Bernstein-Type Operators

Kawa Sardar Mohammad Ali

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Approval of the Institute of Graduate Studies and Research

Prof. Dr. Cem Tanova
Acting Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Sonuç Zorlu Oğurlu
Supervisor

Examining Committee

1. Prof. Dr. Nazim Mahmudov
2. Prof. Dr. Sonuç Zorlu Oğurlu
3. Asst. Prof. Dr. Pembe Sabancıgil

ABSTRACT

In this work we are interested in the approximation of some type of operators called Bernstein-type. For this purpose, the operator $L_n(f,x)(f \in C[0,\infty))$ called the Bernstein-type approximation operator is considered. The aim is to use some probabilistic properties to improve and sharp to operator defined above. Also, the rates of convergence as well as the continuity of the operator are studied. Various methods of approaching the problem are evaluated in this study.

Keywords: Bernstein type operator, probabilistic approach, binomial distribution, rates of convergence.

ÖZ

Bu çalışmada Bernstein - tipi operatörlerin yaklaşımlarıyla ilgilenilimiştir.Bunun için $L_n(f,x)\left(f\in C\left[0,\infty\right)\right)$ olarak belirtilen Bernstein tipi yaklaşım operatörü ele alınmıştır. Yukarıda verilen operatör için bazı olasılıksal metodlar kullanılarak yaklaşım özellikleri çalışılmıştır. Ayrıca, yakınsama hızı yanı sıra operatörün sürekliliği incelenmiş olup farklı yöntemlerle yaklaşım problemi de bu çalışmada değerlendirilmiştir.

Anahtar kelimeler: Bernstein tipi operatörler, olasılıksal yaklaşım, binom dağılımı, yaklaşım hızı.

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Chapter 1

INTRODUCTION

This work is divided in three main chapters as well as the introductory part and the conclusion. An overview of what is developed in each of the chapter is given here.

The idea in chapter 2 is the following. Considering $C\left[0,\infty\right)$ to be the space of continuous functions on the half open interval $\left[0,\infty,\right)$, and given a function $\varphi \in C\left[0,\infty\right)$, Butzer, Hahn and Bleimann $\left[1\right]$ introduced an approximation operator called the Bernstein-type operator for approximation and defined it as follows:

$$L_k\left(\varphi,t\right)=\frac{1}{\left(1+t\right)^k}\left(\sum_{n=0}^{k}\varphi\left(\frac{n}{k-n+1}\right)\right)C_n^k t^n,\ k\in\mathbb{N}. \tag{1.1}$$

They later proved the convergence of $L_k\left(\varphi,t\right)\to\varphi\left(t\right)$ (as $k\to\infty$) for each $t\in\left[0,\infty\right)$. The convergence of the function mentioned above is investigated by the estimation of the quantity $\left|L_k\left(\varphi,t\right)-\varphi\left(t\right)\right|$. This investigation is actually possible provided that the continuity of the function $\varphi\in C_B\left[0,\infty\right)$. The notation $C_B\left[0,\infty\right)$ is set for the space of all the functions which converge uniformly and are bounded on the half open interval $\left[0,\infty\right)$. Some probabilistic arguments are used in what will follows to sharp, and add accuracy to the results mentioned above. The first two module of the continuity are used to establish the rates of convergence.

In the third chapter, the idea developed is the studies of two related approximation problems of the Bernstein polynomials type $B_k\varphi(t)$ if given continuous function φ on the closed interval $[0,1]$: The following two results are investigated. The first is that there is no any Gibbs phenomenon at any jump type discontinuity points of φ and second result is the convergence of the first derivative $(B_k\varphi)'(t)$ of the initial polynomial $B_k\varphi(t)$ towards the first derivative $\varphi'(t)$. The well-known and classical probabilistic arguments are used to prove the above results. The second result for instance is obtained based on the computation of the expectation of a function of random variables.

The fourth chapter which is the third main chapter of this work is centered on the following idea. From the poineenring work of Bernstein, various research results have proven that probabilistic arguments are suitable for any approximation problem which is based on positive linear operators.

The problem is usually defined as follows, given I a real valued interval and given R^t, R_1^t, $\mathrm{R}_2^t,\ldots$, I-valued random variables with their density depending on a parameter $t\in I$. Let us consider two linear operators Y and Y_k positively defined associated to the random variable R^t and R_k^t respectively. By the mean of

$$Y\ (\varphi,t)=E\varphi\left(R^t\right),\ Y_k(\varphi,t)=E\varphi\left(R_k^t\right),\ \varphi\in C_B(I),\ t\in I,$$

with E being the mathematical expectation of a random variable. $C_B(I)$ is the space of all real valued continuous functions which are bounded on the interval I.

Chapter 2

BERNSTEIN OPERATOR

2.1 Introduction

Consider the space $C\left[0,\infty\right)$ of real valued continuous functions defined on the half open interval $\left[0,\infty\right)$. Let us consider also the function $\varphi\in C\left[0,\infty\right)$. The approximation operator of the Bernstein type initially defined by Hahn, Butzer, and Bleimann, is given by

$$L_k\left(\varphi,t\right)=\frac{1}{\left(1+t\right)^k}\left(\sum_{n=0}^{k}\varphi\left(\frac{n}{k-n+1}\right)\right)C_n^k t^n,\ k\in\mathbb{N}. \tag{2.1}$$

They proved that $L_k\left(\varphi,t\right)\to\varphi\left(t\right)$ (as $k\to\infty$) for any $t\in\left[0,\infty\right)$, they also established that the convergence rate can be investigated by estimating $\left|L_k\left(\varphi,t\right)-\varphi\left(t\right)\right|$ of a continuity function $\varphi\in C_B\left[0,\infty\right)$. Some probabilistic arguments are used in what will follows to sharp, and add accuracy to the results mentioned above. The first two module of the continuity are used to establish the rates of convergence. We prove the relation $\left|L_k\left(\varphi,t\right)-\varphi\left(t\right)\right|\le 3\upsilon\left(\varphi,\sqrt{t}\left(1+t\right)^2/k\right)$, where $\upsilon\left(\varphi,\alpha\right)$ stands for the first modulus continuity of φ. What follows is an improvement of the above theorem (inequality)

$$\left|L_k\left(\varphi,t\right)-\varphi\left(t\right)\right|\le 2C\left[\upsilon_2\left(\varphi,\sqrt{\frac{t\left(1+t\right)^2}{k}}\right)+\frac{t\left(1+t\right)^2}{k}\left\|\varphi\right\|\right],$$

where $\upsilon_2(\varphi,\alpha)$ is the second modulus of continuity of $\varphi \in C_B[0,\infty)$, and $\|\varphi\| = \sup_{t\in[0,\infty]} |\varphi(t)|$. An approximation of the limit of L_k is the so called Szasz operator.

2.2 Evaluation of Convergence Rates

Let $W_1, W_2, W_3, \ldots$ be independent random variables with some probability distribution such that $\mathrm{P}(W_i = 1) = p$, $\mathrm{P}(W_i = 0) = q$, where $p = t/(1+t)$ and $q = 1/(1+t)$, $t \in [0,\infty)$. In order to avoid the case where $t = 0$, assume that $t > 0$. The summation $S_k = W_1 + \ldots + W_k$ follows a binomial distribution $b(\mathrm{n}, k, p)$ whose parameters are k and p, and

$$\mathrm{P}(S_k = n) = \binom{k}{n} p^n q^{k-n}, \qquad \mathrm{n} = 0,1,2,\ldots,k. \tag{2.2}$$

Set $T_k = S_k/(k - S_k + 1)$, $\mathrm{k} = 1,2,\ldots,$ it follows from (2.1) that $L_k(\varphi,t) = E\,\varphi(T_k)$, with the character E being the expectation operator. The convergence of $T_k \to p/q = t$ in probability as $k \to \infty$, implies that $L_k(\varphi,t) \to \varphi(t)$ as $\mathrm{k} \to \infty$ by the law of large numbers for $\varphi \in C[0,\infty)$. To get an accurate result, we first calculate ET_k and $ET_k^{\,2}$ and secondly, we estimate the quantity $e_k(t) = E(T_k - t)^2$. It is an easy task to prove that

$$ET_k = t - tp^k \to t \text{ , as } k \to \infty. \tag{2.3}$$

It follows from T_k and (2.2) that

$$ET_k^{\,2} = \sum_{n=0}^{k} \frac{n^2}{(k-n+1)^2}\, p(S_k = n)$$

$$= 0 + \sum_{n=1}^{k} \frac{n^2}{(k-n+1)(k-n+1)} \frac{k!}{(k-n)!n!} p^n q^{k-n}$$

$$= \sum_{n=1}^{k} \frac{n^2}{(k-n+1)(k-n+1)} \frac{k!}{(k-n)!n(n-1)!} p^n q^{k-n}$$

$$= \sum_{n=1}^{k} \frac{n}{(k-n+1)(n-1)!} \frac{k!}{(k-n+1)(k-n)!} p^n q^{k-n}$$

$$= \sum_{n=1}^{k} \frac{n}{(k-n+1)(n-1)!} \frac{k!}{(k-n+1)!} p^n q^{k-n}$$

Letting $n = m+1$

$$= \sum_{m=0}^{k-1} \frac{(m+1)}{(k-m-1+1)(m+1-1)!} \frac{k!}{(k-m+1-1)!} \frac{p^{m+1}}{q^{1+m+k}}$$

$$= \sum_{m=0}^{k-1} \frac{(m+1)}{(k-m)m!} \frac{k!}{(k-m)!} \frac{p^{m+1}}{q^{1+m+k}}$$

$$= \sum_{m=0}^{k-1} \frac{m}{(k-m)m!} \frac{k!}{(k-m)!} \frac{p^{m+1}}{q^{1+m+k}} + \sum_{m=0}^{k-1} \frac{1}{(k-m)m!} \frac{k!}{(k-m)!} \frac{p^{m+1}}{q^{1+m+k}}$$

$$= 0 + \sum_{m=1}^{k-1} \frac{m}{(k-m)m!} \frac{k!}{(k-m)!} \frac{p^{m+1}}{q^{1+m+k}} + \sum_{m=0}^{k-1} \frac{k!}{(k-m)m!(k-m)!} \frac{p^{m+1}}{q^{1+m+k}}$$

$$= \sum_{m=1}^{k-1} \frac{m}{(k-m)m(m-1)!} \frac{k!}{(k-m)!} \frac{p^{m+1}}{q^{1+m+k}} + \sum_{m=0}^{k-1} \frac{k!}{(k-m)m!(k-m)!} \frac{p^{m+1}}{q^{1+m+k}}$$

$$= \sum_{m=1}^{k-1} \frac{k!}{(k-m)(m-1)!(k-m)!} \frac{p^{m+1}}{q^{1+m+k}} + \sum_{m=0}^{k-1} \frac{k!}{(k-m)m!(k-m)!} \frac{p^{m+1}}{q^{1+m+k}}$$

$$= \sum_{m=1}^{k-1} \frac{k!}{(k-m)(m-1)!(k-m)!} p^{m+1} q^{k-m-1} + \frac{k!}{(k-0)0!(k-0)!} p^{0+1} q^{k-0-1}$$

$$+ \sum_{m=1}^{k-1} \frac{k!}{(k-m)m!(k-m)!} p^{m+1} q^{k-m-1}$$

$$= \frac{pq^{k-1}}{k} + \sum_{m=1}^{k-1} \frac{k!}{(k-m)(m-1)!(k-m)!} p^{m+1} q^{k-m-1}$$

$$+\sum_{m=1}^{k-1}\frac{k!}{(k-m)m!(k-m)!}p^{m+1}q^{k-m-1}.$$

$$=\frac{\left(\frac{t}{1+t}\right)\left(\frac{1}{1+t}\right)^{k}\left(\frac{1}{1+t}\right)^{-1}}{k}+\sum_{m=1}^{k-1}\frac{k!}{(k-m)(m-1)!(k-m)!}p^{m+1}q^{k-m-1}$$

$$+\sum_{m=1}^{k-1}\frac{k!}{(k-m)m!(k-m)!}p^{m+1}q^{k-m-1}$$

$$=\frac{t}{(1+t)}\frac{1}{(1+t)^{k}}(1+t)\frac{1}{k}+\sum_{m=1}^{k-1}\frac{k!}{(k-m)(m-1)!(k-m)!}p^{m+1}q^{k-m-1}$$

