Riemann Type Integration for Functions of One Real Variable

Recep Duranay

Submitted to the
Institute of Graduate Studies and Research
in partial fulfillment of the requirements for the Degree of

Master of Science
in
Mathematics

Eastern Mediterranean University
September 2014
Gazimağusa, North Cyprus

Approval of the Institute of Graduate Studies and Research

Prof. Dr. Elvan Yılmaz
Director

I certify that this thesis satisfies the requirements as a thesis for the degree of Master of Science in Mathematics.

Prof. Dr. Nazim Mahmudov
Chair, Department of Mathematics

We certify that we have read this thesis and that in our opinion it is fully adequate in scope and quality as a thesis of the degree of Master of Science in Mathematics.

Prof. Dr. Agamirza Bashirov
Supervisor

Examining Committee

1. Prof. Dr. Agamirza Bashirov
2. Asst. Prof. Dr. Hüseyin Etikan
3. Asst. Prof. Dr. Yücel Tandoğdu

ABSTRACT

In this thesis, process of Riemann integral is tackled. Firstly, theorems and their proofs of Proper Riemann integral are explained. After that, improper Riemann integral with the same proof techniques is handled. Riemann Steiltjes integral with examples and theorems of continuous linear function in Riesz Representation theorem is explained. Finally, Kurzweil-Henstock and Lebesgue integrals are handled with theorems and proofs.

Keywords: Riemann Integral, Riemann Steiltjes Integral, Riesz Representation Theorem, Kurzweil-Henstock and Lebesgue Integral

ÖZ

Bu tezde Riemann integralinin başlangıcından gelişimin günümüze kadar olan süreci işlenmiştir. İlk olarak teoremler ve ispatlarıyla has Riemann integrali açıklanmıştır. Aynı ispat tekniği ile sınırsız alanda has olmayan Riemann İntegrali ele alınmıştır. Sürekli linear fonksiyonların Riesz gösteriminden yardım alarak Riemann Steiltjes integrali anlatılmıştır. Son olarak Kurzweil-Henstock ve Lebesgue'nin uygulamarıyla tezde amaçlanan hedefe ulaşılmıştır.

Anahtar kelimeler: Riemann İntegral'i, Riemann Steiltjes İntegral'i, Riesz Gösterimi, Kurzweil-Henstock ve Lebesgue İntegral'i

ACKNOWLEDGMENTS

I would like to say "thank you" to my supervisor Prof. Dr. Agamirza Bashirov, for his guidance, patience, motivation and contributious. He gave information, clearly and helped all problems about my thesis. I would like to thank all people who helped me int the preparation of this thesis.

TABLE OF CONTENTS

Chapter 1

INTRODUCTION

The idea of integration arose in the works of ancient Greek mathematicians as a calculation of areas of different geometric figures. It was rediscovered by European mathematicians in the seventeenth century. A number of mathematicians contributed to integration. They used different methods and completed theory of integration for functions of a single variable. In this thesis integration of functions of single variable is handled.

A descriptive approach was used by Newton to integrals. If $f'(x)$ is a derivative of the function $F(x)$, then we defined $F(x)$ as an antiderivative of $f(x)$. This leads to the familiar formula

$$F(x) = \int f'(x)\,dx + c.$$

Presently, this is a powerful method of calculation of proper and improper integrals if antiderivative is elementary function. This method does not work for functions such as e^{x^2}, $\sin(x^2)$, $\frac{\sin x}{x}$ etc. since the antiderivatives of them are not elementary.

The other two approaches to integral, developed by Riemann and Lebesgue, respectively, are constructive. They are based on partitioning and integral sums. Riemann's approach uses partitioning in the domain of an integrand, but the Lebesque approach in the range. Both these approaches can be modified so that to cover all possible functions of single variable which can be integrated.

The Lebesgue approach is more advance and generates such branches of mathematics as measure theory, abstract integration, probability theory etc. Modern mathematical analysis is based on Lebesque integration. At the same time, Riemann integration is relatively simple. In this thesis we overview Riemann integration, generalisation of Riemann integration, leading to Riemann–Stieltjes and Henstock–Kurzweil integration.

The definition of the Riemann integral includes two major steps. In the first step, Riemann integral is defined for integrands with bounded domain and range. This is called a proper or definite Riemann integral. In the second step, the integrands which have a finite number of improperness are handled. An improperness may have two forms: unbounded domain from the left and write, and also an infinite behaviour of an integrand about some point in the domain. After reviewing Riemann integral we consider its generalisation in two directions, called Riemann–Stieltjes and Henstock–Kurzweil integrals, respectively.

Chapter 2

PROPER RIEMANN INTEGRAL

2.1 Definition

In this section we define a proper Riemann integral of a function $f(x)$,that is bounded in an interval $[a,b]$, assuming that $-\infty < a < b < \infty$. Shortly, a proper Riemann integral will be called a Riemann integral or an integral. The collection of all bounded functions on $[a,b]$ is denoted by $B(a,b)$.

A partition of the interval $[a,b]$ is a collection of numbers $x_0, x_1, \ldots, x_n$, satisfying

$$a = x_0 < x_1 < \cdots < x_n = b.$$

This partition is denoted by

$$P = \{x_0, x_1, \ldots, x_n\}.$$

The number

$$\|P\| = \max\{x_1 - x_0, \ldots, x_n - x_{n-1}\}$$

is called a mesh or a norm of the partition P. Actually, P is a partition because it splits the interval $[a,b]$ into subintervals

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n],$$

but we think about P as a finite sequence of the end points of these subintervals in the increasing order. A partition Q is called a refinement of the partition P if Q contains all points of P. This is written as $Q \supseteq P$. Clearly, $Q = P_1 \cup P_2$ is a refinement of both partitions P_1 and P_2 of $[a,b]$. For $f \in B(a,b)$ and $P = \{x_0, x_1, \ldots, x_n\}$, the sum

$$S(f,P) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}), \tag{2.1.1}$$

where $c_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$, is said to be a Riemann sum of f. The numbers $c_1, c_2, \ldots, c_n$ are called the tag numbers or simply tags of the partition P.

A proper Riemann integral can be defined in the different equivalent forms. The following definition is one of them.

Definition 2.1.1 *A function f in $B(a,b)$ is said to be integrable in the Riemann sense or, briefly, integrable if there exists a number S such that for all $\varepsilon > 0$ there exists a partition P_ε of the interval $[a,b]$ such that*

$$|S(f,\mathcal{P}) - S| < \varepsilon$$

for every $\mathcal{P} \supseteq \mathcal{P}_\varepsilon$ and for every selection of the tags. This number S is called a Riemann integral or an integral of f and denoted by

$$\int_a^b f(x)\,dx$$

The function f is referred as an integrand. Conventionally,

$$\int_a^a f(x)\,dx = 0 \;\; and \;\; \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Proposition 2.1.2 *The Riemann integral of $f \in B(a,b)$ is unique if it exists.*

Proof. Assume the contrary that S_1 and S_2 are two distinct numbers, satisfying the condition in Definition 2.1.1. Let $\varepsilon = |S_1 - S_2|/2 > 0$. By Definition 2.1.1 there are partitions P_ε and Q_ε of $[a, b]$ such that

$$|S(f, P) - S_1| < \varepsilon \text{ for } P \supseteq P_\varepsilon,$$

and

$$|S(f, Q) - S_2| < \varepsilon \text{ for } Q \supseteq Q_\varepsilon.$$

Then for the refinement $P_\varepsilon \cup Q_\varepsilon$ of P_ε and Q_ε, we obtain the following contradiction:

$$\varepsilon = \frac{|S_1 - S_2|}{2} \leq \frac{|S(f, \mathcal{P}_\varepsilon \cup \mathcal{Q}_\varepsilon) - S_1| + |S(f, P_\varepsilon \cup Q_\varepsilon) - S_2|}{2} < \frac{\varepsilon + \varepsilon}{2} = \varepsilon.$$

This proves the proposition. ■

The collection of all bounded functions that are integrable in the Riemann sense on $[a, b]$ is denoted by $R(a, b)$. Clearly, $R(a, b) \subseteq B(a, b)$. The following examples show that $R(a, b) \neq \emptyset$ and $R(a, b) \neq B(a, b)$.

Example 2.1.3 *Let f be a constant function, that is, $f(x) = c$ for everyl $a \leq x \leq b$. Take any partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$. Then*

$$S(f, P) = \sum_{i=1}^{n} c(x_i - x_{i-1}) = c(b - a).$$

Therefore, $|S(f, P) - c(b - a)| = 0 < \varepsilon$ for all partitions of $[a, b]$ and for all $\varepsilon > 0$, and *this is independent on the tags. Thus,*

$$\int_a^b c\,dx = c(b-a).$$

Example 2.1.4 (Dirichlet function) *Define a function f on $[a,b]$ by*

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

This function is called Dirichlet function and it is not Riemann integrable. To prove, observe that for all partition P of $[a,b]$,

$$S(f,P) = \sum_{i=1}^{n} 1 \cdot (x_i - x_{i-1}) = b-a$$

if the tags are rational, and

$$S(f,P) = \sum_{i=1}^{n} 0 \cdot (x_i - x_{i-1}) = 0$$

if they are irrational. Therefore, if $\varepsilon = \frac{b-a}{2}$, then there is no number S, satisfying

$$|S(f,\mathcal{P}) - S| < \varepsilon$$

for both rational tags and irrational tags. This proves that the Dirichlet function belongs to $B(a,b)$ but not to $R(a,b)$.

These two examples demonstrate that $R(a,b)$ is a nonempty and proper subset of $B(a,b)$.

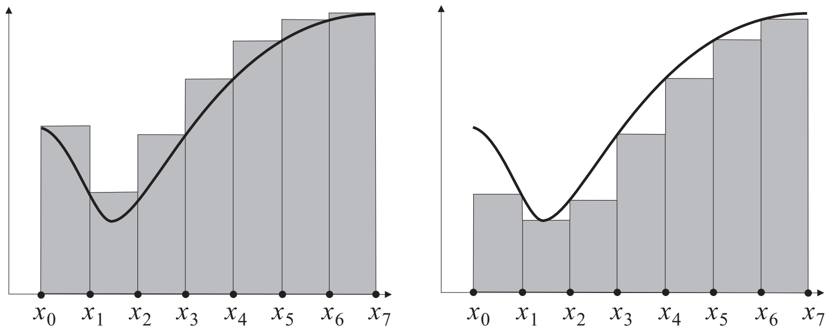

Figure 2.1. Upper and lower Darboux sums.

2.2 Existence

There are several theorems about existence of Riemann integral. In this section these theorems are discussed.

Let $f \in B(a,b)$ and let $P = \{x_0,\dots,x_n\}$ is a partition of $[a,b]$. Since f is bounded, for $i = 1,\dots,n$, we can define the following numbers:

$$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\} \text{ and } m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}. \tag{2.2.1}$$

Furthermore, using these numbers, we can define the sums

$$S^*(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \text{ and } S_*(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}),$$

which said to be the upper and lower Darboux sums of f for the partition P, respectively. In Figure 2.1, $S^*(f,P)$ and $S_*(f,P)$ are shown as the areas of the shaded regions.

Lemma 2.2.1 *Let $f \in B(a,b)$, let P be a partition of $[a,b]$ and let $Q \supseteq P$. Then*

$$S_*(f,P) \le S_*(f,Q) \le S^*(f,Q) \le S^*(f,P).$$

Proof. The second inequality is trivial. The proof of the first and the third inequalities are similar. Therefore, we will prove just one of them, say, the first one.

