,1)$ $\quad$ $\lambda = (2,2)$ $\quad$ $\lambda = (2,1,1)$ $\quad$ $\lambda = (1,1,1,1)$

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$$

$\boldsymbol{\lambda = (4)}$:

$A_\lambda = S_4$

$B_\lambda = \{I\}$

$x_\lambda = e_{\mathrm{I}} + e_{(13)} + e_{(12)} + e_{(14)} + e_{(23)} + e_{(24)} + e_{(34)} + e_{(12)(34)} + e_{(13)(24)} + e_{(14)(23)} + e_{(123)} + e_{(124)} + e_{(132)} + e_{(134)} + e_{(142)} + e_{(143)} + e_{(234)} + e_{(243)} + e_{(1234)} + e_{(1243)} + e_{(1324)} + e_{(1342)} + e_{(1423)} + e_{(1432)}$

$y_\lambda = e_{\mathrm{I}}$

$$\begin{aligned} z_\lambda = x_\lambda y_\lambda = e_{\mathrm{I}} &+ e_{(12)} + e_{(13)} + e_{(14)} + e_{(23)} + e_{(24)} + e_{(34)} + e_{(12)(34)} + e_{(13)(24)} \\ &+ e_{(14)(23)} + e_{(123)} + e_{(124)} + e_{(132)} + e_{(134)} + e_{(142)} + e_{(143)} + e_{(234)} \\ &+ e_{(243)} + e_{(1234)} + e_{(1243)} + e_{(1324)} + e_{(1342)} + e_{(1423)} + e_{(1432)} \end{aligned}$$

Then $\mathbb{C}S_4 . z_\lambda = \langle z_\lambda \rangle$ (since it is invariant under the action of the basis $\mathbb{C}S_4$. So this representation has dimension 1 and it is trivial one.

$\boldsymbol{\lambda = (1+1+1+1)}$:

$A_\lambda = I$

$B_\lambda = S_4$

$x_\lambda = e_{\mathrm{I}}$

$y_\lambda = e_{\mathrm{I}} - e_{(13)} - e_{(12)} - e_{(14)} - e_{(23)} - e_{(24)} - e_{(34)} + e_{(12)(34)} + e_{(13)(24)} + e_{(14)(23)} + e_{(123)} + e_{(124)} + e_{(132)} + e_{(134)} + e_{(142)} + e_{(143)} + e_{(234)} + e_{(243)} - e_{(1234)} - e_{(1243)} - e_{(1324)} - e_{(1342)} - e_{(1423)} - e_{(1432)}$

$$\begin{aligned} z_\lambda = x_\lambda y_\lambda = e_{\mathrm{I}} &- e_{(13)} - e_{(12)} - e_{(14)} - e_{(23)} - e_{(24)} - e_{(34)} + e_{(12)(34)} + e_{(13)(24)} \\ &+ e_{(14)(23)} + e_{(123)} + e_{(124)} + e_{(132)} + e_{(134)} + e_{(142)} + e_{(143)} + e_{(234)} \\ &+ e_{(243)} - e_{(1234)} - e_{(1243)} - e_{(1324)} - e_{(1342)} - e_{(1423)} - e_{(1432)} \end{aligned}$$

Then $\mathbb{C}S_4 . z_\lambda = \langle z_\lambda \rangle$ (since it is invariant under the action of the basis of $\mathbb{C}S_4$. So this representation has dimension 1 and it is the alternating one.

$\boldsymbol{\lambda = (3+1)}$:

$A_\lambda = \{I, (12), (23), (13), (123), (132)\}$

$B_\lambda = \{I, (14)\}$

$x_\lambda = e_{\mathrm{I}} + e_{(12)} + e_{(23)} + e_{(13)} + e_{(123)} + e_{(132)}$

$y_\lambda = e_{\mathrm{I}} - e_{(14)}$

$$\begin{aligned} z_\lambda = x_\lambda y_\lambda = e_{\mathrm{I}} + e_{(12)} - e_{(14)} + e_{(23)} + e_{(23)} - e_{(14)(23)} + e_{(123)} + e_{(132)} - e_{(142)}\mathrm{y} \\ - e_{(143)} - e_{(1423)} - e_{(1432)} \end{aligned}$$