$$+\sum_{m=1}^{k-1}\frac{k!}{(k-m)m!(k-m)!}p^{m+1}q^{k-m-1}$$

$$=\frac{t(1+t)^{-k}}{k}+\frac{p^{m+1}}{q^{1+m+k}}\sum_{m=1}^{k-1}\frac{k!}{(k-m)(m-1)!(k-m)!}$$

$$+\frac{p^{m+1}}{q^{1+m+k}}\sum_{m=1}^{k-1}\frac{k!}{(k-m)m!(k-m)!}.$$

Letting $n=m-1$ and exploiting the binomial distribution condition $p=1-q=t/(1+t)$ it follows that

$$ET_k^{\,2}=\frac{t}{k(1+t)^{k}}+\sum_{n=0}^{k-2}\frac{k!}{(k-n-1)!(n+1-1)!(k-n-1)}\frac{k-n}{k-n}\frac{p^{n+1+1}}{q^{n-k+1+1}}$$

$$+\sum_{n=0}^{k-2}\frac{k!}{(k-n-1)!(n+1)!(k-n-1)}\frac{k-n}{k-n}\frac{p^{n+1+1}}{q^{n-k+1+1}}$$

$$=\frac{t}{k(1+t)^{k}}+\sum_{n=0}^{k-2}\frac{k!}{(k-n-1)(n)!}\frac{k-n}{(k-n)!}\frac{p^{n+2}}{q^{n-k+2}}$$

$$+\sum_{n=0}^{k-2}\frac{k!}{(k-n-1)(n+1)}\frac{k-n}{(k-n)!(n)!}\frac{p^{n+2}}{q^{n-k+2}}$$

$$=\frac{t\left(1+t\right)^{-k}}{k}+\sum_{n=0}^{k-2}\binom{k}{n}\frac{k-n}{\left(k-n-1\right)}\frac{p^{n+2}}{q^{n-k+2}}+\sum_{n=0}^{k-2}\binom{k}{n}\frac{k-n}{\left(k-n-1\right)\left(n+1\right)}\frac{p^{n+2}}{q^{n-k+2}}$$

$$=\frac{t(1+t)^{-k}}{k}+\sum_{n=0}^{k-2}\binom{k}{n}\frac{p^{n+2}}{q^{n-k+2}}\left[\frac{k-n}{\left(k-n-1\right)}+\frac{k-n}{(n+1)(k-n-1)}\right]$$

$$=\frac{t(1+t)^{-k}}{k}+\sum_{n=0}^{k-2}\binom{k}{n}\frac{p^{n+2}}{q^{n-k+2}}\left[\frac{\left(k-n\right)\left(n+1\right)+\left(k-n\right)}{\left(k-n-1\right)\left(n+1\right)}\right]$$

$$=\frac{t(1+t)^{-k}}{k}+\sum_{n=0}^{k-2}\binom{k}{n}\frac{p^{n+2}}{q^{n-k+2}}\left[\frac{\left(k-n\right)\left[\left(n+1\right)+1\right]}{\left(k-n-1\right)\left(n+1\right)}\right]$$

$$=\frac{t(1+t)^{-k}}{k}+\sum_{n=0}^{k-2}\binom{k}{n}\frac{p^{n+2}}{q^{n-k+2}}\frac{\left(n+2\right)\left(k-n\right)}{\left(n+1\right)\left(k-n-1\right)}.$$

Since

$$\frac{(n+2)(k-n)}{(n+1)(k-n-1)}=\frac{\left(n+2-1+1\right)\left(k-n\right)}{\left(k-n-1\right)\left(n+1\right)}$$

$$=\frac{\left(\left(n+1\right)+1\right)\left(k-n\right)}{\left(n+1\right)\left(k-n-1\right)}=\frac{\left(n+1\right)\left(k-n\right)}{\left(n+1\right)\left(k-n-1\right)}+\frac{\left(k-n\right)}{\left(n+1\right)\left(k-n-1\right)}$$

$$=\frac{\left(n+1\right)\left(\left(k-n-1\right)+1\right)}{\left(n+1\right)\left(k-n-1\right)}+\frac{\left(k-n\right)}{\left(n+1\right)\left(k-n-1\right)}$$

$$=\frac{\left(n+1\right)\left(k-n-1\right)}{\left(n+1\right)\left(k-n-1\right)}+\frac{\left(n+1\right)}{\left(n+1\right)\left(k-n-1\right)}+\frac{\left(k-n\right)}{\left(n+1\right)\left(k-n-1\right)}$$

$$=1+\frac{1}{\left(k-n-1\right)}+\frac{\left(k-n\right)-1+1}{\left(n+1\right)\left(k-n-1\right)}$$

$$=1+\frac{1}{\left(k-n-1\right)}+\frac{\left(k-n-1\right)+1}{\left(n+1\right)\left(k-n-1\right)}$$

$$=1+\frac{1}{\left(k-n-1\right)}+\frac{1}{\left(n+1\right)}+\frac{1}{\left(n+1\right)\left(k-n-1\right)}$$

$$=1+\frac{1}{(n+1)}+\frac{(n+1)+1}{(k-n-1)(n+1)}$$

$$=1+\frac{1}{(n+1)}+\frac{(n+2)-1+1}{(n+1)(k-n-1)}$$

$$=1+\frac{1}{(n+1)}+\frac{(n+1)}{(n+1)(k-n-1)}+\frac{1}{(n+1)(k-n-1)}$$

$$=1+\frac{(k-n-1)+(n+1)+1}{(n+1)(k-n-1)}=1+\frac{(k-n-1+n+1+1)}{(n+1)(k-n-1)}$$

$$\frac{(k-n)(n+2)}{(n+1)(k-n-1)}=1+\frac{(k+1)}{(n+1)(k-n-1)},$$

then we have,

$$ET_k^2=\frac{t}{k(1+t)^k}+\sum_{n=0}^{k-2}\binom{k}{n}p^{n+2}q^{k-n-2}+\sum_{n=0}^{k-2}\frac{(k+1)k!}{(k-n)!n!(n+1)(k-n-1)}p^{n+2}q^{k-n-2}$$

$$=\frac{t}{k(1+t)^k}+\sum_{n=0}^{k-2}\binom{k}{n}p^{n+2}q^{k-n-2}+\sum_{n=0}^{k-2}\frac{(k+1)!}{(k-n)!(\mathrm{n}+1)!(k-n-1)}p^{n+2}q^{k-n-2}$$

$$=\frac{t}{k(1+t)^k}+\sum_{n=0}^{k}\binom{k}{n}p^{n+2}q^{k-n-2}-\binom{k}{k-1}p^{k-1+2}q^{k-k+1-2}-\binom{k}{k}p^{k+2}q^{k-k-2}$$

$$+\sum_{n=0}^{k-2}\frac{(k+1)!}{(k-n)!(n+1)!(k-n-1)}p^{n+2}q^{k-n-2}$$

$$=\frac{t}{k(1+t)^k}+\sum_{n=0}^{k}\binom{k}{n}\frac{p^{n+2}}{q^{n-k+2}}-\frac{k!}{(k-k-1)!(k-1)!}\frac{p^{k+1}}{q}$$

$$-\frac{k!}{(k-k)!k!}\frac{p^{n+2}}{q^2}+\sum_{n=0}^{k-2}\frac{(k+1)!}{(k-n)!(n+1)!(k-n-1)}\frac{p^{n+2}}{q^{n-k+2}}$$

$$=\frac{t}{k(1+t)^k}+\sum_{n=0}^{k}\binom{k}{n}\frac{p^{n+2}}{q^{n-k+2}}-\frac{k(k-1)!}{1!(k-1)!}p^{k+1}q^{-1}-\frac{k!}{0!k!}\frac{p^{n+2}}{q^2}$$

$$+\sum_{n=0}^{k-2}\frac{(k+1)!}{(k-n)!(n+1)!(k-n-1)}\frac{p^{n+2}}{q^{n-k+2}}$$

$$=\frac{t}{k(1+t)^k}+\sum_{n=0}^{k}\binom{k}{n}p^{n+2}q^{k-n-2}-p^{k+2}q^{-2}-kp^{k+1}q^{-1}$$

$$+\sum_{n=0}^{k-2}\frac{(k+1)!}{(k-n)!(n+1)!(k-n-1)}p^{n+2}q^{k-n-2}$$

$$=\frac{t}{k(1+t)^k}+\frac{\left(\frac{t}{1+t}\right)^2}{\left(\frac{1}{1+t}\right)^2}\sum_{n=0}^{k}\binom{k}{n}p^{n}q^{k-n}-\left(\frac{t}{1+t}\right)^{k+2}\left(\frac{1}{1+t}\right)^{-2}$$

$$-k\left(\frac{t}{1+t}\right)^{k+1}\left(\frac{1}{1+t}\right)^{-1}+\sum_{n=0}^{k-2}\frac{(k+1)!}{(k-n)!(n+1)!(k-n-1)}p^{n+2}q^{k-n-2}$$

$$=\frac{t}{k(1+t)^k}+\frac{t^2}{(1+t)^2}(1+t)^2-\frac{t^2t^k}{(1+t)^k(1+t)^2}(1-t)^2$$

$$-k\frac{tt^k}{(1+t)^k(1+t)}(1+t)+\sum_{n=0}^{k-2}\frac{(k+1)!}{(k-n)!(n+1)!(k-n-1)}p^{n+2}q^{k-n-2}$$

$$=\frac{t}{k(1+t)^k}+t^2-t^2\left(\frac{t}{1+t}\right)^k-kt\left(\frac{t}{1+t}\right)^k+R, \qquad (2.4)$$

where

$$R=\sum_{n=0}^{k-2}\frac{(k+1)!}{(k-n)!(n+1)!(k-n-1)}p^{n+2}q^{k-n-2}.$$

Let us now analyze the term R. Letting $n+1=m$ we obtain

$$R=\sum_{m=1}^{k-1}\frac{(k+1)!p^{m+1}q^{k-m-1}}{(k+1-m)!m!(k-m)}=\sum_{m=1}^{k-1}\binom{k+1}{m}\frac{p^{m+1}q^{k-m-1}}{(k-m)}.$$

Since $(k-m)^{-1}=\dfrac{(k-m+2)}{(k-m+2)}\dfrac{1}{(k-m)}=(k-m+2)^{-1}\dfrac{(k-m+2)}{(k-m)}$

$$=(k-m+2)^{-1}\frac{(k-m)}{(k-m)}+\frac{2}{(k-m)}=(k-m+2)^{-1}\left(1+\frac{2}{k-m}\right), \qquad 1\le m\le k\text{-}1$$

$$\leq (k-m+2)^{-1}\left(1+\frac{2}{k-k+1}\right)=3/(k-m+2),$$

$$(k-m)^{-1}\leq 3/(k-m+2),$$

$(k\geq 2)$, we obtain

$$R\leq 3\sum_{m=1}^{k-1}\binom{k+1}{m}\frac{p^{m+1}q^{k-m-1}}{(k-m+2)}=3\sum_{m=1}^{k-1}\binom{k+1}{m}\frac{p^{m+1}q^{k-m-1-1+1}}{(k-m+2)}\leq\frac{3p}{q^2}\sum_{m=0}^{k+1}\binom{k+1}{m}\frac{p^m q^{k+1-m}}{(k-m+2)}$$

$$=\frac{3(t/1+t)}{(1/1+t)^2}\sum_{m=0}^{k+1}\binom{k+1}{m}p^m q^{k+1-m}(k-m+2)^{-1},$$

with $\sum_{m=0}^{k+1}b(m;k+1,p)=1$, considering the binomial distribution $b(k,p)$

$$R\leq 3\frac{t}{(1+t)}(1+t)^2 E(k-m+2)^{-1}.$$

$R\leq 3t(1+t)E(k-m+1+1)^{-1}=3t(1+t)E(\xi+1)^{-1}$, where W follows binomial distribution $b(k+1,p)$, and $\xi=k+1-W$ with $b(k+1,q)$, $q=1-p$. Hence a result of Chao and Strawderman [2, p. 430] gives

$$R\leq\frac{3t(1+t)\left(1-p^{k+2}\right)}{(k+2)q}$$

$$=\frac{3t(1+t)\left(1-(t/1+t)^{k+2}\right)}{(k+2)(1/1+t)}$$

$$=\frac{3t(1+t)\left(1-(t)^{k+2}/(1+t)^{k+2}\right)}{(k+2)/(1+t)}$$

$$=3t(1+t)\left(\frac{(1+t)^{k+2}-(t)^{k+2}}{(1+t)^{k+2}}\right)\frac{(1+t)}{(k+2)}$$