Let $P = \{x_0, \ldots, x_n\}$. Then Q is the union of partitions of the intervals $[x_{i-1}, x_i]$, $i = 1, \ldots, n$. Therefore,

$$Q = \{x_{1,0}, \ldots, x_{1,k_1}, x_{2,0}, \ldots, x_{2,k_2}, \ldots, x_{n,0}, \ldots, x_{n,k_n}\},$$

where

$$x_{i-1} = x_{i,0} < \cdots < x_{i,k_i} = x_i \;\; i = 1, \ldots, n.$$

Letting

$$M_{i,j} = \sup\{f(x) : x_{i,j-1} \leq x \leq x_{i,j}\}, \;\; i = 1, \ldots, n, \;\; j = 1, \ldots, k_i,$$

and assuming that M_i, $i = 1, \ldots, n$, are defined in (2.2.1), we obtain

$$\begin{aligned} S^*(f, P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} M_i(x_{i,j} - x_{i,j-1}) \\ &\geq \sum_{i=1}^{n} \sum_{j=1}^{k_i} M_{i,j}(x_{i,j} - x_{i,j-1}) = S^*(f, Q). \end{aligned}$$

This proves the first inequality in the lemma. ■

Lemma 2.2.2 *Let $f \in B(a, b)$. For every two partitions P and Q of $[a, b]$, the following inequality holds:*

$$S_*(f, P) \leq S^*(f, Q).$$

Proof. Consider the refinement $P \cup Q$ of P and Q, by Lemma 2.2.1,

$$S_*(f,P) \le S_*(f,P \cup Q) \le S^*(f,P \cup Q) \le S^*(f,Q).$$

This proves the lemma. ■

By Lemma 2.2.2, the upper Darboux sums of $f \in B(a,b)$ are bounded below and the lower Darboux sums of f are bounded above. Therefore, we can define

$$S^*(f) = \inf_P S^*(f,P) \text{ and } S_*(f) = \sup_P S_*(f,P),$$

where infimum and supremum are taken over all possible partitions of $[a,b]$. $S^*(f)$ and $S_*(f)$ are called the upper and lower Riemann integrals of $f \in B(a,b)$, respectively. Clearly,

$$S_*(f) \le S^*(f).$$

Theorem 2.2.3 (Darboux) *A function $f \in B(a,b)$ is integrable in the Riemann sense and its Riemann integral equals to S if and only if $S^*(f) = S_*(f) = S$.*

Proof. Assume that the Riemann integral of f equals to S. We will prove that $S^*(f) = S$. Then in a similar way it can be proved that $S_*(f) = S$. This will result $S^*(f) = S_*(f) = S$, proving the necessity part of the theorem.

To prove $S^*(f) = S$, assume the contrary, that is, $S^*(f) \ne S$. Denote

$$\varepsilon = \frac{|S^*(f) - S|}{3} > 0.$$

Since $S_*(f) = \sup_P S_*(f,P)$, there exists a partition Q_ε of $[a,b]$, satisfying

$$0 \le S^*(f,Q_\varepsilon) - S^*(f) < \varepsilon.$$

There exists also a partition $\mathcal{P}_\varepsilon$ of $[a,b]$ with

$$|S(f,\mathcal{P}) - S| < \varepsilon$$

for every $P \supseteq P_\varepsilon$ and every tags of P. Particularly, this inequality holds for the refinement $P_\varepsilon \cup Q_\varepsilon$ of P_ε. By Lemma 2.2.1, we also have

$$0 \le S^*(f,P_\varepsilon \cup Q_\varepsilon) - S^*(f) \le S^*(f,Q_\varepsilon) - S^*(f) < \varepsilon.$$

Furthermore, assuming $P_\varepsilon \cup Q_\varepsilon = \{x_0,\ldots,x_n\}$, select $c_i \in [x_{i-1},x_i]$, satisfying

$$M_i - f(\xi_i) < \frac{\varepsilon}{b-a},\ i = 1,\ldots,n,$$

where M_i, $i = 1,\ldots,n$, are defined by (2.2.1). Consider $S(f,P_\varepsilon \cup Q_\varepsilon)$ corresponding to the tags $c_1,\ldots,c_n$. This implies

$$0 \le S^*(f,P_\varepsilon \cup Q_\varepsilon) - S(f,P_\varepsilon \cup Q_\varepsilon) < \frac{\varepsilon}{b-a}\sum_{i=1}^{n}(x_i - x_{i-1}) = \varepsilon.$$

Therefore,

$$
\begin{aligned}
|S^*(f) - S| &\le |S^*(f) - S^*(f, P_\varepsilon \cup Q_\varepsilon)| \\
&\quad + |S^*(f, P_\varepsilon \cup Q_\varepsilon) - S(f, P_\varepsilon \cup Q_\varepsilon)| \\
&\quad + |S(f, P_\varepsilon \cup Q_\varepsilon) - S| \\
&< 3\varepsilon.
\end{aligned}
$$

This contradicts to the definition of ε and proves the necessity.

Conversely, assume $S^*(f) = S_*(f) = S$. Take arbitrary $\varepsilon > 0$. Then there exist partitions P_ε and Q_ε of $[a,b]$ with

$$S^*(f, P_\varepsilon) < S^*(f) + \varepsilon$$

and

$$S_*(f, Q_\varepsilon) > S_*(f) - \varepsilon.$$

Consider $P_\varepsilon \cup Q_\varepsilon$. Then every $P \supseteq P_\varepsilon \cup Q_\varepsilon$ is a refinement of P_ε and Q_ε. By Lemma 2.2.1,

$$S^*(f, P) \le S^*(f, P_\varepsilon) < S^*(f) + \varepsilon$$

and

$$S^*(f, P) \ge S_*(f, Q_\varepsilon) > S_*(f) - \varepsilon.$$

Therefore,

$$S-\varepsilon < S_*(f,\mathcal{P}) \le S(f,\mathcal{P}) \le S^*(f,\mathcal{P}) < S+\varepsilon$$

or

$$|S(f,\mathcal{P})-S| < \varepsilon$$

for every $P \supseteq P_\varepsilon \cup Q_\varepsilon$ and every selection of the tags. Thus, f is Riemann inferable and its integral equals to S. The sufficiency is proved. ■

Theorem 2.2.4 (Riemann) *A function $f \in B(a,b)$ is integrable in the Riemann sense iff for every $\varepsilon > 0$ there exists a partition P_ε of $[a,b]$ with $S^*(f,P_\varepsilon) - S_*(f,P_\varepsilon) < \varepsilon$.*

Proof. Assume $f \in R(a,b)$. By Theorem 2.2.3, $S^*(f) = S_*(f)$. Take arbitrary $\varepsilon > 0$. Then there are partitions P'_ε and P''_ε of $[a,b]$ with

$$S^*(f,P'_\varepsilon) < S^*(f) + \frac{\varepsilon}{2}$$

and

$$S_*(f,P''_\varepsilon) > S_*(f) - \frac{\varepsilon}{2}.$$

Denote $P_\varepsilon = P'_\varepsilon \cup P''_\varepsilon$. Since P_ε is a refinement of P'_ε and P''_ε, by Lemma 2.2.1,

$$\begin{aligned} S^*(f,P_\varepsilon) - S_*(f,P_\varepsilon) &\le S^*(f,P'_\varepsilon) - S_*(f,P''_\varepsilon) \\ &< S^*(f) + \frac{\varepsilon}{2} - S_*(f) + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

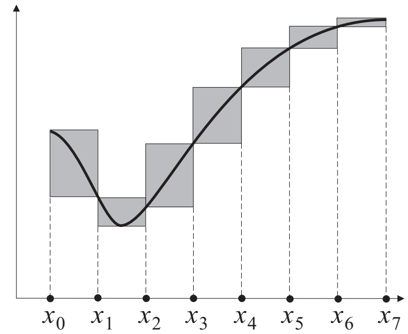

Figure 2.2. The difference of upper and lower Darboux sums.

Thus the necessity part of the theorem is proved.

Conversely, assume that for all $\varepsilon > 0$ there exists a partition P_ε of $[a,b]$ satisfying

$$S^*(f,P_\varepsilon) - S_*(f,P_\varepsilon) < \varepsilon.$$

This implies

$$0 \le S^*(f) - S_*(f) \le S^*(f,P_\varepsilon) - S_*(f,P_\varepsilon) < \varepsilon.$$

Thus from the arbitrariness of $\varepsilon > 0$, we receive $S^*(f) = S_*(f)$. Then by Theorem 2.2.3, we obtain $f \in R(a,b)$. ■

Geometrically, the difference

$$S^*(f,P) - S_*(f,P) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})$$

for $P = \{x_0, \ldots, x_n\}$ is illustrated by the shaded region in Figure 2.2.

Theorem 2.2.5 *A continuous function on* $[a,b]$ *is integrable in the Riemann sense on the interval* $[a,b]$*, that is,* $C(a,b) \subseteq R(a,b)$.

Proof. At first note that a continuous function on the interval $[a,b]$ is bounded. So, $C(a,b) \subseteq B(a,b)$. Take $f \in C(a,b)$. Let $\varepsilon > 0$. Since f is continuous on the interval $[a,b]$ it is uniformly continuous. This means that there is $\delta > 0$ with

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a}$$

for every $x, y \in [a,b]$ satisfying $|x-y| < \delta$. Consider a partition $P_\varepsilon = \{x_0, \ldots, x_n\}$ of $[a,b]$ with the mesh $\|P_\varepsilon\| < \delta$. Since a continuous function takes its maximum and minimum on compact set,

$$M_i = \max_{[x_{i-1},x_i]} f(x) = f(c_i') \text{ and } m_i = \min_{[x_{i-1},x_i]} f(x) = f(c_i'')$$

for some $c_i', c_i'' \in [x_{i-1}, x_i]$. Since $|c_i' - c_i''| < \delta$, we obtain

$$M_i - m_i < \varepsilon/(b-a).$$

This implies

$$S^*(f, P_\varepsilon) - S_*(f, P_\varepsilon) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \frac{\varepsilon}{b-a} \sum_{i=1}^{n} (x_i - x_{i-1}) = \varepsilon.$$

Hence, by Theorem 2.2.4, $f \in R(a,b)$. ■

A function $f : [a,b] \to \mathbb{R}$ is said to be increasing if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$. Similarly, $f : [a,b] \to \mathbb{R}$ is said to be decreasing if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$. A function is said to be monotone if it is either increasing or decreasing.

Theorem 2.2.6 *A monotone function on $[a,b]$ is integrable in the Riemann sense on the interval $[a,b]$.*

Proof. We can assume that $f:[a,b]\to\mathbb{R}$ is increasing. Then $f(a)\le f(b)$. If $f(a)=f(b)$, then f is a constant function and it is integrable in the Riemann sense by Example 2.1.3. Let $f(a)<f(b)$. Take $\varepsilon>0$ and let

$$\delta=\frac{\varepsilon}{f(b)-f(a)}.$$

Take a partition $P_\varepsilon=\{x_0,\dots,x_n\}$ of $[a,b]$ with $\|P_\varepsilon\|<\delta$. If M_i and m_i are defined by (2.2.1), then

$$M_i-m_i\le f(x_i)-f(x_{i-1}).$$

Therefore,

$$\begin{aligned}S^*(f,P_\varepsilon)-S_*(f,P_\varepsilon)&=\sum_{i=1}^{n}(M_i-m_i)(x_i-x_{i-1})\\&<\delta\sum_{i=1}^{n}(f(x_i)-f(x_{i-1}))=\frac{\varepsilon(f(b)-f(a))}{f(b)-f(a)}=\varepsilon.\end{aligned}$$

Hence, by Theorem 2.2.4, f is integrable in the Riemann sense on $[a,b]$. ■ A condition, completely describing the integrable in the Riemann sense functions, belongs to Lebesgue. According to this condition, the Riemann integrable functions are continuous everywhere except a "negligible number" of points. Here, a set of a "negligible number" of elements is a set of measure zero.