Then $\mathbb{C}\boldsymbol{S_4}.\,\boldsymbol{z_\lambda}$:

$e_{\mathrm{I}}(z_\lambda) = z_\lambda$

$e_{(12)}(z_\lambda) = z_\lambda$

$e_{(13)}(z_\lambda) = z_\lambda$

$$e_{(14)}(z_\lambda) = -e_{\mathrm{I}} + e_{(14)} - e_{(23)} - e_{(24)} - e_{(34)} + e_{(124)} + e_{(134)} + e_{(14)(23)} + e_{(1324)} + e_{(1234)} - e_{(234)} - e_{(243)}$$

$e_{(23)}(c_\lambda) = z_\lambda$

$$e_{(24)}(c_\lambda) = e_{(24)} - e_{(12)} + e_{(13)(24)} - e_{(12)(34)} + e_{(142)} + e_{(234)} - e_{(124)} - e_{(123)} + e_{(1342)} + e_{(1423)} - e_{(1234)} - e_{(1243)}$$

$$\vdots$$

This set is spanned by the first forth and sixth vectors, so this representation is a 3 dimensional representation of S_4.

$\boldsymbol{\lambda = (2+2)}$:

$A_\lambda = \{I, (12), (34), (12)(34)\}$

$B_\lambda = \{I, (13), (24), (13)(24)\}$

$x_\lambda = e_{\mathrm{I}} + e_{(12)} + e_{(34)} + e_{(12)(34)}$

$y_\lambda = e_{\mathrm{I}} - e_{(13)} - e_{(24)} + e_{(13)(24)}$

$$z_\lambda = x_\lambda y_\lambda = e_{\mathrm{I}} + e_{(12)} + e_{(34)} + e_{(12)(34)} - e_{(13)} - e_{(24)} + e_{(14)(23)} + e_{(13)(24)} - e_{(124)} - e_{(132)} - e_{(143)} - e_{(234)} - e_{(1234)} - e_{(1432)} + e_{(1324)} + e_{(1423)}$$

Then $\mathbb{C}\boldsymbol{S_4}.\,\boldsymbol{c_\lambda}$:

$e_{\mathrm{I}}(z_\lambda) = z_\lambda$

$e_{(12)}(z_\lambda) = z_\lambda$

$$e_{(13)}(z_\lambda) = e_{(13)} + e_{(123)} + e_{(134)} + e_{(1234)} - e_{\mathrm{I}} - e_{(23)} - e_{(14)} - e_{(14)(23)} - e_{(13)(24)} - e_{(1243)} - e_{(1342)} - e_{(12)(34)} + e_{(24)} + e_{(243)} + e_{(142)} + e_{(1432)}$$

$$e_{(14)}(z_\lambda) = -e_{(123)} - e_{(134)} + e_{(23)} + e_{(14)} + e_{(1243)} + e_{(1342)} - e_{(243)} - e_{(142)} - e_{(12)} - e_{(34)} + e_{(124)} + e_{(132)} + e_{(143)} + e_{(234)} - e_{(1324)} - e_{(1423)}$$