$$=\frac{3t(1+t)^2\left((1+t)^{k+2}-t^{k+2}\right)}{(1+t)^2(1+t)^k(k+2)}=\frac{3t\left((1+t)^{k+2}-t^{k+2}\right)}{(1+t)^k(k+2)}$$

$$= \frac{3t(1+t)^{k+2} - 3t^{k+3}}{(1+t)^k (k+2)} = \frac{3t(1+t)^{k+2}}{(1+t)^k (k+2)} - \frac{3t^{k+3}}{(1+t)^k (k+2)}$$

$$= \frac{3t(1+t)^2}{(k+2)} - \frac{3t^{k+3}}{(1+t)^k (k+2)} \le \frac{3t(1+t)^2}{(k+2)}$$

$$R \le \frac{3t(1+t)^2}{(k+2)}. \tag{2.5}$$

Letting $e_k(t) = E(T_k - t)^2$ it follows from (2.3), (2.4), and (2.5) that

$$e_k(t) = E(T_k - t)^2 = E\left(T_k^2 - 2tT_k + t^2\right)$$

$$e_k(t) = ET_k^2 - 2tET_k + t^2$$

$$e_k(t) = \frac{t}{k(1+t)^k} + t^2 - t^2\left(\frac{t}{1+t}\right)^k - kt\left(\frac{t}{1+t}\right)^k + R - 2t\left(t - tp^k\right) + t^2$$

$$e_k(t) = \frac{t}{k(1+t)^k} + t^2 - t^2\left(\frac{t}{1+t}\right)^k - kt\left(\frac{t}{1+t}\right)^k + R - 2t^2 + 2t^2\left(\frac{t}{1+t}\right) + t^2$$

$$e_k(t) = \frac{t}{k(1+t)^k} + t^2\left(\frac{t}{1+t}\right)^k - kt\left(\frac{t}{1+t}\right)^k + R$$

$$e_k(t) \le \frac{t}{k(1+t)^k} - kt\,\frac{t^k}{(1+t)^k} + t^2\left(\frac{t}{1+t}\right)^k + \frac{3t(1+k)^2}{k+2}$$

$$= \frac{t - k^2 t^{k+1}}{k(1+t)^k} + t^2\left(\frac{t}{1+t}\right)^k + \frac{k}{k+2}\,\frac{3t(1+t)^2}{k}.$$

Since $\left(t/(1+t)\right)^k \le (1+t)/k$, we have

$$e_k(t) \le \frac{t}{k} + \frac{t^2(1+t)}{k} + \frac{3t(1+t)^2}{k}$$

$$=\frac{t}{k}\left(1+t\left(1+t\right)\right)+\frac{3t\left(1+t\right)^2}{k}$$

$$=\frac{t}{k}\left(1+t+t^2\right)+\frac{3t\left(1+t\right)^2}{k}$$

$$\leq\frac{t}{k}\left(1+2t+t^2\right)+\frac{3t\left(1+t\right)^2}{k}$$

$$=\frac{t\left(1+t\right)^2}{k}+\frac{3t\left(1+t\right)^2}{k}=\frac{4t\left(1+t\right)^2}{k}$$

$$e_k\left(t\right)\leq\frac{4t\left(1+t\right)^2}{k}. \qquad (2.6)$$

Consider the space $C_B^*\left[0,\infty\right)$ of continuous function which is bounded on the half open interval $\left[0,\infty\right)$. It is clear that the space, $C_B\left[0,\infty\right)$ defined previously. To produce our first result, let us consider $\varphi\in C_B^*\left[0,\infty\right)$ and set $\upsilon\left(\varphi,\alpha\right)=\sup\left\{\left|\varphi\left(t\right)-\varphi\left(w\right)\right|:\left|t-w\right|\leq\alpha,\ t,w\in\left[0,\infty\right)\right\}$, $\alpha>0$. The convergence rate is obtained in terms of the first modulus of continuity $\upsilon\left(\varphi,\alpha\right)$, as defined in the following theorem.

Theorem 1. Consider $L_k\left(\varphi,t\right)$ be defined by $\left(2.1\right)$ and $\varphi\in C_B^*\left[0,\infty\right)$. Then

$$\left|L_k\left(\varphi,t\right)-\varphi\left(t\right)\right|\leq 3\upsilon\left(\varphi,\sqrt{t\left(1+t\right)^2/k}\right),\qquad \mathrm{k}\geq 1. \qquad (2.7)$$

Proof. Let $\alpha>0$ and $\kappa=\left[\left|T_k-t\right|/\alpha\right]$, $\left[a\right]$ stands for the greatest integer $\leq a$. Obviously,

$$\left|\varphi\left(T_k\right)-\varphi\left(t\right)\right|\leq\upsilon\left(\varphi,\alpha\right)\left(1+\kappa\right)\text{ , and}$$

$$\left|L_k\left(\varphi,t\right)-\varphi\left(t\right)\right|=\left|E\varphi\left(T_k\right)-\varphi\left(t\right)\right|\leq\upsilon\left(\varphi,\alpha\right)\left(1+E\kappa\right).$$

The inequality $E\kappa \le \sqrt{E\kappa^2} \le \sqrt{e_k(t)/\alpha^2}$ add to (2.6) lead to

$$\left|L_k(\varphi,t)-\varphi(t)\right| \le \upsilon(\varphi,\alpha)\left(1+\sqrt{e_k(t)/\alpha^2}\right),$$

$$\left|L_k(\varphi,t)-\varphi(t)\right| \le \upsilon(\varphi,\alpha)\left(1+\sqrt{\frac{4t(1+t)^2}{\mathrm{k}\,\alpha^2}}\right)$$

$$\left|L_k(\varphi,t)-\varphi(t)\right| \le \upsilon(\varphi,\alpha)\left(1+\frac{2(1+t)\sqrt{t}}{\sqrt{k}\;\alpha}\right)$$

and (2.7) is obtained by setting $\alpha=(1+t)\sqrt{t/k}$.

$$\left|L_k(\varphi,t)-\varphi(t)\right| \le \upsilon(\varphi,\alpha)\left(1+\frac{2(1+t)\sqrt{t}}{(1+t)\sqrt{t/k}\,\sqrt{k}}\right)$$

$$\left|L_k(f,x)-\varphi(t)\right| \le \upsilon(\varphi,\alpha)(1+2)$$

$$\left|L_k(\varphi,t)-\varphi(t)\right| \le 3\upsilon(\varphi,\alpha)$$

$$\left|L_k(\varphi,t)-\varphi(t)\right| \le 3\upsilon\left(\varphi,\sqrt{t(1+t)^2/k}\right).$$

The second modulus of continuity can also be exploited to obtain Theorem 1. (cf. [1]).

Let $\|\phi\| = \sup_{t\in[0,\infty)} |\phi(t)|$, where $\phi \in C_B[0,\infty)$, with

$$\mathrm{A}^2\varphi = \varphi(t+2y)-2\varphi(t+y)+\varphi(t) \quad (\varphi\in C_B[0,\infty)).$$

Let us define the second modulus of continuity by

$$\upsilon_2(\varphi,\alpha) = \sup_{y:\,|y|\le\alpha} \left\|\mathrm{A}^2\varphi\right\|, \quad \alpha>0.$$

Setting $g = T_k - t$, it follows from (2.3) that

$$\left|Eg\right| = \left|ET_k - t\right|,$$

with $\quad ET_k = t - tp^k \to t \quad$ as $\quad k \to \infty.$

$$\left|Eg\right| = \left|t - tp^k - t\right|$$

$$\left|Eg\right| = \left|t - t\left(t/(1+t)\right)^k - t\right|$$

$$\left|Eg\right| = \left|-t\left(t/(1+t)\right)^k\right|$$

$$\left|Eg\right| \le t\left(t/(1+t)\right)^k$$

$$\left|Eg\right| \le t\left(\frac{t}{1+t}\right)^k \le t\,\frac{(1+t)}{k}. \tag{2.8}$$

An improved version of a result established by Bleimann, Butzer, and Hahn is given below by dropping the condition $k \ge N(t) = 24(1+t)$ in Theorem 2 of $[1]$. Let us consider the following trivial result. Consider $h \in C_B[0,\infty)$ h' and $h'' \in C_B[0,\infty)$.

With $g = T_k - t$ we note that

$$h(T_k) - h(t) = \int_0^g h'(t+y)\,dy$$

$$u = h'(t+y) \qquad du = h''(t+y)\,dy$$

$$dv = 1 \qquad v = y$$

$$h(T_k) - h(t) = \int_0^g h'(t+y)\,dy = gh'(t+g) - \int_0^g yh''(t+y)\,dy$$

$$h(T_k) - h(t) = gh'(t+g) - \frac{y^2}{2}h''(t+y)\Big|_0^g$$

$$h(T_k) - h(t) = gh'(t+g) - \frac{1}{2}g^2h''(t+g)\,.$$

Taking expectation and using (2.6) and (2.8) it is easy to see that

$$\left|L_k(h,t)-h(t)\right|\le\left|Eg\right|\left\|h'\right\|+\frac{1}{2}Eg^2\left\|h''\right\|$$

$$\left|L_k(h,t)-h(t)\right|\le\frac{t(1+t)}{k}\left\|h'\right\|+\frac{1}{2}E(t_k-t)^2\left\|h''\right\|$$

where $\qquad e_k(t)=E(t_k-t)^2$

$$\left|L_k(h,t)-h(t)\right|\le\left\|h''\right\|\frac{1}{2}e_k(t)+\left\|h'\right\|\frac{t(1+t)}{k}$$

where $\qquad e_k(t)\le\dfrac{4t(1+t)^2}{k}$

$$\left|L_k(h,t)-h(t)\right|\le\frac{t(1+t)}{k}\left\|h'\right\|+\frac{1}{2}\frac{4t(1+t)^2}{k}\left\|h''\right\|.$$

It follows that

$$\left|L_k(h,t)-h(t)\right|\le\frac{2t(1+t)^2}{k}\left(\left\|h'\right\|+\left\|h''\right\|\right). \qquad (2.9)$$

Using (2.9) the following stronger version of the theorem is obtained.

Theorem 2. Consider $\varphi\in C_B[0,\infty)$, $\mathrm{t}\in[0,\infty)$. *Then for* $k=1,\ 2,\ \ldots,$

$$\left|L_k(\varphi,t)-\varphi(t)\right|\le 2C\left[\upsilon_2\left(\varphi,\sqrt{\frac{t(1+t)^2}{k}}\right)+\frac{t(1+t)^2}{k}\left\|\varphi\right\|\right],$$

where C is a constant, with the saturation condition given by $\sup_{t\ge0}\left|L_k(\varphi,t)-\varphi(t)\right|=O\left(k^{-1}\right)$.

The saturation properties depend on φ' and φ'', it is not an improvement of Theorem 2.

2.3 Limitation Property of L_k

Consider the function $\varphi \in C_B[0,\infty)$ and define the Szasz operator by

$$S_j(\varphi,t) = e^{-jt}\sum_{n=0}^{\infty}\varphi\left(\frac{n}{j}\right)\frac{(jt)^n}{n!}, \qquad t \geq 0, \tag{2.10}$$

with j being a positive and fixed integer. $S_j(\varphi,t)$ is a suitable limit function of L_k is an interesting consequence. The limiting property established is proved via the following lemma.