A set $E\subseteq\mathbb{R}$ is said to be of measure zero if for every $\varepsilon>0$, there is a countable number of closed intervals $[a_n,b_n]$, $n=1,2,\dots$, such that $E\subseteq\bigcup_{n=1}^{\infty}[a_n,b_n]$ and $\sum_{n=1}^{\infty}(b_n-a_n)<\varepsilon$.

Example 2.2.7 *A countable set $A=\{x_1,x_2,\dots\}\subseteq\mathbb{R}$ is of measure zero. Indeed, for any*

$\varepsilon > 0$, include the point x_n in to interval $[a_n, b_n]$ with $b_n - a_n < \varepsilon/2^n$. Then

$$A \subseteq \bigcup_{n=1}^{\infty}[a_n, b_n] \ \ \text{and} \ \ \sum_{n=1}^{\infty}(b_n - a_n) \leq \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = \varepsilon.$$

There are uncountable sets of measure zero as well. For example, a famous Cantor ternary set is uncountable set of measure zero.

Theorem 2.2.8 (Lebesgue) *A function $f \in B(a, b)$ is Riemann integrable if and only if it is continuous on $[a, b]$ except the points that form a set of measure zero.*

The proof of this theorem can be found in books on measure and integration. The following is an immediate consequence of Theorem 2.2.8 and useful for proving properties of Riemann integral.

Corollary 2.2.9 *If $f \in R(a, b)$, $g \in R(c, d)$ and $c \leq f(x) \leq d$ for all $x \in [a, b]$, then $(g \circ f) \in R(a, b)$, where $(g \circ f)(x) = g(f(x))$ for $x \in [a, b]$.*

Proof. This follows from the fact that the discontinuity points of f and $g \circ f$ are same. By Theorem 2.2.8, the set of discontinuity points of f form a set of measure zero. Then the same holds for $(g \circ f)$ as well. Thus, $(g \circ f) \in R(a, b)$. ■

2.3 Properties

Theorem 2.3.1 *If $f \in R(a, b)$ and $c \in \mathbb{R}$, then $cf \in R(a, b)$ and*

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Proof. If $c = 0$ then the theorem is trivial. Assume $c \neq 0$. The proof is based on

$$S(cf, P) = cS(f, P),$$

if the same tags are used in the Riemann sums in this equality. Let

$$S = \int_a^b f(x)\,dx.$$

Take $\varepsilon > 0$. Consider the partition P_ε of $[a, b]$ with

$$P \supseteq P_\varepsilon \quad \Rightarrow \quad |S(f, P) - S| < \frac{\varepsilon}{|c|}$$

for all selections of the tags. Then

$$|S(cf, P) - cS| \leq |c||S(f, P) - S| < \frac{\varepsilon |c|}{|c|} = \varepsilon.$$

for all selections of the tags. Hence, $cf \in R(a, b)$ and the equality in the theorem holds.

■

Theorem 2.3.2 *If $f, g \in R(a, b)$, then $f + g \in R(a, b)$ and*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Proof. The theorem is based on

$$S(f + g, P) = S(f, P) + S(g, P),$$

if the same tags are used in the Riemann sums in this equality. Let

$$S_1 = \int_a^b f(x)\,dx \text{ and } S_2 = \int_a^b g(x)\,dx.$$

Take $\varepsilon > 0$. Consider the partitions P_ε and Q_ε of $[a,b]$ with

$$P \supseteq P_\varepsilon \;\Rightarrow\; |S(f,P) - S_1| < \frac{\varepsilon}{2}$$

and

$$P \supseteq Q_\varepsilon \;\Rightarrow\; |S(g,P) - S_2| < \frac{\varepsilon}{2}$$

for all selections of the tags. Then $P \supseteq P_\varepsilon \cup Q_\varepsilon$ implies

$$|S(f+g,P) - S_1 - S_2| \le |S(f,P) - S_1| + |S(g,P) - S_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

for all selections of the tags. Hence, $f+g \in R(a,b)$ and the equality in the theorem holds. ■

Theorem 2.3.3 *If $f,g \in R(a,b)$, then $fg \in R(a,b)$.*

Proof. By Corollary 2.2.9, $f^2 \in R(a,b)$. Then from

$$fg = \frac{(f+g)^2 - (f-g)^2}{4},$$

we conclude that $fg \in R(a,b)$. ■

For $f \in R(a,b)$ and $a \le c < d \le b$, we denote

$$\int_c^d f(x)\,dx = \int_a^b f|_{[c,d]}(x)\,dx,$$

where $f|_{[c,d]}$ denotes the restriction of f to the interval $[c,d]$.

Theorem 2.3.4 *Let $a < c < b$. Then $f \in R(a,b)$ iff $f|_{[a,c]} \in R(a,c)$ and $f|_{[c,b]} \in R(c,b)$. Furthermore,*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{2.3.1}$$

Proof. A subset of a set of measure zero is again a set of measure zero. Therefore, by Theorem 2.2.8, $f \in R(a,b)$ implies $f|_{[a,c]} \in R(a,c)$ and $f|_{[c,b]} \in R(c,b)$. Conversely, the union of two sets of measure zero is again a set of measure zero. Then by the same thorem, $f|_{[a,c]} \in R(a,c)$ and $f|_{[c,b]} \in R(c,b)$ imply $f \in R(a,b)$.

To prove the equality (2.3.1), let

$$S_1 = \int_a^c f(x)\,dx \text{ and } S_2 = \int_c^d f(x)\,dx.$$

Take any $\varepsilon > 0$. Then there exists partitions P_ε and Q_ε of $[a,c]$ and $[c,b]$, respectively, such that

$$P \supseteq P_\varepsilon \;\Rightarrow\; \left|S(f|_{[a,c]},P) - S_1\right| < \frac{\varepsilon}{2}$$

and

$$P \supseteq Q_\varepsilon \quad \Rightarrow \quad \left|S(f|_{[c,b]}, P) - S_2\right| < \frac{\varepsilon}{2}$$

for all selections of the tags. Then $P_\varepsilon \cup Q_\varepsilon$ is a partition of $[a,b]$. Moreover, if $P \supseteq P_\varepsilon \cup Q_\varepsilon$, then $P \cap [a,c] \supseteq P_\varepsilon$ and $P \cap [c,b] \supseteq Q_\varepsilon$. Hence, for every $P \supseteq P_\varepsilon \cup Q_\varepsilon$,

$$|S(f,P) - S_1 - S_2| \le \left|S(f|_{[a,c]}, P \cap [a,c]) - S_1\right| + \left|S(f|_{[c,b]}, P \cap [c,b]) - S_2\right| < \varepsilon$$

for all selections of the tags. This proves the equality (2.3.1). ■

Theorem 2.3.5 *If $f \in R(a,b)$ and $f(x) \ge 0$ for all $a \le x \le b$, then*

$$\int_a^b f(x)\,dx \ge 0.$$

Proof. This follows from $S^*(f,P) \ge S_*(f,P) \ge 0$ for everyl partitions P of $[a,b]$. Hence, $S^*(f) = S_*(f) \ge 0$. ■

Corollary 2.3.6 *If $f, g \in R(a,b)$ and $f(x) \le g(x)$ for all $a \le x \le b$, then*

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Proof. This follows from the application of Theorem 2.3.5 to the function $g - f$ ■

Corollary 2.3.7 *If $f \in R(a,b)$, then $|f| \in R(a,b)$ and*

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Proof. By Corollary 2.2.9, we have $|f| \in R(a,b)$. Then use

$$-|f(x)| \leq f(x) \leq |f(x)|$$

and apply Corollary 2.3.6. ■

Theorem 2.3.8 (Mean-value theorem for integrals) *If* $f \in C(a,b)$, *then there exists* $a \leq c \leq b$ *such that*

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

Proof. Let

$$M = \max\{f(x) : a \leq x \leq b\} \text{ and } m = \min\{f(x) : a \leq x \leq b\},$$

which exist because f is continuous on $[a,b]$. By Corollary 2.3.6,

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a),$$

or

$$m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M.$$

Then by intermediate value theorem, there exists $a \leq c \leq b$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This proves the theorem. ■

For $f \in R(a,b)$, by Theorem 2.3.4, we can define the function

$$F(x) = \int_a^x f(t)\,dt, \; a \le x \le b. \tag{2.3.2}$$

This function has the following properties.

Theorem 2.3.9 (First fundamental theorem of calculus) *For $f \in R(a,b)$ define F by (2.3.2). If f is continuous at the point $c \in [a,b]$, then F is differentiable at the point c and $F'(c) = f(c)$.*

Proof. Take any $\varepsilon > 0$. Since f is continuous at c, there exists $\delta > 0$ such that

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon$$

whenever $|x - c| < \delta$ and $x \in [a,b]$. Take h with $|h| < \delta$ and $c + h \in [a,b]$. Then

$$\int_c^{c+h} (f(c) - \varepsilon)\,dx \le \int_c^{c+h} f(x)\,dx \le \int_c^{c+h} (f(c) + \varepsilon)\,dt.$$

This implies

$$(f(c) - \varepsilon)h \le F(c+h) - F(c) \le (f(c) + \varepsilon)h.$$

Therefore,

$$\left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \varepsilon.$$

This means that F is differentiable at c and $F'(c) = f(c)$. ■

Theorem 2.3.10 (Second fundamental theorem of calculus) *If $f : [a,b] \to \mathbb{R}$ is dif-*

ferentiable and $f' \in R(a,b)$, then

$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{2.3.3}$$

Proof. Consider any partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$. By mean-value theorem of differentiation, there exists $c_i \in (x_{i-1}, x_i)$ such that

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}),\ i = 1, \ldots, n.$$

Therefore,

$$\sum_{i=1}^n f'(c_i)(x_i - x_{i-1}) = \sum_{i=1}^n (f(x_i) - f(x_{i-1})) = f(b) - f(a).$$

Then from

$$S_*(f', P) \le \sum_{i=1}^n f'(c_i)(x_i - x_{i-1}) \le S^*(f', P),$$

we obtain

$$S_*(f', P) \le f(b) - f(a) \le S^*(f', P),$$

implying

$$S_*(f') \le f(b) - f(a) \le S^*(f').$$

Since $f' \in R(a,b)$, we have $S^*(f') = S_*(f')$. This implies (2.3.3). ■

Theorem 2.3.11 (Integration by parts) *If f and g are differentiable on $[a,b]$ and*

$f', g' \in R(a,b)$, then

$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx.$$

Proof. The proof is based on the product rule $(fg)' = f'g + fg'$ of differentiation. Applying Theorem 2.3.10, we obtain

$$\int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx = \int_a^b (fg)'(x)\,dx = f(b)g(b) - f(a)g(a).$$

This proves the theorem. ■

Theorem 2.3.12 (Change of variable) *If g is differentiable on $[a,b]$, $g' \in C(a,b)$ and $f \in C(R(g))$, where $R(g)$ is the range of g, then*

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(t))g'(t)\,dt.$$

Proof. Define

$$G(t) = \int_a^t f(g(x))g'(x)\,dx, \ a \le t \le b,$$

and

$$F(u) = \int_{g(a)}^u f(x)\,dx, \ u \in R(g).$$

By Theorem 2.3.9,

$$G'(t) = f(g(t))g'(t), \ a \le t \le b,$$

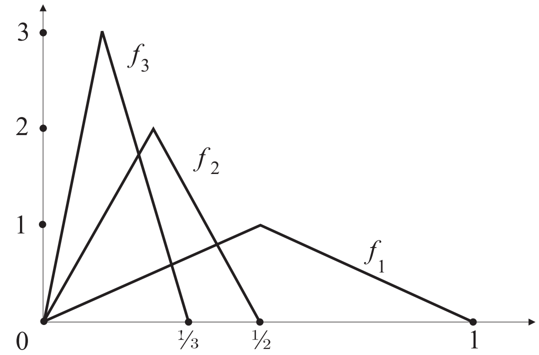

Figure 2.3. Functions f_n from Example 2.4.1.

and

$$F'(u) = f(u),\ u \in R(g).$$

Therefore, $G'(t) = (F \circ g)'(t),\ a \le t \le b$. Then $G(t) - F(g(t)) = \text{const.},\ a \le t \le b$. For $t = a$, we have $G(a) - F(g(a)) = 0$. This implies $G(t) - F(g(t)) = 0,\ a \le t \le b$. Then $G(b) - F(g(b)) = 0$. Theorem is proved. ■

2.4 Dependence on Parameter

Is it possible to interchange the limit and integral, in other words, if $\{f_n\}$ is a sequence of functions in $R(a,b)$ converging pointwise to a function $f : [a,b] \to \mathbb{R}$ as $n \to \infty$ for every $a \le x \le b$, can we assert that

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx?$$

The following example demonstrates that we cannot.