$$= -(e_{\mathrm{I}} + e_{(13)})$$

$e_{(23)}(z_\lambda) = -(z_\lambda + e_{(13)}(z_\lambda))$

$e_{(24)}(z_\lambda) = e_{(13)}(z_\lambda)$

$e_{(34)}(z_\lambda) = z_\lambda$

$e_{(123)}(z_\lambda) = e_{(13)}(z_\lambda)$

$e_{(124)}(z_\lambda) = -(z_\lambda + e_{(13)}(z_\lambda))$

$e_{(132)}(z_\lambda) = -(z_\lambda + e_{(13)}(z_\lambda))$

$e_{(142)}(z_\lambda) = e_{(13)}(z_\lambda)$

$e_{(134)}(z_\lambda) = -(z_\lambda + e_{(13)}(z_\lambda))$

$e_{(143)}(z_\lambda) = -(z_\lambda + e_{(13)}(z_\lambda))$

$e_{(234)}(z_\lambda) = -(z_\lambda + e_{(13)}(z_\lambda))$

$e_{(243)}(z_\lambda) = e_{(13)}(z_\lambda)$

$e_{(12)(34)}(z_\lambda) = z_\lambda$

$e_{(13)(24)}(z_\lambda) = z_\lambda$

$e_{(14)(23)}(z_\lambda) = z_\lambda$

$e_{(1234)}(z_\lambda) = e_{(13)}(z_\lambda)$

$e_{(1243)}(z_\lambda) = -(z_\lambda + e_{(13)}(z_\lambda))$

$e_{(1324)}(z_\lambda) = z_\lambda$

$e_{(1423)}(z_\lambda) = z_\lambda$

$e_{(1432)}(z_\lambda) = e_{(13)}(z_\lambda)$

$e_{(1342)}(z_\lambda) = -(z_\lambda + e_{(13)}(z_\lambda))$

This set is spanned by the first and third vectors, so this representation is 2 dimensional representation of S_4.

5.5 Computing the character table for S_n

We can also compute the character of every irreducible representation with respect to the cycle types.

Theorem 5.5.1 (The Frobenius Formula) [3]

The character formula for the irreducible representation of S_n corresponding to the partition λ is given by,

$$\chi_\lambda(C_i) = \left[\Delta(x) \prod_j P_j(x)^{i_j}\right]_{l_1, l_2, \ldots, l_k}$$

Where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition ,

j is an index through 1 to n ,

i_j is the number of cycles with length j ,

C_i are the conjugacy classes ,

k is the number of independent variables $x_1, x_2, \dots, x_k$,

$P_j(x) = {x_1}^j + {x_2}^j + \cdots + {x_k}^j$

$l_i = \lambda_i + k - i$ where $1 \le i \le k$ and,

$\Delta(x)$ is the discriminant of independent variables such that

$$\Delta(x) = \begin{vmatrix} 1 & x_k & \cdots & {x_k}^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_1 & \cdots & {x_1}^{k-1} \end{vmatrix}$$

If $\Delta(x)\prod_j P_j(x)^{i_j}$ is a polynomial say $g(x)$, then let $[g(x)]_{l_1, l_2, \dots, l_k}$ represent the coefficient of the term ${x_1}^{l_1}{x_2}^{l_2} \dots {x_k}^{l_k}$.

Remark 5.5.2 For cycle type $(12) \in S_4$ $i_1 = 2$ $i_2 = 1$, $i_3 = 0$, $i_4 = 0$.

For cycle type $(12)(34)$ $i_1 = 0$ $i_2 = 2$, $i_3 = 0$, $i_4 = 0$.

For cycle type (123) $i_1 = 1$ $i_2 = 0$, $i_3 = 1$, $i_4 = 0$.

For cycle type (1234) $i_1 = 0$ $i_2 = 0$, $i_3 = 0$, $i_4 = 1$.

Example 5.5.3 Character table for S_4

Corresponding young diagrams are listed below as;

$\lambda = (4)$

1	2	3	4

$\lambda = (3,1)$

1	2	3
4		

$\lambda = (2,2)$

1	2
3	4

$\lambda = (2,1,1)$

1	2
3	
4	

$\lambda = (1,1,1,1)$

1
2
3
4

Dimensions for all partitions are already computed and are 1,3,2,3 and 1 respectively, and we know that these numbers give character number of the identity element.

Lets compute character numbers of cycles,

1. Character numbers of the irreducible representation corresponding to the partition $\boldsymbol{\lambda = (3,1)}$;

Partition of the representation is $\lambda = (3,1)$

Here $= 2$, $\Delta(x) = \begin{vmatrix} 1 & x_2 \\ 1 & x_1 \end{vmatrix} = x_1 - x_2$,

To calculate character of $\lambda = (3,1)$ at $(12)(3)(4)$,

$i_1 =$ number of cycles with length $1 = 2$, $i_2 = 1$, $i_3 = 0$, $i_4 = 0$.

$l_1 = \lambda_1 + 2 - 1 = 3 + 2 - 1 = 4$

$l_2 = \lambda_1 + 2 - 2 = 1 + 2 - 2 = 1$

So with respect to l_i we can say that the coefficient of ${x_1}^4{x_2}^1$ is the character number of the irreducible representation at (12).