Lemma. *Let* $\mathrm{O}_n(jk,t) = \binom{jk}{n}(t/k)^n(1+t/k)^{-jk}$, $n = 0, 1, \ldots, jk$, *and*

$$\gamma_n(jt) = \exp(-jt)(jt)^n / n!, \qquad n = 0, 1, 2, \ldots \text{ Then}$$

$$(i)\ \gamma_n(jt)\exp\left(-n(n-1)/(jk-n+1)\right) \leq \mathrm{O}_n(jk,t)$$
$$\leq \gamma_n(jt)\exp\left(jt^2/(n+t)\right),$$

$$(ii)\ \sum_{n=0}^{jk}\left|\mathrm{O}_n(jk,t) - \gamma_n(jt)\right| \to 0 \text{ as } k \to \infty,$$

$$(iii)\ \max_{0\leq n\leq jk}\left|\mathrm{O}_n(jk,t) - \gamma_n(jt)\right| \to 0 \text{ as } k \to \infty.$$

Proof. Since $e^{-y} \geq 1 - y\ (0 \leq y \leq 1)$, it follows that

$$\mathrm{O}_n(jk,t) = \binom{jk}{n}\left(\frac{t}{k}\right)^n\left(1+\frac{t}{k}\right)^{-jk}$$

$$\mathrm{O}_n(jk,t) = \binom{jk}{n}\left(\frac{t}{k}\right)^n\left(\frac{k+t}{k+t}\right)^n\left(1+\frac{t}{k}\right)^{-jk}$$

$$\mathrm{O}_n(jk,t) = \binom{jk}{n}\left(\frac{t}{k+t}\right)^n\left(\frac{k+t}{k}\right)^n\left(\frac{k+t}{k}\right)^{-jk}$$

$$O_n\left(jk,t\right)=\binom{jk}{n}\left(\frac{t}{k+t}\right)^n\left(\frac{k}{k+t}\right)^{-n}\left(\frac{k}{k+t}\right)^{jk}$$

$$O_n\left(jk,t\right)=\binom{jk}{n}\left(\frac{t}{k+t}\right)^n\left(\frac{k+t-t}{k+t}\right)^{jk-n}$$

$$O_n\left(jk,t\right)=\binom{jk}{n}\left(\frac{t}{k+t}\right)^n\left(\frac{k+t}{k+t}-\frac{t}{k+t}\right)^{jk-n}$$

$$O_n\left(jk,t\right)=\binom{jk}{n}\left(\frac{t}{k+t}\right)^n\left(1-\frac{t}{k+t}\right)^{-n}\left(1-\frac{t}{k+t}\right)^{jk}$$

$$O_n\left(jk,t\right)=\binom{jk}{n}\left(\frac{t}{k+t}\right)^n\left(\frac{k+t-t}{k+t}\right)^{-n}\left(1-\frac{t}{k+t}\right)^{jk}$$

$$O_n\left(jk,t\right)=\binom{jk}{n}\frac{t^n}{\left(k+t\right)^n}\frac{\left(k+t\right)^n}{k^n}\left(1-\frac{t}{k+t}\right)^{jk}$$

$$O_n\left(jk,t\right)=\frac{\left(jk\right)!}{\left(jk-n\right)!n!}\frac{t^n}{k^n}\left(1-\frac{t}{k+t}\right)^{jk}$$

Since $\left(jk\right)!/\left(jk-n\right)!<\left(jk\right)^n$, we have

$$O_n\left(jk,t\right)\leq\frac{\left(jk\right)^n}{n!}\frac{t^n}{k^n}\left(1-\frac{t}{k+t}\right)^{jk}=\left(\frac{k+t-t}{k+t}\right)^{jk}\frac{\left(jt\right)^n}{n!}$$

$$O_n\left(jk,t\right)\leq\left(1-\frac{t}{k+t}\right)^{jk}\frac{\left(jt\right)^n}{n!},$$

where $\gamma_n\left(jt\right)=e^{-jt}\frac{\left(jt\right)^n}{n!}, \qquad n=0,\ 1,\ 2,\ \ldots$

$$\frac{\left(jt\right)^n}{n!}=\gamma_n\left(jt\right)e^{jt}$$

$$O_k\left(jk,t\right)\leq\gamma_n\left(jt\right)e^{jt}\left(1-\frac{t}{k+t}\right)^{jk}.$$

Since $e^{-y}\geq 1-y\ \left(0\leq y\leq 1\right)$, it follows that

$$\mathrm{O}_n(jk,t) \le \gamma_n(jt)\exp(jt)\exp\left(-(t)\,jk/(k+t)\right)$$

$$= \gamma_n(jt)\exp\left(jt - jkt/(k+t)\right)$$

$$= \gamma_n(jt)\exp\left(\left(jt(k+t) - jkt\right)/(k+t)\right)$$

$$= \gamma_n(jt)\exp\left(\left(jkt + jt^2 - jkt\right)/(k+t)\right)$$

$$\mathrm{O}_n(jk,t) \le \gamma_n(jt)\exp\left(jt^2/(k+t)\right).$$

Since $(1-\beta) \ge \exp\left(-\beta/(1-\beta)\right)$ $(0 \le \beta < 1)$, it follows that

$$\mathrm{O}_n(jk,t) = \binom{jk}{n}\left(\frac{t}{n}\right)^n\left(1+\frac{t}{k}\right)^{-jk}$$

$$\mathrm{O}_n(jk,t) = \frac{(jk)!}{(jk-n)!n!}\frac{t^n}{k^n}\left(\frac{k+t}{k}\right)^{-jk}$$

$$\mathrm{O}_n(jk,t) = \frac{(jk)!}{(jk-n)!n!}\frac{j^n}{j^n}\frac{t^n}{k^n}\left(\frac{k}{k+t}\right)^{jk}$$

$$\mathrm{O}_n(jk,t) = \frac{j^n t^n}{n!}\frac{(jk)!}{(jk-n)!\,j^n k^n}\left(\frac{k+t-t}{k+t}\right)^{jk}$$

$$\mathrm{O}_n(jk,t) = \frac{(jt)^n}{n!}\frac{(jk)!}{(jk-n)!(jk)^n}\left(\frac{k+t}{k+t}-\frac{t}{k+t}\right)^{jk}$$

$$\mathrm{O}_n(jk,t) = \frac{(jt)^n}{n!}\frac{(jk)(jk-1)\ldots(jk-n+1)(jk-n)!}{(jk-n)!(jk)^n}\left(1-\frac{t}{k+t}\right)^{jk}$$

$$\mathrm{O}_n(jk,t) = \frac{(jt)^n}{n!}\frac{(jk)(jk-1)\ldots(jk-n+1)}{(jk)^n}\left(1-\frac{t}{k+t}\right)^{jk}$$

$$\mathrm{O}_n(jk,t) = \frac{(jt)^n}{n!}\left(\frac{(jk)}{(jk)}\frac{(jk-1)}{(jk)}\cdots\frac{jk-(n-1)}{jk}\right)\left(1-\frac{t}{k+t}\right)^{jk}$$

$$O_n(jk,t)=\frac{(jt)^n}{n!}\prod_{i=1}^{n}\left(1-\frac{i-1}{jk}\right)\left(1-\frac{t}{k+t}\right)^{jk}$$

$$O_n(jk,t)\geq\frac{(jt)^n}{n!}\left(1-\frac{n-1}{jk}\right)^n\left(1-\frac{t}{k+t}\right)^{jk}.$$

Since $\gamma_n(jt)=\exp(-jt)(jt)^n/n!,\quad n$ being a positive integer,

Thus,

$$\frac{(jt)^n}{n!}=\gamma_n(jt)\exp(jt).$$

$$O_n(jk,t)\geq\gamma_n(jt)\exp(jt)\left(1-\frac{n-1}{jk}\right)^n\left(1-\frac{t}{k+t}\right)^{jk}.$$

Since $(1-\beta)\geq\exp(-\beta/(1-\beta)),\ for\ 0\leq\beta<1,$

$$O_n(jk,t)\geq\gamma_n(jt)\exp(jt)\exp\left(\frac{-(n-1)n}{jk}\bigg/\left(1-\frac{n-1}{jk}\right)\right)$$

$$\exp\left(\frac{-(t)\,jk}{k+t}\bigg/\left(1-\frac{t}{k+t}\right)\right)$$

$$O_n(jk,t)\geq\gamma_n(jt)\exp(jt)\exp\left(\frac{-(n-1)n}{jk}\bigg/\left(\frac{jk-n+1}{jk}\right)\right)$$

$$\exp\left(\frac{-(t)\,jk}{k+t}\bigg/\left(\frac{k}{k+t}\right)\right)$$

$$O_n(jk,t)\geq\gamma_n(jt)\exp(jt)\exp\left(\frac{-(n-1)n}{(jk)}\frac{(jk)}{jk-n+1}\right)$$

$$\exp\left(\frac{-jkt}{(k+t)}\frac{(k+t)}{k}\right)$$

$$\mathrm{O}_n(jk,t) \ge \gamma_n(jt)\exp(jt)\exp\left(\frac{-n(n-1)}{jk-n+1}\right)\exp(-jt)$$

$$\mathrm{O}_n(jk,t) \ge \gamma_n(jt)\exp\big(-n(n-1)/(jk-n+1)\big).$$

To prove (ii) let $u_n = \mathrm{O}_n(jk,t) - \gamma_n(jt)$, $\xi_n = \gamma_n(jt)\Big(\exp\big(jt^2/(k+t)\big)-1\Big)$, and

$\eta_n = \gamma_n(jt)\Big(1-\exp\big(-n(n-1)/(jk-n+1)\big)\Big)$.

Since $-\eta_n \le u_n \le \xi_n$ from (i), and $|u_n| \le \xi_n + \eta_n$, we have

$$\sum_{n=0}^{jk}|u_n| \le \sum_{n=0}^{jk}\xi_n + \sum_{n=0}^{jk}\eta_n. \qquad (2.11)$$

It is obvious that

$$\xi_n = \gamma_n(jt)\Big(\exp\big(jt^2/(k+t)\big)-1\Big)$$

$$\sum_{n=0}^{jk}\xi_n = \sum_{n=0}^{jk}\gamma_n(jt)\Big(\exp\big(jt^2/(k+t)\big)-1\Big)$$

$$\sum_{n=0}^{jk}\xi_n = \Big(\exp\big(jt^2/(k+t)\big)-1\Big)\sum_{n=0}^{jk}\gamma_n(jt)$$

$$\sum_{n=0}^{jk}\xi_n = \Big(\exp\big(jt^2/(k+t)\big)-1\Big)\sum_{n=0}^{jk}\exp(-jk)(jt)^n/n!$$

$$\sum_{n=0}^{jk}\xi_n = \Big(\exp\big(jt^2/(k+t)\big)-1\Big)\exp(-jk)\sum_{n=0}^{jk}(jt)^n/n!$$

$$\sum_{n=0}^{jk}\xi_n \le \Big(\exp\big(jt^2/(k+t)\big)-1\Big)\exp(-jk)\sum_{n=0}^{\infty}(jt)^n/n!,$$

where $\displaystyle\sum_{n=0}^{\infty}(jt)^n/n! = \exp(jt)$

$$\sum_{n=0}^{jk}\xi_n \le \Big(\exp\big(jt^2/(k+t)\big)-1\Big)\exp(-jk)\exp(jt)$$

$$\sum_{n=0}^{jk}\xi_n \le \left(\exp\left(jt^2/(k+t)\right)-1\right)\to 0 \quad \text{as} \quad k\to\infty. \tag{2.12}$$

$$\eta_n = \gamma_n(jt)\left(1-\exp\left(\frac{-n(n-1)}{(jk-n+1)}\right)\right)$$

$$\sum_{n=0}^{jk}\eta_n = \sum_{n=0}^{jk}\gamma_n(jt)\left(1-\exp\left(\frac{-n(n-1)}{(jk-n+1)}\right)\right),$$

since $\exp(-\upsilon)\ge 1-\min(1,\upsilon)\ (\upsilon\ge 0)$, it follows that

$$\sum_{n=0}^{jk}\eta_n \le \sum_{n=0}^{jk}\gamma_n(jt)\left(1-\left(1-\min\left(1,\frac{n(n-1)}{jk-n+1}\right)\right)\right)$$

$$=\sum_{n=0}^{jk}\gamma_n(jt)\left(1-1+\min\left(1,\frac{n(n-1)}{jk-n+1}\right)\right)$$

$$\sum_{n=0}^{jk}\eta_n \le \sum_{n=0}^{jk}\gamma_n(jt)\min\left(1,\frac{n(n-1)}{jk-n+1}\right)$$

$$\le \sum_{n=0}^{jk}\gamma_n(jt)\left(1-\exp\left(\frac{-n(n-1)}{jk-n+1}\right)\right),$$

where $\dfrac{n(n-1)}{jk-n+1}<1$. Using,

$$n(n-1)-(jk-n+1)<0$$

$$n<\sqrt{jk+1}$$

we have

$$\sum_{n=0}^{jk}\eta_n \le 0+\sum_{n=2}^{\sqrt{jk+1}}\gamma_n(jt)\frac{n(n-1)}{jk-n+1}$$

$$\sum_{n=0}^{jk}\eta_n \le 0+\sum_{n=2}^{\sqrt{jk+1}}\gamma_n(jt)\frac{n(n-1)}{jk-n+1}+\sum_{\sqrt{jk+1}+1}^{\infty}\gamma_n(jt)\frac{n(n-1)}{jk-n+1}$$

$$\sum_{n=0}^{jk}\eta_n \le 0+\sum_{n=2}^{\sqrt{jk+1}}\gamma_n(jt)\frac{n(n-1)}{jk-n+1}+\sum_{\sqrt{jk+1}+1}^{\infty}\exp(-jk)\frac{(jt)^n}{n!}\frac{n(n-1)}{jk-n+1},$$

where $\quad P(U=n)=\exp(-jt)\dfrac{(jt)^n}{n!}$

$$\sum_{n=0}^{jk}\eta_k \le \sum_{n=2}^{\left[\sqrt{jk+1}\right]}\gamma_n(jt)\frac{n(n-1)}{jk-n+1}+P\left(U>\sqrt{jk+1}\right),$$

with U being the Poisson random variable with mean jt. One can easily check the following relation

$$\sum_{n=0}^{jk}\eta_n \le \frac{(jt)^2}{\left(jk+1-\sqrt{jk+1}\right)}+P\left(U>\sqrt{jk+1}\right)\to 0,\text{ as } k\to\infty. \qquad (2.13)$$

(ii) results from (2.11), (2.12), and (2.13), and (ii) implies (iii).