Example 2.4.1 *Define*

$$f_n(x) = \begin{cases} 2n^2 x & \text{if } 0 \le x \le 1/2n, \\ 2n - 2n^2 x & \text{if } 1/2n < x \le 1/n, \\ 0 & \text{if } 1/n < x \le 1. \end{cases}$$

The graphs of f_1, f_2 and f_3 are given. The function f_n increases on $[0, 1/2n]$ linearly, gets a peak at $x = 1/2n$, decreases on $[1/2n, 1/n]$ linearly and vanishes on $[1/n, 1]$. The graph of f_n and x-axis form a triangle, that has the area to be $1/2$. Therefore,

$$\int_0^1 f_n(x)\,dx = \frac{1}{2}.$$

On the other hand, $\lim_{n\to\infty} f_n(x) = f(x) = 0$ for all $0 \le x \le 1$ because $1/n \to 0$ and $f_n(0) = 0$. Thus,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \frac{1}{2} \neq 0 = \int_0^1 f(x)\,dx.$$

Therefore, an additional condition is required for the interchange of the limit and integral. This condition is a uniform convergence.

Definition 2.4.2 *A sequence of functions $f_n : [a, b] \to \mathbb{R}$ is said to be uniformly convergent to $f : [a, b] \to \mathbb{R}$ if for every $\varepsilon > 0$, there exists a positive integer N such that for all $n > N$ and for all $a \le x \le b$, $|f_n(x) - f(x)| < \varepsilon$.*

Theorem 2.4.3 (Interchange of limit and integral) *If a sequence $\{f_n\}$ of functions in $R(a, b)$ converges uniformly to f on $[a, b]$ as $n \to \infty$, then $f \in R(a, b)$ and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Proof. Take any $\varepsilon > 0$. Since f_n converges to f uniformly, there is N such that for all $n > N$,

$$f_n(x) - \varepsilon \leq f(x) \leq f_n(x) + \varepsilon, \text{ for all } a \leq x \leq b.$$

Therefore, $f \in B(a,b)$ and

$$\int_a^b (f_n(x) - \varepsilon)\, dx \leq S_*(f) \leq S^*(f) \leq \int_a^b (f_n(x) + \varepsilon)\, dx. \tag{2.4.1}$$

This implies

$$0 \leq S^*(f) - S_*(f) \leq 2\varepsilon(b-a).$$

Since $\varepsilon > 0$ is an arbitrary positive number, we conclude that $S^*(f) = S_*(f)$, i.e., $f \in R(a,b)$. Moreover, from (2.4.1), for every $n > N$, we have

$$\left| \int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx \right| \leq \varepsilon(b-a).$$

Hence the limit in the theorem holds. ■

Theorem 2.4.4 (Continuity under the integral) *Let $f \in C([a,b] \times [c,d])$. Define*

$$F(y) = \int_a^b f(x,y)\, dx, \; c \leq y \leq d.$$

Then $F \in C(c,d)$, that is, for all $y_0 \in [c,d]$,

$$\lim_{y \to y_0} \int_a^b f(x,y)\, dx = \int_a^b \lim_{y \to y_0} f(x,y)\, dx = \int_a^b f(x,y_0)\, dx.$$

Proof. The continuity of f on $[a,b]\times[c,d]$ implies its uniform continuity. Therefore, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x,y) - f(x_0,y_0)| < \frac{\varepsilon}{b-a}$$

for all pairs $(x,y) \in [a,b]\times[c,d]$ satisfying

$$(x-x_0)^2 + (y-y_0)^2 < \delta^2.$$

This holds if $x = x_0$ and $|y-y_0| < \delta$ as well. Therefore,

$$|F(y) - F(y_0)| \leq \int_a^b |f(x,y) - f(x,y_0)|\,dx \leq \varepsilon.$$

This means that F is continuous at arbitrary y_0. Hence $F \in C(c,d)$. ■

Theorem 2.4.5 (Interchange of differentiation and integration) *Assume that a function $f : [a,b]\times[c,d]$ is so that $f(\cdot,y) \in R(a,b)$ for all $y \in [c,d]$ and $f'_y \in C([a,b]\times[c,d])$. Then the function*

$$F(y) = \int_a^b f(x,y)\,dx, \ c \leq y \leq d.$$

is differentiable on $[c,d]$ and

$$F'(y) = \int_a^b f'_y(x,y)\,dx, \ c \leq y \leq d.$$

Proof. Take any $y_0 \in [c,d]$ and $y \in [c,d] \setminus \{y_0\}$. By mean value theorem of differentiation, we have

$$\frac{F(y)-F(y_0)}{y-y_0} = \int_a^b \frac{f(x,y)-f(x,y_0)}{y-y_0}\,dx = \int_a^b f'_y(x,z)\,dx,$$

for some number z between y and y_0. Here, $z \to y_0$ when $y \to y_0$. Therefore. by continuity of f'_y on $[a,b]\times[c,d]$, we can apply Theorem 2.4.4 to the last integral and complete the proof. ■

Theorem 2.4.6 (Interchange the order of integration) *Let $f \in C([a,b]\times[c,d])$ and define*

$$F(y) = \int_a^b f(x,y)\,dx,\ c \le y \le d,$$

and

$$G(x) = \int_c^d f(x,y)\,dy,\ a \le x \le b.$$

Then $F \in R(c,d)$ and $G \in R(a,b)$ and

$$\int_c^d F(y)\,dy = \int_a^b G(x)\,dx.$$

Respectively,

$$\int_a^b \Big(\int_c^d f(x,y)\,dy\Big)dx = \int_c^d \Big(\int_a^b f(x,y)\,dx\Big)dy.$$

Proof. According to Theorem 2.4.4, we have

$$F \in C(c,d) \subseteq R(c,d)$$

and

$$G \in C(a,b) \subseteq R(a,b).$$

Define functions $\bar{F} : [a,b] \to \mathbb{R}$ and $\bar{G} : [a,b] \to \mathbb{R}$ by

$$\bar{F}(t) = \int_a^t \left(\int_c^d f(x,y)\,dy \right) dx,$$

and

$$\bar{G}(t) = \int_c^d \left(\int_a^t f(x,y)\,dx \right) dy.$$

By Theorems 2.4.5 and 2.3.9, $\bar{F}$ and $\bar{G}$ are differentiable and

$$F_0'(t) = G_0'(t) = \int_c^d f(t,y)\,dy.$$

Hence, $\bar{F}(t) = \bar{G}(t)$ for all $a \le t \le b$ since $\bar{F}(a) = \bar{G}(a).$ This implies $\bar{F}(b) = \bar{G}(b).$ This proves the theorem. ■

Chapter 3

Improper Riemann Integral

3.1 First Kind Improper Integrals

Proper Riemann integral can be extended to unbounded integrands on unbounded intervals in the following way.

Definition 3.1.1 (First kind improper integral) *Let I be an interval of one the form $[a,\infty)$ or $(-\infty,b]$ and let f be a function on the interval I such that f is properly integrable in the Riemann sense on every compact subinterval of I. Denote*

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \ \ \text{if} \ \ I=[a,\infty),$$

and

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx \ \ \text{if} \ \ I=(-\infty,b].$$

These are called first kind improper integrals of f on I. If the respective limit exists,then the improper integral is said to be convergent. Otherwise, it is said to be divergent. In the convergent cases f is said to be improperly Riemann integrable on I.

First kind Improper integrals are continuous analogs of series. Therefore, many theorems about series valid for them as well.

Theorem 3.1.2 (Comparison test for improper integrals) *Assume that either $I=[a,\infty)$*

or $I=(-\infty,b]$, f *and* g *are functions on the interval* I *that are properly Riemann on every compact subinterval of* I, *and*

$$0 \le |f(x)| \le g(x),\ x \in I,$$

If the improper integral of g *on* I *is convergent, then the improper integral of* f *on* I *is also convergent. If the improper integral of* $|f|$ *on* I *is divergent, then the improper integral of* g *on* I *is also divergent.*

Proof. Consider the case $I=[a,\infty)$. Denote

$$F(y)=\int_a^x |f(x)|\,dx \ \text{ and } \ G(y)=\int_a^y g(x)\,dx,\ y \ge a.$$

Here, F and G are increasing functions with $F(y) \le G(y)$ and $\lim_{y\to\infty} G(y)$ exists. Therefore, F is an increasing and bounded function on $[a,\infty)$. By monotone bounded convergence theorem, $\lim_{y\to\infty} F(y)$ exists. Thus,

$$\int_a^\infty |f(x)|\,dx = \lim_{y\to\infty} F(y)$$

is convergent. Define

$$f^+(x)=\begin{cases} f(x) & \text{if } f(x)\ge 0, \\ 0 & \text{if } f(x)<0 \end{cases} \quad \text{and} \quad f^-(x)=\begin{cases} -f(x) & \text{if } f(x)\le 0, \\ 0 & \text{if } f(x)>0 \end{cases}$$

The following relations are obvious:

$$f(x)=f^+(x)-f^-(x),\ \ 0 \le f^+(x) \le |f(x)|,\ \ 0 \le f^-(x) \le |f(x)|.$$

Thus

$$\int_a^\infty f^+(x)\,dx \text{ and } \int_a^\infty f^-(x)\,dx$$

are convergent. This implies that

$$\int_a^\infty f(x)\,dx = \lim_{y\to\infty}\int_a^y f^+(x)\,dx - \lim_{y\to\infty}\int_a^y f^-(x)\,dx$$

is also convergent. The case $I = (\infty, b]$ can be proved similarly. ■

Theorem 3.1.3 (Integral test) *Let $f : [1,\infty) \to \mathbb{R}$ be a positive decreasing function. Then the improper integral*

$$\int_1^\infty f(x)\,dx$$

converges if and only if the series $\sum_{n=1}^\infty f(n)$ converges.

Proof. Introduce the functions g and h by

$$g(x) = f(n) \text{ and } h(x) = f(n+1) \text{ if } n \le x < n+1,\ n = 1,2,\ldots.$$

Then

$$0 \le h(x) \le f(x) \le g(x),\ x \ge 1.$$

Therefore, it remains to apply Theorem 3.1.2 to complete the proof. ■

Example 3.1.4 *It is known that the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$. *Therefore, by Theorem 3.1.3, the improper integral*

$$\int_1^{\infty} \frac{dx}{x^p}$$

converges if and only if $p > 1$.