$\chi_{(3,1)}(12) = [(x_1 - x_2)(x_1 + x_2)^2({x_1}^2 + {x_2}^2)^1]_{4,1}$ by directly computing the coefficient of ${x_1}^4{x_2}^1$ we found 1.

We can calculate the characters numbers with respect to all cycle types by using the Frobenius formula,

$\chi_{(3,1)}(12)(34) = [(x_1 - x_2)({x_1}^2 + {x_2}^2)^2]_{4,1}$ by directly computing the coefficient of ${x_1}^4{x_2}^1$ we found -1.

$\chi_{(3,1)}(123) = [(x_1 - x_2)(x_1 + x_2)^1({x_1}^3 + {x_2}^3)^1]_{4,1}$ by directly computing the coefficient of ${x_1}^4{x_2}^1$ we found 0.

$\chi_{(3,1)}(1234) = [(x_1 - x_2)({x_1}^4 + {x_2}^4)^1]_{4,1}$ by directly computing the coefficient of ${x_1}^4{x_2}^1$ we found -1.

By looking at these character numbers with respect to cycle types, we can say that $\lambda = (3,1)$ is the standard representation.(by Table 4)

2. Character numbers of the irreducible representation corresponding to the partition $\boldsymbol{\lambda} = (\mathbf{2}, \mathbf{2})$

Partition of the representation is $\lambda = (2,2)$ Here $= 2$,

$\Delta(x) = \begin{vmatrix} 1 & x_2 \\ 1 & x_1 \end{vmatrix} = x_1 - x_2$.

To calculate character of $\lambda = (2,2)$ at $(12)(3)(4)$,

$i_1 =$ number of cycles with length $1 = 2$, $i_2 = 1$, $i_3 = 0$, $i_4 = 0$.

$l_1 = \lambda_1 + 2 - 1 = 2 + 2 - 1 = 3$

$l_2 = \lambda_2 + 2 - 2 = 2 + 2 - 2 = 2$

So with respect to l_i we can say that the coefficient of ${x_1}^3{x_2}^2$ is the character number of the irreducible representation at (12)

$\chi_{(2,2)}(12) = [(x_1 - x_2)(x_1 + x_2)^2({x_1}^2 + {x_2}^2)^1]_{3,2}$ by directly computing the coefficient of ${x_1}^3{x_2}^2$, we find 0

We can calculate the characters numbers with respect to all cycle types in a similar way by the using the Frobenius formula,

$\chi_{(2,2)}(12)(34) = [(x_1 - x_2)({x_1}^2 + {x_2}^2)^2]_{3,2}$ by directly computing the coefficient of ${x_1}^3{x_2}^2$ we find 2.

$\chi_{(2,2)}(123) = [(x_1 - x_2)(x_1 + x_2)^1({x_1}^3 + {x_2}^3)^1]_{3,2}$ by directly computing the coefficient of ${x_1}^3{x_2}^2$ we find -1.

$\chi_{(2,2)}(1234) = [(x_1 - x_2)({x_1}^4 + {x_2}^4)^1]_{4,1}$ by directly computing the coefficient of ${x_1}^3{x_2}^2$ we find 0.

By looking these character numbers with respect to cycle types we can say that $\lambda =$ (2,2) is the irreducible representation , $W^`$.(Table 4)

3. Character numbers of the irreducible representation corresponding to the partition$\boldsymbol{\lambda} = (\mathbf{2}, \mathbf{1}, \mathbf{1})$

Partition of the representation is $\lambda = (2,1,1)$ Here $= 3$,

$$\Delta(x) = \begin{vmatrix} 1 & x_3 & {x_3}^2 \\ 1 & x_2 & {x_2}^2 \\ 1 & x_1 & {x_1}^2 \end{vmatrix} = {x_1}^2x_3 - x_1{x_2}^2 - {x_1}^2x_3 + {x_2}^2x_3 + x_1{x_3}^2 - x_2{x_3}^2 \quad ,$$

To calculate character of $\lambda = (2,1,1)$ at $(12)(3)(4)$,

$i_1 =$ number of cycles with length $1 = 2$, $i_2 = 1$, $i_3 = 0$, $i_4 = 0$.