Theorem 3. Consider L_k and S_j defined by (2.1) and (2.10) respectively for $\varphi\in C_B[0,\infty)$. It follows that for each $t\in[0,\infty)$ and for any fixed integer j,

$$L_{jk}\left(\varphi\left(\frac{kt}{1+t}\right),\frac{t}{k}\right)\to S_j(\varphi,t),\qquad \text{as}\quad k\to\infty. \qquad (2.14)$$

Proof. From (2.1) and (2.10) we have

$$L_{jk}\left(\varphi\left(\frac{kt}{1+t}\right),\frac{t}{k}\right)=\sum_{n=0}^{jk}\varphi\left(\frac{kn/jk-n+1}{1+n/jk-n+1}\right)\binom{jk}{n}\left(\frac{t}{k}\right)^n\left(1+\frac{t}{k}\right)^{-jk}$$

$$L_{jk}\left(\varphi\left(\frac{kt}{1+t}\right),\frac{t}{k}\right)=\sum_{n=0}^{jk}\varphi\left(\frac{kn/jk-n+1}{jk-n+1+n/jk-n+1}\right)\binom{jk}{n}\left(\frac{t}{k}\right)^n\left(1+\frac{t}{k}\right)^{-jk}$$

$$L_{jk}\left(\varphi\left(\frac{kt}{1+t}\right),\frac{t}{k}\right)=\sum_{n=0}^{jk}\varphi\left(\frac{kn}{(jk-n+1)}\frac{(jk-n+1)}{jk+1}\right)\binom{jk}{n}\left(\frac{t}{k}\right)^n\left(1+\frac{t}{k}\right)^{-jk}$$

$$L_{jk}\left(\varphi\left(\frac{kt}{1+t}\right),\frac{t}{k}\right)=\sum_{n=0}^{jk}\varphi\left(\frac{kn}{jk+1}\right)\binom{jk}{n}\left(\frac{t}{k}\right)^n\left(1+\frac{t}{k}\right)^{-jk}=\sigma_k(j,t)$$

$$L_{jk}\left(\varphi\left(\frac{kt}{1+t}\right),\frac{t}{k}\right)=\sum_{n=0}^{jk}\varphi\left(\frac{kn}{jk+1}\right)\mathrm{O}_n\left(jk,t\right)=\sigma_k\left(j,t\right),$$

and

$$S_j\left(\varphi,t\right)=e^{-jt}\sum_{n=0}^{\infty}\varphi\left(\frac{n}{j}\right)\frac{\left(jt\right)^n}{n!},\qquad t\geq 0$$

$$S_j\left(\varphi,t\right)=\sum_{n=0}^{\infty}\varphi\left(\frac{n}{j}\right)\exp\left(-jk\right)\frac{\left(jt\right)^n}{n!}$$

$$S_j\left(\varphi,t\right)=\sum_{n=0}^{jk}\varphi\left(\frac{n}{j}\right)\exp\left(-jt\right)\frac{\left(jt\right)^n}{n!}+\sum_{n=jk+1}^{\infty}\varphi\left(\frac{n}{j}\right)\exp\left(-jt\right)\frac{\left(jt\right)^n}{n!}$$

$$S_j\left(\varphi,t\right)=\sum_{n=0}^{jk}\varphi\left(\frac{n}{j}\right)\gamma_n\left(jt\right)+\sum_{n=jk+1}^{\infty}\varphi\left(\frac{n}{j}\right)\gamma_n\left(jt\right)$$

$$=Q_k\left(j,t\right)+R_k\left(j,t\right).$$

Thus

$$\left|L_{jk}\left(\varphi\left(\frac{kt}{1+t}\right),\frac{t}{k}\right)-S_j\left(\varphi,t\right)\right|=\left|\sigma_k\left(j,t\right)-Q_k\left(j,t\right)-R_k\left(j,t\right)\right|$$

$$\left|L_{jk}\left(\varphi\left(\frac{kt}{1+t}\right),\frac{t}{k}\right)-S_j\left(\varphi,t\right)\right|\leq\left|\sigma_k\left(j,t\right)-Q_k\left(j,t\right)\right|+\left|R_k\left(j,t\right)\right|. \qquad (2.15)$$

The function φ is bounded, thus the following relation holds

$$\left|R_k\left(j,t\right)\right|=\left|\sum_{n=jk+1}^{\infty}\varphi\left(\frac{n}{j}\right)\gamma_n\left(jt\right)\right|\leq M\sum_{n=jk+1}^{\infty}\gamma_n\left(jt\right)\to 0\quad \text{as}\quad k\to\infty. \quad (2.16)$$

Furthermore,

$$\left|\sigma_k\left(j,t\right)-Q_k\left(j,t\right)\right|=\left|\sum_{n=0}^{jk}\varphi\left(\frac{kn}{jk+1}\right)\mathrm{O}_n\left(jk,t\right)-\sum_{n=0}^{jk}\varphi\left(\frac{n}{j}\right)\gamma_n\left(jt\right)\right|$$

$$=\left|\sum_{n=0}^{jk}\varphi\left(\frac{kn}{jk+1}\right)\mathrm{O}_n(jk,t)-\sum_{n=0}^{jk}\varphi\left(\frac{kn}{jk+1}\right)\gamma_n(jt)\right.$$

$$\left.+\sum_{n=0}^{jk}\varphi\left(\frac{kn}{jk+1}\right)\gamma_n(jt)-\sum_{n=0}^{jk}\varphi\left(\frac{n}{j}\right)\gamma_n(jt)\right|$$

$$\leq\left|\sum_{n=0}^{jk}\varphi\left(\frac{kn}{jk+1}\right)\mathrm{O}_n(jk,t)-\sum_{n=0}^{jk}\varphi\left(\frac{kn}{jk+1}\right)\gamma_n(jt)\right|$$

$$+\left|\sum_{n=0}^{jk}\varphi\left(\frac{kn}{jk+1}\right)\gamma_n(jt)-\sum_{n=0}^{jk}\varphi\left(\frac{n}{j}\right)\gamma_n(jt)\right|$$

$$=\sum_{n=0}^{jk}\left|\varphi\left(\frac{kn}{jk+1}\right)\right|\left|\mathrm{O}_n(jk,t)-\gamma_n(jt)\right|$$

$$+\sum_{n=0}^{jk}\left|\varphi\left(\frac{kn}{jk+1}\right)-\varphi\left(\frac{n}{j}\right)\right|\gamma_n(jt)$$

$$\leq M\sum_{n=0}^{jk}\left|\mathrm{O}_n(jk,t)-\gamma_n(jt)\right|$$

$$+\sum_{n=0}^{jk}\left|\varphi\left(\frac{kn}{jk+1}\right)-\varphi\left(\frac{n}{j}\right)\right|\gamma_n(jt).$$

Since the function φ is uniformly continuous on the interval $[0,\infty)$, it follows that, for a given $\varepsilon>0$ there exists an integer N_0 such that for any integer $k\geq N_0$,

$$\sum_{n=0}^{jk}\left|\varphi\left(\frac{kn}{jk+1}\right)-\varphi\left(\frac{n}{j}\right)\right|\gamma_n(jt)\leq\varepsilon.$$

Thus for $k\geq N_0$ we have

$$\left|\sigma_k(j,t)-Q_k(j,t)\right|\leq M\sum_{n=0}^{jk}\left|\mathrm{O}_n(jk,t)-\gamma_n(jt)\right|+\varepsilon.$$

We obtain by the previous Lemma that

$$\left|\sigma_k(j,t)-Q_k(j,t)\right|\to 0 \quad \text{as} \quad \mathrm{k}\to\infty, \tag{2.17}$$

and (2.14) follows from (2.15), (2.16), and (2.17).

Chapter 3

APPROXIMATION PROPERTIES OF BERNSTEIN POLYNOMIALS VIA PROBABLISTIC TOOLS

3.1 Introduction

Consider a real valued function φ, defined on the closed interval $[0,1]$. Consider $S_{k,t}$ to be a Binomial random variable whose parameters are k and t, and let $E[T]$ being the expectation value of the random operator T. In the previous chapter we proved the derivative of the convergence rate of the Bernstein polynomial from the large deviation theory as follows:

$$B_k \varphi(t) = \sum_{j=0}^{k} \binom{k}{j} t^j (1-t)^{k-j} \varphi\left(\frac{j}{k}\right) = E\varphi\left(\frac{S_{k,t}}{k}\right). \tag{3.1}$$

The optimal convergence rates of Lipshitz function is $k^{-1/2}$. For a Hölder continuous function with exponent γ for some $0 < \gamma \le 1$, the convergence rate is $k^{\frac{-\gamma}{2}}$.

The following two questions come to mind when comparing the given approximation of φ. The first is the behavior of $B_k \varphi(t_0)$ when a jump discontinuity appears at a given point t_0. Secondly does the Gibbs phenomenon appear also?

The probabilistic tools are still used in this chapter for the approximation properties of $B_k \varphi(t)$ when the function φ is not continuous. We showed approximating smooth function defined piecewise with left and right derivative, by Bernstein polynomials there is no appearance of the Gibbs phenomenon. Let us consider a jump function and let us prove that $B_k\varphi(t_0) \to \frac{1}{2}\left(\varphi(t_0+0)+\varphi(t_0-0)\right)$ as $k \to \infty$, with a monotone convergence on both left and right sides of the discontinuity. We also prove that for a given bounded function φ, the derivative function $(B_k\varphi)'(t)$ converges to $\varphi'(t)$ wherever $\varphi'(t)$ is defined.

Consider φ_k which approximate φ in a piecewise form and which possesses left and right derivatives at every point. For the general case, the Gibbs phenomenon is described as follows:

(i) If t_0 is a discontinuity point of φ, then

$$\lim_{k\to\infty}\varphi_k(t) = \frac{\varphi(t+0)+\varphi(t-0)}{2}.$$

(ii) On any closed sub interval $[t_1,t_2]$ on which the function is continuous, the function is uniformly convergent:

$$\lim_{k\uparrow\infty}\max_{t_1\le t\le t_2}\left|\varphi_k(t)-\varphi(t)\right| = 0.$$

(iii) On any subinterval containing a single discontinuity t_0 of the function, we have Gibbs phenomenon: for small $\alpha>0$

$$\lim_{k\uparrow\infty}\left(\max_{|t_0-t|\le\alpha}\varphi_k(t)-\min_{|t_0-t|\le\alpha}\varphi_k(t)\right) = Q\left|\varphi(t_0+0)-\varphi(t_0-0)\right|,$$

where

$$Q=\frac{2}{\pi}\int_0^{\pi}\left(\frac{\sin t}{t}\right)dt\approx 1.18.$$

In what follows we will prove that the Bernstein Polynomial approximant satisfies the conditions (i) and (ii) above also that (iii) holds under the condition $Q=1$ whenever φ is a finite sum of jump functions. Finally we write $(B_k\varphi)'(t)$ in the form of the expectation of a function of a binomial variable and then we investigate its' rate of convergence to $\varphi'(t)$.

3.2 Investigation of No Gibbs Phenomenon of Bernstein Polynomials

Consider a simple jump function

$$\varphi(t)=\begin{cases} c & \mathrm{t<t_0} \\ d\ (>c) & \mathrm{t}\geq t_0 \end{cases}. \tag{3.2}$$

The Bernstein polynomial is given by

$$B_k\varphi(t)=cP\left(\frac{S_{k,t}}{k}<t_0\right)+dP\left(\frac{S_{k,t}}{k}\geq t_0\right)=c+(d-c)P\left(\frac{S_{k,t}}{k}\geq t_0\right),$$

and the following boundedness condition holds:

$$c=B_k\varphi(0)\leq B_k\varphi(t)\leq B_k\varphi(1)=d. \tag{3.3}$$

The function $B_k\varphi$ increases because of that if $t\leq w$ for $0\leq n\leq k$ we have

$$P\left(S_{k,t}\geq n\right)=\sum_{j\geq n}\binom{k}{j}t^j(1-t)^{k-j}\leq\sum_{j\geq n}\binom{k}{j}w^j(1-w)^{k-j}=P\left(S_{k,w}\geq n\right). \tag{3.4}$$

Left and right sides of (3.4) are all equal to 1 for $n=0$.