3.2 Second Kind Improper Integrals

Definition 3.2.1 (*2nd kind improper integral*) *Let* I *be an interval of one the form* $[a,b)$ *or* $(a,b]$ *and let* f *be a function on the interval* I *such that* f *is unbounded on* I *but properly Riemann integrable on every compact subinterval of* I. *Denote*

$$\int_a^b f(x)\,dx = \lim_{c\to b-} \int_a^c f(x)\,dx \ \text{ if } \ I = [a,b),$$

and

$$\int_a^b f(x)\,dx = \lim_{c\to a+} \int_c^b f(x)\,dx \ \text{ if } \ I = (a,b].$$

These are called second kind improper integral of f *on* I. *If the respective limit exists, the improper integral is said to be convergent. Otherwise, it is said to be divergent. In the convergent cases* f *is said to be improperly Riemann integrable on* I.

An analog of Theorem 3.1.2 is valid for second kind improper integrals as well in the form

Theorem 3.2.2 (Comparison test for improper integrals) *Assume that either $I = [a,b)$ or $I = (a,b]$, f and g are functions on I that are properly Riemann on every compact subinterval of I, and*

$$0 \le |f(x)| \le g(x),\ x \in I,$$

If the improper integral of g on I is convergent, then the improper integral of f on I is also convergent. If the improper integral of $|f|$ on I is divergent, then the improper integral of g on I is also divergent.

Proof. This is similar to the Theorem 3.1.2. ■

Example 3.2.3 *Consider the second kind improper integral*

$$\int_0^1 \frac{dx}{x^p},$$

noticing that for $p \le 0$ it is a proper integral and has a finite value. If $p = 1$, then

$$\int_0^1 \frac{dx}{x} = \lim_{y\to 0^+} \int_y^1 \frac{dx}{x} = \lim_{y\to 0^+} \ln x\Big|_y^1 = -\lim_{y\to 0^+} \ln y = \infty.$$

Therefore, the given improper integral diverges for $p = 1$. Let $p > 0$ and $p \ne 0$. Then

$$\int_0^1 \frac{dx}{x^p} = \lim_{y\to 0^+} \int_y^1 \frac{dx}{x^p} = \lim_{y\to 0^+} \frac{x^{1-p}}{1-p}\Big|_y^1 = \lim_{y\to 0^+} \frac{1-y^{1-p}}{1-p}.$$

This limit equals to $1/(1-p)$ if $0 < p < 1$ and to ∞ if $p > 1$. Thus the given integral converges iff $p < 1$.

In case if a function has a finite number of improperness of the first or second kind, then the interval I is devided into finite number of subintervals so that the given function has a single improperness on each subinterval. If the improper integrals of the given function on all these subintervals are convergent, then the total improper integral is said to be convergent. If at least one of them is divergent, then the total improper integral is said to be divergent.

Example 3.2.4 *The improper integral*

$$\int_0^\infty \frac{dx}{x^p}$$

is divergent for all values of p. Indeed it has two improperness and can be divided into two improper integrals with single improperness:

$$\int_0^\infty \frac{dx}{x^p} = \int_0^1 \frac{dx}{x^p} + \int_1^\infty \frac{dx}{x^p}.$$

By Example 3.1.4, the second improper integral in the right side is divergent if $p \leq 1$, and, by Example 3.2.3, the first improper integral in the right side is divergent if $p \geq 1$. Anyway, the total improper integral is divergent.

3.3 Absolute and Conditional Convergence

According to Theorems 3.1.2 and 3.2.2, the convergence of the first or second kinds improper integrals of $|f|$ implies the convergence of the respective improper integral for f. But the converse is not always true. Respectively, we give the following definition.

Definition 3.3.1 *A first or second kind improper integral of the function f is said to be absolutely convergent if the respective improper integral of $|f|$ converges. If the* *improper integral of f converges while the respective improper integral of $|f|$ diverges, then the improper integral of f is said to be conditionally convergent.*

Example 3.3.2 *A convergent improper integral of a positive function is obviously absolutely convergent since in this case $f = |f|$. A conditionally convergent improper integral can be constructed by use of relationship between improper integrals and series.*

Take, for example, the conditionally convergent numerical series

$$\sum_{n=1}^{\infty}(-1)^n/n.$$

Consider the improper integral

$$\int_1^{\infty} f(x)\,dx,$$

where the function $f : [1,\infty) \to \mathbb{R}$ is defined by

$$f(x) = \frac{(-1)^n}{n} \;\; if \;\; x \in [n, n+1),\; n = 1, 2, \ldots$$

Then

$$\int_1^{\infty} f(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

and

$$\int_1^{\infty} |f(x)|\,dx = \sum_{n=1}^{\infty} \frac{1}{n}.$$

Therefore, given improper integral is conditionally convergent.

Chapter 4

The Riemann–Stieltjes Integral

4.1 Definition

Assume that $-\infty < a < b < \infty$. The definition of the Riemann–Stieltjes integral differs from the definition of Riemann integral by replacement of the linear function $u(x) = x$, $a \le x \le b$, with a general function u on $[a,b]$.

Let $f, u \in B(a,b)$ and consider a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$. Define the Riemann–Stieltjes sum similar to Riemann sums by

$$S(f,u,P) = \sum_{i=1}^{n} f(c_i)(u(x_i) - u(x_{i-1})),$$

where $c_i, \ldots, c_n$ are the tags of the partition P.

Definition 4.1.1 *A function $f \in B(a,b)$ is said to be integrable in the Riemann-Stieltjes sense with respect to $u \in B(a,b)$ or, briefly, integrable if there is a number S such that for all $\varepsilon > 0$ there is a partition P_ε of $[a,b]$ with*

$$|S(f,u,\mathcal{P}) - S| < \varepsilon$$

for every $P \supseteq P_\varepsilon$ and for every selection of tags. This number S is called a Riemann–Stieltjes integral of f with respect to u and denoted by

$$\int_a^b f(x)\,du(x).$$

The functions f and u are referred as integrand and integrator, respectively. Conventionally,

$$\int_b^a f(x)\,du(x) = -\int_a^b f(x)\,du(x),$$

Comparing the definitions of Riemann and Riemann–Stieltjes integrals, it is easily seen that the integral of f in the Riemann sense of f is the Riemann–Stieltjes sense with respect to the function $u(x) = x$, $a \le x \le b$. Notice that unlike the integral in the Riemann sense,

$$\int_a^a f(x)\,du(x) \neq 0$$

since u may have a discontinuity at a.

The collection of all pairs (f, u) of functions $f, u \in B(a, b)$, for which the Riemann–Stieltjes integral of f with respect to u exists, is denoted by $RS(a, b)$. For every $f \in B(a, b)$ and for a constant function u on $[a, b]$, we have $(f, u) \in RS(a, b)$ because $S(f, u, P) = 0$ for every partition P of $[a, b]$ and for all tags. At the same time, $(f, u) \notin RS(a, b)$ if f is Dirichlet function from Example 2.1.4 and $u(x) = x$. Therefore, $RS(a, b)$ is not a rectangle (a set of the form $A \times B$) in $B(a, b) \times B(a, b)$. Therefore, it is important to find a sufficiently large rectangle $A \times B$ in $B(a, b) \times B(a, b)$ such that $A \times B \subseteq RS(a, b)$.

4.2 Properties

Properties of the integrals in the Riemann–Stieltjes sense in comparison to properties ofintegral in the Riemann sense can be devided into three groups:

(a) Those which are same as the respective property of Riemann integral.

(b) Those which essentially generalize the respective property of Riemann integral.

(c) Those which have not an analog in Riemenn integration.

The next three theorems are same as in Riemann integration.

Theorem 4.2.1 *If* $(f,u) \in RS(a,b)$ *and* $c \in \mathbb{R}$, *then* $(cf,u) \in RS(a,b)$ *and*

$$\int_a^b cf(x)\,du(x) = c\int_a^b f(x)\,du(x).$$

Theorem 4.2.2 *If* $(f,u),(g,u) \in RS(a,b)$, *then* $(f+g,u) \in RS(a,b)$ *and*

$$\int_a^b (f(x)+g(x))\,du(x) = \int_a^b f(x)\,du(x) + \int_a^b g(x)\,du(x).$$

Theorem 4.2.3 *Let* $a < c < b$. *Then* $(f,u) \in RS(a,b)$ *if and only if* $(f|_{[a,c]}, u|_{[a,c]}) \in RS(a,c)$ *and* $(f|_{[c,b]}, u|_{[c,b]}) \in R(c,b)$. *Furthermore,*

$$\int_a^b f(x)\,du(x) = \int_a^c f(x)\,du(x) + \int_c^b f(x)\,du(x).$$

Theorems 4.2.1 and 4.2.2 are valid with regards to u as well which have no analog in Riemann integration.

Theorem 4.2.4 *If* $(f,u) \in RS(a,b)$ *and* $c \in \mathbb{R}$, *then* $(f,cu) \in RS(a,b)$ *and*

$$\int_a^b f(x)\,d(cu(x)) = \int_a^b f(x)\,du(x).$$

Theorem 4.2.5 *If* $(f,u),(f,v) \in RS(a,b)$*, then* $(f,u+v) \in RS(a,b)$ *and*

$$\int_a^b f(x)\,d(u(x)+v(x)) = \int_a^b f(x)\,du(x) + \int_a^b f(x)\,dv(x).$$

The next theorem is regarded as the integration by parts formula for the Riemann–Stieltjes integrals and it is an essential generalization of the integration by parts formula for the Riemann integral.

Theorem 4.2.6 *If* $(f,u) \in RS(a,b)$*, then* $(u,f) \in RS(a,b)$ *and*

$$\int_a^b f(x)\,du(x) + \int_a^b u(x)\,df(x) = f(b)u(b) - f(a)u(a).$$

Proof. Take arbitrary $\varepsilon > 0$. Let $P_\varepsilon = \{x_0,\dots,x_n\}$ be a partition of $[a,b]$ with $P \supseteq P_\varepsilon$ implies

$$\left| S(f,u,\mathcal{P}) - \int_a^b f(x)\,du(x) \right| < \varepsilon.$$

Consider arbitrary tags $c_1,\dots,c_n$ of P. Then

$$S(u,f,P) = \sum_{i=1}^{n} u(c_i)f(x_i) - \sum_{i=1}^{n} u(c_i)f(x_{i-1})$$

and

$$f(x)u(x)\big|_a^b = \sum_{i=1}^{n} f(x_i)u(x_i) - \sum_{i=1}^{n} f(x_{i-1})u(x_{i-1}).$$

Therefore,

$$f(x)u(x)|_a^b - S(u, f, \mathcal{P}) = \sum_{i=1}^{n} f(x_i)(u(x_i) - u(c_i)) + \sum_{i=1}^{n} f(x_{i-1})(u(c_i) - u(x_{i-1})).$$

One can see that the right side is the Riemann–Stieltjes sum $S(f, u, Q_\varepsilon)$ for the partition

$$Q_\varepsilon = \{x_0, c_1, x_1, c_2, \dots, c_n, x_n\},$$

if the tags are selected as $x_0, x_1, x_1, \dots, x_{n-1}, x_{n-1}, x_n$. Here $Q_\varepsilon \supseteq P \supseteq P_\varepsilon$. Therefore,

$$\left| f(x)u(x)|_a^b - S(u, f, P) - \int_a^b f(x)\,du(x) \right| = \left| S(f, u, Q_\varepsilon) - \int_a^b f(x)\,du(x) \right| < \varepsilon,$$

proving the theorem. ■

The next theorem is a reduction formula of the Riemann–Stieltjes integral to the Riemann integral and has no analog in Riemann integration.