$l_1 = \lambda_1 + 3 - 1 = 2 + 3 - 1 = 4$

$l_2 = \lambda_2 + 3 - 2 = 1 + 3 - 2 = 2$

$l_3 = \lambda_3 + 3 - 3 = 1 + 3 - 3 = 1$

So with respect to l_i we can say that the coefficient of ${x_1}^4{x_2}^2x_3$ is the character number of the irreducible representation at (12)

$\chi_{(2,1,1)}(12) = [({x_1}^2x_3 - x_1{x_2}^2 - {x_1}^2x_3 + {x_2}^2x_3 + x_1{x_3}^2 - x_2{x_3}^2)(x_1 + x_2 + x_3)^2({x_1}^2 + {x_2}^2 + {x_3}^2)]_{4,2,1}$ by computing the coefficient of ${x_1}^4{x_2}^2x_3$ we find -1.

We can calculate the character numbers with respect to all cycle types by using the Frobenius formula,

$\chi_{(2,1,1)}(12)(34) = [({x_1}^2x_3 - x_1{x_2}^2 - {x_1}^2x_3 + {x_2}^2x_3 + x_1{x_3}^2 - x_2{x_3}^2)({x_1}^2 + {x_2}^2 + {x_3}^2)^2]_{3,2}$ by directly computing the coefficient of ${x_1}^4{x_2}^2x_3$ we find -1.

$\chi_{(2,1,1)}(123) = [({x_1}^2x_3 - x_1{x_2}^2 - {x_1}^2x_3 + {x_2}^2x_3 + x_1{x_3}^2 - x_2{x_3}^2)(x_1 + x_2 + x_3)^1({x_1}^3 + {x_2}^3 + {x_3}^3)^1]_{4,2,1}$ by computing the coefficient of ${x_1}^4{x_2}^2x_3$ we find 0.

$\chi_{(2,1,1)}(1234) = [({x_1}^2x_3 - x_1{x_2}^2 - {x_1}^2x_3 + {x_2}^2x_3 + x_1{x_3}^2 - x_2{x_3}^2)({x_1}^4 + {x_2}^4 + {x_3}^4)^1]_{4,2,1}$ by directly computing the coefficient of ${x_1}^4{x_2}^2x_3$ we find 1.

By looking these character numbers with respect to cycle types we can say that $\lambda = (2,1,1)$ is the irreducible representation, W.(Table 4)

Chapter 6

CONCLUSION

Representation theory arises in wide variety of areas including mathematical physics, number theory, combinatorics, engineering, mathematical chemistry and coding theory.

As emphasized under introduction, it proves to be an extremely useful branch of pure mathematics, as it reduces complex questions of abstract algebra to easier ones under linear algebra.

This thesis is aimed at bringing together the most important notions of representation theory under one source, and hence it proves to be a useful read as a first introduction to the subject.

REFERENCES

[1] Serre, J. P. (1977) , *Linear Representations of Finite Groups* (Translated from the French by Leonard L. Scott), Graduate Texts in Mathematics 42, Springer, New-York.

[2] Fulton W. (1997), *Young Tableaux with Applications to Representation Theory and Geometry*, London Mathematical Society Student Text 35. Cambridge University Press, New York

[3] Fulton W., Harris J. (1991) , *Representation Theory*, Graduate Text in Mathematics, Springer, New York

[4] Gallian J. A., *Contemporary Abstract Algebra*, Sixth Edition, Houghton Mifflin Company, Boston, New York

[5] Prof. Dr. Çallıalp F. (1981), *Soyut Cebir I*, Hacettepe Universitesi Basimevi.

[6] http://www.math.uchicago.edu/~may/VIGRE/VIGRE2010/REUPapers/Kao.pdf

[7] Friedberg S. H. Friedberg, Insel A. J., Lawrence E. Spence, *Linear Algebra*, 2nd Ed. Prentice-Hall International, Inc.

[8] Pinter C. C. , *A Book of Abstarct Algebra* , 2nd edition , McGraw-Hill, Inc.