A great technique to obtain (3.4) is to choose many copies of the bivariate 0 – 1-valued independent variables $\left(T_{j,}W_{j}\right)$, $1\leq j\leq k$, with joint distribution defined by the following probabilities $P\left(T_{j}=W_{j}=1\right)=t$, $P\left(T_{j}=1,\ \mathrm{W}_{j}=0\right)=0$ and $P\left(T_{j}=W_{j}=0\right)=1-w$. If we consider Z and W to be respectively $Z=\sum_{j=1}^{k}T_{j}$ and $Y=\sum_{j=1}^{k}W_{j}$, it follows that the distribution of Z and W are respectively those of, $S_{k,t}$ and $S_{k,w}$, one can therefore construct $\{Z\geq n\}\subset\{Y\geq n\}$, such that

$$P\left(S_{k,t}\geq n\right)=P\left(Z\geq n\right)\leq P\left(Y\geq n\right)=P\left(S_{k,w}\geq n\right).$$

Let us now focus on the uniform convergence on the interval $[t_1,t_2]\subset[0,1]-\{t_0\}$. Let us first consider $t_2<t_0$. It follows by Chebyschev's inequality that

$$P\left(\frac{S_{k,t}}{k}\geq t_0\right)\leq P\left(\left|\frac{S_{k,t}}{t}-t\right|\geq t_0-t\right)$$

$$\leq\frac{t\left(1-t\right)}{k\left(t_0-t\right)^2}\leq\frac{1}{4k\left(t_0-t_2\right)^2}, \qquad (3.5)$$

implying that $B_k\varphi(t)\rightarrow c=\varphi(t)$, uniformly on the interval $[t_1,t_2]$. The same argument holds when we choose $t\in[t_1,t_2]$ such a way that $t_0<t_1$.

On the other side, if $t=t_0$ then

$$B_k\varphi(t_0)=c+(d-c)P\left(\frac{S_{k,t_0}}{k}\geq t_0\right).$$

If we now consider $T_j, 1\leq j\leq k,$ as a sequence of independent Bernoulli random variables with parameter t_0 it follows:

$$P\left(S_{k,t_0}/k \geq t_0\right) = P\left(\sum_{j=1}^{k}\left(T_j - t_0\right) \geq 0\right) = P\left(\frac{\sum_{j=1}^{k}\left(T_j - t_0\right)}{\sqrt{kt_0\left(1-t_0\right)}} \geq 0\right) \to \frac{1}{2} \quad \text{as } k \to \infty,$$

using the Central Limit Theorem (CLT), and the relation

$$\left|B_k\varphi(t_0) - \frac{c+d}{2}\right| = |c-d|\left|P\left(\frac{S_{k,t_0}}{k} \geq t_0\right) - \frac{1}{2}\right| \to 0 \qquad \text{as} \qquad k \to \infty,$$

this means that $B_k\varphi(t_0)$ is convergent to the average left and right limits of the function φ at t_0.

The computations above verify the conditions (i) and (ii) for a jump function as described in (2).

To check whether (iii) holds with the value $Q=1$, i.e., we consider the relation

$$\max_{|t_0-t|\leq\alpha} B_k\varphi(t) - \min_{|t_0-t|\leq\alpha} B_k\varphi(t) = B_k\varphi(t_0+\alpha) - B_k\varphi(t_0-\alpha),$$

if $n=\infty$ then from (ii) on $t_0-\alpha$ and $t_0+\alpha$ it follows that

$$\lim_{k\uparrow\infty}\left(\max_{|t_0-t|\leq\alpha} B_k\varphi(t) - \min_{|t_0-t|\leq\alpha} B_k\varphi(t)\right) = \left|\varphi(t_0+\alpha) - \varphi(t_0-\alpha)\right|$$

$$= \left|\varphi(t_0+0) - \varphi(t_0-0)\right|.$$

Comments After all the investigation done so far, the conclusion is that the approximation by Bernstein polynomials has no Gibbs phenomenon.

If t_0 is a discontinuity of φ then

$$\lim_{\alpha\to 0}\lim_{k\to\infty}\left(\max_{|t_0-t|\leq\alpha} B_k\varphi(t) - \min_{|t_0-t|\leq\alpha} B_k\varphi(t)\right) = \left|\varphi(t_0+0) - \varphi(t_0-0)\right|.$$

The equation above lead to the following situation (problem) that seems to have no answer: Iis there a special type of sequence which cause overshoot phenomenon when we are approximating a jump? Is the approximation by orthogonal polynomial cause a jump? If the answer to second question is yes, let us not forget that Bernstein polynomial does not cause a jump. Is it because such polynomials are smooth? From the questions stated above, one can conclude that there is no general formula to approximate the solution. Nevertheless, Gibbs phenomenon may occur when the approximation is at an arbitrary order. The approximation by Bernstein polynomials holds only up to order k^{-1}, though it is smooth polynomial.

We did not focus in our previous chapter on the convergence speed. That is one of our purpose of interest in this section.

Lemma. For a binomial variable $S_{k,t}$ with $c > 0$ chosen arbitrarily, we have

$$P\left(\left|S_{k,t} - kt\right| > c\right) \leq 2e^{-\frac{2c^2}{k}}. \tag{3.6}$$

It follows for instance that the convergence bound $N\left(\frac{1}{k}\right)$ given in (ii) computed by Chebyschev's inequality in (3.5) can be increased up to the exponential bound $2e^{-2k(t_0-t)^2}$. Similarly, the condition (iii) can be checked and it can be proved that the convergence speed limit is exponential.

3.3 The Convergences Speed of $(B_k\varphi)(t)'$ towards $\varphi'(t)$

We previously showed that the function increases $B_k\varphi$ is as the jump type function φ increases using the binomial distribution. A general result is that when $\varphi(t)$

increases (respectively. decreases), then the derivative $(B_k\varphi)'(t)$ of $(B_k\varphi)(t)$ is positive (respectively. negative), thus $B_k\varphi$ also increases (respectively. decreasing).

Proposition 1. The following is the Bernstein polynomial $B_k\varphi$ can be expressed derivative

$$(B_k\varphi)'(t)=E\left[\left(\frac{S_{k,t}-kt}{\sqrt{kt(1-t)}}\right)^2 G(k,\varphi)(t)\right], \qquad (3.7)$$

where

$$G(k,\varphi)(t)=\frac{\varphi\left(\frac{s_{k,t}}{k}\right)-\varphi(t)}{\frac{s_{k,t}}{k}-t}.$$

Proof. Deriving (1.1) with respect to t we obtain

$$(B_k\varphi)'(t)=\sum_{j=0}^{k}\binom{k}{j} j\varphi\left(\frac{j}{k}\right)t^{j-1}(1-t)^{k-j}-\sum_{j=0}^{k}\binom{k}{j}(k-j)\varphi\left(\frac{j}{k}\right)t^{j}(1-t)^{k-j-1},$$

with $j=S_{k,t}$

$$(B_k\varphi)'(t)=\frac{1}{t}ES_{k,t}\varphi\left(\frac{S_{k,t}}{k}\right)-\frac{1}{1-t}E\left(k-S_{k,t}\right)\varphi\left(\frac{S_{k,t}}{k}\right)$$

$$(B_k\varphi)'(t)=\frac{1-t}{1-t}\frac{1}{t}ES_{k,t}\varphi\left(\frac{S_{k,t}}{k}\right)-\frac{t}{t}\frac{1}{1-t}E\left(k-S_{k,t}\right)\varphi\left(\frac{S_{k,t}}{k}\right)$$

$$(B_k\varphi)'(t)=\frac{1}{t(1-t)}\left[(1-t)ES_{k,t}\varphi\left(\frac{S_{k,t}}{k}\right)-tE\left(k-S_{k,t}\right)\varphi\left(\frac{S_{k,t}}{k}\right)\right]$$

$$(B_k\varphi)'(t)=\frac{1}{t(1-t)}E\left[\left(S_{k,t}-tS_{k,t}-kt+tS_{k,t}\right)\varphi\left(\frac{S_{k,t}}{k}\right)\right]$$

$$(B_k\varphi)'(t)=\frac{1}{t(1-t)}E\left[\left(S_{k,t}-kt\right)\varphi\left(\frac{S_{k,t}}{k}\right)\right]$$

$$\left(B_k\varphi\right)'(t)=\frac{1}{t\,(1-t)}E\left[\left(S_{k,t}-kt\right)\left(\varphi\left(\frac{S_{k,t}}{k}\right)-\varphi(t)+\varphi(t)\right)\right]$$

$$\left(B_k\varphi\right)'(t)=\frac{1}{t\,(1-t)}E\left(S_{k,t}-kt\right)\varphi\left(\frac{S_{k,t}}{k}-\varphi(t)\right)+\frac{1}{t\,(1-t)}E\left(S_{k,t}-kt\right)\varphi(t)$$

$$\left(B_k\varphi\right)'(t)=\frac{1}{t\,(1-t)}E\left(S_{\mathrm{k,t}}-kt\right)\varphi\left(\frac{S_{\mathrm{k,t}}}{k}-\varphi(t)\right)$$

$$+\frac{1}{t\,(1-t)}E\left(S_{\mathrm{k,t}}-kt\right)\varphi(t),$$

since $E\left(S_{k,t}\right)=kt$,

$$\left(B_k\varphi\right)'(t)=\frac{1}{t\,(1-t)}E\left(S_{k,t}-kt\right)\varphi\left(\frac{S_{k,t}}{k}-\varphi(t)\right)+\frac{1}{t\,(1-t)}\left(kt-kt\right)\varphi(t)$$

$$\left(B_k\varphi\right)'(t)=\frac{1}{t\,(1-t)}E\left[\left(S_{k,t}-kt\right)\left(\varphi\left(\frac{S_{k,t}}{k}\right)-\varphi(t)\right)\right]$$

$$\left(B_k\varphi\right)'(t)=\frac{1}{t\,(1-t)}E\left[\left(S_{k,t}-kt\right)\frac{S_{k,t}-kt}{S_{k,t}-kt}\left(\varphi\left(\frac{S_{k,t}}{k}\right)-\varphi(t)\right)\right]$$

$$\left(B_k\varphi\right)'(t)=E\left[\frac{\left(S_{k,t}-kt\right)^2}{t\,(1-t)}\left(\frac{\varphi\left(S_{k,t}/k\right)-\varphi(t)}{S_{k,t}-kt}\right)\right]$$

$$\left(B_k\varphi\right)'(t)=E\left[\frac{\left(S_{k,t}-kt\right)^2}{t\,(1-t)}\left(\frac{\varphi\left(S_{k,t}/k\right)-\varphi(t)}{k\left(S_{k,t}/k-t\right)}\right)\right]$$

$$\left(B_k\varphi\right)'(t)=E\left[\frac{\left(s_{k,t}-kt\right)^2}{kt\,(1-t)}\left(\frac{\varphi\left(S_{k,t}/k\right)-\varphi(t)}{\left(S_{k,t}/k\right)-t}\right)\right]$$

$$\left(B_k\varphi\right)'(t)=E\left[\left(\frac{S_{k,t}-kt}{\sqrt{kt\,(1-t)}}\right)^2\left(\frac{\varphi\left(S_{k,t}/k\right)-\varphi(t)}{\left(S_{k,t}/k\right)-t}\right)\right]$$

$$\left(B_k\varphi\right)'(t)=E\left[\left(\frac{S_{k,t}-kt}{\sqrt{kt(1-t)}}\right)^2 G(k,\varphi)(t)\right],$$

As assumed.

It is therefore obvious that $\left(B_k\varphi\right)'(t)$ is positive (negative) when $G(k,\varphi)(t)$ is positive (negative), and also that this exists only if φ increases (decreases).

Moreover, if the first derivative is defined at a given point t $G(k,\varphi)(t)\to\varphi'(t)$ as $k\to\infty$, and also if the squared part of the integral (3.7), from limit theory it is convergent to the square of the standard normal random variable. It will follow that $\left(B_k\varphi\right)'(t)\approx\varphi'(t)$ when k is choose to be large. The probabilistic proof is used for rigor.