Theorem 4.2.7 *Assume that $f \in R(a,b)$ and u is differentiable on $[a,b]$ with $u' \in R(a,b)$. Then $(f,u) \in RS(a,b)$ and*

$$\int_a^b f(t)\,du(x) = \int_a^b f(x)u'(x)\,dx.$$

Proof. Take any partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$. By mean value theorem of differentiation,

$$u(x_i) - u(x_{i-1}) = \int_{x_{i-1}}^{x_i} u'(x)\,dx.$$

Therefore,

$$S(f,u,P) = \sum_{i=1}^{n} f(c_i) \int_{x_{i-1}}^{x_i} u'(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(c_i)u'(x)\,dx$$

for the arbitrary tags $c_1, \dots, c_n$ of P. This implies

$$\left| S(f,u,P) - \int_a^b f(x)u'(x)\,dx \right| \le \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(c_i) - f(x)||u'(x)|\,dx$$

If $M = \sup_{[a,b]} |u'(x)|$, then

$$\begin{aligned} \left| S(f,u,P) - \int_a^b f(x)u'(x)\,dx \right| &\le M \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(c_i) - f(x)|\,dx \\ &\le M(S^*(f,P) - S_*(f,P)), \end{aligned}$$

Now take any $\varepsilon > 0$ and choose partition P_ε of $[a,b]$, satisfying

$$S^*(f,P_\varepsilon) - S_*(f,P_\varepsilon) < \frac{\varepsilon}{M}.$$

Then for every $P \supseteq P_\varepsilon$, we have

$$\begin{aligned} \left| S(f,u,P) - \int_a^b f(x)u'(x)\,dx \right| &\le M(S^*(f,P) - S_*(f,P)) \\ &\le M(S^*(f,P_\varepsilon) - S_*(f,P_\varepsilon)) < \varepsilon. \end{aligned}$$

This proves the theorem. ■

Finally, we present mean value theorems for Riemann–Stieltjes integrals.

Theorem 4.2.8 *If $f \in C(a,b)$ and u is an increasing function on $[a,b]$, then there is*

$c \in [a,b]$ such that

$$\int_a^b f(x)\,du(x) = f(c)(u(b) - u(a)).$$

Proof. . The theorem is trivial if u is a constant function. Therefore we assume $u(b) > u(a)$. Since $f \in C(a,b)$, we can let

$$M = \sup\{f(x) : a \le x \le b\}$$

and

$$m = \inf\{f(x) : a \le x \le b\}.$$

Then from $m \le f(x) \le M$ we obtain

$$m(u(b) - u(a)) \le \int_a^b f(x)\,du(x) \le M(u(b) - u(a)).$$

This implies

$$m \le \frac{1}{u(b) - u(a)} \int_a^b f(x)\,du(x) \le M.$$

Therefore, by intermediate value theorem, there is $c \in [a,b]$ such that

$$f(c) = \frac{1}{u(b) - u(a)} \int_a^b f(x)\,du(x).$$

This proves the theorem. ■

Theorem 4.2.9 *If f is an increasing function on $[a,b]$ and $u \in C(a,b)$, then there is*

$c \in [a,b]$ *with*

$$\int_a^b f(x)\,du(x) = f(a)(u(c)-u(a)) + f(b)(u(b)-u(c)).$$

Proof. By Theorem 4.2.6,

$$\int_a^b f(x)\,du(x) = f(b)u(b) - f(a)u(a) - \int_a^b u(x)\,df(x).$$

By Theorem 4.2.8, there exists of $c \in [a,b]$ such that

$$\int_a^b u(x)\,df(x) = u(c)(f(b)-f(a)).$$

Combining, we obtaion

$$\begin{aligned}\int_a^b f(x)\,du(x) &= f(b)u(b) - f(a)u(a) - u(c)(f(b)-f(a)) \\ &= f(a)(u(c)-u(a)) + f(b)(u(b)-u(c)).\end{aligned}$$

This proves the theorem. ■

4.3 Existence

Assume that u is an increasing function on $[a,b]$ and $f \in B(a,b)$. Consider a partition $P = \{x_0,\ldots,x_n\}$ and let

$$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\},\ i = 1.\ldots,n,$$

and

$$m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\},\ i = 1\ldots,n.$$

Define the upper and lower Darboux sums by

$$S^*(f,u,P) = \sum_{i=1}^{n} M_i(u(x_i) - u(x_{i-1}))$$

and

$$S_*(f,u,P) = \sum_{i=1}^{n} m_i(u(x_i) - u(x_{i-1})).$$

Let

$$S^*(f,u) = \inf_P S^*(f,u,P)$$

and

$$S_*(f,u) = \sup_P S_*(f,u,P).$$

Here infimum and supremum are over all partitions P of $[a,b]$. Theorems similar to Theorems 2.2.3 and 2.2.4 can be proved for Riemann–Stieltjes integral as well.

Theorem 4.3.1 (Darboux) *Assume that u is an increasing function on $[a,b]$ and $f \in B(a,b)$. Then $(f,u) \in RS(a,b)$ and its Riemann-Stieltjes integral equals to S if and only if $S^*(f) = S_*(f) = S$.*

Theorem 4.3.2 (Riemann) *Let u be an increasing function on $[a,b]$ and $f \in B(a,b)$.*

Then $(f,u) \in RS(a,b)$ *if and only if for every* $\varepsilon > 0$ *there exists a partition* P_ε *of* $[a,b]$ *such that* $S^*(f,P_\varepsilon) - S_*(f,P_\varepsilon) < \varepsilon$.

The proof of these theorems are similar to the proofs of Theorems 2.2.3 and 2.2.4. An analog of Theorems 2.2.5 can also be proved for Riemann–Stieltjes integral. For this we need in the following.

Definition 4.3.3 *A function* $u : [a,b] \to \mathbb{R}$ *is said to have a bounded variation if it can be shown as a difference of two increasing functions on* $[a,b]$. *The collection of all functions of bounded variation on* $[a,b]$ *is denoted by* $BV(a,b)$.

Theorem 4.3.4 $C(a,b) \times BV(a,b) \subseteq RS(a,b)$ *and* $BV(a,b) \times C(a,b) \subseteq RS(a,b)$.

Proof. By Theorem 4.2.6, it suffices to prove only $C(a,b) \times BV(a,b) \subseteq RS(a,b)$ and by Definition 4.3.3 and Theorems 4.2.4 and 4.2.5 it suffices to prove that if $f \in C(a,b)$ and u is increasing, then $(f,u) \in RS(a,b)$. The proof in this case is similar to the proof of Theotrem 2.2.5. ■

Remark 4.3.5 *While everything in Riemann–Stieltjes integration is going parallel to Riemann integration, there are issues in Riemann–Stieltjes integration which do not arise in Riemann integration. One of them is the following. The points of discontinuity of* f *and* u *must be consistent in order the Riemann–Stieltjes integral*

$$\int_a^b f(x)\,du(x)$$

to be existent. More specifically, if u *is an increasing function on* $[a,b]$ *and* $f \in B(a,b)$ *so that* f *and* u *have a right discontinuity at the same number* $c \in [a,b)$, *that is,* $f(c) \neq$

$f(c+)$ and $u(c) \neq u(c+)$, then the Riemann–Stieltyes integral of f with respect to u does not exist. The same happens if f and u have a left discontinuity at the same number $c \in (a,b]$, that is, $f(c) \neq f(c-)$ and $u(c) \neq u(c-)$. This problem does not arise in Riemann integration since in this case $u(x) = t$ is a continuous function.

4.4 Riesz Representation

One of important applications of Riemann–Stieltyes integration is a representation of continuous linear functionals in the space $C(a,b)$. More specifically, we give the following.

Definition 4.4.1 *A function F from a Banach space E to $\mathbb{R}$ is said to be additive functional if*

$$F(x+y) = F(x) + F(y) \ \textit{ for every } \ x,y \in E,$$

homogenous functional if

$$F(ax) = aF(x) \ \textit{ for every } \ x \in E \ \textit{ and } \ a \in \mathbb{R},$$

and a linear functional if it is additive and homogenous.

A linear functional may be continuous or not. A simple necessary and sufficient condition for continuity of the linear functional $F : E \to \mathbb{R}$ is the existence of $c > 0$ such that

$$|F(x)| \leq c\|x\| \ \text{ for all } \ x \in E.$$

To prove that a given functional is linear and continuous it is sufficient to prove the above mentioned inequality and the additivity because the homogeneity is a consequence from them.

Example 4.4.2 *Fix* $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$*. The function*

$$F(x) = \sum_{i=1}^{k} x_i y_i, \; x = (x_1, \ldots, x_k) \in \mathbb{R}^k, \tag{4.4.1}$$

is a continuous linear functional on $\mathbb{R}^k$*. The linearity can be verified easily. The continuity follows from the Cauchy–Schwarz inequality*

$$\left|\sum_{i=1}^{k} x_i y_i,\right| \leq \sqrt{\sum_{i=1}^{k} x_i^2} \sqrt{\sum_{i=1}^{k} y_i^2},$$

where

$$||x|| = ||(x_1, \ldots, x_k)|| = \sqrt{\sum_{i=1}^{k} x_i^2}$$

is the Euclidean norm in $\mathbb{R}^k$*.*

It turns out that every linear continuous functional on $\mathbb{R}^k$ *can be described in the form (4.4.1) for some* $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$*. For this, let* G *be any linear functional on* $\mathbb{R}^k$*. Denote*

$$e_1 = (1, 0, \ldots, 0), \; e_2 = (0, 1, \ldots, 0), \; \ldots, \; e_k = (0, 0, \ldots, 1).$$

Define

$$y_1 = G(e_1), y_2 = G(e_2), \ldots, y_k = G(e_k).$$

Then for every $x = (x_1, \dots, x_k) \in \mathbb{R}^k$,

$$G(x) = G\Big(\sum_{i=1}^{k} x_i e_i\Big) = \sum_{i=1}^{k} x_i G(e_i) = \sum_{i=1}^{k} x_i y_i.$$

This proves the representation (4.4.1) for G. Moreover, this proves that every linear functional on $\mathbb{R}^k$ *is continuous.*

Following this example remind that the Riemann–Stieltjes integral

$$\int_a^b f(x)\,du(x)$$

is linear functional in $f \in C(a,b)$ for fixed $u \in BV(a,b)$, and in $u \in BV(a,b)$ for fixed $f \in C(a,b)$. Note that $C(a,b)$ is a Banach space with the norm

$$\|f\|_C = \max\{f(x) : a \le x \le b\}.$$

Also, for $u \in BV(a,b)$ we can define its variation on $[a,b]$ by

$$V(f;a,b) = \sup_P \sum_{i=1}^{n} (u(x_i) - u(x_{i-1}),$$

where supremum is taken over all partitions $P = \{x_0, \dots, x_n\}$ of $[a,b]$. Then $BV(a,v)$ is a Banach space with the norm

$$\|u\|_{BV} = |u(a)| + V(f;a,b).$$

Lemma 4.4.3 *For every* $(f,u) \in C(a,b) \times BV(a,b)$, *the following inequality holds:*

$$\Big|\int_a^b f(x)\,du(x)\Big| \le \|f\|_C \|u\|_{BV}. \tag{4.4.2}$$

Proof. For the partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$, we have

$$|S(f,u,P)| \leq \|f\|_C \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})| \leq \|f\|_C V(u;a,b).$$

Therefore this inequality holds for the limit as well, producing (4.4.2). ∎

Example 4.4.4 *Fix $u \in BV(a,b)$. Then the function*

$$F(f) = \int_a^b f(x)\,du(x), \;\; f \in C(a,b), \tag{4.4.3}$$

is a continuous linear functional on $C(a,b)$. The linearity was mentioned previously. The continuity follows from the inequality (4.4.2).

Example 4.4.5 *Fix $f \in C(a,b)$. Then the function*

$$F(u) = \int_a^b f(x)\,du(x), \;\; u \in BV(a,b), \tag{4.4.4}$$

is a continuous linear functional on $BV(a,b)$. The linearity was mentioned previously. The continuity follows from the inequality (4.4.2).

The following theorem stating the form of linear continuous functionals in $C(a,b)$ due to Riesz is spectacular.