Proposition 2. Consider $\sup_{t\in[0,1]}\varphi(t)=M<\infty$. it follows for any t such that $\varphi'(t)$ is defined that

$$\lim_{k\to\infty}\left(B_k\varphi\right)'(t)=\varphi'(t).$$

Proof. Define $H(k,\varphi)(t)=G(k,\varphi)(t)-\varphi'(t)$. We have

$$\left(B_k\varphi\right)'(t)=E\left[\left(\frac{S_{k,t}-kt}{\sqrt{kt(1-t)}}\right)^2 G(k,\varphi)(t)\right]$$

$$\left(B_k\varphi\right)'(t)=E\left[\left(\frac{S_{k,t}-kt}{\sqrt{kt(1-t)}}\right)^2 \left(H(k,\varphi)(t)+\varphi'(t)\right)\right]$$

$$\left(B_k\varphi\right)'(t)=E\left(\frac{S_{k,t}-kt}{\sqrt{kt(1-t)}}\right)^2 H(k,\varphi)(t)+E\left(\frac{S_{k,t}-kt}{\sqrt{kt(1-t)}}\right)^2\varphi'(t)$$

$$(B_k\varphi)'(t)=\varphi'(t)+E\left[\left(\frac{S_{k,t}-kt}{\sqrt{kt(1-t)}}\right)^2 H(k,\varphi)(t)\right], \tag{3.8}$$

since $E\left(\frac{S_{k,t}-kt}{\sqrt{kt(1-t)}}\right)^2=1.$

Because $\varphi'(t)$ is defined for $\zeta>0,$ there is $\alpha>0$ such that if $\left|\frac{S_{k,t}}{k}-t\right|<\alpha$, then $\left|H(k,\varphi)(t)\right|<\zeta.$ We show that the absolute value in (3.8) goes to zero if we split and bound it as follows:

$$E\left[\left(\frac{S_{k,t}-kt}{\sqrt{kt(1-t)}}\right)^2 \left|H(k,\varphi)(t)\right| 1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|<\alpha\right\}}\right]$$

$$+E\left[\left(\frac{S_{k,t}-kt}{\sqrt{kt(1-t)}}\right)^2 \left|H(k,\varphi)(t)\right| 1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|<\alpha\right\}}\right], \tag{3.9}$$

where 1_H is the indicator function of the set H. The first summand in (3.9) can be bounded by

$$\zeta E\left[\left(\frac{S_{k,t}-kt}{\sqrt{kt(1-t)}}\right)^2 1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|<\alpha\right\}}\right]\le\zeta, \tag{3.10}$$

since $E\left(\frac{S_{k,t}-kt}{\sqrt{kt(1-t)}}\right)^2=\operatorname{var}(Z)$ as $k\to 0$, by the Central Limit Theorem.

2^{nd} summand can be evaluated as follows:

$$E\left[\left(\frac{S_{k,t}-kt}{\sqrt{kt(1-t)}}\right)^2 \left|H(k,\varphi)(t)\right| 1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\ge\alpha\right\}}\right]$$

$$= E\left[\left(\frac{S_{k,t}-kt}{\sqrt{kt(1-t)}}\right)^2 \left(\left(G(k,\varphi)\right)(t)-(\varphi)'(t)\right)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\ge\alpha\right\}}\right]$$

$$= E\left[\left(\frac{S_{k,t}-t}{\sqrt{kt(1-t)}}\right)^2 G(k,\varphi)(t)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\ge\alpha\right\}}\right]$$

$$-E\left[\left(\frac{S_{k,t}-t}{\sqrt{kt(1-t)}}\right)^2 \varphi'(t)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\ge\alpha\right\}}\right]$$

$$= \frac{k}{t(1-t)}E\left[\left(\frac{S_{k,t}}{k}-t\right)^2 G(k,\varphi)(t)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\ge\alpha\right\}}\right]$$

$$-\frac{k}{t(1-t)}E\left[\left(\frac{S_{k,t}}{k}-t\right)^2 \varphi'(t)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\ge\alpha\right\}}\right],$$

where $G(k,\varphi)(t)=\dfrac{\varphi\left(\frac{S_{k,t}}{k}\right)-\varphi(t)}{\frac{S_{k,t}}{k}-t}$.

$$= \frac{k}{t(1-t)}E\left[\left(\frac{S_{k,t}}{k}-t\right)^2 \frac{\varphi\left(\frac{S_{k,t}}{k}\right)-\varphi(t)}{\frac{S_{k,t}}{k}-t}1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\ge\alpha\right\}}\right]$$

$$-\frac{k}{t(1-t)}E\left[\left(\frac{S_{k,t}}{k}-t\right)^2 \varphi'(t)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\ge\alpha\right\}}\right]$$

$$= \frac{k}{t(1-t)}E\left[\left(\frac{S_{k,t}}{k}-t\right)\left(\varphi\left(\frac{S_{k,t}}{k}\right)-\varphi(t)\right)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\ge\alpha\right\}}\right]$$

$$-\frac{k}{t(1-t)}E\left[\left(\frac{S_{k,t}}{k}-t\right)^2 \varphi'(t)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\ge\alpha\right\}}\right]$$

$$
=\frac{k}{t(1-t)}E\left[\left(\frac{S_{k,t}}{k}-t\right)\varphi\left(\frac{S_{k,t}}{k}\right)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\geq\alpha\right\}}\right]
$$

$$
-\frac{k}{t(1-t)}E\left[\left(\frac{S_{k,t}}{k}-t\right)\varphi(t)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\geq\alpha\right\}}\right]
$$

$$
-\frac{k}{t(1-t)}E\left[\left(\frac{S_{k,t}}{k}-t\right)^2\varphi'(t)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\geq\alpha\right\}}\right]
$$

$$
=\left|\frac{k}{t(1-t)}E\left(\frac{S_{k,t}}{k}-t\right)\varphi\left(\frac{S_{k,t}}{k}\right)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\geq\alpha\right\}}\right.
$$

$$
-\frac{k}{t(1-t)}E\left(\frac{S_{k,t}}{k}-t\right)\varphi(t)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\geq\alpha\right\}}
$$

$$
\left.-\frac{k}{t(1-t)}E\left(\frac{S_{k,t}}{k}-t\right)^2\varphi'(t)1_{\left\{\left|\frac{St}{k}-t\right|\geq\alpha\right\}}\right|
$$

$$
\leq\left|\frac{k}{t(1-t)}E\left(\frac{S_{k,t}}{k}-t\right)\varphi\left(\frac{S_{k,t}}{k}\right)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\geq\alpha\right\}}\right|
$$

$$
+\left|\frac{k}{t(1-t)}E\left(\frac{S_{k,t}}{k}-t\right)\varphi(t)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\geq\alpha\right\}}\right|
$$

$$
+\left|\frac{k}{t(1-t)}E\left(\frac{S_{k,t}}{k}-t\right)\varphi'(t)1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\geq\alpha\right\}}\right|
$$

$$
\leq\frac{k}{t(1-t)}\left|\frac{S_{k,t}}{k}-t\right|\varphi\left(\frac{S_{k,t}}{k}\right)E\left(1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\geq\alpha\right\}}\right)
$$

$$
+\frac{k}{t(1-t)}\left|\frac{S_{k,t}}{k}-t\right|\varphi(t)E\left(1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\geq\alpha\right\}}\right)
$$

$$+\frac{k}{t(1-t)}\left(\frac{S_{k,t}}{k}-t\right)^2\varphi'(t)E\left(1_{\left\{\left|\frac{S_{k,t}}{k}-t\right|\geq\alpha\right\}}\right),$$

where $\left|\frac{S_{k,t}}{k}-t\right|\leq\frac{1}{\alpha}$, and $E(1_H)=P(H)$

$$\leq\frac{k}{t(1-t)}\frac{1}{\alpha}MP\left(\left|\frac{S_{k,t}}{k}-t\right|\geq\alpha\right)+\frac{k}{t(1-t)}\frac{1}{\alpha}MP\left(\left|\frac{S_{k,t}}{k}t\right|\geq\alpha\right)$$

$$+\frac{k}{t(1-t)}\left(\frac{S_{k,t}}{k}-t\right)^2\varphi'(t)P\left(\left|\frac{S_{k,t}}{k}-t\right|\geq\alpha\right)$$

$$\leq\frac{k}{t(1-t)}\left(\frac{2M}{\alpha}+\varphi'(t)\right)P\left(\left|\frac{S_{k,}}{k}-t\right|\geq\alpha\right), \qquad (3.11)$$

where the last inequality uses the fact that the square of the distance of the points $\frac{S_{k,t}}{k}$ and t in the interval $[0,1]$ is less than 1. Now (3.11) can be bounded, using (3.6) by

$$\leq\frac{k}{t(1-t)}\left(\frac{2M}{\alpha}+\varphi'(t)\right)2e^{-2\alpha^2}$$

$$=\frac{k}{t(1-t)}\left(\frac{2M}{\alpha}+\varphi'(t)\right)\frac{2}{e^{2\alpha^2}},$$

the relation above tends to zero as $k\to\infty$.

Chapter 4

LIMITING PROPERTIES OF SOME BERNSTEIN-TYPE OPERATORS

4.1 Introduction

From its establishment by Bernstein, probabilistic methods have been widely used for the approximation purpose.

To prepare our mind for understanding, let us consider I to be a real valued interval and let us consider R^t, R_1^t, $\mathrm{R}_2^t, \ldots,$ I-valued random variables with probability distribution depending upon the parameter $t \in I$. Assume that Y and Y_k are two positive operators which are linear and associated with R^t and R_k^t respectively using

$$Y\ (\varphi,t) = E\,\varphi\left(R^t\right),\ Y_k\ (\varphi,t) = E\,\varphi\left(R_k^t\right),\ \varphi \in CB\,(I),\ t \in I.$$

We now state the following theorem.

Theorem 1. $\forall\ t \in I$ the following statements are equivalent:

(a) R_k^t *Converges in distribution to* $R^t\ (k \to \infty)$.

(b) $Y_k(\varphi, t) \to Y(\varphi, t)(k \to \infty)$ for *all* φ belonging to the space of all continuous and bounded functions on I.

(c) The space in the assertion (b) changed by real valued uniform, bounded and continuous function on the interval I.

Modes of convergence of probability theory are enough to approach the problems of approximation.

On the other hand, to determine the rate of convergence, we can estimate $\left| \mathrm{Y}_k(\varphi,t)-Y(\varphi,t)\right|$, the probabilistic technique work also for this purpose. The following are the operators and notations involved in the mentioned result above.

For any $k \in \mathbb{N},\ t \geq 0,$ and $\varphi \in C_B[0,\infty)$ define

$$L_k(\varphi,t)=(1+t)^{-k}\sum_{n=0}^{k}\varphi\left(\frac{n}{k-n+1}\right)\binom{k}{n}t^n,$$

which is the operator introduction by Bleimann, Butzer, and Hahn.

Similarly

$$S_k(\varphi,t)=e^{-kt}\sum_{n=0}^{\infty}\varphi(n/k)\frac{(kt)^n}{n!}$$

is the Szasz operator.

The Baskakov operator is defined by

$$B_k^*(\varphi,t)=(1+t)^{-k}\sum_{n=0}^{\infty}\varphi(n/k)\binom{k+n-1}{n}\left(\frac{t}{1+t}\right)^n,$$

and

$$G_k(\varphi,t)=\begin{cases}\dfrac{t^{-n}}{(k-1)!}\displaystyle\int_0^\infty \varphi(h/k)h^{k-1}e^{-h/t}dh, & \text{if } t>0\\ \varphi(0), & \text{if } t=0\end{cases}$$

is the well known Gamma operator.

Finally, for $k \in \mathbb{N},\ 0\leq t\leq 1,\ \varphi\in C[0,1],\ B_k(\varphi,t)$ define by

$$B_k(\varphi,t) = \sum_{n=0}^{k} \varphi(n/k)\binom{k}{n} t^n (1-t)^{k-n}$$

is called the Bernstein operator.

4.2 Limiting Properties

Theorem 2. Let us choose j to be an integer and let us consider the function φ to be real valued, bounded and continuous function on $[0,+\infty)$. The following statements hold for each $t \geq 0$ as $k \to \infty$.

(a) $L_{jk}\left(\varphi(kh/(1+h)),t/k\right) \to S_j(\varphi,t)$,

(b) $B_{jk}\left(\varphi(kh),t/k\right) \to S_j(\varphi,t)$,

(c) $B^*_{jk}\left(\varphi(kh),t/k\right) \to S_j(\varphi,t)$,

(d) $B^*_j\left(\varphi(h/k),kt\right) \to G_j(\varphi,t)$.