Theorem 4.4.6 (Riesz) *Every continuous linear functional F on the Banach space $C(a,b)$ has a representation in the form of Riemann–Stieltjes integral (4.4.3) for some $u \in BV(a,b)$.*

Remark 4.4.7 *It should be noticed that while every continuous linear functional on* $C(a,b)$ *can be represented in the form (4.4.3) as a Riemann–Stieltjes integral, the same does not hold about representation (4.4.4) on* $BV(a,b)$.

Chapter 5

Kurzweil–Henstock Integral

5.1 Definition

Riemann integral was defined in two steps for the proper and improper cases. Making a change in the definition of the Riemann integral, we can join these cases into one and, additionally, cover all the functions that can be integrable in general. To present this extension let us start from the easy case of bounded interval $[a,b]$ for $-\infty < a < b < \infty$ and consider the following condition for the proper Riemann integrability.

Theorem 5.1.1 *A function $f \in B(a,b)$ is integrable in the Riemann sense on $[a,b]$ and its Riemann integral equals to S if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every partition P of $[a,b]$ with $\|P\| < \delta$,*

$$S(f,P) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \tag{5.1.1}$$

holds independently on the tags.

Proof. We first prove the sufficiency part of the theorem. Take $\varepsilon > 0$ and let $\delta > 0$ be so that (5.1.1) holds for every partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ with $\|P\| < \delta$ independently on the tags. Denote by P_ε one of such partitions. Then $P \supseteq P_\varepsilon$ implies $\|P\| < \delta$. Therefore, (5.1.1) holds for every $P \supseteq P_\varepsilon$ independently on the tags. Then by Definition 2.1.1, $f \in R(a,b)$ and its Riemann integral equals to S.

Now consider the necessity part. Let $f \in R(a,b)$ and

$$S = \int_a^b f(x)\,dx.$$

Then $S^*(f) = S_*(f) = S$, reminding that $S^*(f)$ and $S_*(f)$ are the upper and lower Riemann integrals of f on $[a,b]$, respectively. Take arbitrary $\varepsilon > 0$ and select $\sigma > 0$ in the following way. Denote the change of f by

$$d = \sup_{[a,b]} f - \inf_{[a,b]} f.$$

Since

$$S = S^*(f) = \inf_P S^*(f,P),$$

we can find a partition

$$P_\varepsilon = \{x'_0, \ldots, x'_m\}$$

of $[a,b]$ such that

$$S^*(f,P_\varepsilon) < S + \frac{\varepsilon}{2}.$$

Let

$$\sigma = \frac{\varepsilon}{2m(d+1)}.$$

Now consider any partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ with $\|P\| < \sigma$. Let

$$Q = P \cup P_\varepsilon = \{x''_0, \ldots, x''_k\}.$$

Since $Q \supseteq P_\varepsilon$, we have

$$S^*(f,Q) \le S^*(f,P_\varepsilon) < S + \frac{\varepsilon}{2}.$$

Furthermore, denote

$$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$$

and

$$M''_j = \sup\{f(x) : x''_{j-1} \le x \le x''_j\}.$$

If we eliminate the equal terms in $S^*(f,P)$ and $S^*(f,Q)$, the difference

$$S^*(f,P) - S^*(f,Q) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) - \sum_{j=1}^{k} M''_j(x''_j - x''_{j-1})$$

equals to the sum of no more that $m-1$ terms and each term is smaller than σd. Hence, we have

$$S^*(f,P) - S^*(f,Q) < (m-1)\sigma d = \frac{(m-1)\varepsilon d}{2m(d+1)} < \frac{\varepsilon}{2}.$$

This implies

$$S^*(f,P) - S = S^*(f,P) - S^*(f,Q) + S^*(f,Q) < S + \varepsilon.$$

Similarly, we can find $\sigma' > 0$ such that for all partition P of $[a,b]$ satisfying $\|P\| < \sigma'$,

$$S - \varepsilon < S_*(f,P).$$

Letting $\delta = \min\{\sigma, \sigma'\}$, we arrive to

$$S - \varepsilon < S_*(f,P) \le S(f,P) \le S^*(f,P) < S + \varepsilon,$$

that is, (5.1.1) holds for all partition P of $[a,b]$ with $\|P\| < \delta$ independently on the tags. This completes the proof. ■

By Theorem 5.1.1, we can write

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} S(f,P). \tag{5.1.2}$$

But this limit is complicated since the Riemann sum $S(f,P)$ depends the tags as well. Therefore, under (5.1.2), we mean that that this limit is independent on the tags. More precisely, for all $\varepsilon > 0$, there is $\delta > 0$ such that for every partitions P with $\|P\| < \delta$ and for all possible tags, the inequality (5.1.1) holds.

In Kurzweil–Henstock integration δ is selected dependently on the tags. This allows for essential enlargement of the class $R(a,b)$. In definition of the Kurzweil–Henstock integral the concepts of gauge and tagged partition play a central role.

Definition 5.1.2 *Any function* $\delta : [a,b] \to (0,\infty)$ *is said to be a gauge on the interval* $[a,b]$. *A partition* $P = \{x_0, \ldots, x_n\}$ *is called a tagged partition if it employs one fixed* *choice of tags $c_1, \ldots, c_n$. The symbol*

$$\hat{P} = \{x_0, \ldots, x_n; c_1, \ldots, c_n\}$$

is used for the tagged partition $P = \{x_0, \ldots, x_n\}$ together with the fixed tags $c_1, \ldots, c_n$. The tagged partition $\hat{\mathcal{P}} = \{x_0, \ldots, x_n; c_1, \ldots, c_n\}$ is said to be δ-fine if $x_i - x_{i-1} \le \delta(c_i)$ for all $i = 1, \ldots, n$.

The first question toward Kurzweil–Henstock integral is whether a δ-fine tagged partition exits for a given gauge δ. The following positively answers to this question.

Theorem 5.1.3 *Given a gauge δ on $[a,b]$, there is a δ-fine tagged partition of $[a,b]$.*

Proof. Take any gauge δ on $[a,b]$. Define by A a set of all $x \in (a,b]$ such that a $\delta|_{[a,x]}$-fine tagged partition of $[a,x]$ exists. Then for

$$x_1 = \min\{b, a + \delta(a)\},$$

the tagged partition $\hat{P} = \{a, x_1; a\}$ on $[a, x_1]$ is $\delta|_{[a,x_1]}$-fine. This implies $x_1 \in A$, that is, $A \neq \emptyset$. Furthermore, b is clearly an upper bound of A. Therefore, $c = \sup A$ exists. We assert that $c = b$.

To prove this assertion assume the contrary, that is, $c < b$. Denote by

$$\hat{Q} = \{a, x_1, \ldots, x_n; c_1, \ldots, c_n\}$$

a $\delta|_{[a,x_n]}$-fine tagged partition of $[a,x_n]$ assuming that

$$x_n = \max\{x_1, c-\delta(c)/2\}.$$

The existence of such a tagged partition follows from $x_n \in A$. Then

$$\hat{R} = \{a, x_1, \dots, x_n, x_{n+1}; c_1, \dots, c_n, c\}$$

is a $\delta|_{[a,x_{n+1}]}$-fine tagged partition of $[a,x_{n+1}]$, assuming that

$$x_{n+1} = \min\{b, c+\delta(c)/2\}.$$

Therefore, $x_{n+1} \in A$ and this contradicts to $c < x_{n+1}$. This proves that $c = b$. In a similar manner it can be proved that $c \in A$. Thus, there exists a δ-fine tagged partition of $[a,b]$.

■

Based on this theorem the Kurzweil–Henstock integral is defined in the following way.

Definition 5.1.4 *A bounded or unbounded function* $f : [a,b] \to \mathbb{R}$ *is said to be integrable in the Kurzweil–Henstock sense and its Kurzweil–Henstock integral is equal to* S *if for every* $\varepsilon > 0$ *there exists a gauge* $\delta : [a,b] \to (0,\infty)$ *such that for all* δ*-fine tagged partition*$\hat{P}$ *of* $[a,b]$*,*

$$\left| S(f,\hat{P}) - S \right| < \varepsilon.$$

The set of all Kurzweil–Henstock integrable functions on $[a,b]$ *is denoted by* $KH(a,b)$*.*

Similar to the case of Riemann integral, it can be proved that if the Kurzweil–Henstock integral of f on $[a,b]$ exists, then it is unique. To prove, assume the contrary. Then there are numbers S_1 and S_2 with $S_1 \neq S_2$, satisfying. Let

$$\varepsilon = \frac{|S_1 - S_2|}{2}.$$

Then there are two gauges δ_1 and δ_2 with

$$\left|S(f,\hat{P}) - S_1\right| < \frac{\varepsilon}{2}$$

whenever $\hat{P}$ is δ_1-fine, and

$$\left|S(f,\hat{P}) - S_2\right| < \frac{\varepsilon}{2}$$

whenever $\hat{P}$ is δ_2-fine. Denote

$$\delta(x) = \min\{\delta_1(x), \delta_2(x)\}.$$

Take arbitrary δ-fine tagged partition $\hat{P}$. Obviously, $\hat{P}$ is δ_1- and δ_2-fine. This implies the following contradiction:

$$\varepsilon = \frac{|S_1 - S_2|}{2} \leq \frac{\left|S(f,\hat{P}) - S_1\right| + \left|S(f,\hat{P}) - S_2\right|}{2} < \frac{\varepsilon + \varepsilon}{2} = \varepsilon.$$

Assuming that the gauge δ is a constant function, let $\|P\| < \delta$. Therefore, all tagged partitions $\hat{P}$, constricted over P is δ-fine. Hence, by Theorem 5.1.1, all properly Riemann integrable functions are Kurzweil–Henstock integrable. In other words, $R(a,b) \subseteq KH(a,b)$, and the Riemann and Kurzweil–Henstock integrals of them are equal. Therefore, there is no ambiguity in using the same notation

$$\int_a^b f(x)\,dx$$

for Kurzweil–Henstock integral of f. Similar to Riemann integration, we also conventionally let

$$\int_a^a f(x)\,dx = 0 \quad \text{and} \quad \int_b^a f(x)\,dx = -\int_a^a f(x)\,dx.$$

The following examples give some ideas about wideness of $KH(a,b)$ in comparison to $R(a,b)$.

Example 5.1.5 *By Example 2.1.4, the Dirichlet's function $f:[a,b]\to\mathbb{R}$, defined by*

$$f(x) = \begin{cases} 1 & \textit{if } x \textit{ is rational,} \\ 0 & \textit{if } x \textit{ is irrational.} \end{cases}$$

is not integrable in the Riemann sense. But it is integrable in the Kurzweil–Henstock sense and its Kurzweil–Henstock integral equals to 0. To porove take any $\varepsilon > 0$. Denote by $\mathbb{Q}$ the system of rational numbers. Since $[a,b]\cap\mathbb{Q}$ is a countably infinite set, we can write

$$[a,b]\cap\mathbb{Q} = \{a_1, a_2, \ldots\}.$$

Define the gauge

$$\delta(x) = \begin{cases} 1 & \textit{if } x \in [a,b]\setminus\mathbb{Q}, \\ \varepsilon/2^{k+1} & \textit{if } x = a_k. \end{cases}$$

Let

$$\hat{P} = \{x_0, \ldots, x_n; c_1, \ldots, c_n\}$$

be a δ-fine tagged partition of $[a,b]$*. Then*

$$\begin{aligned}\left|\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})\right| &\leq \sum_{c_i \in [a,b]\cap\mathbb{Q}} (x_i - x_{i-1}) \\ &\leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

This means that $f \in KH(a,b)$ while $f \notin R(a,b)$. Thus $R(a,b)$ is a proper subset of $KH(a,b)$.