Proof. To prove (a) first observe that

$$L_{jk}\left(\varphi\left(\frac{kh}{1+h}\right),t/k\right) = E\varphi\left(\left(j+k^{-1}\right)^{-1} U_k^t\right)$$

and

$$S_j(\varphi,t) = E\varphi\left(j^{-1}R^t\right),$$

where U_k^t is a binomial distribution random variable with parameters jk and $p = p(k) = t(k+t)^{-1}$, and R^t is a Poisson distribution random variable having the mean jt. Since $jkp(k) \to jt$ (as $k \to \infty$) it follows that U_k^t is converges in law to Z^t which implies that $\left(j+k^{-1}\right)^{-1} U_k^t$ converges in law to $j^{-1}R^t$. We can then conclude using Theorem 1.

Similarly we can prove (b). At this point let us consider $k \geq t$ and the fact that

$$B_{jk}\left(\varphi(kh), t/k\right) = E\,\varphi\left(j^{-1}V_k^t\right),$$

with V_k^t having a binomial distribution with parameters jk ; $p = t/k$. We now deal with (c); (d). It easy follow that

$$B_{jk}^{*}\left(\varphi(kh), t/k\right) = E\,\varphi\left(j^{-1}W_k^t\right)$$

and

$$B_j^{*}\left(\varphi(h/k), kt\right) = E\,\varphi\left(j^{-1}k^{-1}Y_k^t\right),$$

with W_k^t having a negative binomial distribution of parameter jk ; $p = k\,(k+t)^{-1}$ and Y_k^t having a negative binomial distribution of parameter j ; $p = (1+kt)^{-1}$. The characteristic functions of W_k^t and $k^{-1}\,Y_k^t$ are defined respectively by

$$\varphi_k(y) = \left(1 + k^{-1}t\left(1 - e^{iy}\right)\right)^{-jk}$$

and

$$\psi_k(y) = \left(1 + kt\left(1 - e^{iy/k}\right)\right)^{-j}$$

since

$$\varphi_k(y) \to \exp\left(jt\left(e^{iy} - 1\right)\right)$$

and

$$\psi_k(y) \to (1 - ity)^{-j}$$

(as $k \to \infty$), the continuity of Levy theorem is used to conclude that W_k^t ($k^{-1}\,Y_k^t$) is convergent to a Poisson random variable with mean jt (a gamma random variable of parameters $1/t$; j, if $t > 0$, or that the distribution will degenerate at 0 if the time $t = 0$).

4.3 Convergence Rates

Theorem 3. Let us consider j to be an integer and let us consider φ to be a real valued continuous and bounded function on the interval $[0,+\infty)$. Then we have:

(a) $\forall\ k \geq t \geq 0,$

$$\left|B_{jk}\left(\varphi(kh),t/k\right)-S_j\left(\varphi,t\right)\right| \leq \frac{t}{k}2\|\ \varphi\|\min\left(2,jt\right).$$

(b) For $t \geq 0$ and $k = 1,\ 2,\ \ldots$

$$\left|L_{jk}\left(\varphi\left(\frac{kh}{1+h}\right),t/k\right)-S_j\left(\varphi,t\right)\right| \leq 2\upsilon\left(\varphi,t/jk\right)+\left(t\left(1+t\right)/k\right)2j\|\varphi\|\left(1+2jt\right)e^{2jt}$$

with $\|\varphi\|$ being the sup norm of the function φ and $\upsilon(\varphi,\alpha)$ being the first modulus of continuity of the function φ.

Proof. The notations are the same with those used in Theorem 2 to prove the relation (a) we first observe that, for any $k \geq t \geq 0,$

$$\left|B_{jk}\left(\varphi(kh),t/k\right)-S_j\left(\varphi,t\right)\right| = \sum_{n=0}^{\infty}\varphi\left(\frac{n}{k}\right)\binom{jk}{n}\left(t/k\right)^n\left(1-\frac{t}{k}\right)^{jk-n}$$

$$-\ e^{-jt}\sum_{n=0}^{\infty}\varphi\left(\frac{n}{k}\right)\frac{(jt)^n}{n!}$$

$$\left|B_{jk}\left(\varphi(kh),t/k\right)-S_j\left(\varphi,t\right)\right| \leq \sum_{n=0}^{\infty}\left|\varphi\left(\frac{n}{k}\right)\right|\ \left|P_n\left(k\right)-\pi_n\right|$$

$$\leq \|\varphi\|\sum_{n=0}^{\infty}\left|P_n\left(k\right)-\pi_n\right|,$$

where, for n is an positive integer

$$P_n\left(k\right)=P\left(V_k^t=n\right)=\binom{jk}{n}\left(t/k\right)^n\left(1-\frac{t}{k}\right)^{jk-n}$$

and

$$\pi_n = P\left(R^t = n\right) = e^{-jt}\frac{(jt)^n}{n!}.$$

Now, using Prokhorov's inequality it follows that

$$\sum_{n=0}^{\infty}\left|P_n(k)-\pi_n\right| \le C_1(\kappa)\,p,$$

where

$$C_1(\kappa) = 2\min(2,\kappa),\ \ \kappa = mean = jk\,\frac{t}{k} = jt \ \text{ and } P = \frac{t}{k}$$

$$\sum_{n=0}^{\infty}\left|P_n(k)-\pi_n\right| \le C_1(\kappa)\,p = \frac{t}{k}2\min(2,\kappa)$$

$$\left|B_{jk}\left(\varphi(kh),t/k\right)-S_j(\varphi,t)\right| \le \frac{t}{k}2\|\varphi\|\min(2,jt)$$

whence the result.

To prove (b) observe that, for $t \ge 0$ and $k = 1,\ 2,\ ...,$ it is possible to write

$$\begin{aligned}\left|L_{jk}\left(\varphi\left(\frac{kh}{1+h}\right),t/k\right)-S_j(\varphi,t)\right| &= \left|E\varphi\left(\left(j+k^{-1}\right)^{-1}U_k^t\right)-E\varphi\left(j^{-1}R^t\right)\right| \\ &\le E\left|\ \varphi\left(\left(j+k^{-1}\right)^{-1}U_k^t\right)-\varphi\left(j^{-1}U_k^t\right)\right| \\ &\quad +\left|E\varphi\left(j^{-1}U_k^t\right)-E\varphi\left(j^{-1}R^t\right)\right|. \qquad (4.1)\end{aligned}$$

We first estimate the right hand side of the term in (4.1) separately. Let us consider $\alpha > 0$ and define

$$\kappa = \left[\alpha^{-1}\left|\left(j+k^{-1}\right)^{-1}U_k^t - j^{-1}U_k^t\right|\right]$$

$$\kappa = \left[\alpha^{-1}j^{-1}(jk+1)^{-1}U_k^t\right],$$

It follows obviously that

$$\left| \varphi\left(\left(j+k^{-1}\right)^{-1}U_k^t\right)-\varphi\left(j^{-1}U_k^t\right)\right| \le \upsilon(\varphi,\alpha)(1+\kappa)$$

thus

$$\begin{aligned} E\left| \varphi\left(\left(j+k^{-1}\right)^{-1}U_k^t\right)-\varphi\left(j^{-1}U_k^t\right)\right| &\le E\,\upsilon(\varphi,\alpha)(1+\kappa) \\ &\le \upsilon(\varphi,\alpha)(1+E\,\kappa) \\ &\le \upsilon(\varphi,\alpha)\left(1+E\left[\alpha^{-1}j^{-1}(jk+1)^{-1}U_k^t\right]\right) \\ &\le \upsilon(\varphi,\alpha)\left(1+\alpha^{-1}j^{-1}(jk+1)^{-1}EU_k^t\right), \end{aligned}$$

where

$$\left(j+k^{-1}\right)^{-1}U_k^t \to j^{-1}R^t$$

$$E\left(j+k^{-1}\right)^{-1}U_k^t \to Ej^{-1}R^t$$

$$EU_k^t \to \frac{jk+1}{jk}ER^t,$$

with R^t being a Poisson distribution random variable with a mean jt.

$$EU_k^t \to \frac{(jk+1)}{jk}jt$$

$$EU_k^t = \frac{(jk+1)t}{k}$$

$$\begin{aligned} &\le \upsilon(\varphi,\alpha)\left(1+\alpha^{-1}j^{-1}(jk+1)^{-1}EU_k^t\right) \\ &\le \upsilon(\varphi,\alpha)\left(1+\alpha^{-1}j^{-1}(jk+1)^{-1}\frac{(jk+1)t}{k}\right) \\ &\le \upsilon(\varphi,\alpha)\left(1+t\,(jk)^{-1}\alpha^{-1}\right). \end{aligned}$$

On taking $\alpha = t/jk$ we obtain

$$E\left|\ \varphi\left(\left(j+k^{-1}\right)^{-1}U_k^t\right)-\varphi\left(j^{-1}U_k^t\right)\right| \le 2\upsilon\left(\varphi, t/jk\right). \tag{4.2}$$

Let us now estimate the second term on the right of (4.1) similar to (a).

$$\left|E\,\varphi\left(j^{-1}U_k^t\right)-E\,\varphi\left(j^{-1}R^t\right)\right| \le \|\varphi\| \sum_{n=0}^{\infty}\left|P_n'(k)-\pi_n\right|. \tag{4.3}$$

With, for $n = 0,\ 1, 2, \ldots,$

$$P_n'(k) = P\left(U_k^t = n\right) = \binom{jk}{n}\left(\frac{t}{k+t}\right)^n\left(\frac{k}{k+t}\right)^{jk-n}.$$

It follows that

$$\sum_{n=0}^{\infty}\left|P_n'(k)-\pi_n\right| \le \left(2+4jkt\left(k+t\right)^{-1}\right)e^{jt}Q\left(k,t\right), \tag{4.4}$$

with

$$Q\left(k,t\right) = \sup\left\{\left|\left[jky\right]t\left(k+t\right)^{-1}-jty\right| :\ 0 \le y \le 1\right\},$$

since, for any $0 \le y \le 1$

$$\left|\left[jky\right]t\left(k+t\right)^{-1}-jty\right| = \left|\left[jky\right]t\left(k+t\right)^{-1}-jyt\left(k+t\right)\left(k+t\right)^{-1}\right|$$

$$= \left|\left[jky\right]t\left(k+t\right)^{-1}-jkyt\left(k+t\right)^{-1}-jyt^2\left(k+t\right)^{-1}\right|$$

$$= \left|\left[jky\right]t-jkyt-jyt^2\right|\left(k+t\right)^{-1}$$

$$= \left|jkyt-jkyt-jyt^2\right|\left(k+t\right)^{-1}$$

$$\le \left(jyt^2+t\right)\left(k+t\right)^{-1}$$

$$\le j\left(t^2+t\right)k^{-1},$$

we have

$$Q(k,t) \le jt(1+t)k^{-1}, \tag{4.5}$$

and so the result (b) comes from the inequalities $(4.1)-(4.5)$.

$$\left| L_{jk}\left(\varphi\left(\frac{kh}{1+h}\right),t/k\right) - S_j(\varphi,t)\right|$$

$$\le 2\upsilon(\varphi,t/jk) + \left(t(1+t)/k\right)2j\,\|\varphi\|(1+2jt)e^{2jt}.$$

A consequence of Theorem 3 is given by the following corollary.

Corollary.

(a) Considering any real valued bounded and continuous function φ on $[0,+\infty)$ the convergence of

$$B_{jk}(\varphi(kh),t/k) \to S_j(\varphi,t)\ (k\to\infty)$$

is a uniform convergence on each bounded subinterval $[0,a]$.

(b) For any real valued uniform, bounded and continuous function φ on the interval $[0,+\infty)$ the convergence

$$L_{jk}\left(\varphi\left(\frac{kh}{1+h}\right),t/k\right) \to S_j(\varphi,t) \qquad (k\to\infty)$$

is uniform on bounded subinterval $[0,a]$. Moreover, the convergence rates come from Theorem 3.

Chapter 5

CONCLUSION

The aim of our work was to study the properties of Bernstein-type operators. The following results are observed.

We considered and evaluated two similar problems based on the approximation using Bernstein polynomials $B_k\varphi(t)$ of a given continuous function φ on the interval $[0,1]$. We also proved that this method leads to an absence of the Gibbs phenomenon even at a jump point, due to the smoothness of the Bernstein polynomials. We established the convergence rate of $(B_k\varphi)'(t)$ towards $\varphi'(t)$. All the results mentioned above were also obtained using probabilistic assumption and computing expected value of a function of some special random variable.

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