Example 5.1.6 *Define the function f by*

$$f(x) = \begin{cases} 1/\sqrt{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

From

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0+} (2 - 2\sqrt{a}) = 2,$$

this function is improperly Riemann integrable on $(0,1]$ *and its improper integral on* $(0,1]$ *equals to 2.*

Let us prove that $f \in KH(a,b)$ and its Kurzweil–Henstock integral on $[0,1]$ *equals to*

2. For this, take any $\varepsilon > 0$ and define the gauge

$$\delta(x) = \begin{cases} \varepsilon^2/64 & \text{if } x = 0, \\ \min\{x/2, \varepsilon x\sqrt{x}/8\} & \text{if } 0 < x \le 1. \end{cases}$$

Take a δ-fine tagged partition $\hat{P} = \{x_0, \ldots, x_n; c_1, \ldots, c_n\}$ on $[0,1]$. If $c_i \neq 0$, then we have that

$$|x - c_i| \le \delta(c_i)$$

implies

$$x \ge c_i - \delta(c_i) \ge \frac{c_i}{2}.$$

Therefore,

$$\left|\frac{1}{\sqrt{c_i}} - \frac{1}{\sqrt{x}}\right| = \frac{|x - c_i|}{\sqrt{xc_i}(\sqrt{x} + \sqrt{c_i})} \le \frac{\delta(c_i)}{x\sqrt{c_i}} \le \frac{2\delta(c_i)}{c_i\sqrt{c_i}} \le \frac{\varepsilon}{4}.$$

Then $x_i - x_{i-1} \le \delta(c_i)$ implies

$$\frac{1}{\sqrt{c_i}} - \frac{\varepsilon}{4} \le \frac{1}{\sqrt{x_i}} \le \frac{2}{\sqrt{x_i} + \sqrt{x_{i-1}}} \le \frac{1}{\sqrt{x_{i-1}}} \le \frac{1}{\sqrt{c_i}} + \frac{\varepsilon}{4}.$$

This implies

$$|f(c_i)(x_i - x_{i-1}) - 2(\sqrt{x_i} - \sqrt{x_{i-1}})| \le \frac{(x_i - x_{i-1})\varepsilon}{4} < \frac{(x_i - x_{i-1})\varepsilon}{2}$$

for $c_i \neq 0$. Additionally, for $c_1 = 0$, we have $x_1 \le \delta(0) = \varepsilon^2/64$, producing

$$|f(c_1)(x_1 - x_0) - 2(\sqrt{x_1} - \sqrt{x_0})| = 2\sqrt{x_1} \le \frac{\varepsilon}{4} < \frac{\varepsilon}{2}.$$

The last two inequalities yield

$$\left|\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) - 2\right| \le \sum_{i=1}^{n} |f(c_i)(x_i - x_{i-1}) - 2(\sqrt{x_i} - \sqrt{x_{i-1}})| < \varepsilon.$$

This means that $f \in KH(0,1)$*, that is, the Kurzweil–Henstock integral of the improperly Riemann integrable function* f *exists and equals to 2.*

All 2nd kind improperly Riemann integrable functions are Kurzweil–Henstock integrable. This is a consequence of the following theorem .

Theorem 5.1.7 (Hake) *The following statements hold:*

(a) $f \in KH(a,b)$ *if and only if* $f|_{[c,b]} \in KH(c,b)$ *for every* $a < c < b$*, and*

$$\lim_{c \to a+} \int_c^b f(x)\,dx$$

exists.

(b) $f \in KH(a,b)$ *if and only if* $f|_{[a,c]} \in KH(a,c)$ *for every* $a < c < b$*, and*

$$\lim_{c \to b-} \int_c^b f(x)\,dx$$

exists.

In both these cases

$$\lim_{c \to a+} \int_c^b f(x)\,dx = \lim_{c \to b-} \int_a^c f(x)\,dx = \int_a^b f(x)\,dx.$$

By this theorem, the limits, producing the 2nd kind improper Riemann integrals, do not provide further extension of $KH(a,b)$. This means that all 2nd kind improperly Riemann integrable functions on the intervals $(a,b]$ and $[a,b)$ belong to $KH(a,b)$. Just these functions should be considered on $[a,b]$ and for this an arbitrary value should be assigned at a or b to these functions.

By operation with the gauges on the extended real line $\overline{\mathbb{R}} = [-\infty,\infty]$, Kurzweil–Henstock integral can be allowed to the 1st kind improperly Riemann integrable functions as well.

Now a function $\delta : \overline{\mathbb{R}} \to (0,\infty)$ will be called a gauge. Instead of tagged partition, now we will consider tagged subpartition of $\mathbb{R}$, that is $\hat{P} = \{x_0,\ldots,x_n;c_1,\ldots,c_n\}$ with $-\infty < x_0 < \cdots < x_n < \infty$ and $c_i \in [x_{i-1},x_i]$ for $i = 1,\ldots,n$. A tagged subpartition $\hat{P} = \{x_0,\ldots,x_n;c_1,\ldots,c_n\}$ of $\mathbb{R}$ is δ-fine if $x_i - x_{i-1} \le \delta(c_i)$ for every $i = 1,\ldots,n$ and

$$x_0 \le -\frac{1}{\delta(-\infty)} \text{ and } \frac{1}{\delta(\infty)} \le x_n.$$

Now assume that I equals to one of the intervals $[a,b]$, $[a,\infty)$, $(-\infty,b]$ and $(-\infty,\infty)$. Let $f : I \to \mathbb{R}$ be given. Extend f to $\mathbb{R}$ by making it vanish outside of I. Then f is said Kurzweil–Henstock integrable on I and its Kurzweil–Henstock integral equals to S if for every $\varepsilon > 0$ there is a gauge δ on $\overline{\mathbb{R}}$ such that for all δ-fine tagged subpartition $\hat{P} = \{x_0,\ldots,x_n;c_1,\ldots,c_n\}$ of $\overline{\mathbb{R}}$,

$$\left|\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) - S\right| < \varepsilon$$

By modification of the Hake's theorem to infinite intervals, it is seen that this definition covers all properly as well as 1st and 2nd kind improperly Riemann integrable functions. This general Kurzweil–Henstock integral on I is still denoted by

$$\int_I f(x)\,dx.$$

5.2 Properties

The following three properties of Kurzweil–Henstock integral are similar to Riemann integral.

Theorem 5.2.1 *Let I be a closed subinterval of $\mathbb{R}$. If $f \in KH(I)$ and $c \in \mathbb{R}$, then $cf \in KH(I)$ and*

$$\int_I cf(x)\,dx = c\int_I f(x)\,dx.$$

Theorem 5.2.2 *Let I be a closed subinterval of $\mathbb{R}$. If $f, g \in KH(I)$, then $(f+g) \in KH(I)$ and*

$$\int_I (f(x)+g(x))\,dx = \int_I f(x)\,dx + \int_I g(x)\,dx.$$

Theorem 5.2.3 *Let I be a closed subinterval of $\mathbb{R}$ and let c be an interior point of I. Denote $I_1 = I \cap (-\infty, c]$ and $I_2 = I \cap [c, \infty)$. Then $f \in KH(I)$ if and only if $f|_{I_1} \in KH(I_1)$ and $f|_{I_2} \in KH(I_2)$. Moreover,*

$$\int_I f(x)\,dx = \int_{I_1} f(x)\,dx + \int_{I_2} f(x)\,dx.$$

For the fundamental theorem of calculus for the Kurzweil–Henstock integral, we will consider the case of finite interval $[a,b]$. It is said that the property $\mathcal{P}(x)$, which depends on $x \in [a,b]$, holds almost everywhere (shortly a.e.) on $[a,b]$ if $\mathcal{P}(x)$ holds for all $x \in [a,b] \setminus E$, where $E \subseteq [a,b]$ is a set of measure zero. In particular, f is a.e. differentiable on $[a,b]$ if it is differentiable at every $x \in [a,b]$ except the set of points of measure zero.

Theorem 5.2.4 (First fundamental theorem of calculus) *If $f \in (C(a,b)$ is differentiable on $[a,b]$ except a countable set of points, then $f' \in KH(a,b)$ and*

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Theorem 5.2.5 (Second fundamental theorem of calculus) *Let $f \in KH(a,b)$ and*

$$F(x) = \int_a^x f(x)\,dx.$$

Then $F \in C(a,b)$, F is a.e. differentiable on $[a,b]$ and $F'(x) = f(x)$ at every point x of continuity of f.

5.3 Lebesgue Integral

Lebesgue integral can be obtained as a particular Kurzweil–Henstock integral. For this, we first consider the following example.

Example 5.3.1 *Consider the function f defined on $[0,1]$ by*

$$f(x)=\begin{cases} 2 & \textit{if } 1/2<x\le 1,\\ -3 & \textit{if } 1/3<x\le 1/2,\\ 4 & \textit{if } 1/4<x\le 1/3,\\ -5 & \textit{if } 1/5<x\le 1/4,\\ \cdots & \cdots \quad \cdots \quad \cdots\\ 0 & \textit{if } x=0.\end{cases}$$

We have

$$\int_{1/n}^{1}=\sum_{i=1}^{n-1}(-1)^{i+1}(i+1)\Big(\frac{1}{i}-\frac{1}{i+1}\Big)=\sum_{i=1}^{n-1}\frac{(-1)^{i+1}}{i}$$

This is a partial sum of alternating harmonic series and it is convergent. Therefore, $f\in KH(0,1)$ and

$$\int_0^1 f(x)\,dx=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}.$$

On the other hand,

$$|f(x)|=\begin{cases} 2 & \textit{if } 1/2<x\le 1,\\ 3 & \textit{if } 1/3<x\le 1/2,\\ 4 & \textit{if } 1/4<x\le 1/3,\\ 5 & \textit{if } 1/5<x\le 1/4,\\ \cdots & \cdots \quad \cdots \quad \cdots\\ 0 & \textit{if } x=0,\end{cases}$$

and, therefore, similarly to above calculations we have

$$\int_0^1 f(x)\,dx = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Thus unlike Riemann integral, we obtain that $f \in KH(0,1)$ *while* $|f| \notin KH(0,1)$. *This example suggests the following definition.*

Definition 5.3.2 *If* $f \in KH(a,b)$ *and also* $|f| \in KH(a,b)$, *then* f *is said to be Lebesgue integrable and the Kurzweil–Henstock integral of* f *is also called the its Lebesgue integral. The collection of all Lebesgue integrable function on* $[a,b]$ *is denoted by* $L(a,b)$.

The relations between the sets of functions integrable in the Riemann, Lebesgue and Kurzweil–Henstock senses can be given by

$$R(a,b) \subset L(a,b) \subset KH(a,b),$$

noticing that $R(a,b)$ is a proper subset of $L(a,b)$ and $L(a,b)$ is also a proper subset of $KH(a,b)$. $L(a,b)$ is an important class of functions between $R(a,b)$ and $KH(a,b)$. It is possible to define a powerful norm in $L(a,b)$, making it a Banach space, while there is no an efficient norm in $R(a,b)$ and $KH(a,b)$. The Lebesgue integration is indeed another kind of developments in integration leading to such important topics in mathematics as measure theory , probability theory etc. These subjects are out of the scope of this thesis.

REFERENCES

[1] A. E. Bashirov, *Mathematical Analysis Fundamentals*, Elsevier, (2014).

[2] S. A. Douglass, *Introduction to Mathematical Analysis*, Addision-Wesley, New York, (1996).

[3] S. A. Lay,*Analysis with an introduction to Proof*, Pearsion, (2005).

[4] W. A. J. Kosmala, *A Friendly Introduction to Analysis,Single and Multivariable*, Pearsion, (2005).

[5] R. G. Bartle and D.R. Sherbert, *Introduction to Real Analysis*, John Wiley and Sons, (2000).

[6] M. Rosenlicht, *Introduction to Analysis*, Dover, (1968).